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SOME LIMIT THEOREMS FOR FUNCTIONALS OF THE BROWNIAN SHEET[1]

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We study properties of functionals of the Brownian sheet: in particular, we construct square-integrable functionals of R^d-valued Brownian sheets, so that k-intersection local times on R^{dk} can be obtained as specific examples. Results of central limit theorem type on sums of functionals of Brownian sheets, give us new fields, and we investigate which properties of the Brownian sheet (e.g., Markovianess, renormalizability) these limit fields inherit.

1. Introduction. In a previous paper [Adler and Epstein (1987)] we studied the construction in law of generalized Gaussian fields and their functionals via limit theorems for sums of functionals of Markov processes. The initial motivation for this came from the "well known" relationship in mathematical physics between the free field of Euclidean quantum field theory and Brownian motion [cf. Symanzik (1969)] as well as related papers in the probability literature [e.g., Dynkin (1980, 1983, 1984a, b), Wolpert (1978a, b) and Albeverio and Høegh-Krohn (1984)]. Our primary aim, however, was not so much to develop this relationship further, as to exploit it to study properties of generalized Gaussian fields of the kind studied by Dobrushin (1979), Dobrushin and Kel'bert (1983a, b), Major (1981), etc. The general approach of these (as, in fact, most) authors to Gaussian fields was via the spectrum, which is often a clumsy tool leading to results which, while correct, are unintuitive. (A spectral approach, of course, also limits one to the study of stationary fields only.) In Adler and Epstein (1987) we showed that via the Markov process–Gaussian field link it was not only easy to drop stationarity assumptions, but that the exploitation of well known results on Markov processes led to easy and natural results on Gaussian fields, related, for example, to properties such as Markovianess, locality and renormalizability. Related questions were further studied in Adler (1989), Walsh [(1986), Chapter 8] and Adler and Epstein (1988).

Over the past few years the type of quantum field theory that was based on the notion of random particles has moved to a base of random strings, and the paths of particles have been replaced with the so-called "world sheet" of the string, an object most easily understood in probability language as a random surface [cf. Schwarz (1982), Polyakov (1981), Eguchi (1980), Gross, Harvey, Martinec and Rohm (1985) and the Schwarz collection (1985)]. From this grew the present paper and our interest in replacing the Markov processes in Adler

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and Epstein (1987) by random surfaces to see what would happen. In fact, we shall concentrate on just one random surface, which is the natural generalization to surfaces of the Brownian motion. This is the Brownian sheet, and is defined as the R^d-valued two-parameter random field $B^d(t, s)$, $(t, s) \in R^2_+ := [0, \infty) \times [0, \infty)$, whose $d \geq 1$ components $B^d_1, \dots, B^d_d$ are independent Gaussian random fields with zero mean and common covariance function

$$(1.1) \qquad E\{B^d_i(t, s) B^d_i(\tau, \sigma)\} = \min(s, \sigma) \cdot \min(t, \tau), \qquad (t, s), (\tau, \sigma) \in R^2_+.$$

Note that $B^d \equiv 0$ on the axes of R^2_+, a condition akin to that of regular Brownian motion starting at zero. Since this is a restriction that we do not want, we shall work with the "randomly started" sheet given by

$$(1.2) \qquad W\colon W^d_i(t, s) = z_i + B^d_i(t, s), \qquad i = 1, \dots, d,$$

where $\mathbf{z} = (z_1, \dots, z_d)$ is distributed on R^d according to some initial measure $m(d\mathbf{z}) = m(dz_1) \times \cdots \times m(dz_d)$, independently of B^d.

The choice of $m(d\mathbf{z})$ to be a Lebesgue measure on R^d makes the calculations on W^d very neat, and we work with it until the time for limit theorems comes. At that point we introduce a sequence of independent Brownian sheets starting at the points of a Poisson point process on R^d, that is typically used when describing such phenomena as "randomly located points." In this way we can work with probability measures, and so are in the correct setting to state results on weak convergence.

Our primary aim in this paper is, following the example of the link between regular Brownian motion and the free Gaussian field, to first define, via construction, the square integrable functionals of several Brownian sheets, establish limit theorems for sums of these, and then investigate the properties of the limit fields. Of particular interest, although it turns out to be just a special case of the theory that we shall present, is the local time and intersection local time of several sheets, which is of interest in the study of interactions in string theories.

Our general method of construction of square integrable functionals for the Brownian sheet is close to that of Dynkin for Markov processes [cf. Dynkin (1981, 1984b) and Adler and Epstein (1987), Section 2]. In that theory, particles must either "die" after a random, but finite, time or somehow have their contribution to any functional dampened over time. Mathematically, both of these procedures turn out to be equivalent. In the Brownian sheet theory, "killing" is not so easy (primarily because of the lack of total order in the parameter space) and damping is the order of the day. Consider a simple example, the local time at $\mathbf{x} \in R^d$ of the Brownian sheet, defined symbolically as

$$(1.3) \qquad L_{\mathbf{x}} = \int_{R^2_+} ds\, dt\, \theta(t, s) \delta(W_1(t, s) - x_1) \cdots \delta(W_d(t, s) - x_d),$$

where δ denotes the Dirac delta function and θ remains, for the moment, undefined.

We want $L_{\mathbf{x}}$ to satisfy $E(L^2_{\mathbf{x}}) < \infty$. It is easy to see, from the neighborhood recurrence of the W_i, that this means that $\theta(t, s)$ must be small for large s and t. It turns out that θ must be exponentially small. What is not so obvious is that there is also a divergence in $E(L_{\mathbf{x}}^2)$ due to that part of the integral (1.3) where s and t are small. This comes from the fact that the W_i are constant for s or $t = 0$ and vary only slowly in the neighborhood of the axes. Thus, there too, $\theta(t, s)$ must be small. A convenient choice of θ turns out to be

$$\theta(t, s) = se^{-s}e^{-st}. \tag{1.4}$$

[Interestingly, the exponential factor st is simply the "area to the left below the point (t, s)," and area damping is a common phenomenon in string theories.] Precisely why (1.4) is not only convenient but actually quite natural will become clear in Section 5 when we consider problems of renormalization.

