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RATES FOR THE CLT VIA NEW IDEAL METRICS

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Let $(B, \|\ \|)$ be a separable Banach space and $\mathscr{X} := \mathscr{X}(B)$ the vector space of all random variables defined on a probability space $(\Omega, \mathscr{A}, P)$ and taking values in B. It is shown that new ideal metrics for $\mathscr{X}$ may be used to obtain refined rates of convergence of normalized sums to a stable limit law. The rates hold uniformly in n and are expressed in terms of a variety of uniform metrics on $\mathscr{X}$. In the Banach space setting the rates hold with respect to the total variation metric and in the Euclidean space setting the rates hold with respect to uniform metrics between density and characteristic functions. The main result provides a sharp order estimate of the rate of convergence in local limit theorems with respect to the uniform distance between densities. The method is based on the theory of probability metrics, especially those of convolution type.

0. Introduction. Let $(B, \|\ \|)$ be a separable Banach space, $\mathscr{B}$ the usual Borel sets and $\mathscr{X} := \mathscr{X}(B)$ the vector space of all random variables defined on a probability space $(\Omega, \mathscr{A}, P)$ and taking values in B. New ideal metrics for the space $\mathscr{X}$ are introduced and it is shown that they provide refined rates of convergence of normalized sums to a stable limit law. The rates hold uniformly in n and are expressed in terms of a variety of uniform metrics on $\mathscr{X}$. In the Banach space setting the rates hold with respect to the total variation metric, improving and extending upon existing results. In the classical Euclidean space setting the approach allows a determination of convergence rates with respect to uniform metrics between density and characteristic functions. The main result, Theorem 3.3, provides a sharp order estimate of the rate of convergence in local limit theorems with respect to the uniform distance between densities. This is done by extending the classic method of Bergström, which provides the rate of convergence in the CLT with respect to the uniform metric [cf. Bergström (1945)].

The method is based on the theory of probability metrics, especially those of convolution type; for earlier work on convolution metrics see Rachev and Ignatov (1984) and Yukich (1985). Here it is shown that convolution-type metrics are particularly well suited for the rate-of-convergence problem. As indicated, the metrics are ideal, which by itself is of interest since the only ideal metric known thus far is the ζ_r-metric discovered by Zolotarev (1976, 1977, 1979). The properties of this metric have been analyzed in Ignatov and Rachev (1983). Ideal

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metrics of the convolution type turn out to be extremely versatile and provide rates of convergence with respect to a variety of metrics. In fact, each ideal convolution metric on $\mathscr{X}$ can be used to provide rates of convergence in the general CLT in terms of some uniform metric corresponding to the proposed ideal metric. This remark constitutes one of the central themes in this article. That the method works is essentially due to the fact that ideal convolution metrics satisfy crucial smoothing inequalities not available for ζ_r (see especially Lemmas 2.6 and 2.7). Moreover, the convolution metrics are weaker with respect to all known pseudomoments appearing in the right-hand side of the Berry–Esseen-type estimates [see Senatov (1980) and Maejima and Rachev (1987)]. The construction of the new ideal metrics yields sharp estimates of the rate of convergence in the local limit theorem (see Theorem 3.3).

1. Metrics and their properties. Let $\mathscr{L}(\mathscr{X})$ be the space of all laws P_X, $X \in \mathscr{X}(B)$. The metrics μ on $\mathscr{L}(X)$ used in this article take the form

$$\mu(P_{X_1}, P_{X_2}) = \nu(P_{X_1} - P_{X_2}),$$

where ν is a subadditive $[0, \infty]$-valued functional on the space $\mathscr{M} := \mathscr{M}(B)$ of bounded signed measures on B. In the sequel put

$$\rho := P_{X_1} - P_{X_2},$$

$$\int f\,d\rho = \mathbb{E}(f(X_1) - f(X_2)),$$

$$F_\rho(x) := F_{X_1}(x) - F_{X_2}(x),$$

where $F_X(x) := P(X \le x)$, $X \in \mathscr{X}(\mathbb{R}^n)$, and if $n = 1$ let

$$F_{r,\rho}(x) := \int_{-\infty}^{x} \frac{(x-t)^{r-1}}{(r-1)!}\,dF_\rho(t), \qquad r = 1, 2, \dots.$$

A subadditive $[0, \infty]$-valued functional ν on $\mathscr{M}$ is called *ideal* of order r if the following two conditions are satisfied:

(i) Regularity: $\nu(\rho * \rho_0) \le \nu(\rho)|\rho_0|$ for all $\rho, \rho_0 \in \mathscr{M}$, where $|\rho_0|$ is the total variation of ρ_0.

(ii) Homogeneity of order $s \ge 0$:

$$\nu(\rho^{(c)}) = |c|^s \nu(\rho) \quad \text{for } c \ne 0,$$

where $\rho \in \mathscr{M}$ and $\rho^{(c)}(\cdot) = \rho(\cdot / c)$ [cf. Zolotarev (1976) and Maejima and Rachev (1987)].

