The Annals of Probability
1989, Vol. 17, No. 2, 723–736

A RENEWAL THEORY WITH VARYING DRIFT[1]

BY CUN-HUI ZHANG

State University of New York at Stony Brook and Rutgers University

Let R be the excess over the boundary in renewal theory. It is well known that ER has a limit r when the drift of the random walk $\mu \geq 0$. We study renewal theorems with varying μ. Conditions are given under which the tail $ER - r$ is uniformly dominated by a decreasing integrable function for μ in a compact interval in $(0, \infty)$. Conditions are also given under which the derivative of the tail $(\partial/\partial\mu)(ER - r)$ is uniformly dominated by a directly Riemann integrable function.

1. Introduction. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with $EX = 0$ and $\operatorname{Var}(X) = \sigma^2 > 0$. Put $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Define for each $c \geq 0$ and $\mu \geq 0$,

$$T = T(c, \mu) = \inf\{n: S_n + n\mu > c\} \tag{1.1}$$

and

$$R = R(c, \mu) = S_T + \mu T - c. \tag{1.2}$$

For fixed $\mu \geq 0$, the asymptotic behavior of R and T is well described by the classical renewal theory. See, for example, Feller (1971). This article concerns uniform renewal theorems with varying drift μ.

Let us first consider the following nonlinear boundary crossing time that motivates our investigation. Define

$$T_A = \inf\{n: S_n > A(n)\},$$

where $A(t)$ is a decreasing function. To approximate probabilistic quantities involving the stopping time T_A and the overshoot $S_{T_A} - A(T_A)$ as the boundary tends to infinity, Woodroofe (1976) and Lai and Siegmund (1977, 1979) suggested that we consider the corresponding linear problems involving the stopping time $T(c_A, \mu_A)$ and the overshoot $R(c_A, \mu_A)$, where $c_A = -bA'(b)$, $\mu_A = -A'(b)$ and $b = \sup\{t: A(t) = 0\} = ET + O(1)$. Applying classical renewal theorems, they obtained nonlinear renewal theorems for the case $A'(b) =$ constant. Noting that

$$T(c_A, \mu_A) = \inf\{n: S_n > A'(b)(n - b)\}$$

and $A'(b)(t - b)$ is the Taylor expansion of $A(t)$ near ET, we realize that a linear renewal theory with varying drift is needed to study more general and deeper nonlinear renewal theorems for two reasons: (a) to study the case

Received November 1986; revised April 1988.
[1]Research supported by National Science Foundation Grant DMS-88-57774.
AMS 1980 *subject classifications*. 60K05, 60G40.
Key words and phrases. Renewal theory, uniform convergence, excess over the boundary, Fourier transformation.

$A'(b) \neq$ constant in the limit process; (b) to get high-order approximations by considering

$$T_A^* = \inf\{n: S_n > A^*(n)\},$$

where $A^*(t)$ is the linear interpolation of $A(t)$. In Zhang (1988) various nonlinear renewal theorems were obtained by using the results in this article.

We shall state the main results in this section and prove Theorems 2 and 3 in Sections 2 and 3, respectively.

A random variable Z is arithmetic if $P\{Z/d = \text{integer}\} = 1$ for some $0 < d < \infty$. In this case, we shall also say that Z has an arithmetic distribution. The span of an arithmetic random variable Z, denoted by $\operatorname{span}(Z)$, is the largest real number $d < \infty$ such that $P\{Z/d = \text{integer}\} = 1$. If Z is not arithmetic we define $\operatorname{span}(Z) = 0$. Also, a random variable Z has a lattice distribution if $Z + \mu$ is arithmetic for some real number μ.

Let $Y(\mu)$, $Y_n(\mu)$, $n \geq 1$, be i.i.d. random variables for each μ with $Y = Y(\mu) = R(0, \mu)$. Define

$$m_1 = m_1(\mu) = EY(\mu), \tag{1.3}$$

$$r = r(\mu) = EY^2(\mu)/(2m_1(\mu)), \tag{1.4}$$

$$G = G(x, \mu) = \int_x^\infty P\{Y(\mu) > y\}\, dy/m_1 \tag{1.5}$$

and

$$f = f(u, \mu) = E\exp[iuY(\mu)]. \tag{1.6}$$

Our first theorem is a uniform Blackwell-type renewal theorem.

THEOREM 1. *Let $\mu > 0$. Suppose that $X + \mu$ does not have an arithmetic distribution. Let c_k and u_k be two sequences of constants such that $c_k \to \infty$ and $0 \leq u_k \to \mu$. Then for any real number $x \geq 0$,*

$$\lim_k \left[\sum_{n=1}^\infty P\{c_k < Y_1(u_k) + \cdots + Y_n(u_k) \leq c_k + x\} - \frac{x}{m_1(u_k)} \right] = 0, \tag{1.7}$$

$$\lim_k \sum_{n=1}^\infty P\{c_k < S_n + u_k n \leq c_k + x\} = x/\mu \tag{1.8}$$

and

$$\lim_k P\{R(c_k, u_k) > x\} = G(x, \mu). \tag{1.9}$$

Furthermore, if $EX^2 = \sigma^2 < \infty$, then for any $x \geq 0$ and t,

$$\begin{aligned} &\lim_k P\left\{R(c_k, u_k) > x,\ T(c_k, u_k) > c_k/u_k + t\sqrt{c_k\sigma^2/u_k^3}\right\} \\ &\quad = G(x, \mu)P\{N(0,1) > t\}, \end{aligned} \tag{1.10}$$

where $N(0, 1)$ is standard normal.