Previous studies of Brownian sheet local time [Adler (1981) and Rosen (1984, 1986), among others] avoided (1.4) in a number of ways. For example, both Adler and Rosen restricted (t, s) to lie within the unit square and thus required no damping for large values of the parameter. Adler (1981) avoided the small value problem by considering only (t, s) for which $\min(s, t) \geq \varepsilon > 0$. Rosen (1986), in a study of self-intersection local time, handled the small (t, s) problem with a damping factor st [cf. also Imkeller (1984a, b) and Wong and Zakai (1974)].

The remainder of the paper is organized as follows. In Section 2 we construct general square-integrable functionals of the Brownian sheet. Section 3 concentrates on some specific examples, in particular, intersection local times. Section 4 is devoted to limit theorems for the sum of the functionals of Section 2, based on a general result of Dynkin and Mandelbaum (1983). In Section 5 we study the limiting Gaussian fields that arise in this fashion.

To conclude our introduction, we return to the physics that motivated us (at least partially), i.e., the study of the fields defined via the intersections of an infinite system of Brownian sheets. The physics, built on interacting random surfaces of the kind that we shall consider, falls under the general subject of gauge theories. It is believed there, that the interesting cases are for space-time dimensions $8 > d \geq 4$ and that the Hausdorff dimensions of a random surface are 4 [cf. Parisi (1979)]. However, by Adler [(1981) page 251], the Hausdorff dimension of the Brownian sheet $W\colon R_+^2 \to R^d$ is 4 for $d \geq 4$, and according to results of this paper, the Gaussian field that arises from summing the local times of infinite system of Brownian sheets is generalized for $d \geq 4$. Moreover, we can build the theory of interactions of the two independent Brownian sheets only for $d < 8$, the "magic" upper bound to most gauge theories [see Parisi (1979) for details].

2. Functionals of independent Brownian sheets. We start with some special functions, and some spaces of functions. Let $C = 0.57721\ldots$ denote Euler's constant, and let $E_1(\tau)$ denote the exponential-integral function

$$E_1(\tau) = -Ei(-\tau) = \int_\tau^\infty e^{-t}t^{-1}\,dt, \qquad \tau > 0 \tag{2.1}$$

[see Gradshteyn and Ryzhik (1965) for definitions and properties of E_1]. Define

$$\rho(\tau) := e^{-\tau}\tilde{\rho}(\tau), \qquad \tilde{\rho}(\tau) := \ln\tau + e^{\tau}E_1(\tau) + C, \qquad \tau > 0. \tag{2.2}$$

(This choice of function ρ will become clear in the proof of Proposition 2.1.)

Let

$$p_t(\mathbf{x}, \mathbf{y}) \equiv p_t(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi t)^{d/2}} e^{-|\mathbf{x}-\mathbf{y}|^2/2t}, \qquad \mathbf{x}, \mathbf{y} \in R^d,$$

be the transition density of the standard Brownian motion on R^d and define the function g that, for the Brownian sheet, will play a role analogous to that of the Green function for Markov processes:

$$g(\mathbf{x}) := \int_0^\infty \rho(\tau) p_\tau(\mathbf{x})\, d\tau, \qquad \mathbf{x} = (x_1, \dots, x_d) \in R^d. \tag{2.3}$$

This function defines a bilinear form

$$\langle f, h \rangle := \iint f(\mathbf{x}) g(\mathbf{x} - \mathbf{y}) h(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \tag{2.4}$$

for functions from the class

$$M^d \equiv M^d(g) := \{ f \colon f \text{ integrable on } R^d \text{ and } \langle |f|, |f| \rangle < \infty \}. \tag{2.5}$$

The main result of this section will be an extension to the case of measures of

PROPOSITION 2.1. *For $f \in M^d$ the functional F_f of the Brownian sheet $W \colon R_+^2 \to R^d$, defined by*

$$F_f \equiv F_f(W) := \iint_{R_+^2} \theta(t, s) f(W(t, s))\, dt\, ds \tag{2.6}$$

with function $\theta(t, s) = se^{-s}e^{-st}$ is square-integrable and

$$E(F_f F_h) = \iint_{R^{2d}} d\mathbf{x}\, f(\mathbf{x}) g(\mathbf{x} - \mathbf{y}) h(\mathbf{y})\, d\mathbf{y} = \langle f, h \rangle, \qquad f, h \in M^d. \tag{2.7}$$

We shall establish the proof of Proposition 2.1 later.

Now we look more closely at the covariance kernel g that will play a major role in our theory.

LEMMA 2.2. *The Fourier transform of g is given by*

$$\hat{g}(\mathbf{k}) = \frac{\ln(1 + |\mathbf{k}|^2/2)}{(|\mathbf{k}|^2/2)(1 + |\mathbf{k}|^2/2)}, \tag{2.8}$$

$$|\mathbf{k}|^2 = (k_1)^2 + \cdots + (k_d)^2, \mathbf{k} = (k_1, \dots, k_d) \in R^d.$$

PROOF. Since

$$\hat{p}_t(\mathbf{k}) = \int_{R^d} d\mathbf{x}\, p_t(\mathbf{x}) e^{i(\mathbf{k}\cdot\mathbf{x})} = e^{-|\mathbf{k}|^2 t/2}, \tag{2.9}$$

by straightforward computation and formulas (6.224.1) and (4.331.1) of

Gradshteyn and Ryzhik (1965),

$$\hat{g}(\mathbf{k}) = \int_0^\infty e^{-|\mathbf{k}|^2\tau/2} e^{-\tau}(C + e^\tau E_1(\tau) + \ln\tau)\,d\tau$$

$$= \frac{C}{(1+|\mathbf{k}|^2/2)} + \int_0^\infty e^{-|\mathbf{k}|^2\tau/2} E_1(\tau)\,d\tau - \frac{1}{(1+|\mathbf{k}|^2/2)}\left(C + \ln\left(1 + \frac{|\mathbf{k}|^2}{2}\right)\right)$$

$$= \frac{C}{(1+|\mathbf{k}|^2/2)} + \frac{1}{|\mathbf{k}|^2/2}\ln\left(1 + \frac{|\mathbf{k}|^2}{2}\right) - \frac{1}{(1+|\mathbf{k}|^2/2)}\left(C + \ln\left(1 + \frac{|\mathbf{k}|^2}{2}\right)\right)$$

$$= \frac{\ln(1+|\mathbf{k}|^2/2)}{(|\mathbf{k}|^2/2)(1+|\mathbf{k}|^2/2)}. \qquad \square$$