Zolotarev (1976) showed the existence of an ideal metric of a given order $r \ge 0$ and defined the ideal metric

$$\zeta_r(\rho) := \sup\left\{\left|\int f\,d\rho\right| : |f^{(m)}(x) - f^{(m)}(y)| \le \|x - y\|^\beta\right\}, \tag{1.1}$$

where $m \in \mathbb{N}^+$ and $\beta \in (0,1]$ satisfy $m + \beta = r$; $f^{(m)}$ denotes the mth Fréchet derivative of f for $m \geq 0$ and $f^{(0)}(x) := f(x)$. He also obtained an upper bound for ζ_k, $k \in \mathbb{N}^+$, in terms of the so-called difference pseudomoment κ_k, where for $r > 0$,

$$\kappa_r(\rho) := \sup\left\{\left|\int f\,d\rho\right| : |f(x) - f(y)| \leq \|x\|x\|^{r-1} - y\|y\|^{r-1}\|\right\}.$$

If $B = \mathbb{R}$, then $\|x\| = |x|$ and

$$\kappa_r(\rho) = r\int_{-\infty}^{\infty} |x|^{r-1}|F_\rho(x)|\,dx, \qquad r > 0. \tag{1.2}$$

In this article new ideal metrics are introduced on the spaces $\mathscr{X}$ and $\mathscr{X}^*(\mathbb{R}^n)$, where $\mathscr{X}^*(\mathbb{R}^n)$ denotes those random variables with densities. It is shown that the metrics provide improved rates of convergence for CLTs in Banach spaces involving a stable limit law.

More precisely, let Y_α denote a strictly stable symmetrical random variable with parameter $\alpha \in (0,2]$, that is, $Y_\alpha =_{\mathscr{D}} -Y_\alpha$ and $Y_1' + \cdots + Y_n' =_{\mathscr{D}} n^{1/\alpha}Y_\alpha$, where Y_i', $1 \leq i \leq n$, are i.i.d. random variables with the same distribution as Y_α. Letting $X, X_1, X_2, \ldots$ denote i.i.d. random variables and $S_n := n^{-1/\alpha}(X_1 + \cdots + X_n)$, we use ideal metrics of convolution type to describe the rate of convergence $S_n \to_{\mathscr{D}} Y_\alpha$ with respect to the following uniform metrics on $\mathscr{M}$.

Total variation metric:

$$\begin{aligned}\operatorname{Var}(\rho) &:= \sup\left\{\left|\int f\,d\rho\right| : f\colon B \to \mathbb{R} \text{ is measurable}\right. \\ &\qquad\qquad \left.\text{and } \|f\|_\infty := \sup_{x\in B}|f(x)| \leq 1\right\} \\ &= 2\sup_{A\in\mathscr{B}}|\rho(A)|, \qquad \rho \in \mathscr{M}(B).\end{aligned}$$

In $\mathscr{M}(\mathbb{R}^n)$ we have $\operatorname{Var}(\rho) = \int|\,dF_\rho|$.

Uniform metric between characteristic functions: $\chi(\rho) := \sup_t|\varphi_\rho(t)|$, where φ_ρ denotes the characteristic function of ρ.

Uniform metric between densities [p_ρ denotes the density for $\rho \in \mathscr{M}(\mathbb{R}^n)$]:

$$l(\rho) := \operatorname*{ess\,sup}_x |p_\rho(x)|.$$

The metric χ is topologically weaker than Var, which is itself topologically weaker than l by Scheffé's theorem; see Billingsley (1968), page 224.

For any $f\colon B \to \mathbb{R}$, $\|f\|_\infty$ denotes the supremum, and when $B = \mathbb{R}^n$, $\|f\|_p$ denotes the L^p norm. Let $\|f\|_L := \sup\{|f(x) - f(y)|/\|x - y\|\colon x \neq y\}$ be the usual Lipschitz norm.

We use the following metrics on $\mathscr{M}$.

Kolmogorov metric:

$$K(\rho) := \sup_{x\in\mathbb{R}} |F_\rho(x)|, \qquad \rho \in \mathscr{M}(\mathbb{R}).$$

L^p version of ζ_m:

$$\zeta_{m,p}(\rho) := \sup\left\{\left|\int f\,d\rho\right| : \|f^{(m+1)}\|_q \le 1\right\}, \qquad \frac{1}{p} + \frac{1}{q} = 1,\ m \in \mathbb{N}^+.$$

[If $\zeta_{m,p}(\rho) < \infty$, then $\zeta_{m,p}(\rho) = \|F_{m+1,\rho}\|_p$; also $\zeta_{m,1}(\rho) = \zeta_m(\rho)$.]
Generalized Kantorovich–Wasserstein metric:

$$W_p^p(\rho) := \sup\left\{\left|\int f\,dF_{X_1} + \int g\,dF_{X_2}\right| : \|f\|_\infty + \|f\|_L < \infty,\right.$$

$$\left.\|g\|_\infty + \|g\|_L < \infty \text{ and } f(x) + g(y) \le \|x-y\|^p\ \forall\, x, y \in B\right\}, \qquad p \ge 1.$$

If Δ denotes a metric on $\mathscr{M}$, then assume throughout that $\Delta(X_1, X_2) := \Delta(P_{X_1}, P_{X_2})$ for all $X_1, X_2 \in \mathscr{X}$.

