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TIME INHOMOGENEOUS MARKOV PROCESSES AND THE POLARITY OF SINGLE POINTS[1]

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Kesten (1969) gives necessary and sufficient conditions for single points to be polar for Lévy processes. In this article we investigate when single points are polar for processes obtained by adding functions to stable processes.

1. Introduction. Let $\{X_\alpha(t): t \geq 0\}$ be a stable process on R^1 with exponent $\alpha \in (0, 2]$. It has long been known that when $\alpha < 1$ the single points are polar while for $\alpha > 1$ the single points are nonpolar. When α is equal to 1 (Cauchy processes), the single points are polar in the symmetric case but not in the asymmetric cases. Kesten (1969) gives necessary and sufficient conditions for a Lévy process to have single points nonpolar and the above facts follow from these [see Kesten (1969) or Bretagnolle (1971)]. If a nonconstant linear term is added to a Lévy process the resulting process is also a Lévy process. Kesten's conditions show that all the processes obtained by adding nonconstant linear terms to stable processes have singletons nonpolar except in the case of the symmetric Cauchy process. In this article we examine the effect of adding general functions to stable processes. For nonlinear functions, the resulting process will no longer be a Lévy process so Kesten's conditions will no longer tell whether single points are polar.

Graverson (1982) showed that in two dimensions there exist functions which when added to a planar Brownian motion result in a modified process for which singletons are nonpolar. He further showed that for each $\beta < \frac{1}{2}$, one could choose the function to be Hölder continuous of order β. Le Gall (1987) showed that the addition of any $\frac{1}{2}$-Hölder continuous function to planar Brownian motion results in a process which has singletons polar. Throughout this article we will use the term hits points to describe a process for which the single points are not polar.

In this article we show

PROPOSITION 1. *Let $\{X_\alpha(t): t \geq 0\}$ be a stable process with $\alpha > 1$ and let f be any (possibly noncontinuous) Borel function. Then for any fixed $x \in R^1$, $\{(X_\alpha + f)(t): t \geq 0\}$ hits x with positive probability.*

Single points are polar for the stable processes with $\alpha < 1$, so Proposition 1 cannot hold when $\alpha < 1$. We show

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PROPOSITION 2. *Let* $f\colon R^1_+ \to R^1$ *be continuous and nonconstant and let* X_α *be a one-dimensional stable process of index* $\alpha < 1$. *Then a.s. the range of* $\{(X_\alpha + f)(t)\colon t \geq 0\}$ *has positive Lebesgue measure.*

It will be shown that proving this proposition is equivalent to showing that single points are not polar for the processes obtained by adding nonconstant functions to stable processes with $\alpha < 1$.

The Cauchy processes do not behave as might be expected given these results. Since the singletons are polar for the stable processes with $\alpha < 1$ and for the symmetric Cauchy process, we might expect that the symmetric Cauchy process would extend Proposition 2. However, as has already been noted, single points remain polar when a linear drift is added to a symmetric Cauchy process. We generalize this fact.

PROPOSITION 3.1. *Let* X *be a symmetric Cauchy process and* f *a Lipschitz function. Then for each singleton* $\{x\}$, $P(\exists\, t > 0\colon X(t) + f(t) = x) = 0$.

There do exist functions which when added to the symmetric Cauchy process result in a modified process having the singletons nonpolar.

PROPOSITION 3.2. *Let* X *be a symmetric Cauchy process. For each* $\beta < 1$, *there exists a function* f^β *such that*:

(i) f^β *is Hölder of order* β.
(ii) *The process* $\{X(t) + f^\beta(t)\colon t \geq 0\}$ *hits points.*

Proposition 3.2 cannot be extended to the set of all functions which fail to be β-Hölder continuous everywhere for some fixed β.

PROPOSITION 3.3. *Let* X *be a symmetric Cauchy process. There exists a function* $f\colon [0,1] \to R^1$ *such that at each point* $t \in [0,1]$, f *is not Hölder continuous for any* $\beta > 0$ *and for which the process* $\{X(t) + f(t)\colon t \geq 0\}$ *does not hit points.*

Single points are nonpolar for the asymmetric Cauchy process. So one might hope that this process could extend Proposition 1. This is not so.

PROPOSITION 3.4. *Let* X *be an asymmetric Cauchy process. Then there exists a function* f *such that* $P(X(t) + f(t)$ *hits points*$) = 0$.