We shall not provide a proof of Theorem 1 here since it can be proved by modifying the existing proofs of the Blackwell renewal theorem. For example, noting that $\operatorname{span}(X + u_k) \to 0$, we can modify the methods of Woodroofe (1982), pages 113–116, to prove (1.8) (and therefore the rest of the theorem), since we only need to consider functions $\hat{g}$ with compact support there. For the nonlattice case Theorem 1 can be easily proved by the coupling methods. See Lindvall (1977). If $m_1(0) < \infty$, then (1.7) and (1.9) still hold when $u_k \to 0$. Sufficient and necessary conditions for $m_1(0) < \infty$ were obtained by Chow (1986).

Assume that $\sigma^2 < \infty$. Let $0 < a \le b < \infty$ and

$$M = \sup\left\{ \sum_{n=1}^{\infty} P\{c < S_n + n\mu \le c + 1\} : a \le \mu \le b, -\infty < c < \infty \right\}.$$

By (1.8) $M < \infty$. Since the right-hand side of

$$\begin{aligned} P\{R(c,\mu) > x\} &= \sum_{n=1}^{\infty} P\{T(c,\mu) \ge n, S_n + n\mu > x + c\} \\ &\le \int_{x-\mu}^{\infty} \sum_{n=1}^{\infty} P\{x + c - y - \mu < S_{n-1} + (n-1)\mu \le c\}\, dP\{X \le y\} \\ &\le 2M \int_{x-b}^{\infty} (1 + b + y - x)\, dP\{X \le y\} \end{aligned}$$

is integrable and independent of μ, $\{R(c,\mu)\colon c \ge 0,\ a \le \mu \le b\}$ is uniformly integrable, so that (1.9) implies $\lim_k ER(c_k, u_k) = r(\mu)$. However, under higher moment and smoothness conditions we have the following stronger results.

THEOREM 2. *Suppose that $E|X|^3 < \infty$ and for some $k < \infty$ and $0 < a \le b < \infty$,*

$$\sup\left\{ \int_{-\infty}^{\infty} |f(u,\mu)|^k\, du : a \le \mu \le b \right\} < \infty. \tag{1.11}$$

Then

$$\int_0^{\infty} \sup\{|ER(y,\mu) - r(\mu)| : y > x,\ a \le \mu \le b\}\, dx < \infty. \tag{1.12}$$

The Lebesgue integrability of $ER(x,\mu) - r(\mu)$ on $[0,\infty)$ for fixed $\mu > 0$ was established by Stone (1965) under the moment condition $E|X|^3 < \infty$. The smoothness condition (1.11) is similar to the condition C under which a renewal theorem for curved boundaries was obtained by Woodroofe (1976).

A function $h(\cdot)$ is directly Riemann-integrable if it is Riemann-integrable on any compact interval and

$$\sum_{n=-\infty}^{\infty} \sup\{|h(x)| : n \le x < n + 1\} < \infty.$$

Note that $h(x) \geq 0$ is directly Riemann-integrable if $h(x)$ is continuous, integrable and eventually monotone at both tails.

THEOREM 3. *Suppose that the density function $p(x)$ of X exists and is bounded by a directly Riemann-integrable function. Then* (1.11) *is valid. Furthermore, suppose that in addition $E|X|^4 < \infty$. Then for any $0 < a \leq b < \infty$,*

$$\sum_{n=0}^{\infty} \sup\{|(\partial/\partial\mu)(ER(x,\mu) - r(\mu))|: n \leq x < n+1,\ a \leq \mu \leq b\} < \infty. \tag{1.13}$$

This theorem is essential for the expansion of variances of nonlinear boundary crossing times in Zhang (1988). The methods in the following sections may also be used to obtain other uniform renewal theorems with varying μ or with respect to $\partial/\partial\mu$ when the needs for such theorems arise. Uniform renewal theorems have been studied by Lai (1976) and Kartashov (1980). Although they considered uniform convergences for larger classes of underlying distribution functions, their results are not delicate enough to meet our needs for nonlinear renewal theory. Also, the convergence in Theorem 3 has not been considered in previous studies. Chow and Lai (1979) investigated the behavior of the stopping times $T(0, \mu)$ as $\mu \to 0$ to obtain results on driftless random walks.

We conclude this section with a proposition which gives an expression for $(\partial/\partial\mu)r(\mu)$.