2. Ideal convolution metrics and their properties. Associated with Var, χ and l we define the following metrics of convolution type.

(i) Let $\theta \in \mathscr{X}(B)$, $r > 0$ and define [cf. Rachev and Ignatov (1984)]

$$\nu_{\theta,r}(\rho) := \sup_{h\in\mathbb{R}} |h|^r \operatorname{Var}(\rho * F_{h\theta}), \qquad \rho \in \mathscr{M}(B).$$

Thus each θ generates a metric $\nu_{\theta,r}$, $r > 0$.

(ii) Define for $r \in \mathbb{N}^+$,

$$\chi_r(\rho) := \sup_{t\in\mathbb{R}} |t|^{-r}|\varphi_\rho(t)|, \qquad \rho \in \mathscr{M}(\mathbb{R}).$$

χ_r has a convolution-type structure, since with a slight abuse of notation $\chi_r(\rho) = \chi(\rho * p_r)$, where $p_r(t) := t^r/r!I_{\{t>0\}}$ is the density of an unbounded positive measure on $[0,\infty)$; see Zolotarev (1981).

(iii) Let $\theta \in \mathscr{X}(\mathbb{R}^n)$, $r > 0$ and define

$$\mu_{\theta,r}(\rho) := \sup_{h\in\mathbb{R}} |h|^r l(\rho * F_{h\theta}), \qquad \rho \in \mathscr{M}(\mathbb{R}^n).$$

Lemma 2.1 below shows that $\nu_{\theta,r}$, χ_r and $\mu_{\theta,r}$ are ideal. In general, $\nu_{\theta,r}$ and $\mu_{\theta,r}$ are actually only pseudometrics, but this distinction is not important in what follows and so we omit it.

When θ is an α-stable random variable we will write $\nu_{\alpha,r}$ and $\mu_{\alpha,r}$ (or simply ν_r and μ_r when it is understood) in place of $\nu_{\theta,r}$ and $\mu_{\theta,r}$. Also, if θ has a density g, then $\mu_{\theta,r}$ represents a generalization of the convolution metric $d_g(\rho) := l(\rho * F_{h\theta})$ used in Yukich (1985).

LEMMA 2.1. (i) *For all* $\theta \in \mathscr{X}$ *and* $r > 0$, $\nu_{\theta, r}$ *is an ideal metric of order* r.
(ii) *For all* $r > 0$, χ_r *is an ideal metric of order* r.
(iii) *For all* $\theta \in \mathscr{X}$ *and* $r > 0$, $\mu_{\theta, r}$ *is an ideal metric of order* $r - 1$.

The proof of this lemma is straightforward. The following lemmas show that both $\nu_{\theta, r}$ and $\mu_{\theta, r}$ may be bounded above by the difference pseudomoments. Moreover, these lemmas show that $\nu_{\theta, r}$ and $\mu_{\theta, r}$ are weaker than the Zolotarev metrics ζ_r and $\zeta_{m, p}$, respectively. Thus rates in terms of the convolution metrics are superior to rates in terms of ζ_r and $\zeta_{m, p}$.

LEMMA 2.2. *Let* $r \in \mathbb{N}^+$, $\rho \in \mathscr{M}(\mathbb{R})$ *and suppose that* $\int x^j \, d\rho = 0$ *for* $j = 1, \ldots, r - 2$. *Then for every* $\theta \in \mathscr{X}(\mathbb{R})$ *with a density* g *which is* $r - 1$ *times differentiable*,

$$\mu_{\theta, r}(\rho) \le \frac{\|g^{(r-1)}\|_\infty}{(r-1)!} \kappa_{r-1}(\rho).$$

The next lemma is a refinement of Lemma 2.2. We defer the proof to the Appendix.

LEMMA 2.3. *For every* $\theta \in \mathscr{X}^*(\mathbb{R})$ *with a density* g *which is* m *times differentiable and for all* $X_1, X_2 \in \mathscr{X}(\mathbb{R})$,

$$\mu_{\theta, r}(\rho) \le C(m, p, g) \zeta_{m-1, p}(\rho),$$

where $r = m + 1/p$, $m \in \mathbb{N}^+$, $p \in [1, \infty)$ *and*

$$C(m, p, g) := \|g^{(m)}\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

The proof of the following lemma follows from straightforward modification of the techniques of Senatov (1980).

LEMMA 2.4. *Under the hypotheses of Lemma* 2.3

$$\nu_{\theta, r}(\rho) \le C(r, g) \zeta_r(\rho),$$

where $C(r, g)$ *is a finite constant*, $r \in \mathbb{N}^+$.