In Section 3 we conjecture that for stable processes with $\alpha < 1$, the modified process

$$\{X_\alpha(t) + f(t)\colon t \geq 0\}$$

will have single points polar if and only if the range of the function has positive Lebesgue measure. In Section 5 we demonstrate that no part of this conjecture can hold for arbitrary Lévy processes for which single points are polar.

PROPOSITION 4. *There exist a Lévy process $\{Z(t): t \geq 0\}$ which does not hit points and a fixed nonrandom function f whose range has zero Lebesgue measure such that $\{0\}$ is nonpolar for the process $\{Z(t) + f(t): t \geq 0\}$.*

2. The stable process with $\alpha > 1$. The following lemma gives a criterion for there to be no exceptional points which are not polar.

LEMMA 1. *Consider a process $\{Z(t): t \geq 0\}$ with the properties*

(i) *$Z(t)$ has a strictly positive density at strictly positive times.*
(ii) *$Z(t)$ has independent increments.*

Then $P(Z(t) = 0$ for any $t > 0) = 0$ if and only if $P(Z(t) = x$ for any $t > 0) = 0$ for all x in R^1.

PROOF. $P(Z(t) = 0$ for any $t > 0) = 0$ if and only if $P(Z(t) = 0$ for any $t > \delta) = 0$ for all δ in R^1. By the independent increments property, it follows that for fixed $\delta > 0$, this last expression equals

$$\int f_\delta(x) P\big(Z^\delta(t) = -x \text{ for } t > 0\big)\, dx,$$

where the process $\{Z^\delta(t): t \geq 0\}$ is equal to the process $\{Z(t + \delta) - Z(\delta): t \geq 0\}$ and $f_\delta(\cdot)$ is the density of $Z(\delta)$. Since $f_\delta(\cdot)$ is strictly positive, the above integral is zero if and only if

$$\int P\big(Z^\delta(t) = -x \text{ for } t > 0\big)\, dx = 0.$$

This integral equals

$$\int P\big(Z^\delta(t) = y - x \text{ for } t > 0\big)\, dx$$

which is zero if and only if

$$\int f_\delta(x) P\big(Z^\delta(t) = y - x \text{ for } t > 0\big)\, dx$$

is zero [using the strict positivity of $f_\delta(\cdot)$ again]. This last integral equals $P(Z(t) = y$ for any $t > \delta)$. This shows that $\forall\, \delta$, $P(Z(t) = 0$ for any $t > \delta) = 0$ if and only if $P(Z(t) = y$ for any $t > \delta) = 0$. □

Any process $\{Z(t) = X(t) + f(t): t \geq 0\}$, where $X(t)$ is a stable process and $f(\cdot)$ is a fixed nonrandom curve, satisfies the hypotheses of Lemma 1 except when X is a one-sided stable process. But in this case $\{Z(t) = X(t) + f(t): t \geq 0\}$ satisfies the hypotheses below.

LEMMA 2. *Consider a process* $\{Z(t): t \geq 0\}$ *with the properties*

either

(i) $Z(t)$ *has a strictly positive density on* $\{x > g(t)\}$ *at strictly positive times for some* $g(t)$

or

(i′) $Z(t)$ *has a strictly positive density on* $\{x < g(t)\}$ *at strictly positive times for some* $g(t)$

and

(ii) $Z(t)$ *has independent increments.*

If $P(Z(t) = x$ *for some* $t > 0) > 0$, *then* $P(Z(t) = y$ *for some* $t > 0) > 0$ *for all* $y >$ (*respectively* $<$) x.

The proof is similar to that of Lemma 1.

Using Lemma 1 or 2 and Fubini's theorem, we see that for $\{Z(t) = X(t) + f(t): t \geq 0\}$, where X is a stable process,

$$P\big(\lambda(\{X(t): t \geq 0\}) = 0\big) = 1 \quad \text{if and only if } P\big(Z(t) = x \text{ for some } t \geq 0\big) = 0, \qquad \forall x \in R^1,$$

where λ denotes the Lebesgue measure of a set. If either (and hence both) of these conditions fails, then by the strong Markov property and the 0–1 law, $P(\lambda(\{X(t): t \geq 0\}) = 0) = 0$.