LEMMA 2.3. *$g(\mathbf{x})$ is finite for all $\mathbf{x} \in R^d$, for $d = 1,2,3$. For $d \geq 4$, $g(\mathbf{0}) = \infty$.*

PROOF. Note that

$$p_t(\mathbf{x}) \equiv \frac{1}{(2\pi t)^{d/2}} e^{-|\mathbf{x}|^2/2t} \leq p_t(\mathbf{0}),$$

$$g(\mathbf{x}) = \int_0^\infty \rho(\tau) p_\tau(\mathbf{x})\,d\tau \leq g(\mathbf{0}),$$

for every $\mathbf{x} \in R^d$, and therefore, it is enough to consider finiteness or infiniteness of $g(\mathbf{0})$. An inverse Fourier transform and Lemma 2.2 give us

$$g(\mathbf{0}) = \frac{1}{(2\pi)^d}\int_{R^d} \hat{g}(\mathbf{k})\,d\mathbf{k}$$

$$= \frac{1}{(2\pi)^d}\int_{R^d} \frac{\ln(1+|\mathbf{k}|^2/2)}{(|\mathbf{k}|^2/2)(1+|\mathbf{k}|^2/2)}\,d\mathbf{k}$$

$$= \frac{1}{(\pi)^d}\int_{R^d} \frac{\ln(1+|\mathbf{k}|^2)}{(|\mathbf{k}|^2)(1+|\mathbf{k}|^2)}\,d\mathbf{k}.$$

Transforming to polar coordinates (for $d > 1$), this last integral converges or diverges with $\int_0^\infty r^{d-3}[\ln(1+r^2)]/(1+r^2)\,dr$ and this is finite for $d = 1,2,3$ and infinite for $d \geq 4$. This completes the proof. $\square$

In the next lemma we prove that $\langle f, f \rangle \geq 0$ for all functions in M^d so that the form $\langle\ ,\ \rangle$ is an inner product on the space of functions M^d.

LEMMA 2.4. *The function $g(\mathbf{x},\mathbf{y}) \equiv g(\mathbf{y},\mathbf{x}) := g(\mathbf{x}-\mathbf{y})$ is positive-definite on $R^d \times R^d$.*

COROLLARY 2.5. *There exists a Gaussian random field on M^d with zero mean and covariance kernel $g(\mathbf{x},\mathbf{y})$.*

REMARK. Since for $d \geq 4$, g is not finite, we understand its positive-definiteness in the sense that $g(\mathbf{x})\,d\mathbf{x}$ is a positive-definite measure on R^d, that is [cf. Argabright and Gil de Lamadrid (1974)],

$$\int_{R^d} g(\mathbf{x})\,d\mathbf{x}\left(\int_{R^d} f(\mathbf{y})f(\mathbf{y}-\mathbf{x})\,d\mathbf{y}\right) \geq 0 \quad \text{for all } f \in S_d, \tag{2.10}$$

where S_d denotes the Schwartz space of C^∞ functions of rapid decrease. We shall show that (2.10) holds for all $f \in M^d$, so that $\langle f, f \rangle \geq 0$ for all $f \in M^d$. (For g finite this definition is equivalent to a usual one, i.e., for all $n = 1, 2, \dots, C_1, \dots, C_n$ real, $\sum_{i,j=1}^n g(\mathbf{x}_i - \mathbf{x}_j)C_iC_j \geq 0$.)

PROOF OF LEMMA 2.4. By (2.3) and the Chapman–Kolmogorov equation for the Brownian motion transition density

$$\int_{R^d} g(\mathbf{x})\,d\mathbf{x}\left(\int_{R^d} f(\mathbf{y})f(\mathbf{y}-\mathbf{x})\,d\mathbf{y}\right)$$

$$\equiv \iint_{R^{2d}} d\mathbf{y} f(\mathbf{y}) g(\mathbf{y}-\mathbf{x}) f(\mathbf{x})\,d\mathbf{x}$$

$$= \int_0^\infty \rho(\tau)\,d\tau \iint f(\mathbf{y}) p_\tau(\mathbf{y},\mathbf{x}) f(\mathbf{x})\,d\mathbf{x}\,d\mathbf{y} \tag{2.11}$$

$$= \int_0^\infty \rho(\tau)\,d\tau \int_{R^d} d\mathbf{z} \int_{R^d} f(\mathbf{y}) p_{\tau/2}(\mathbf{y},\mathbf{z})\,d\mathbf{y} \int_{R^d} f(\mathbf{x}) p_{\tau/2}(\mathbf{z},\mathbf{x})\,d\mathbf{x}$$

$$= \int_0^\infty \rho(\tau)\,d\tau \int_{R^d} d\mathbf{z} \left(\int_{R^d} f(\mathbf{x}) p_{\tau/2}(\mathbf{x},\mathbf{z})\,d\mathbf{x}\right)^2 \geq 0,$$

so the lemma is proven. □

LEMMA 2.6. $S_d \subset M^d$.

PROOF. Use (2.3), the Fubini theorem and formulas (4.352(4)) and (6.223) of Gradshteyn and Ryzhik (1965) to see that $g(\mathbf{x})$ is in $L^1(R^d)$. Then by Reed and Simon [(1975), page 28], $\iint g(\mathbf{x}-\mathbf{y})|f(\mathbf{y})|\,d\mathbf{x}\,d\mathbf{y} < \infty$ for every $f \in L^1(R^d)$. Since all f from S_d are bounded, it follows that $\iint |f(\mathbf{x})| g(\mathbf{x}-\mathbf{y})|f(\mathbf{y})|\,d\mathbf{x}\,d\mathbf{y} < \infty$, so that we are done. □

Lemma 2.6 shows that the class of functions f for which a square-integrable functional $F_f(W)$ of a Brownian sheet W: $R_+^2 \to R^d$ is defined by (2.6) is wide enough to include all $f \in S_d$. However, we would like to extend the result of Proposition 2.1 in two directions:

1. To define a functional of several independent Brownian sheets.
2. To define a functional on the space of measures, so that, in particular, the local time $L_\mathbf{x}$ considered in the introduction will be well-defined [for $L_\mathbf{x}$, $f(W_{ts}) = \delta(W_1^d(t,s) - x_1) \cdots \delta(W_d^d(t,s) - x_d)$].