LEMMA 2.5 [cf. Theorem 2 of Maejima and Rachev (1987)]. *Let* $m \in \mathbb{N}^+$ *and suppose* $\int x^j \, d\rho = 0$ *for* $j = 0, 1, \ldots, m$. *Then for* $p \in [1, \infty)$,

$$\zeta_{m, p}(\rho) \le \begin{cases} \kappa_1^{1/p}(\rho), & m = 0, \\ \dfrac{\Gamma(1 + 1/p)}{\Gamma(1 + r)} \kappa_r(\rho), & r = m + \dfrac{1}{p}, \ m = 1, 2, \ldots. \end{cases}$$

Next we show that the ideal metrics satisfy crucial smoothing inequalities with respect to the corresponding uniform metrics (see especially Lemmas 2.6 and 2.7); such inequalities are not available for ζ_r. These smoothing inequalities will yield sharp order estimates for the rate of convergence in the CLT. Throughout, recall that Y_α (or Y) denotes a strictly stable symmetrical random variable with parameter $\alpha \in (0,2]$. If $Y_\alpha \in \mathscr{X}(\mathbb{R})$, then assume that Y_α has the characteristic function $\varphi_Y(t) = \exp\{-|t|^\alpha\}$, $t \in \mathbb{R}$.

LEMMA 2.6. (i) *For any* $\rho \in \mathscr{M}(B)$ *and* $\sigma > 0$,

$$\operatorname{Var}(\rho * P_{\sigma Y}) \le \sigma^{-r}\nu_r(\rho).$$

(ii) *For any* $\rho \in \mathscr{M}(\mathbb{R})$, $\sigma > 0$ *and* $r > \alpha$,

$$\chi(\rho * P_{\sigma Y}) \le C_r \sigma^{-r}\chi_r(\rho), \qquad C_r := \left(\frac{r}{\alpha e}\right)^{r/\alpha}.$$

(iii) *For any* $\rho \in \mathscr{M}(\mathbb{R}^n)$ *and* $\sigma > 0$,

$$l(\rho * P_{\sigma Y}) \le \sigma^{-r}\mu_r(\rho).$$

PROOF. (i) Since Y and $-Y$ have the same distribution

$$\operatorname{Var}(\rho * P_{hY}) \le h^{-r} \sup_{h>0} h^r \operatorname{Var}(\rho * P_{hy}) = h^{-r}\nu_r(X_1, X_2).$$

(ii) We have

$$\begin{aligned}\chi(\rho * P_{\sigma Y}) &= \sup_{t\in\mathbb{R}} \{|\varphi_\rho(t)|\}\exp\{-|\sigma t|^\alpha\}\\ &\le \sup_{t\in\mathbb{R}} |\sigma t|^{-r}|\varphi_\rho(t)| \sup_{u>0} u^r e^{-u^\alpha}\\ &= C_r\sigma^{-r}\chi_r(\rho),\end{aligned}$$

since $C_r = \sup_{u>0} u^r e^{-u^\alpha}$ by a simple computation.

(iii) $l(\rho * P_{\sigma Y}) = \sigma^{-r}\sigma^r l(\rho * P_{\sigma Y}) \le \sigma^{-r}\mu_r(X_1, X_2)$. □

Lemma 2.6(i) resembles Lemma 1 of Senatov (1980) for the metric ζ_r. Lemma 2.7(i), which follows, resembles Lemma 2 of Senatov (1980) proved for $B = \mathbb{R}^n$. Estimates of this sort have been used by Sazonov (1972) and Sazonov and Ul'yanov (1979).

LEMMA 2.7. (i) *For any* $\rho \in \mathscr{M}(B)$ *and* $U, V \in \mathscr{X}(B)$,

$$\operatorname{Var}(\rho * P_U) \le \operatorname{Var}(\rho)\operatorname{Var}(U, V) + \operatorname{Var}(\rho * P_V).$$

(ii) *For any* $\rho \in \mathscr{M}(\mathbb{R})$ *and* $U, V \in \mathscr{X}(\mathbb{R})$,

$$\chi(\rho * P_U) \le \chi(\rho)\chi(U, V) + \chi(\rho * P_V).$$

(iii) *For any* $X_1, X_2, U, V \in \mathscr{X}^*(\mathbb{R}^n)$,

$$l(\rho * P_U) \le l(\rho)\operatorname{Var}(U, V) + l(\rho * P_V).$$

PROOF. We only prove (i), since (iii) is similar and (ii) is trivial. To prove (i), use the triangle inequality to obtain

$$\begin{aligned}\operatorname{Var}(\rho * P_U) &= \sup\{|(\rho * P_U)(f)|: \|f\|_\infty \le 1\}\\ &\le \sup\{|\rho(\bar f)|: \|f\|_\infty \le 1\} + \operatorname{Var}(\rho * P_U),\end{aligned}$$

where $\bar f(x) := \int_B f(u + x)(P_U - P_V)\,du$. Since $\|f\|_\infty \le 1$ and

$$\begin{aligned}\|\bar f\|_\infty &= \sup_{x\in B}\left|\int f(u + x)(P_U - P_V)\,du\right|\\ &\le \left|\int_B f(u)(P_U - P_V)\,du\right| \le \operatorname{Var}(U, V),\end{aligned}$$

we see that $\sup\{|\rho(\bar f)|: \|f\|_\infty \le 1\}$ is bounded by

$$\begin{aligned}&\le \sup\{|\rho(g)|: \|g\|_\infty \le \operatorname{Var}(U, V)\}\\ &= \operatorname{Var}(\rho)\operatorname{Var}(U, V).\end{aligned}$$

□

We conclude the discussion of the properties of μ_r and ν_r by noting that they satisfy the same weak convergence properties as do the Kantorovich–Wasserstein distance W_r and the pseudomoments κ_r.