PROPOSITION 1. *Let* $\{X_\alpha(t): t \geq 0\}$ *be a stable process with* $\alpha > 1$ *and let* f *be any (possibly noncontinuous) Borel function. Then for any fixed* $x \in R^1$, $\{(X_\alpha + f)(t): t \geq 0\}$ *hits* x *with positive probability.*

PROOF. We use an argument from Kahane (1968) to prove that the Lebesgue measure of the range of $X_\alpha + f$ is positive with probability 1. By the remarks above this will be sufficient to prove the proposition.

Define the (random) occupation measure μ of $X_\alpha + f$ by $\mu(A) = \lambda_{[0,1]}((X + f)^{-1}(A))$, where $\lambda_{[0,1]}$ is the Lebesgue measure restricted to the interval $[0, 1]$. If a.s. μ is absolutely continuous with respect to the Lebesgue measure, then (since the range has full μ-measure) the range must have positive Lebesgue measure. This sufficient condition will certainly be satisfied if μ has a square integrable density a.s., which by Plancherel's theorem will hold if

$$\text{(1)} \qquad \text{a.s.} \quad \int_{-\infty}^{\infty} |\bar{\mu}(u)|^2 \, du < \infty,$$

where $\bar{\mu}(\cdot)$ is the Fourier transform of μ; condition (1) holds if

$$\text{(2)} \qquad E\int_{-\infty}^{\infty} |\bar{\mu}(u)|^2 \, du = \int_{-\infty}^{\infty} E|\bar{\mu}(u)|^2 \, du < \infty.$$

We complete the proof by showing (2).

The Fourier transform of μ is given by

$$\bar{\mu}(u) = \int_0^1 e^{iu(f(t)+X_\alpha(t))}\,dt$$

so

$$|\bar{\mu}(u)|^2 = \int_0^1\int_0^1 e^{iu(f(t)+X_\alpha(t))}e^{-iu(f(s)+X_\alpha(s))}\,dt\,ds$$
$$= \int_0^1\int_0^1 e^{iu(f(t)-f(s))}e^{iu(X_\alpha(t)-X_\alpha(s))}\,dt\,ds.$$

Hence

$$E|\bar{\mu}(u)|^2 = \int_0^1\int_0^1 e^{iu(f(t)-f(s))}E\left[e^{iu(X_\alpha(t)-X_\alpha(s))}\right]dt\,ds$$
$$\le \int_0^1\int_0^1 e^{-|cu|^\alpha|t-s|}\,dt\,ds.$$

Integrating with respect to u, we get

$$\int_{-\infty}^{\infty} E|\bar{\mu}(u)|^2 \le \int_0^1\int_0^1\int_{-\infty}^{\infty} e^{-|cu|^\alpha|t-s|}\,du\,ds\,dt$$
$$\le K_\alpha\int_0^1\int_0^1 \frac{1}{|t-s|^{1/\alpha}}\,dt\,ds < \infty. \qquad \square$$

3. The stable process with α < 1.

PROPOSITION 2. *Let $f\colon R^1_+ \to R^1$ be continuous and nonconstant and let X_α be a one-dimensional stable process of index $\alpha < 1$. Then a.s. the range of $\{(X_\alpha + f)(t)\colon t \ge 0\}$ has positive Lebesgue measure.*

REMARK. The proof to be given assumes that X_α is two-sided but with obvious modifications can be made to deal with the one-sided case too. In the two-sided case, given the continuous density of X_α at positive times, the proposition states that $X_\alpha + f$ hits 0 with positive probability at some strictly positive time.

PROOF. We prove that for X_α started uniformly on $[0,1]$, the process $\{(X_\alpha + f)(t)\colon t \ge 0\}$ hits any fixed point x with positive probability. By the remarks in Section 2, proof of this statement is sufficient.

We shall prove our result if we can show that given x, during some fixed finite time interval, $X_\alpha + f$ hits the interval $[x, x + 2^{-n}]$ for each n with probability bounded away from 0. Our strategy is to find about 2^{-n} time points $\{s_1, s_2, \dots, s_{m(n)}\}$ so that the events

$$A_i = \{(X_\alpha + f)(s_i) \in [x, x + 2^{-n}]\}, \qquad 0 \le i \le m,$$

are almost independent. We then consider the random variable $W_n = \sum_{i=0}^m I_{A_i}$. If we can show (I) $E(W_n)$ is bounded away from 0 and (II) $E(W_n^2)$ is bounded away from ∞, then $P(W_n > 0)$ is bounded away from 0. This will complete our proof.