PROPOSITION 1. *Let $r(\mu)$ be defined by* (1.4). *Suppose that $EX^2 = \sigma^2 < \infty$ and X is not lattice. Then*

$$(\partial/\partial\mu)r(\mu) = (1 - \sigma^2/\mu^2)/2 + \sum_{n=1}^{\infty} P\{S_n + \mu n \leq 0\}.$$

2. Proof of Theorem 2. Since the theorems with $b = 2a$ can be proved by considering X_n/a, $n \geq 1$, we shall assume without loss of generality that the domain of μ is $[1, 2]$ in Sections 2 and 3.

Let f be defined by (1.6). Then, by the Wiener–Hopf factorization [Feller (1971), pages 604–610]

$$f = 1 + (g - 1)U, \tag{2.1}$$

where

$$g = g(u, \mu) = E\exp[iu(X + \mu)] \tag{2.2}$$

and

$$U = U(u, \mu) = \exp\left[\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0+} e^{iux}\, dP\{S_n + n\mu \leq x\}\right]. \tag{2.3}$$

Since

$$\sum_{n=1}^{\infty} n^{-1}P\{S_n + n\mu \le 0\} = \log[EY(\mu)/E(X+\mu)] = \log ET(0,\mu) \le \log ET(0,1), \tag{2.4}$$

we have by (2.1)

$$|g(u,\mu) - 1|/ET(0,1) \le |f(u,\mu) - 1| \le |g(u,\mu) - 1|ET(0,1), \tag{2.5}$$

$$g(u,\mu) \ne 1 \quad \text{when } u \ne 0, \tag{2.6}$$

and for small u,

$$|g - 1| \ge |1 - e^{iu\mu}| - |1 - Ee^{iuX}| \ge \mu|u|/2 \ge |u|/2. \tag{2.7}$$

Therefore, for any $M \ge 0$,

$$\sup\{|(f(u,\mu) - 1)/u|^{-1}: 0 \le |u| \le M, 1 \le \mu \le 2\} < \infty. \tag{2.8}$$

We shall consider Borel functions on $(-\infty,\infty) \times [1,2]$ in the space

$$L^* = \left\{ h = h(x,\mu): \sum_{n=-\infty}^{\infty} \sup_{\substack{n \le x < n+1 \\ 1 \le \mu \le 2}} |h(x,\mu)| < \infty \right\} \tag{2.9}$$

and introduce the following notation. For any complex-valued Borel functions h, h_1 and h_2 on $(-\infty,\infty) \times [1,2]$ for which the corresponding operations are applicable, define

$$Dh = Dh(u,\mu) = \begin{cases} (iu)^{-1}(h(u,\mu) - h(0,\mu)), & u \ne 0, \\ \lim_{u\to 0} Dh(u,\mu), & u = 0, \end{cases}$$

$$D_u h = D_u h(u,\mu) = -i(\partial h/\partial u)(u,\mu),$$

$$D_\mu h = D_\mu h(u,\mu) = (\partial h/\partial \mu)(u,\mu),$$

$$\hat{h} = \hat{h}(u,\mu) = (h)\hat{}\,(u,\mu) = \int_{-\infty}^{\infty} e^{iux} h(x,\mu)\,dx,$$

$$\check{h} = \check{h}(x,\mu) = (h)\check{}\,(x,\mu) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-ixu} h(u,\mu)\,du,$$

$$h_1 * h_2 = h_1 * h_2(x,\mu) = \int_{-\infty}^{\infty} h_1(x-y,\mu)h_2(y,\mu)\,dy,$$

$$h^{*n} = h * h^{*(n-1)}, \qquad n \ge 1,\ h^{*1} = h,$$

and $\hat{L}^* = \{\hat{h}: h \in L^*\}$, where L^* is defined by (2.9).

By elementary calculus, we immediately have

$$\begin{aligned} &D\int_{-\infty}^{\infty} e^{iux}\,dH(x) \\ &\quad = \int_0^{\infty} e^{iux}(H(\infty) - H(x))\,dx - \int_{-\infty}^{0} e^{iux}(H(x) - H(-\infty))\,dx \end{aligned} \tag{2.10}$$

for any monotone function $H(x)$ with $|\int_{-\infty}^{\infty}|x|\,dH(x)| < \infty$,

$$D_u Dh = DD_u h - D^2 h \quad \text{if} \int_{-\infty}^{\infty} |x\check{h}(x,\mu)|\,dx < \infty \tag{2.11}$$

and

$$\{h_1, h_2\} \subset \hat{L}^* \text{ implies } \{h_1 + h_2, h_1 - h_2, h_1 h_2\} \subset \hat{L}^*. \tag{2.12}$$

Define $f_0 = f$, $g_0 = g$, $U_0 = U$,

$$f_n = Df_{n-1}, \qquad g_n = Dg_{n-1}, \qquad U_n = DU_{n-1}, \qquad n \geq 1, \tag{2.13}$$

and

$$\xi(c,\mu) = \begin{cases} ER(c,\mu) - r(\mu) + m_1^{-1}(\mu)\int_c^{\infty}\int_x^{\infty} P\{Y(\mu) \geq y\}\,dy\,dx, \\ \qquad c \geq 0, \\ 0, \qquad c < 0. \end{cases} \tag{2.14}$$

Then by Definition $f_n(0,\mu) = EY^n(\mu)/n!$, $n \geq 1$.