Theorem 2.7 gives the construction of square-integrable functionals of k independent Brownian sheets in its most general form.

Let N_k^d be a family of finite measures on R^{dk}, satisfying

$$\langle \gamma, \gamma \rangle = \int_{R^{2dk}} \gamma(d\mathbf{x}_1, \dots, d\mathbf{x}_k) g(\mathbf{x}_1 - \mathbf{y}_1) \cdots \times g(\mathbf{x}_k - \mathbf{y}_k) \gamma(d\mathbf{y}_1, \dots, d\mathbf{y}_k) < \infty, \tag{2.12}$$

$$\gamma(d\mathbf{x}_1, \dots, d\mathbf{x}_k) = \gamma(d\mathbf{x}_{\pi_1}, \dots, d\mathbf{x}_{\pi_k}) \quad \text{on } R^{dk}, \tag{2.13}$$

for each permutation $(\pi_1, \dots, \pi_k)$ of the set $\{1, \dots, k\}$.

THEOREM 2.7. *For $\gamma \in N_k^d$, such that*

$$\gamma(d\mathbf{x}_1, \dots, d\mathbf{x}_k) = f(\mathbf{x}_1, \dots, \mathbf{x}_k)\, d\mathbf{x}_1 \cdots d\mathbf{x}_k, \qquad \mathbf{x}_1, \dots, \mathbf{x}_k \in R^d, \tag{2.14}$$

define the following functional of k independent R^d-valued Brownian sheets [living on some infinite measure space (Ω, F, P')]:

$$\begin{aligned} F_\gamma \equiv F_f &= F_f(W^1, \dots, W^k) \\ &= \iint_{(R_+^2)^k} \theta(t_1, s_1) \cdots \theta(t_k, s_k) \\ &\qquad \times f\big(W^1(t_1, s_1), \dots, W^k(t_k, s_k)\big)\, dt_1\, ds_1 \cdots dt_k\, ds_k, \end{aligned} \tag{2.15}$$

$\theta(t, s) = se^{-s}e^{-st}$. Then $F_\gamma \in L^2(P')$. For $\gamma \in N_k^d$ that is not absolute continuous with respect to Lebesgue measure on $(R^d)^k$, define the "smoothed" density

$$\begin{aligned} &f_{\gamma, \delta}(\mathbf{x}_1, \dots, \mathbf{x}_k) \\ &\quad := e^{-(\delta_1 + \cdots + \delta_k)} \int_{R^{dk}} p_{\delta_1}(\mathbf{x}_1, \mathbf{y}_1) \cdots p_{\delta_k}(\mathbf{x}_k, \mathbf{y}_k) \gamma(d\mathbf{y}_1, \dots, d\mathbf{y}_k). \end{aligned} \tag{2.16}$$

Then, as $\delta = (\delta_1, \dots, \delta_k) \downarrow 0$, the functionals $F_{f_{\gamma,\delta}}$ converge in $L^2(P')$ to a functional, say F_γ, $F_\gamma \in L^2(P')$, which satisfies

$$\begin{aligned} E(F_\gamma)^2 &= \int_{R^{2dk}} \gamma(d\mathbf{x}_1, \dots, d\mathbf{x}_k) g(\mathbf{x}_1 - \mathbf{y}_1) \cdots \\ &\qquad \times g(\mathbf{x}_k - \mathbf{y}_k) \gamma(d\mathbf{y}_1, \dots, d\mathbf{y}_k) \\ &= \langle \gamma, \gamma \rangle. \end{aligned} \tag{2.17}$$

To prove Theorem 2.7, we need to start with a technical lemma that provides a conditional density function for one $R_+^2 \to R^1$ Brownian sheet. We then look at the simplest case, considered in Proposition 2.1, and finally the general case follows easily.

Let W_1 be the Brownian sheet from R_+^2 to R^1, as defined in (1.2), with $d = 1$.

LEMMA 2.8.

$$P'\big(W_1(t, s) \in dx, W_1(\tau, \delta) \in dy\big) = p_T(x, y)\, dx\, dy, \qquad x, y \in R^1, \tag{2.18}$$

where the density function $p_T(x, y)$ is the transition density function of the standard Brownian motion on R^1, i.e.,

$$p_T(x, y) \equiv p_T(x - y) = \frac{1}{\sqrt{2\pi T}} e^{-(x-y)^2/2T}, \qquad x, y \in R^1, T > 0, \tag{2.19}$$

and

$$T = \begin{cases} ts - \tau\sigma, & t > \tau, s > \sigma, \\ ts + \tau\sigma - 2t\sigma, & t < \tau, s > \sigma, \\ ts + \tau\sigma - 2\tau s, & t > \tau, s < \sigma, \\ \tau\sigma - ts, & t < \tau, s < \sigma. \end{cases} \tag{2.20}$$

This result follows from straightforward calculations on Brownian sheets, and we leave its verification to the reader.

PROOF OF PROPOSITION 2.1. We get the formula (2.7) by straightforward computation [remember that the components of the vector $W = (W_1^d, \dots, W_d^d)$ are independent $R_+^2 \to R^1$ Brownian sheets] using Lemma 2.8; viz.