We shall take 0 as the fixed point and without loss of generality assume that $f(0) = 0$ and $f(1) = 1$. Let $t_i = \inf\{t: f(t) = i/2^n\}$ for $i = 0, 1, \dots, 2^n$, and $T = \bigcup\{t_i\}$. We define a subset S of T as follows:

$$s_0 = t_0 = 0.$$

Given that $s_i = t_j$, $s_{i+1} = \inf\{t_v > t_j: t_v - t_j \le 2(v-j)/2^n\}$.

We continue to define the s_i until either

1. s_i is equal to t_{2^n} or
2. $1 - s_i > 2(1 - f(s_i))$. In this case $s_m = 1$.

Let $S = \bigcup_{i=0}^m \{s_i\}$ and B (for bad) $= T \setminus S$. We now notice:

(a) The interval $[0,1]$ contains $\bigcup_{i=0}^{m-1} [s_i, s_{i+1}]$, so $1 \ge \sum_{i=0}^m (s_{i+1} - s_i)$.

(b) In each interval $[s_i, s_{i+1}]$, the number of elements of B is less than or equal to $2^n(s_{i+1} - s_i)/2$.

(c) Every element of B is contained in some interval $[s_i, s_{i+1}]$.

These observations imply that $|B| \le 2^{n-1}$ so $|S| > 2^{n-1}$. The crucial property of S is

$$\forall j > i, \qquad f(s_j) - f(s_i) \ge \tfrac{1}{2}(s_j - s_i)$$

with the possible exception of $j = m$. As it makes no difference to the essentials of the argument, we shall assume that this property also holds for $j = m$.

We now show the A_i are independent enough. Recall $A_i = \{(X_\alpha + f)(s_i) \in [0, 2^{-n}]\}$ for $i \le m$. As we assumed that X_α was started uniformly on $[0,1]$, it is easy to see that for each j, $P(A_j)$ is of the order of 2^{-n}, so $\sum_{i=0}^m P(A_i)$ is bounded away from 0. This completes the proof of (I); it only remains to show (II). For fixed i, we wish to bound $\sum_{j>i} P(A_j | A_i)$. It is easily seen that this expression is bounded by

$$\sum_{j=1}^{2^n} P\big(X_\alpha(j/2^{n+1}) \in [j/2^n, (j+1)/2^n]\big),$$

which by the stability of X_α is equal to

$$\sum_{j=1}^{2^n} P\Big(X_\alpha(1/2) \in \big[(j/2^n)^{1/\alpha - 1}, ((j+1)/2^n)^{1/\alpha - 1}\big]\Big);$$

this in turn is bounded by

$$\begin{aligned}
&\sum_{j=1}^{2^n} 2^{-n}(j/2^n)^{-1/\alpha}\big[(j/2^n)^{(1/\alpha - 1)(1+\alpha)}\big] \\
&\quad = 2^{-n} \sum_{j=1}^{2^n} (j/2^n)^{-\alpha} \\
&\quad = 2^{-(1-\alpha)n} \sum_{j=1}^{2^n} (j)^{-\alpha}.
\end{aligned}$$

As this last expression is clearly bounded in a way not depending on n, the proof of (II) is complete. □

An examination of the above proof shows that much the same arguments would work to prove the above proposition when f is a function from R_+ to R^1, such that there exists a closed set $E \subset R^1_+$ on which f is continuous, increasing and has range of positive Lebesgue measure. We use this observation to extend the above result.

LEMMA. *Let f be an increasing function on R^1_+ whose range has positive Lebesgue measure. Then there is a closed subset E of R^1_+ on which f is continuous and has range with positive Lebesgue measure.*

PROOF. We may without loss of generality suppose that f maps $[0,1]$ to $[0,1]$ and that its range on this interval has positive Lebesgue measure. Define the measure on $[0,1]\mu_f$ by $\mu_f(A) = \lambda_1(f(A))$ [$\lambda_1(\cdot)$ is Lebesgue measure]. We form the set E by taking the intersection of a nested sequence of unions of intervals. First pick disjoint intervals $I_1, I_2, \dots$ of length at most $\frac{1}{10}$ so that $\mu_f(f^{-1}(\cup I_i)) > \|\mu_f\|/2$. We may choose compacts $K_1, K_2, \dots$ so that

(i) for all $i\, K_i \subset f^{-1}(I_i)$,
(ii) $\mu_f(\cup K_i) > \|\mu_f\|/2$.