LEMMA 1. *Let R and ξ be defined by* (1.2) *and* (2.14), *respectively. If $E|X|^3 < \infty$, then*

$$\hat{\xi}(u,\mu) = f_2^2(u,\mu)/(f_1(u,\mu)m_1(\mu)), \tag{2.15}$$

where f_1, f_2 and m_1 are defined by (2.13) *and* (1.3).

A proof of Lemma 1 may be found in Carlsson (1983). Indeed, (2.15) is the first displayed equation on page 149 of that article. We state without proof the following lemma which follows easily from the argument of Stone (1965), page 334.

LEMMA 2. *For any positive integer $k \geq 2$ there exists a universal constant $A_k < \infty$ such that for any probability distribution function $H(x)$,*

$$\int_{-1}^{1} |D_u^{k-1} D^2 h(u)|\,du \leq A_k \int_{-\infty}^{\infty} |x|^k\,dH(x),$$

where $h(u) = \int_{-\infty}^{\infty} e^{iux}\,dH(x)$.

Our next lemma is the key to the proofs of Theorems 2 and 3. Since the argument is to be used repeatedly, we define the space $L(k,c)$ for each $c \geq -\infty$ and integer $k \geq 0$ as follows:

$$L(k,c) = \{h\colon h \text{ satisfies } (2.17)\text{–}(2.20)\}, \tag{2.16}$$

where

$$\left(f^n f_1^j \hat{h}\right)^{\vee}(x,\mu) I\{x \geq c\} \in L^*, \qquad n \geq 0, 0 \leq j \leq k, \tag{2.17}$$

$$\sup_{(u,\mu)} |D_u^j \hat{h}(u,\mu)| < \infty, \qquad j = 0, 1, \tag{2.18}$$

$$\sup\left\{|D_u^2 \hat{h}(u,\mu)|\colon |u| \geq 1, 1 \leq \mu \leq 2\right\} < \infty \tag{2.19}$$

and

$$\sup_{\mu} \int_{-1}^{1} |D_u^2 \hat{h}(u, \mu)|\, du < \infty. \tag{2.20}$$

And we define $\hat{L}(k, c) = \{\hat{h}\colon h \in L(k, c)\}$.

LEMMA 3. *Let f_n, $n \geq 0$, be defined by* (2.13). *Suppose that $E|X|^3 < \infty$ and* (1.11) *is satisfied. Then*

$$\left((f_2/f_1)^k \hat{h}\right)^{\vee}(x, \mu) I\{x \geq c\} \in L^*$$

for any $h \in L(k, c)$, $c \geq -\infty$ and $k \geq 0$. In particular, $f_2 \in \hat{L}(k, -\infty)$ for any $k \geq 0$ and

$$f_2^2/f_1 \in \hat{L}^*.$$

PROOF. Since for $x \geq 0$,

$$P\left\{\sup_{1 \leq \mu \leq 2} Y(\mu) \geq x + 2\right\} \leq P\left\{\max_{j \leq T(0,1)} X_j \geq x\right\} \leq P\{X \geq x\} ET(0,1), \tag{2.21}$$

by (2.10), (2.12) and (2.13) for $n \geq 0$ and $j \geq 0$,

$$f^n f_1^j f_2 = m_1^j f_2 + (f_1 - m_1) D(f^n f_1^j) \in \hat{L}^*.$$

It follows from (2.11) and Lemma 2 that

$$\check{f}_2 \in L(k, -\infty) \text{ defined by (2.16)}, \qquad k \geq 0. \tag{2.22}$$

By (1.11) there exists an integer $m > 0$ such that

$$\sup_{\mu} \int_{-\infty}^{\infty} |f(u, \mu)|^{m-2}\, du < \infty. \tag{2.23}$$

Since X is strongly nonlattice by (1.11) and (2.5),

$$\sup\{|g(u, \mu)| = |Ee^{iux}|\colon |u| \geq 1, \mu\} < 1$$

and by (2.5)–(2.8)

$$\sup_{(u, \mu)} |f_1(u, \mu)(1 + |u|)|^{-1} < \infty. \tag{2.24}$$

By algebra

$$f_2/f_1 = (m_1 - f_1)/(1 - f) = (1 + f + \cdots + f^{m-1})(m_1 - f_1) + f^m f_2/f_1.$$

It follows that

$$(f_2/f_1)^k \hat{h} = (1 + f + \cdots + f^{m-1})^k (m_1 - f_1)^k \hat{h} + f^m Q_0 \hat{h}, \tag{2.25}$$

where Q_0 is a polynomial of f, f_1 and f_2/f_1. By (2.17) and (2.12)

$$\left[(1 + f + \cdots + f^{m-1})^k (m_1 - f_1)^k \hat{h}\right]^{\vee}(x, \mu) I\{x \geq c\} \in L^*. \tag{2.26}$$