$$\begin{aligned} E(F_f \cdot F_h) &= E\left\{ \iint_{R_+^2} \theta(t, s) f\big(W_1^d(t, s), \dots, W_d^d(t, s)\big)\, dt\, ds \right. \\ &\qquad \left. \times \iint_{R_+^2} \theta(\tau, \sigma) h\big(W_1^d(\tau, \sigma), \dots, W_d^d(\tau, \sigma)\big)\, d\tau\, d\sigma \right\} \\ &= \iint_{R^{2d}} d\mathbf{x}\, f(\mathbf{x})\, d\mathbf{y}\, h(\mathbf{y}) \\ &\qquad \times \iint_{R_+^4} \theta(t, s)\, dt\, ds\, \theta(\tau, \sigma)\, d\tau\, d\sigma\, p_T(x_1, y_1) \cdots p_T(x_d, y_d) \\ &\equiv \iint_{R^{2d}} d\mathbf{x}\, f(\mathbf{x}) g(\mathbf{x} - \mathbf{y}) h(\mathbf{y})\, d\mathbf{y}, \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} g(\mathbf{x} - \mathbf{y}) &= \iint_{R_+^4} \theta(t, s)\, dt\, ds\, \theta(\tau, \sigma)\, d\tau\, d\sigma \\ &\qquad \times p_T(x_1 - y_1) \cdots p_T(x_d - y_d) \end{aligned} \tag{2.22}$$

and p_T is defined by (2.19)–(2.20). Note that $p_T(x_1 - y_1) \cdots p_T(x_d - y_d) = p_T(\mathbf{x} - \mathbf{y})$, and then

$$\begin{aligned} g(\mathbf{x} - \mathbf{y}) &= 2\left\{ \iint_{\left\{ \substack{t > \tau \\ s > \sigma} \right\}} dt\, ds\, d\tau\, d\sigma\, \theta(t, s) \theta(\tau, \sigma) p_{ts - \tau\sigma}(\mathbf{x} - \mathbf{y}) \right. \\ &\qquad \left. + \iint_{\left\{ \substack{t < \tau \\ s > \sigma} \right\}} dt\, ds\, d\tau\, d\sigma\, \theta(t, s) \theta(\tau, \sigma) p_{ts + \tau\sigma - 2t\sigma}(\mathbf{x} - \mathbf{y}) \right\} \\ &:= 2\{\mathrm{I} + \mathrm{II}\}. \end{aligned}$$

To finish the proof we show that g can be written in the form (2.2)–(2.3). This follows from straightforward manipulation of the integrals, as follows:

1. Apply the transformation $s-\sigma=\alpha_1$, $\sigma=\beta_1$, $ts-\tau\sigma=\alpha_2$, $\sigma\tau=\beta_2$, $|J^{-1}|=s\sigma=(\alpha_1+\beta_1)\beta_1$ to the integral I, to obtain

$$\mathrm{I}=\int_0^\infty d\alpha_2\int_0^\infty d\beta_2\int_0^\infty d\alpha_1\int_{\beta_2\alpha_1/\alpha_2}^\infty d\beta_1\frac{1}{(\alpha_1+\beta_1)\beta_1}$$
$$\times\theta\left(\frac{\alpha_2+\beta_2}{\alpha_1+\beta_1},\alpha_1+\beta_1\right)\theta\left(\frac{\beta_2}{\beta_1},\beta_1\right)p_{\alpha_2}(\mathbf{x}-\mathbf{y}).$$

2. We obtain a similar form for the integral II, using the transformation $s-\sigma=\alpha_1$, $\sigma=\beta_1$, $ts+\tau\sigma-2t\sigma=\alpha_2$, $t\sigma=\beta_2$, $|J^{-1}|=\sigma^2=\beta_1^2$:

$$\mathrm{II}=\int_0^\infty d\alpha_2\int_0^\infty d\beta_2\int_0^\infty d\alpha_1\int_{\beta_2\alpha_1/\alpha_2}^\infty d\beta_1\frac{1}{\beta_1^2}\theta\left(\frac{\beta_2}{\beta_1},\alpha_1+\beta_1\right)$$
$$\times\theta\left(\frac{\alpha_2+\beta_2(1-\alpha_1/\beta_1)}{\beta_1},\beta_1\right)p_{\alpha_2}(\mathbf{x}-\mathbf{y}).$$

3. Rewrite g in the form (2.3):

$$g(\mathbf{x}-\mathbf{y})=2\{\mathrm{I}+\mathrm{II}\}=\int_0^\infty\rho(\alpha_2)p_{\alpha_2}(\mathbf{x}-\mathbf{y})\,d\alpha_2,$$

with

$$\rho(\alpha_2)=2\int_0^\infty d\beta_2\int_0^\infty d\alpha_1\int_{\beta_2\alpha_1/\alpha_2}^\infty d\beta_1\frac{1}{(\alpha_1+\beta_1)\beta_1}$$
$$\times\left\{\theta\left(\frac{\alpha_2+\beta_2}{\alpha_1+\beta_1},\alpha_1+\beta_1\right)\theta\left(\frac{\beta_2}{\beta_1},\beta_1\right)\right.$$
$$\left.+\left(\frac{\alpha_1+\beta_1}{\beta_1}\right)\theta\left(\frac{\beta_2}{\beta_1},\alpha_1+\beta_1\right)\theta\left(\frac{\alpha_2+\beta_2(1-\alpha_1/\beta_1)}{\beta_1},\beta_1\right)\right\}.$$

4. Put $\theta(t,s)=se^{-s}e^{-st}$ into $\rho(\alpha_2)$ to get

$$\rho(\alpha_2)=e^{-\alpha_2}\int_0^\infty e^{-2\beta_2}\,d\beta_2\int_0^\infty e^{-\alpha_1}\,d\alpha_1\int_{\beta_2\alpha_1/\alpha_2}^\infty 2e^{-2\beta_1}\left(\frac{\alpha_1}{\beta_1}+2\right)d\beta_1$$
$$=e^{-\alpha_2}\int_0^\infty e^{-2\beta_2}\,d\beta_2$$
$$\times\int_0^\infty e^{-\alpha_1}\,d\alpha_1\left\{\alpha_1\cdot 2\int_{\beta_2\alpha_1/\alpha_2}^\infty\frac{e^{-2\beta_1}}{2\beta_1}d(2\beta_1)+2\int_{\beta_2\alpha_1/\alpha_2}^\infty e^{-2\beta_1}\,d(2\beta_1)\right\}$$
$$=e^{-\alpha_2}\int_0^\infty e^{-2\beta_2}\,d\beta_2\int_0^\infty e^{-\alpha_1}\,d\alpha_1\left\{2\alpha_1E_1\left(\frac{2\beta_2\alpha_1}{\alpha_2}\right)+2e^{-2(\beta_2\alpha_1/\alpha_2)}\right\}$$
$$=e^{-\alpha_2}\int_0^\infty e^{-2\beta_2}\,d\beta_2\left\{\mathrm{I}(\beta_2)+\frac{2}{1+2\beta_2/\alpha_2}\right\}.$$