Now successively choose the intervals $I_{i_1, i_2, \dots, i_r}$ and compacts $K_{i_1, i_2, \dots, i_r}$ so that

(a) the length of every $I_{i_1, i_2, \dots, i_r}$ is $< 10^{-r}$,
(b) $K_{i_1, i_2, \dots, i_r} \subset I_{i_1, i_2, \dots, i_r}$,
(c) $K_{i_1, i_2, \dots, i_r} \subset K_{i_1, i_2, \dots, i_{r-1}}$, $I_{i_1, i_2, \dots, i_r} \subset I_{i_1, i_2, \dots, i_{r-1}}$,
(d) $\mu_f(\cup K_{i_1, i_2, \dots, i_r}) > \|\mu_f\|/2$.

Now as the $K_{i_1, i_2, \dots, i_r}$ are compact and disjoint d_r is strictly positive where $d_r = \inf\{|x - y|: x \in K_{i_1, i_2, \dots, i_r} y \in K_{j_1, j_2, \dots, j_r}$ and $i_1, i_2, \dots, i_r \neq j_1 j_2 \dots j_r\}$. Take E equal to $\cap_r \cup_{i_1, i_2, \dots, i_r} K_{i_1, i_2, \dots, i_r}$. By condition (d) above $\mu_f(E) > 0$ and by the definition of d_r if $x, y \in E$ satisfy $|x - y| < d_r$, then $|f(x) - f(y)| < 10^{-r}$. □

This lemma and the remarks preceding it give the following corollary.

COROLLARY 1. *Let $f: R^1_+ \to R^1$ be increasing on a set E on which its range has positive Lebesgue measure and let X_α be a one-dimensional stable process of index $\alpha < 1$. Then a.s. the range of $\{(X_\alpha + f)(t): t \geq 0\}$ has positive Lebesgue measure.*

The proofs of Proposition 2 or Corollary 1 do not carry over to arbitrary functions with range of positive Lebesgue measure. But it is difficult to see how the extra wildness of discontinuous nonmonotone functions should upset the above conclusions. We therefore conjecture:

CONJECTURE. Let $Z(t) = X(t) + f(t)$ where X is stable of order $\alpha < 1$. Then the range of Z a.s. has positive Lebesgue measure if and only if the range of f has positive Lebesgue measure.

4. The Cauchy processes. In this section we first look at the symmetric stable process of index 1 and then at the asymmetric processes. We can summarize the behavior of the stable processes of index 1 by saying that they do not behave as might have been expected given Propositions 1 and 2.

PROPOSITION 3.1. *Let X be a symmetric Cauchy process and f a Lipschitz function. Then for each singleton $\{x\}$, $P(\exists\, t > 0: X(t) + f(t) = x) = 0$.*

REMARK. The following proof uses ideas from Le Gall (1987); see also Dvoretzky, Erdös and Kakutani (1961).

PROOF. It suffices to prove the following: For fixed x, $P(X(t) + f(t) = x$ for some $t \in [1,2]) = 0$. Define $p_n = P(\exists\, t \in [1,2]: X(t) + f(t) \in (x - 2^{-n}, x + 2^{-n}))$; the proposition will be established if we can show p_n tends to zero as $n \to \infty$.

At time t, the process $X + f$ has density

$$\frac{1}{\pi}\frac{t}{t^2 + (f(t) - x)^2},$$

which is bounded for $t \in [1,3]$, so

$$E_n = E\left[\int_1^3 I_{\{(X+f)(t) \in (x \pm 2^{-n})\}}\, dt\right]$$

satisfies $E_n/2^{-n} \le C$ for some C and all n. Let $T_n = \inf\{t > 1: (X + f)(t) \in (x - 2^{-n}, x + 2^{-n})\}$, so p_n is simply the probability that $T_n < 2$. Let F_{T_n} be the σ-field generated by events preceding T_n. As in Le Gall (1987), we have

$$E\left[E_n | F_{T_n}\right] = \int_{T_n}^{\max(T_n,3)}\left(\int_{x-2^{-n}}^{x+2^{-n}} \frac{t - T_n}{(t - T_n)^2 + (y - Z(T_n) + f(T_n) - f(t))^2}\frac{dy}{\pi}\right) dt.$$