To bound $(f^m Q_0 \hat{h})^\vee$, we consider

$$D_u^2(f^m Q_0 \hat{h}) = f^{m-2}\big(Q_1 D_u^2 \hat{h} + Q_2(D_u^2 f_2)/f_1 + Q_3\big), \tag{2.27}$$

where Q_1, Q_2 and Q_3 are polynomials of $D_u^j \hat{h}$, $j = 0, 1$, $D_u^j f_n$, $0 \le n \le 2$, $0 \le j \le 2$, $j + n \le 3$, and $(D_u^j f_n)/f_1$, $n = 1, 2$, $0 \le j \le 2$, $n + j \le 3$. Since $E|X|^3 < \infty$, by (2.10), (2.11), (2.21) and (2.24)

$$\sup_{(u,\mu)} |(D_u^j f_n)/f_1| < \infty \quad \text{for all } n = 1, 2, 0 \le j \le 2,\ n + j \le 3,$$

so that by (2.18), (2.10) and (2.21)

$$\sup_{(u,\mu)} |Q_n(u, \mu)| < \infty, \qquad 0 \le n \le 3.$$

By (2.19), (2.11) and (2.24)

$$\sup\big\{|D_u^2 \hat{h}(u, \mu)| + |f_1^{-1}(u, \mu) D_u^2 f_2(u, \mu)| : |u| \ge 1, 1 \le \mu \le 2\big\} < \infty.$$

By (2.20), (2.24) and Lemma 2

$$\sup_\mu \int_{-1}^{1} \big(|D_u^2 \hat{h}(u, \mu)| + |f^{-1}(u, \mu) D_u^2 f_2(u, \mu)|\big)\, du < \infty.$$

Therefore, by (2.27) and (2.23)

$$\sup_\mu \int_{-\infty}^{\infty} |D_u^2(f^m Q_0 \hat{h})(u, \mu)|\, du < \infty.$$

By the inversion formula for Fourier transformations

$$\begin{aligned}(1 + x^2)(f^m Q_0 \hat{h})^\vee(x, \mu) &\\ = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixu}(1 + D_u^2)(f^m Q_0 \hat{h})(u, \mu)\, du.&\end{aligned}$$

It follows that

$$\begin{aligned}&\sup_{(x,\mu)} |(1 + x^2)(f^m Q_0 \hat{h})^\vee(x, \mu)| \\ &\le (2\pi)^{-1}\left[\left(\sup_\mu \int_{-\infty}^{\infty} |f|^m\, du\right) \sup_{(u,\mu)} |Q_0 \hat{h}| + \sup_\mu \int_{-\infty}^{\infty} |D_u^2(f^m Q_0 \hat{h})|\, du\right] \\ &< \infty.\end{aligned} \tag{2.28}$$

Hence, $f^m Q_0 \hat{h} \in \hat{L}^*$ and by (2.25) and (2.26)

$$\big((f_2/f_1)^k \hat{h}\big)^\vee(x, \mu) I\{x \ge c\} \in L^*. \qquad \square$$

PROOF OF THEOREM 2. By (2.10)

$$(Df^n)^\vee(x, \mu) = P\{Y_1 + \cdots + Y_n > x\} I\{x \ge 0\},$$

$$(D^2 f^n)^\vee(x, \mu) = \int_x^{\infty} P\{Y_1 + \cdots + Y_n > y\}\, dy\, I\{x \ge 0\}.$$

It follows from (2.21) that there exists a function $M(x)$ not depending on μ such

that $M(x) = 0$ for $x < 0$, $M(x)$ is decreasing for $x \geq 0$, $\int_0^\infty M(x)\,dx < \infty$ and

$$|(D^j f^n)^\vee(x,\mu)| \leq M(x), \qquad j = 1,2,\ 1 \leq n \leq m. \tag{2.29}$$

Since

$$f_2 f^n = f_2 + (f_1 - m_1)Df^n = D^2 f + (Df - m_1)Df^n, \tag{2.30}$$

$(1 + f + \cdots + f^{m-1})(m_1 - f_1)f_2$ is a polynomial of $D^j f^n$, $j = 1,2$, $1 \leq n < m$, so that by (2.29) and the fact that $M * M(x) \leq 2M(x/2)\int_0^\infty M(y)\,dy$

$$\int_0^\infty \sup\left\{|((1 + f + \cdots + f^{m-1})(m_1 - f_1)f_2)^\vee(y,\mu)| : y \geq x, 1 \leq \mu \leq 2\right\} dx < \infty.$$

Replace $\hat{h}$ by f_2 and take $k = 1$ in (2.25) and (2.28), and we have by (2.22)

$$\int_0^\infty \sup\left\{|(f_2^2/f_1)^\vee(y,\mu)| : y \geq x, 1 \leq \mu \leq 2\right\} dx < \infty. \tag{2.31}$$

The proof is complete since (1.12) follows from (2.31), Lemma 1, (2.21), (2.14) and (2.29). □