THEOREM 2.8. *Let $k \in \mathbb{N}^+$, $0 < \alpha \le 2$ and $X_n, U \in \mathscr{X}(\mathbb{R})$ with $EX_n^j = EU^j$, $j = 1, \dots, k - 2$. If k is odd, then the following are equivalent as $n \to \infty$:*

(i) $\mu_{\alpha,k}(X_n, U) \to 0$,
(ii) (a) $X_n \to_{\mathscr{D}} U$ *and* (b) $\mathbb{E}X_n^{k-1} \to \mathbb{E}U^{k-1}$,
(iii) $W_{k-1}(X_n, U) \to 0$,
(iv) $\kappa_{k-1}(X_n, U) \to 0$ *and*
(v) $\nu_{\alpha,k-1}(X_n, U) \to 0$.

The proof may be found in the Appendix.

3. Main results. By exploiting the homogeneity and special structure properties of ideal metrics, especially Lemmas 2.6 and 2.7, we show that each ideal metric on $\mathscr{X}$ provides rates of convergence in terms of the corresponding uniform metric. The first result (Theorem 3.1) shows that ν_r may be used to obtain a refined convergence rate in the CLT with respect to Var. The method for the ideal metric ν_r is simple and yet general enough to extend to other cases of interest. For example, χ_r describes the rate of convergence in the CLT with respect to the uniform metric χ (Theorem 3.2). More importantly, μ_r and ν_r when taken together, describe convergence rates in terms of the uniform metric l (Theorem 3.3).

In stating the results we adhere to the notation of Section 1.

THEOREM 3.1. *Let Y be an α-stable random variable in $\mathscr{X}(B)$. Let $r > \alpha$, $A := 2(2^{r/\alpha - 1} + 3^{r/\alpha})$ and $a := 2^{-r/\alpha}A$. If $X \in \mathscr{X}(B)$ satisfies*

$$\tau_0 := \tau_0(X, Y) := \max\{\mathrm{Var}(X, Y), \nu_{\alpha, r}(X, Y)\} \le a, \tag{3.1}$$

then for all $n \ge 1$,

$$\mathrm{Var}(S_n, Y) \le A\tau_0 n^{1-r/\alpha} \le 2^{-r/\alpha} n^{1-r/\alpha}. \tag{3.2}$$

REMARKS. (i) A result of this type was proved by Sazonov and Ul'yanov (1979) for the case $B = \mathbb{R}^n$ and $\alpha = 2$; similar results have also been obtained by Zolotarev (1976, 1977), Paulauskas (1973, 1976) and Senatov (1980).

(ii) Concerning hypothesis (3.1), note that the estimate (3.2) necessarily implies $\tau_0 < \infty$; if (3.2) holds for $\alpha = 2$, then $\kappa_r(X, Y) < \infty$ [using the inequality $2K \le \mathrm{Var}$ and Theorem 3.4.1 of Ibragimov and Linnik (1971)] and if $r = 3$, then $\nu_{\alpha, r}(X, Y) < \infty$ by Lemma 2.4. For general α, see also Hall (1981) for two-sided bounds for K.

The next result provides rates of convergence in terms of χ.

THEOREM 3.2. *Let Y be an α-stable random variable in $\mathscr{X}(R)$. Let $r > \alpha$, $C_r := (r/\alpha e)^{r/\alpha}$, $B := \max(3^{r/\alpha}, 2C_r(2^{r/\alpha - 1} + 3^{r/\alpha}))$ and $b := 1/2^{r/\alpha}B$. If $X \in \mathscr{X}(\mathbb{R})$ satisfies*

$$\tau_r := \tau_r(X, Y) := \max\{\chi(X, Y), \chi_r(X, Y)\} \le b, \tag{3.3}$$

then for all $n \ge 1$,

$$\chi(S_n, Y) \le B\tau_r n^{1-r/\alpha} \le 2^{-r/\alpha} n^{1-r/\alpha}. \tag{3.4}$$

REMARKS. (i) In comparing conditions (3.1) and (3.3) it is useful to note that the metric χ is topologically weaker than Var, that is, $\mathrm{Var}(X_n, Y) \to 0$ implies $\chi(X_n, Y) \to 0$, but not conversely. Also, it is easy to show that if $r = m + \beta$, $m \in \mathbb{N}^+$ and $\beta \in (0, 1)$, then

$$\chi_r \le C_\beta \zeta_r, \quad \text{where } C_\beta := \sup_t |t^{-\beta}(1 - e^{it})|; \tag{3.5}$$

if $r = m$, $m \in N^+$, then $\chi_r \le \zeta_r$.

(ii) Banys (1976) has obtained a result similar to Theorem 3.2 but weaker; he only considers the sup norm difference between characteristic functions over finite intervals depending on n. Additionally, his result is expressed in terms of the so-called rth absolute pseudomoment

$$\bar{\nu}_r(X_1, X_2) := \int |x|^r| \, dF_\rho(x)|.$$

Since χ_r is weaker than $\bar{\nu}_r$ (more precisely, there are constants C_1 and C_2 such that $\chi_r \le C_1\zeta_r \le C_2\bar{\nu}_r$), the estimate (3.4) is more refined.