5. To calculate

$$I(\beta_2) = \int_0^\infty (2\alpha_1) e^{-\alpha_1} E_1\left(\frac{2\beta_2\alpha_1}{\alpha_2}\right) d\alpha_1,$$

differentiate $I(\beta_2)$, recalling that $(E_1(z))' = -e^{-z}/z$ [Gradshteyn and Ryzhik (1965), page xxxiii] to get

$$\begin{aligned} I'(\beta_2) &= (-2)\int_0^\infty \alpha_1 e^{-\alpha_1} \frac{e^{-2(\beta_2\alpha_1/\alpha_2)}}{\beta_2} d\alpha_1 \\ &= -\frac{2}{\beta_2(1 + 2\beta_2/\alpha_2)^2} = (-2\alpha_2^2)\frac{1}{\beta_2(\alpha_2 + 2\beta_2)^2}. \end{aligned}$$

Therefore, $I(\beta_2)$ is given by

$$\begin{aligned} I(\beta_2) &= (-2\alpha_2^2)\int \frac{d\beta_2}{\beta_2(\alpha_2 + 2\beta_2)^2} + \text{const.} \\ &= (-2\alpha_2^2)\left(\frac{1}{\alpha_2(\alpha_2 + 2\beta_2)} - \frac{1}{\alpha_2^2}\ln\left(\frac{\alpha_2 + 2\beta_2}{\beta_2}\right)\right) + \text{const.} \end{aligned}$$

6. Since $I(\infty) = 0$, const. $= -2\ln 2$, so that we finally get

$$\begin{aligned} \rho(\alpha_2) &= e^{-\alpha_2}\int_0^\infty e^{-2\beta_2}\left\{2\ln\left(\frac{\alpha_2 + 2\beta_2}{\beta_2}\right) - 2\ln 2\right\} d\beta_2 \\ &= e^{-\alpha_2}\{\ln\alpha_2 + e^{\alpha_2}E_1(\alpha_2) + C\}, \end{aligned}$$

and we are done. □

PROOF OF THEOREM 2.7. Assume for a moment $k = 1$. To establish convergence in L^2 of a sequence $\{F_{f_{\gamma,\delta}}\}$ (we put $F_\delta \equiv F_{f_{\gamma,\delta}}$ for convenience, in this proof) to some limit, we look at

$$E(F_\delta - F_\varepsilon)^2 = E(F_\delta F_\delta) - 2E(F_\delta F_\varepsilon) + E(F_\varepsilon F_\varepsilon).$$

Then to guarantee $E(F_\delta - F_\varepsilon)^2 \to_{\delta,\varepsilon\downarrow 0} 0$, it is enough to show that $E(F_\delta F_\varepsilon)$ approaches some limit that is independent of the way that ε and δ approach zero. The last expectation can be obtained directly from (2.7), (2.16), (2.2)–(2.3) and the Chapman–Kolmogorov equation:

$$\begin{aligned} E(F_\delta \cdot F_\varepsilon) &= \iint_{R^{2d}} d\mathbf{x}\, f_{\gamma,\delta}(\mathbf{x}) g(\mathbf{x} - \mathbf{y}) f_{\gamma,\varepsilon}(\mathbf{y})\, d\mathbf{y} \\ &= \iint_{R^{2d}} d\mathbf{x}\, d\mathbf{y}\, e^{-\delta}\int_{R^d} p_\delta(\mathbf{x},\mathbf{u})\gamma(d\mathbf{u}) e^{-\varepsilon} \\ &\quad \times \int_{R^d} p_\varepsilon(\mathbf{y},\mathbf{v})\gamma(d\mathbf{v}) \int_0^\infty e^{-\tau}\tilde{\rho}(\tau)\cdot p_\tau(\mathbf{x},\mathbf{y})\, d\tau \\ &= \iint_{R^{2d}} \gamma(d\mathbf{u})\gamma(d\mathbf{v}) \int_0^\infty e^{-(\tau+\varepsilon+\delta)}\tilde{\rho}(\tau) p_{\tau+\varepsilon+\delta}(\mathbf{u},\mathbf{v})\, d\tau. \end{aligned}$$

The function

$$\tilde{\rho}(\tau) = e^{\tau}E_1(\tau) + \ln \tau + C$$

is monotone-increasing on R^1_+, since its derivative is positive:

$$\frac{d}{d\tau}\tilde{\rho}(\tau) = \frac{d}{d\tau}(e^{\tau}E_1(\tau)) + \frac{1}{\tau} = e^{\tau}E_1(\tau) > 0, \qquad \tau > 0.$$

Then, for $\alpha = \varepsilon + \delta \downarrow 0$ as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$ we have

$$\int_0^{\infty} e^{-(\tau+\varepsilon+\delta)}\tilde{\rho}(\tau)p_{\tau+\varepsilon+\delta}(\mathbf{u},\mathbf{v})\,d\tau \equiv \int_{\alpha}^{\infty} e^{-t}p_t(\mathbf{u},\mathbf{v})\tilde{\rho}(t-\alpha)\,dt$$

$$\uparrow \int_0^{\infty} e^{-t}p_t(\mathbf{u},\mathbf{v})\tilde{\rho}(t)\,dt \equiv g(\mathbf{u},\mathbf{v}).$$

So, by the Lebesgue monotone convergence theorem,

$$\lim_{\alpha=\delta+\varepsilon\downarrow 0} E(F_{\delta}\cdot F_{\varepsilon}) = \int \gamma(d\mathbf{u})g(\mathbf{u}-\mathbf{v})\gamma(d\mathbf{v}),$$

and this finishes the proof for $k = 1$.