3. Proof of Theorem 3. We split the proof into several lemmas.

LEMMA 4. *Let $p(x)$ be the density function of X and define*

$$q = q(x,\mu) = \sum_{n=1}^\infty p^{*n}(x - n\mu)I\{x \leq 0\}. \tag{3.1}$$

Suppose that $p(x)$ is bounded by a directly Riemann-integrable function. Then

$$\sup_{(x,\mu)} q(x,\mu) < \infty.$$

PROOF. Let $M > 0$ be an upper bound for $p(x)$ and $\sum_{n=1}^\infty P\{k < S_n + n\mu \leq k + 1\}$ for all $1 \leq \mu \leq 2$ and k. Theorem 1 ensures such an upper bound $M < \infty$. Given $x \leq 0$,

$$\begin{aligned} q(x,\mu) &= p(x - \mu) + \sum_{n=1}^\infty \int_{-\infty}^\infty p(x - y - \mu)\,dP\{S_n + n\mu \leq y\} \\ &\leq M + \left[\sup_k \sum_{n=1}^\infty P\{k \leq S_n + n\mu < k + 1\}\right] \sum_k \sup_{k \leq y < k+1} p(x - y - \mu) \\ &\leq M\left[1 + 2\sum_{-\infty}^\infty \sup_{k \leq y < k+1} p(y)\right] < \infty. \end{aligned}$$

□

LEMMA 5. *Let h and h_1 be two Borel functions on $(-\infty,\infty) \times [1,2]$ such that $h(x,\mu)$ vanishes on $(0,\infty) \times [1,2]$. Suppose that $\sup_{(x,\mu)}|h(x,\mu)| < \infty$ and $\int_x^\infty |h_1(y,\mu)|\,dy\,I\{x \geq 0\} \in L^*$. Then*

$$h * h_1(x,\mu)I\{x \geq 0\} \in L^*.$$

PROOF. Clearly,

$$\begin{aligned}|h * h_1(x,\mu)| &= \left|\int_{-\infty}^{\infty} h(x-y,\mu)h_1(y,\mu)\,dy\right| \\ &\le \int_x^{\infty} |h(x-y,\mu)h_1(y,\mu)|\,dy \\ &\le \left[\sup_{(y,\mu)} |h(y,\mu)|\right]\int_x^{\infty} |h_1(y,\mu)|\,dy \in L^*. \end{aligned}$$ □

LEMMA 6. *If* $E|X|^3 < \infty$, *then*

$$D_\mu f_1 = f_1(iu\hat{q} - q(0,\mu)) + gU,$$
$$D_\mu f_2 = f_1\hat{q} - f_2 q(0,\mu) + f_1 + U_1$$

and

$$D_\mu f_3 = f_2\hat{q}(0,\mu) + f_1 D\hat{q} - f_3 q(0,\mu) + f_2 + U_2,$$

where f_n, g, U_n, $n \ge 0$, *and* q *are defined by* (2.13) *and* (3.1).

PROOF. By (2.1)–(2.3)

$$D_\mu g = iug, \quad D_\mu g_1 = g, \quad f_1 = g_1 U \quad \text{and} \quad D_\mu f_1 = g_1 D_\mu U + U D_\mu g_1.$$

Since by (2.3)

$$D_\mu U = U D_\mu \sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0} e^{iux} p^{*n}(x - n\mu)\,dx = U(iu\hat{q} - q(0,\mu)),$$

we have

$$D_\mu f_1 = f_1(iu\hat{q} - q(0,\mu)) + gU.$$

And the expressions for $D_\mu f_n$ may be derived from

$$D_\mu f_n = D_\mu D f_{n-1} = D D_\mu f_{n-1}.$$ □

PROOF OF THEOREM 3. (i) Let us first prove that (1.11) holds. By (2.1)–(2.3)

$$\begin{aligned}(\log U)^{\vee}(x,\mu) &= \sum_{n=1}^{\infty} n^{-1} p^{*n}(x - n\mu) I\{x \le 0\} \\ &\le q(x,\mu) \le M < \infty \end{aligned}$$

and

$$\begin{aligned}((\log U)^{\vee})^{*n}(x,\mu) &\le M\left(\int_{-\infty}^{\infty} (\log U)^{\vee}(y,\mu)\,dy\right)^{n-1} \\ &\le M\left(\sum_{j=1}^{\infty} j^{-1} P\{S_j + j \le 0\}\right)^{n-1} < \infty. \end{aligned}$$

By (2.4)

$$(3.2)\qquad \begin{aligned}(U-1)^{\vee}(x,\mu) &\le \sum_{n=1}^{\infty}(n!)^{-1}((\log U)^{\vee})^{*n}(x,\mu)\\ &\le MET(0,1) < \infty.\end{aligned}$$

It follows from (2.1) that $f = g + g(U-1) - (U-1)$ and

$$\check{f}(x,\mu) = \left[p(x-\mu) + \int_{-\infty}^{\infty}(U-1)^{\vee}(x-y,\mu)p(y-\mu)\,dy\right]I\{x \ge 0\}.$$

By Lemma 5, (3.2), (2.10) and (2.21)

$$(3.3)\qquad \check{f}_n \in L^*, \quad \text{if } E|X|^n < \infty,\ n \ge 1.$$

Therefore, (1.11) is satisfied with $k = 2$ by Parseval's identity and the uniform boundedness of $\check{f}(x,\mu)$.