(iii) The estimate (3.4) may be used to obtain strong approximation theorems for sums of random vectors, see, for example, Theorem 2 of Berkes, Dabrowski, Dehling and Phillip (1986).

The main result provides rates of convergence in terms of l. For $1 < \alpha < 2$ Basu (1976) obtained density convergence results but without rates; see also

Basu and Maejima (1980) for $0 < \alpha < 2$. For $\alpha = 2$ Theorem 4.5.1 of Ibragimov and Linnik (1971) provides rates of l-convergence, but only up to unspecified constant factors.

THEOREM 3.3. *Let Y be an α-stable random variable in $\mathscr{X}(\mathbb{R}^n)$. Let $r > \alpha$, $A := 2(2^{((r+1)/\alpha)-1} + 3^{(r+1)/\alpha})$, $a := 1/2^{r/\alpha}A$ and $D := 3^{1/\alpha}2^{r/\alpha}$. Let $X \in \mathscr{X}^*$ and $\tau := \tau(X, Y) := \max\{l(X, Y), \mu_{\alpha, r+1}(X, Y)\} < \infty$. If*

$$\tau_0 := \tau_0(X, Y) := \max\{\mathrm{Var}(X, Y), \nu_{\alpha, r}(X, Y)\} \le 1/AD, \tag{3.6}$$

then for all $n \ge 1$,

$$l(S_n, Y) \le A\tau n^{1-r/\alpha}. \tag{3.7}$$

If, in addition, $\tau \le a$, then (trivially)

$$l(S_n, Y) \le 2^{-r/\alpha}n^{1-r/\alpha}.$$

REMARKS. (i) Condition (3.6) describes the domain of attraction of a stable random variable Y; in fact it guarantees l-closeness of order $n^{1-r/\alpha}$ between Y and the normalized sums S_n.

(ii) From Lemmas 2.2, 2.3 and 2.5 we know that $\mu_{r+1}(X, Y)$, $r = m + 1/p$, $m \in \mathbb{N}^+$, can be approximated from above by κ_r whenever X and Y share the same first m moments; also, if r is integral, then the same is true for $\nu_r(X, Y)$ by Lemma 2.4. Thus condition (3.6) could be expressed in terms of difference pseudomoments, which amounts to conditions on the tails of X.

(iii) As explained above, if (3.7) holds for $\alpha = 2$, $2 < r \le 3$, then $\kappa_r(X, Y) < \infty$ and $\mu_{\alpha, r+1}(X, Y) < \infty$ by Lemmas 2.2 and 2.3; additionally, if $r = 3$, then $\nu_{\alpha, r}(X, Y) < \infty$ by Lemma 2.4.

(iv) The metrics μ_r and ν_r may also be used to obtain rates of convergence in the local limit theorem on the group of random motions on $\mathbb{R}^n$; see Rachev and Yukich (1988) for details.

4. Proofs of main results.

We provide the proofs of Theorems 3.1, 3.2 and 3.3. Actually, from the method of proof for Theorem 3.3 it will become clear how to prove Theorems 3.1 and 3.2 and so we only provide the details for Theorem 3.3. The proof of Theorem 3.3 is achieved by combining Theorem 3.1 with the smoothing Lemma 2.7(iii). Throughout, $Y_1, Y_2, \dots$ denote i.i.d. copies of Y and $V_n := n^{-1/\alpha}(Y_1 + \cdots + Y_n)$. For $1 \le m \le n$, let $\underline{V}_m := \underline{V}_{n,m} := n^{-1/\alpha}(Y_1 + \cdots + Y_m)$ and $\overline{V}_m := \overline{V}_{n,m} := n^{-1/\alpha}(Y_{m+1} + \cdots + Y_n)$. Define $\underline{S}_m$ and $\overline{S}_m$ similarly in terms of X.

PROOF OF THEOREM 3.3. We proceed by induction; for $n = 1$ the assertion of the theorem is trivial since $l(X, Y) \le Al(X, Y) \le A\tau(X, Y)$. For $n = 2$ the assertion follows from the (-1) ideality of l,

$$\begin{aligned} l(S_2, Y) &= 2^{1/\alpha}l(X_1 + X_2, Y_1 + Y_2) \\ &\le 2^{(1+r)/\alpha}\tau 2^{1-r/\alpha} \le A\tau 2^{1-r/\alpha}. \end{aligned}$$

A similar calculation holds for $n = 3$.

Proceeding by induction, assume that

$$l(S_j, Y) \le A\tau j^{1-r/\alpha} \tag{4.1}$$

holds for all $j < n$.