The proof for general $k > 1$ follows from similar arguments involving longer formulas but no new mathematics and we feel free to omit it. □

3. An example—intersections of two independent Brownian sheets. In the introduction we define a functional $L_{\mathbf{x}}$ that "measures how much time the Brownian sheet W^d spends at point $\mathbf{x}$." Using Theorem 2.7 we can now define this functional precisely, for $d < 4$, as a functional $F_{\gamma_{\mathbf{x}}}$ with $\gamma_{\mathbf{x}}(d\mathbf{y}) = \delta(\mathbf{y}-\mathbf{x})\,d\mathbf{y}$. To define an intersection local time for two independent Brownian sheets, we have to consider the measure on R^{2d}, given by

$$\text{(3.1)} \qquad \gamma(d\mathbf{x}^1, d\mathbf{x}^2) = \phi(\mathbf{x}^1)\delta(\mathbf{x}^1 - \mathbf{x}^2)\,d\mathbf{x}^1\,d\mathbf{x}^2, \qquad \phi \in S_d,$$

and to show that Theorem 2.7 can be applied to it. In this section, we shall show that for $d < 8$, $\gamma \in N_2^d$ and so F_{γ} can be defined. On the intuitive level this means that the two independent Brownian sheets W^1, W^2: $R^2_+ \to R^d$ have nontrivial intersections for $d < 8$.

The subject of intersections of independent Brownian sheets is not new. It is closely related to the problem of multiple points for a single sheet, as studied by Rosen (1984, 1986). Rosen (1984), for example, showed that W^d has nontrivial double self-intersections, with probability 1, as long as $d < 8$.

Now we shall formulate and prove

PROPOSITION 3.1. *The measure, γ, defined by* (3.1), *belongs to the family N_2^d, as long as $d < 8$.*

COROLLARY 3.2. *Two independent Brownian sheets have nontrivial intersections if $d < 8$. Their intersection local time, weighted by function ϕ, is defined as a functional F_{γ}, measure γ given by* (3.1).

PROOF. Note that

$$\langle \gamma, \gamma \rangle = \iint_{R^{4d}} \gamma(d\mathbf{x}^1, d\mathbf{x}^2) g(\mathbf{x}^1 - \mathbf{y}^1) g(\mathbf{x}^2 - \mathbf{y}^2) \gamma(d\mathbf{y}^1, d\mathbf{y}^2)$$
$$(3.2) \qquad = \iint_{R^{2d}} \phi(\mathbf{x}) g^2(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}.$$

By Reed and Simon (1975), $\widehat{f g} = \hat{f} * \hat{g}$, where the sign $*$ stands for convolution of functions. Therefore,

$$(3.3) \qquad \widehat{g^2} = \hat{g} * \hat{g}.$$

Furthermore, if $\hat{g} \in L^2(R^d)$, then from Proposition 0.2.1 of Butzer and Nessel (1971) it follows that $\widehat{g^2}$ is continuous and bounded, so that

$$\int_{R^d} |\hat{\phi}(\mathbf{k})|^2 \widehat{g^2}(\mathbf{k})\, d\mathbf{k} < \infty.$$

Then by a Parseval-type equality

$$\langle \gamma, \gamma \rangle = \iint_{R^{2d}} \phi(\mathbf{x}) g^2(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$$
$$= \int_{R^d} |\hat{\phi}(\mathbf{k})|^2 \widehat{g^2}(\mathbf{k})\, d\mathbf{k} < \infty.$$

Thus, to complete the proof, it is enough to show that $\hat{g} \in L^2(R^d)$ for $d < 8$.

By a straightforward calculation

$$(3.4) \qquad \int_{R^d} |\hat{g}(\mathbf{k})|^2\, d\mathbf{k} = \text{const.} \int_0^\infty x^{d/2-3} \frac{\ln^2(1+x)}{(1+x)^2}\, dx,$$

the last integral being finite for $1 \le d < 8$ [we use a comparison argument and formulas (4.291) and (4.293) of Gradshteyn and Ryzhik (1965)]. So we are done. □

We conclude this section with the comment that, for $d \ge 8$, $\hat{g}$ is not in $L^2(R^d)$:

$$\int_{R^d} |\hat{g}(\mathbf{k})|^2\, d\mathbf{k} \ge \text{const.} \int_{e-1}^\infty \tfrac{1}{4} x^{d/2-5}\, dx = \infty \quad \text{for } d \ge 8.$$

4. The limit theorems. In this section we study the behaviour of an infinite system of independent Brownian sheets.

Let (Ω, F, P) be a probability space and Π^λ be a Poisson point process on R^d of intensity λ. This means that the number of points in a Borel set $A \subset R^d$ is a Poisson random variable with mean $\lambda|A|$ (we use $|\cdot|$ to denote Lebesgue measure), and the numbers in disjoint sets are independent. The points of Π^λ can be ordered by their magnitude and we shall denote them by $W_0^1, \dots, W_0^i, \dots$. We take $W^1, \dots, W^i, \dots$ to be an infinite system of independent Brownian sheets with initial values given by $W_0^1, \dots, W_0^i, \dots$. This setting is similar to one of the Poisson particle systems introduced by Martin-Löf (1976), although we consider an evolution of Brownian sheets instead of Markov processes. Walsh (1986) studies a Poisson system of branching Brownian motions and Adler and Epstein (1988) establish limit theorems for intersection local time for a Poisson system of planar Brownian motions. Our aim in the current and the next section is to study the limits of sums of functionals for a Poisson system of Brownian sheets.

Let us assume that the sheets are of two types by assigning a random sign to each of them, i.e., we take $\sigma_1, \dots, \sigma_i, \dots$ to be an infinite system of independent random variables (signs) defined by $P(\sigma_i = 1) = P(\sigma_i = -1) = \frac{1}{2}$ (the sequence is independent of W^i's and Π^λ).

We are interested in the limit behaviour, as $\lambda \to \infty$, of the average amount of time the system of Brownian sheets spends at a point $\mathbf{x} \in R^d$. That is, we study the weak convergence, as $\lambda \to \infty$, of the sum

$$\begin{aligned} \phi_\lambda(f) &:= \lambda^{-1/2} \sum_i \sigma_i F_f(W^i) \\ &\equiv \lambda^{-1/2} \sum_i \sigma_i \iint_{R_+^2} \theta(t,s) f(W^i(t,s))\, dt\, ds, \qquad f \in S_d, \end{aligned} \tag{4.1}$$

where we sum over an infinite number of individuals in our collection.