(ii) Since by (2.10), (2.14) and Lemma 1 the Fourier transformation of $ER(x,\mu) - r(\mu)$ on the positive half-axis is $f_2^2/(f_1m_1) - f_3/m_1$, we shall consider $(f_2/f_1)D_\mu f_2$, $(f_2/f_1)^2D_\mu f_1$ and $D_\mu f_3$. By (2.1), $f_1 = g_1U$ and $\dot{U}_1 = DU = \mu^{-1}(f_2 - Ug_2)$. It follows from Lemma 6 that

$$(3.4)\qquad (f_2/f_1)D_\mu f_2 = f_2(\hat{q}+1) + (\mu^{-1} - q(0,\mu))f_2^2/f_1 - \mu^{-1}(f_2/f_1)g_2U.$$

Since $E|X|^3 < \infty$ and $U - 1$ is the Fourier transformation of a nonnegative integrable function,

$$(3.5)\qquad \begin{aligned}\sup_{(u,\mu)}|D_u^2U(u,\mu)| &= \sup_{\mu}|D_u^2U(0,\mu)|\\ &= \sup_{\mu}2U_2(0,\mu)\\ &= \sup_{\mu}2\mu^{-1}(Df_2 - D(Ug_2))(0,\mu) < \infty.\end{aligned}$$

By (3.2), Lemma 5, (2.10), (2.21), (2.29) and (3.5)

$$(3.6)\qquad U - 1 \in \hat{L}(k,0), \qquad k = 1,2.$$

Since $g_2U = g_2(U-1) + g_2$, by (3.5), (3.6) and Lemma 2

$$(3.7)\qquad g_2U \in \hat{L}(k,0), \qquad k = 1,2.$$

By (3.3) and Lemmas 4 and 5, $(f_2(\hat{q}+1))^{\vee}I\{x \ge 0\} \in L^*$. Therefore, we have by (3.4), Lemma 3 and (3.7)

$$(3.8)\qquad ((f_2/f_1)D_\mu f_2)^{\vee}(x,\mu)I\{x \ge 0\} \in L^*.$$

For $(f_2/f_1)^2D_\mu f_1$ we have by Lemma 6 and the definition of D,

$$(3.9)\qquad (f_2/f_1)^2D_\mu f_1 = f_2\hat{q} - m_1(f_2/f_1)\hat{q} - q(0,\mu)f_2^2/f_1 + (f_2/f_1)^2gU.$$

Since $gU = f + (U-1)$,

$$(3.10)\qquad gU \in \hat{L}(k,0),\ k = 1,2, \quad \text{by (3.3) and (3.6).}$$

It follows from Lemmas 4, 5, 3 and (3.10) that

$$(3.11)\qquad \left(f_2\hat{q} - q(0,\mu)f_2^2/f_1 + (f_2/f_1)^2 gU\right)^{\vee}(x,\mu)I\{x \geq 0\} \in L^*.$$

Let us use the assumption that $E|X|^4 < \infty$. By Stone (1965)

$$(3.12)\qquad \int_{-\infty}^{0} |x| \sum_{n=1}^{\infty} P\{S_n + n \leq x\}\, dx < \infty,$$

which implies that

$$(3.13)\qquad |D_u^2\hat{q}(u,\mu)| \leq 2\int_{-\infty}^{0} |x| \sum_{n=1}^{\infty} P\{S_n + n \leq x\}\, dx < \infty.$$

By Lemmas 3, 4, 5, (3.3) and (3.13)

$$(3.14)\qquad ((f_2/f_1)\hat{q})^{\vee}(x,\mu)I\{x \geq 0\} \in L^*.$$

By (3.9), (3.11), (3.14) and (2.12)

$$(3.15)\qquad \left((f_2/f_1)^2 D_\mu f_1\right)^{\vee}(x,\mu)I\{x \geq 0\} \in L^*.$$

Now let us consider $D_\mu m_1$. By Lemma 6

$$D_\mu m_1 = D_\mu f_1(0,\mu) = -q(0,\mu)m_1 + U(0,\mu),$$

which implies that

$$(3.16)\qquad \sup_{\mu} |D_\mu m_1(\mu)| < \infty.$$

Since $(m_1(\mu))^{-1}$ is bounded, by Lemma 1, (3.8), (3.15), (3.16), Theorem 2 and (2.12)

$$(3.17)\quad \left(D_\mu \hat{\xi}\right)^{\vee}(x,\mu)I\{x \geq 0\} = \left(D_\mu\left(f_2^2/(f_1 m_1)\right)\right)^{\vee}(x,\mu)I\{x \geq 0\} \in L^*.$$