By Lemma 2.7(iii) for any $n \ge 4$ and $m = [n/2]$, where [ ] denotes integer part,

$$l := l(S_n, Y_n) = l(S_n, V_n) \le I_1 + I_2 + I_3, \tag{4.2}$$

where

$$I_1 := l(\underline{S}_m, \underline{V}_m)\operatorname{Var}(\bar{S}_m, \bar{V}_m),$$
$$I_2 := l(\underline{S}_m + \bar{V}_m, \underline{V}_m + \bar{V}_m),$$
$$I_3 := l(\underline{V}_m + \bar{S}_m, \underline{V}_m + \bar{V}_m).$$

We first estimate I_2 and I_3. Using the relation $V_n =_{\mathscr{D}} Y_1$, Lemma 2.6(iii) and the ideality of μ_{r+1} we obtain

$$\begin{aligned} I_2 &= l\left(\underline{S}_m + \left(\frac{n-m}{n}\right)^{1/\alpha} Y, \underline{V}_m + \left(\frac{n-m}{n}\right)^{1/\alpha} Y\right) \\ &\le \left(\frac{n-m}{n}\right)^{-(r+1)/\alpha} \mu_{r+1}(\underline{S}_m, \underline{V}_n) \\ &\le 2^{(r+1)/\alpha} m \mu_{r+1}(n^{-1/\alpha}X_1, n^{-1/\alpha}Y_1) \\ &\le 2^{((r+1)/\alpha)-1} n^{1-r/\alpha} \mu_{r+1}(X, Y). \end{aligned}$$

Analogously,

$$\begin{aligned} I_3 &= l\left(\underline{S}_{n,n-m} + \left(\frac{m}{n}\right)^{1/\alpha} Y, \underline{V}_{n,n-m} + \left(\frac{m}{n}\right)^{1/\alpha} Y\right) \\ &\le \left(\frac{m}{n}\right)^{-(r+1)/\alpha} \mu_{r+1}(\underline{S}_{n,n-m}, \underline{V}_{n,n-m}) \\ &\le 3^{(r+1)/\alpha} n^{1-r/\alpha} \mu_{r+1}(X, Y). \end{aligned}$$

Combining, we deduce

$$I_2 + I_3 \le (2^{((r+1)/\alpha)-1} + 3^{(r+1)/\alpha}) \mu_{r+1}(X, Y) n^{1-r/\alpha}. \tag{4.3}$$

To estimate I_1, we use the (-1) ideality of l and (4.1) to obtain

$$\begin{aligned} l(\underline{S}_m, \underline{V}_m) &= l\left(\left(\frac{m}{n}\right)^{1/\alpha} \underline{S}_{m,m}, \left(\frac{m}{n}\right)^{1/\alpha} \underline{Y}_{m,m}\right) \\ &= \left(\frac{n}{m}\right)^{1/\alpha} l(\underline{S}_{m,m}, \underline{Y}_{m,m}) \\ &\le \left(\frac{n}{m}\right)^{1/\alpha} A\tau m^{1-r/\alpha} \\ &\le 3^{1/\alpha} 2^{r/\alpha - 1} A\tau n^{1-r/\alpha}. \end{aligned} \tag{4.4}$$

By (3.6), $\tau_0 \le a$ and thus Theorem 3.1 allows us to estimate the second factor in I_1 by the upper bound

$$\begin{aligned}\operatorname{Var}(\underline{S}_{n-m,\,n-m}, \underline{V}_{n-m,\,n-m}) &\le A\tau_0(n-m)^{1-r/\alpha}\\ &\le A\tau_0 \le 1/D.\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) and using the definitions of A and D, we deduce

$$I_1 \le \frac{A}{2D} 3^{1/\alpha} 2^{r/\alpha} \tau n^{1-r/\alpha} = \frac{A}{2} \tau n^{1-r/\alpha}.$$

This estimate, together with (4.3), yields

$$\begin{aligned} l &\le \left(\frac{A}{2} + 2^{(r+1)/\alpha - 1} + 3^{(r+1)/\alpha} \right) \tau n^{1-r/\alpha} \\ &\le A\tau n^{1-r/\alpha}. \end{aligned}$$

□

The proof of Theorem 3.1 (respectively, Theorem 3.2) is very similar to that of Theorem 3.3 and makes critical use of Lemmas 2.6(i) and 2.7(i) [respectively, Lemmas 2.6(ii) and 2.7(ii)]. The details are left to the reader.

APPENDIX

We provide the proofs of some of the unproved results.

Proof of Lemma 2.3. For any $r > 0$ and X_1, X_2, we have, using integration by parts and Hölder's inequality [cf. Rachev and Ignatov (1984)],

$$\begin{aligned} &\mu_{\theta,r}(X_1, X_2) \\ &\quad = \sup_{h>0} h^r \sup_{x\in\mathbf{R}} |p_{X_1+h\theta}(x) - p_{X_2+h\theta}(x)| \\ &\quad = \sup_{h>0} h^{r-1} \sup_{x\in\mathbf{R}} \left| \int_{-\infty}^{\infty} g\left(\frac{x-y}{h}\right) dF_\rho(y) \right| \\ &\quad = \sup_{h>0} h^{r-2} \sup_{x\in\mathbf{R}} \left| \int_{-\infty}^{\infty} g^{(1)}\left(\frac{x-y}{h}\right) F_\rho(y)\, dy \right| \\ &\quad = \sup_{h>0} h^{r-m-1} \sup_{x\in\mathbf{R}} \left| \int_{-\infty}^{\infty} g^{(m)}\left(\frac{x-y}{h}\right) F_{m,\rho}(y)\, dy \right| \\ &\quad \le \sup_{h>0} h^{r-m-1} \sup_{x\in\mathbf{R}} \left[\int_{-\infty}^{\infty} \left| g^{(m)}\left(\frac{x-y}{h}\right) \right|^q dy \right]^{1/q} \left[\int_{-\infty}^{\infty} |F_{m,\rho}(y)|^p\, dy \right]^{1/p} \\ &\quad = \sup_{h>0} h^{r-m-1+(1/q)} \|g^{(m)}\|_q \|F_{m,\rho}\|_p. \end{aligned}$$