On $\{T_n < 2\}$, this quantity is greater than

$$K2^{-n}\int_{2^{-n}}^{1}\frac{t\,dt}{t^2 + (2^{-n} + Mt)^2} = K'n2^{-n},$$

from which we easily obtain the desired result,

$$p_n \le \frac{E_n}{E[E_n | T_n < 2]} \le \frac{C}{k'n}. \qquad \square$$

In fact this proof shows that if f satisfies the weaker condition

$$\forall s, t, \qquad |f(t+s) - f(t)| \le \alpha(s),$$

where

$$\int_{0+} \frac{s\,ds}{s^2 + \alpha^2(s)} = \infty,$$

then $X + f$ does not hit points.

The following proposition shows that Proposition 3.1 cannot be improved much.

PROPOSITION 3.2. *Let X be a symmetric Cauchy process. For each $\beta < 1$, there exists a function f^β such that:*

(i) *f^β is Hölder of order β.*
(ii) *The process $\{X(t) + f^\beta(t): t \ge 0\}$ hits points.*

REMARK. Here we use ideas from Graverson (1982).

PROOF. Following Kahane (1968), one can see that $\forall \alpha < 1$, there exists a Gaussian process Y_α which satisfies:

1. $\forall s, t \in [0,1]$, $E[(Y_\alpha(s) - Y_\alpha(t))^2] \ge |(t-s)|^{2\alpha}$.
2. $E[(Y_\alpha(t) - Y_\alpha(s))] = 0$.

Separable versions of this process are necessarily Hölder continuous for all $\alpha' < \alpha$. Consider a symmetric Cauchy process X defined on a probability space (Ω, F, P) and the Gaussian process Y_α defined on a separate probability space (Ω', F', P'). As in Graverson (1982), we define the process $X + Y_\alpha$ on the product probability space to be

$$(X + Y_\alpha)(t, (\omega, \omega')) = X(t, \omega) + Y_\alpha(t, \omega').$$

By the standard argument used in Proposition 1, we will have shown that a.s. the range of this process has positive Lebesgue measure if we can show

$$\int_{-\infty}^{\infty} E\left[|\bar{\mu}(u)|^2\right] du < \infty.$$

As before this quantity is equal to

$$\int_0^1 \int_0^1 \int_{-\infty}^{\infty} E\left[e^{iu(X(t)-X(s))} e^{iu(Y_\alpha(t)-Y_\alpha(s))}\right] du\,ds\,dt.$$

By the independence of X and Y_α, this expression is equal to

$$\int_0^1 \int_0^1 \int_{-\infty}^{\infty} E\left[e^{iu(X(t)-X(s))}\right] E\left[e^{iu(Y_\alpha(t)-Y_\alpha(s))}\right] du\,ds\,dt,$$

which is majorized by

$$\int_0^1\int_0^1\int_{-\infty}^{\infty}\left|E\left[e^{iu(Y_\alpha(t)-Y_\alpha(s))}\right]\right|du\,ds\,dt$$

$$< \int_0^1\int_0^1\int_{-\infty}^{\infty} e^{-(u^2|t-s|^{2\alpha})/2}\,du\,ds\,dt$$

$$= C\int_0^1\int_0^1 \frac{dt\,dt}{|t-s|^\alpha} \quad \text{for some } C.$$

Since this last quantity is clearly finite for $\alpha < 1$, we establish that a.s. the range of $X + Y_\alpha$ has positive Lebesgue measure.

By Fubini's theorem, P' a.s. the range of the (Ω, F, P) process $\{X(t,\omega) + Y_\alpha(t,\omega'): t \in [0,1]\}$ (here ω' is considered fixed) has positive Lebesgue measure. We also know that for $t \in [0,1]$, $Y_\alpha(t,\omega')$ is Hölder continuous for all $\beta < \alpha P'$ a.s. So P' a.s. the function $Y_\alpha(t,\omega')$ satisfies conditions 1 and 2 of Proposition 3.2. □

We now establish that the lack of suitable Hölder continuity conditions is not sufficient to cause the symmetric Cauchy process to hit points.