Finally, let us consider $D_\mu f_3$. Again by Lemma 6

$$D_\mu f_3 = f_2\hat{q}(0,\mu) + f_1 D\hat{q} - f_3 q(0,\mu) + f_2 + U_2.$$

Since

$$(U_2)^{\vee}(x,\mu)I\{x \geq 0\} = 0$$

and

$$(D\hat{q})^{\vee}(x,\mu) = \sum_{n=1}^{\infty} P\{S_n + n\mu \leq x\}I\{x \leq 0\},$$

we have by (3.3) and Lemmas 5 and 4

$$(3.18)\qquad \left(D_\mu(f_3/m_1)\right)^{\vee}(x,\mu)I\{x \geq 0\} \in L^*.$$

Let $h_0 = I\{x \geq 0\}$. It follows from (2.14), (2.10), (3.17) and (3.18) that

$$(3.19)\quad \left(D_\mu\left[((ER - r)h_0)^{\wedge}\right]\right)^{\vee} h_0 = \left(D_\mu\left(f_2^2/(f_1 m_1) - f_3/m_1\right)\right)^{\vee} h_0 \in L^*.$$

Let $h_1(x) = x^2 \exp[-|x|]$ and h_2 be the right-hand side of (3.19). Then

$$\begin{aligned}(h_2h_1)^\wedge &= (2\pi)\big(D_\mu[((ER-r)h_0)^\wedge]\big) * \big((h_0h_1)^\wedge\big)\\ &= (2\pi)D_\mu[((ER-r)h_0)^\wedge * (h_0h_1)^\wedge]\\ &= D_\mu[((ER-r)h_0h_1)^\wedge].\end{aligned}$$

Hence, by Lebesgue's dominated convergence theorem

$$\begin{aligned}(ER-r)h_0 &= (1/h_1)\big((D_\mu)^{-1}[(h_2h_1)^\wedge]\big)^\vee\\ &= (1/h_1)(D_\mu)^{-1}(h_2h_1) \quad \big(\text{exchange integral } D_\mu^{-1} \text{ and } {}^\wedge\big)\\ &= (D_\mu)^{-1}h_2\end{aligned}$$

and the proof is complete by (3.19). □

PROOF OF PROPOSITION 1. By (2.1)–(2.3) $f_1 = g_1U$ and $f_2 = g_2U + \mu U_1$. Since $f_1(0, \mu) = EY(\mu)$, $f_2(0, \mu) = EY(\mu)^2/2$, $g_1(0, \mu) = \mu$ and $g_2(0, \mu) = (EX^2 + \mu^2)/2$,

$$r(\mu) = (2\mu)^{-1}(\sigma^2 + \mu^2) - \sum_{n=1}^{\infty} n^{-1}E(S_n + n\mu)^-.$$

It follows that

$$\begin{aligned}D_\mu r(\mu) &= (1 - \sigma^2/\mu^2)/2 - \sum_{n=1}^{\infty} n^{-1}D_\mu\int_{\mu n}^{\infty} P\{S_n \le -x\}\,dx\\ &= (1 - \sigma^2/\mu^2)/2 + \sum_{n=1}^{\infty} P\{S_n + \mu n \le 0\}. \qquad \square\end{aligned}$$

Acknowledgments. This article is based on a part of my Ph.D. dissertation. I am grateful to my advisor Professor Yuan Shih Chow and to Professor Steven P. Lalley for many invaluable suggestions. Also, I would like to thank the Associate Editor and a referee for helpful comments.

REFERENCES

BLACKWELL, D. H. (1948). A renewal theorem. *Duke Math. J.* **15** 145–150.

CARLSSON, H. (1983). Remainder term estimates of the renewal function. *Ann. Probab.* **11** 143–157.

CHOW, Y. S. (1986). On moments of ladder height variables. *Adv. in Appl. Math.* **7** 46–54.

CHOW, Y. S. and LAI, T. L. (1979). Moments of ladder variables for driftless random walks. *Z. Wahrsch. verw. Gebiete* **48** 253–257.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

KARTASHOV, N. V. (1980). A uniform asymptotic renewal theorem. *Theory Probab. Appl.* **25** 589–592.

LAI, T. L. (1976). Uniform Tauberian theorems and their applications to renewal theory and first passage problems. *Ann. Probab.* **4** 628–643.

LAI, T. L. and SIEGMUND, D. (1977). A nonlinear renewal theory with applications to sequential analysis. I. *Ann. Statist.* **5** 946–954.

LAI, T. L. and SIEGMUND, D. (1979). A nonlinear renewal theory with applications to sequential analysis. II. *Ann. Statist.* **7** 60–76.

LINDVALL, T. (1977). A probabilistic proof of Blackwell's renewal theorem. *Ann. Probab.* **5** 482–485.

STONE, C. (1965). On characteristic functions and renewal theory. *Trans. Amer. Math. Soc.* **120** 327–342.

WOODROOFE, M. (1976). A renewal theorem for curved boundaries and moments of first passage times. *Ann. Probab.* **4** 67–80.

WOODROOFE, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis*. SIAM, Philadelphia.

ZHANG, C.-H. (1988). A nonlinear renewal theory. *Ann. Probab.* **16** 793–824.

DEPARTMENT OF STATISTICS
HILL CENTER FOR THE MATHEMATICAL SCIENCES
BUSCH CAMPUS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903