By Theorem 1 of Maejima and Rachev (1987), $\zeta_{m-1,p}(X_1, X_2) < \infty$ implies $\zeta_{m-1,p}(X_1, X_2) = \|F_{m,\rho}\|_p$, completing the proof of Lemma 2.3. □

Proof of Theorem 2.8. We note that (ii) ⇔ (iii) follows immediately from Theorem 4.1 of Rachev (1985), Theorem 1 of Rachev (1984b), Theorem 2 of

Rachev (1984a) and the identity $EX^{k-1} = E|X|^{k-1}$ for k odd. Also, (ii) $\Leftrightarrow$ (iv) follows from Rachev (1982); (iv) $\Rightarrow$ (i) follows from Lemma 2.2 and (iv) $\Rightarrow$ (v) from Lemmas 2.4 and 2.5; thus the only new results here are the implications (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii).

Now (i) $\Rightarrow$ (ii)(a) follows easily from Fourier transform arguments since the characteristic function φ_Y never vanishes. Similarly, if (v) holds, then $X_n + Y \to_{\mathscr{D}} U + Y$ and thus (ii)(a) follows. To prove (i) $\Rightarrow$ (ii)(b), we need a lemma.

Lemma. *Let $0 < \alpha \leq 2$ and for all k let $\mu_k := \mu_{k,\alpha}$. Then there is a constant $\beta := \beta(\alpha, k) < \infty$ such that for all $\rho \in \mathscr{M}(\mathbb{R})$,*

$$(*) \qquad \mu_k(\rho) \geq \beta \left| \int_{-\infty}^{\infty} F_{k-2,\rho}(z)\, dz \right|.$$

Using equality of the first $k - 2$ moments and applying $(*)$ to X_n and U yields

$$\begin{aligned}\beta^{-1}\mu_k(\rho) &\geq \left| \int_{-\infty}^{\infty} F_{k-2,\rho}(z)\, dz \right| \\ &= \left| \int_{-\infty}^{0} (\ )\, dz + \int_0^{\infty} (\ )\, dz \right| \\ &:= |I_1 + I_2|,\end{aligned}$$

where $\rho := F_{X_n} - F_U$. To compute I_1 and I_2, we first note that

$$F_{k-2,\rho}(\infty) = \mathbb{E}(z - X_n)^{k-2} - \mathbb{E}(z - U)^{k-2} = 0.$$

Using this equality and Fubini's theorem gives

$$I_1 = \int_{-\infty}^{0} \int_t^0 \frac{(z-t)^{k-2}}{(k-2)!}\, dz\, dF_\rho(t) = F_{k-1,\rho}(0).$$

Analogously,

$$I_2 = \int_0^{\infty} \frac{t^{k-1}}{(k-1)!}\, dF_\rho(t).$$

Combining the above gives

$$\beta^{-1}\mu_k(\rho) \geq \frac{1}{(k-1)!} |\mathbb{E}(X_n^{k-1} - U^{k-1})|,$$

which gives the desired implication (i) $\Rightarrow$ (ii)(b).

To prove (v) $\Rightarrow$ (ii)(b), integrate by parts to obtain

$$\nu_k(\rho) \geq \left| \int_{-\infty}^{\infty} p_Y^{(k)}(x)\, dx \right| |\mathbb{E}(X_n^k - U^k)|,$$

thus completing the proof of Theorem 2.8. It only remains to prove the lemma.

PROOF OF LEMMA. Integration by parts shows that

$$\beta^{-1}\mu_k(\rho) \geq \beta^{-1} \sup_{x \in \mathbf{R}} \left| \int_{-\infty}^{\infty} F_{k-2,\rho}(x-z) \lim_{h \to \infty} \left(h^k p_{hY}^{(k-1)}(z) \right) dz \right|$$

$$= \sup_{x \in \mathbf{R}} \left| \int_{-\infty}^{\infty} F_{k-2,\rho}(x-z)\, dz \right|$$

since

$$2\pi |h^k p_{hY}^{(k-1)}(z)| = \left| h^k \int_{-\infty}^{\infty} (it)^{k-1} e^{itz - |ht|^{\alpha}}\, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} (it)^{k-1} e^{itz/h - |t|^{\alpha}}\, dt \right|$$

and

$$\beta := \beta(\alpha, k) := \frac{1}{2\pi} \int_{-\infty}^{\infty} |it|^{k-1} e^{-|t|^{\alpha}}\, dt < \infty. \qquad \square$$

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