PROPOSITION 3.3. *Let X be a symmetric Cauchy process. There exists a function $f: [0,1] \to R^1$ such that at each point $t \in [0,1]$, f is not Hölder continuous for any $\beta > 0$ and for which the process $\{X(t) + f(t): t \geq 0\}$ does not hit points.*

PROOF. The plan of the proof is to construct a suitable sequence of functions f_i which converge uniformly to a function f having the property claimed. We assume that all the small numbers $(\varepsilon_i, \delta_i)$ are reciprocals of integers and fix $x \neq 0$.

Let f_1 be defined by $f_1(t) = t$ on $[0,1]$. We first choose δ_1 such that $P(f_1(t) + X(t) \text{ hits } (x - \delta_1, x + \delta_1)) < 2^{-2}$. Next we choose ε_1 such that $\varepsilon_1^{2^{-2}} < 10^{-2}\delta_1$ and define the function g_2 by

$$g_2(t) = 10^{-2}\delta_1 \quad \text{if } t = i\varepsilon_1 \text{ with } i \text{ even},$$
$$g_2(t) = 0 \quad \text{if } t = i\varepsilon_1 \text{ with } i \text{ odd},$$

and g_2 linear elsewhere. On $[0,1]$ the function $f_2 = f_1 + g_2$ is Lipschitz, so by Proposition 3.1 we can find δ_2 $(< 10^{-2}\delta_1)$ such that $P(f_2(t) + X(t)$ hits $(x - \delta_2, x + \delta_2) < 2^{-2\cdot 2^2}$. Now choose ε_2 such that $\varepsilon_2^{2-2\cdot 2^2} < 10^{-2}\delta_2$ and define g_3 to be

$$g_3(t) = 10^{-2}\delta_2 \quad \text{if } t = i\varepsilon_2 \text{ with } i \text{ even},$$
$$g_3(t) = 0 \quad \text{if } t = i\varepsilon_2 \text{ wth } i \text{ odd},$$

and g_3 linear elsewhere. As with f_2, the function $f_3 = f_2 + g_3$ is Lipschitz on $[0,1]$. Again by Proposition 3.1, we can find δ_3 $(< 10^{-2}\delta_2)$ and continue defining f_n, ε_n and δ_n in this manner, always ensuring $\varepsilon_i^{2-2i^2} < \delta_i$, $\delta_{i+1} < 10^{-2}\delta_i$ and $P(f_i(t) + X(t) \in (x - \delta_i, x + \delta_i) \text{ for } t \in [0,1]) < 2^{-2i^2}$.

This method of defining the f_i ensures $\| f_{i+1} - f_i \| < 10^{-2}\delta_1$. So f_i converges uniformly to some function f. Since by design $\| f - f_i \| < \delta_i/99$, we have

$$P(f(t) + X(t) \in (x - 0.99\delta_i, x + 0.99\delta_i) \text{ for } t \in [0,1]) < 2^{-2i^2},$$

which easily yields the polarity of $\{x\}$ for $X + f$. However, it is easy to see that at no point in $[0,1]$ is the function f Hölder continuous for any $\alpha > 0$. □

We now turn to the asymmetric case.

PROPOSITION 3.4. *Let X be an asymmetric Cauchy process. Then there exists a function f such that $P(X(t) + f(t)$ hits points$) = 0$.*

PROOF. Consider an independent Cauchy process Y which has the same distribution as X. Then the process $\{X(t,\omega) - Y(t,\omega'): t \geq 0\}$ is a symmetric Cauchy process. Accordingly, for almost all paths of $Y(\cdot,\omega')$, the range of the process $\{X(t,\omega) - Y(t,\omega'): t \geq 0\}$ has Lebesgue measure 0. Letting f be one of these fixed paths $Y(t,\omega')$ gives the result. □

5. On the conjecture of Section 3. In Section 3, we conjectured that when a function whose range has positive Lebesgue measure is added to a stable process of index $\alpha < 1$, the range of the resulting process a.s. has positive Lebesgue measure. However, the conjecture cannot be extended to arbitrary Lévy processes which do not hit points. We give an example of a Lévy process which does not hit points but which can be made to hit points by adding a deterministic function to it whose range has zero Lebesgue measure.

PROPOSITION 4. *There exist a Lévy process $\{Z(t): t \geq 0\}$ which does not hit points and a fixed nonrandom function f whose range has zero Lebesgue measure such that $\{0\}$ is nonpolar for the process $\{Z(t) + f(t): t \geq 0\}$.*

PROOF. Our method of proof is to exhibit two independent Lévy processes Y and Z, neither of which hits points but whose sum is a Lévy process which does. We then let f be an appropriate path $Y(\cdot,\omega')$.

Let $\{X(t): t \geq 0\}$ be a symmetric stable process of index $\alpha > 1$. It is well known that X hits points. The Lévy measure of X is given by

$$\mu_X(dx) = \frac{c}{2}\frac{dx}{|x|^{\alpha+1}}.$$

We will henceforth take c to be equal to 1. By suitably dividing the Lévy measure, we create two Lévy processes Y and Z which sum to X. A Lévy process $\{W(t): t \geq 0\}$ has characteristic function given by

$$E\left[e^{iuW(t)}\right] = e^{-t\phi_W(u)}.$$

For the process X, the function ϕ_X is given by

$$\phi_X(u) = \int_0^\infty \frac{1 - \cos(ux)}{x^{\alpha+1}}dx = |u|^\alpha.$$

Kesten (1969) proved that W hits points if and only if

$$\operatorname{Re}\int \frac{du}{\beta + \phi(u)} < \infty$$

for each real $\beta > 0$. This fact will be crucial in our division of μ_X.

Consider the function

$$\phi_w(u) = \int_{1/w}^{\infty} \frac{1 - \cos(ux)}{x^{\alpha+1}} dx.$$

We choose u_0 greater than 2, it is easily seen that for u_0, $\phi_{u_0}(u) \leq C/u_0^{\alpha}$, so we can find u_0' such that

$$\int_{u_0}^{u_0'} \frac{du}{2 + \phi_{u_0}(u)} > 2. \tag{1}$$

We can also find v_0 so large that

$$\int_0^{1/v_0} \frac{1 - \cos(ux)}{x^{\alpha+1}} dx < 1 \tag{2}$$

for all $u \leq u_0'$. We may also ensure that $v_0 \geq u_0'$. Obviously, the procedure giving v_0 and u_0' from u_0 does not uniquely define them but abusing notation, we write $v_0 = g(u_0)$ and $u_0' = h(u_0)$ and then recursively define $u_{i+1} = g(v_i)$, $v_i' = h(v_i)$, and $v_i = g(u_i)$, $u_i' = h(u_i)$. We now define the Lévy measures of Y and Z to be

$$\mu_Y = \frac{1}{2} \frac{dx}{|x|^{\alpha+1}} I_{\{|x| \geq 1/u_0\}} + \sum_{i \geq 0} \frac{1}{2} \frac{dx}{|x|^{\alpha+1}} I_{\{|x| \in (1/u_{i+1}, 1/v_i)\}},$$

$$\mu_Z = \sum_{i \geq 0} \tfrac{1}{2} \frac{dx}{|x|^{\alpha+1}} I_{\{|x| \in (1/v_i, 1/u_i)\}}.$$

Note that $\mu_Y + \mu_Z = \mu_X$.

As a consequence of our definition of $\{u_i\}$ and $\{v_i\}$, we know that if

$$\phi_Y(u) = \int 1 - \cos(ux) \mu_Y(dx),$$

then for $u \in (u_i, u_i')$,

$$\phi_Y(u) \leq \int_{1/u_i} \frac{(1 - \cos(ux))}{x^{\alpha+1}} dx + 1$$

and so

$$\int_{u_i}^{u_i'} \frac{du}{1 + \phi_Y(u)} > 1$$

for each i, so

$$\int \frac{du}{1 + \phi_Y(u)} = \infty.$$

Similarly, if

$$\phi_Z(u) = \int \frac{1 - \cos(ux)}{x^{\alpha+1}} \mu_Z(dx),$$

then

$$\int \frac{du}{1 + \phi_Z(u)} = \infty.$$

By Kesten's condition, neither Y nor Z hits points, but the sum of independent copies does. So almost all paths $Y(\cdot, \omega')$ of Y have the property that the range has zero Lebesgue measure and a.s. the range of the process $Y(t, \omega') + Z(t)$ has positive Lebesgue measure. Taking $Y(\cdot, \omega')$ as f completes the proof. □

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