In our intuitive interpretation, we have $f = \delta_{\mathbf{x}}$ and $F_f = L_{\mathbf{x}}$, which works well if $d \le 3$. For higher dimensions we have to consider functionals F_γ, $\gamma \in N_1^d$, so that we define also

$$\phi_\lambda(\gamma) := \lambda^{-1/2} \sum_i \sigma_i F_\gamma(W^i), \qquad \gamma \in N_1^d. \tag{4.2}$$

$\phi_\lambda(f)$ and $\phi_\lambda(\gamma)$ can be considered as random fields on S_d and N_1^d, correspondingly.

We now define the Gaussian field $\{\phi(f),\ f \in S_d\}$ with

$$E\phi(f) = 0, \qquad E\{\phi(f) \cdot \phi(h)\} = \langle f, h \rangle, \qquad f, h \in S_d, \tag{4.3}$$

and the Gaussian field $\{\phi(\gamma),\ \gamma \in N_1^d\}$ with

$$E\phi(\gamma) = 0, \qquad E\{\phi(\gamma) \cdot \phi(\mu)\} = \langle \gamma, \mu \rangle, \qquad \gamma, \mu \in N_1^d. \tag{4.4}$$

Theorem 4.1. *As $\lambda \to \infty$, the field $\{\phi_\lambda(\gamma),\ \gamma \in N_1^d\}$ converges in the sense of weak convergence of finite-dimensional distributions to the Gaussian field $\{\phi(\gamma),\ \gamma \in N_1^d\}$.*

Theorem 4.2. *As $\lambda \to \infty$, the field $\phi_\lambda(f)$ on S_d converges in the sense of weak convergence of finite-dimensional distributions to the Gaussian field $\{\phi(f),\ f \in S_d\}$.*

The proof of these theorems appears at the end of this section.

We stated these two theorems separately because of their simplicity and because for convergence to the Gaussian limit it is enough to show moment convergence. For the general weak-convergence result, which includes Theorems 4.1 and 4.2 as a special case, we need to recall the Dynkin–Mandelbaum construction of Wiener integrals on an arbitrary measure space (X, B, ν). [See Dynkin and Mandelbaum (1983).]

Define a Gaussian family $\{I_1(f),\ f \in L^2(\nu)\}$ with

$$(4.5)\quad E\{I_1(f)\} = 0, \qquad E\{I_1(f)I_1(h)\} = \nu(f,h) := \int_X f(x)h(x)\nu(dx).$$

Let $H_k = L^2_{\text{symm}}(\nu \times \cdots \times \nu)$ be a space of symmetric functions $h_k(x_1,\dots,x_k)$, such that

$$(4.6)\qquad \nu^k(h_k^2) = \int h_k^2(x_1,\dots,x_k)\nu(dx_1)\cdots\nu(dx_k) < \infty.$$

The multiple Wiener integral of order k, associated with the Gaussian family I_1, is defined as a linear mapping I_k from H_k into the space of random variables, which are functionals of the Gaussian family $I_1(f)$. The mapping is defined uniquely by the following conditions:

Condition A. For functions of the form

$$(4.7)\qquad h_k^f(x_1,\dots,x_k) = f(x_1)\cdots f(x_k), \qquad f \in L^2(\nu),$$

we have

$$(4.8)\qquad I_k(h_k^f) = \left(\nu(f^2)\right)^{k/2} E_k\left(\frac{I_1(f)}{\sqrt{\nu(f^2)}}\right),$$

where E_k is the Hermite polynomial of degree k with leading coefficient 1.

Condition B. For $h_k \in H_k$, $E\{I_k^2(h_k)\} = k!\nu^k(h_k^2)$.

Theorem 4.3 [Dynkin and Mandelbaum (1983)]. *Let $X_1, X_2, \dots$ be independent and identically distributed random variables taking values in (X, B), and with distribution ν. For $\lambda > 0$ let N_λ be a Poisson variable with mean λ independent of the X_i. For $k = 1,2,\dots,$ take $h_k \in H_k$ and let $I_k(h_k)$ be its multiple Wiener integral. If*

$$(4.9)\qquad \sum_{k=1}^{\infty} \frac{1}{k!} E\{h_k^2(X_1,\dots,X_k)\} < \infty$$

and

$$(4.10)\qquad E\{h_k(x_1,\dots,x_{k-1},X_k)\} \equiv \int_X h_k(x_1,\dots,x_{k-1},y)\nu(dy) = 0,$$

then the random variables

$$Z_\lambda(h_1,h_2,\dots) := \sum_{k=1}^{\infty} \lambda^{-k/2} \sum_{1\le i_1<\cdots<i_k\le N_\lambda}\!\!\cdots\sum h_k(X_{i_1},\dots,X_{i_k})$$

converge in distribution, as $\lambda\to\infty$, *to*

$$Z(h_1,h_2,\dots) := \sum_{k=1}^{\infty}\frac{1}{k!}I_k(h_k).$$

Remarks in Section 1.5 of Dynkin and Mandelbaum (1983) allow us to drop the condition (4.10) by inserting the random signs $\sigma_1,\sigma_2,\dots$ into the definition of Z_λ and to view a sample of a Poisson size $X_1,\dots,X_{N_\lambda}$ as a Poisson point process.

Now we are ready to formulate the main result of this section. For $\gamma_k\in N_k^d$ define

$$\psi_\lambda(\gamma_k) := \lambda^{-k/2}\sum_{i_1<\cdots<i_k}\!\!\cdots\sum \sigma_{i_1}\cdots\sigma_{i_k}F_{\gamma_k}(W^{i_1},\dots,W^{i_k}).$$

Theorem 4.4. *As* $\lambda\to\infty$, *the pair* $\langle\phi_\lambda(\gamma),\psi_\lambda(\gamma_k)\rangle$ *on* $N_1^d\times N_k^d$ *converge, in the sense of weak convergence of finite-dimensional distributions, to the pair* $\langle\phi(\gamma),(1/k!)\psi(\gamma_k)\rangle$, *where* $\psi(\gamma_k)$ *is the multiple Wiener integral of order* k *associated with the Gaussian family* $\{\phi(\gamma),\ \gamma\in N_1^d\}$.

Proof. We apply Theorem 4.3 to $X_i = W^i$, $X=(R^d)^{R_+^2}$; ν is the probability distribution W^i induces on the Borel σ-algebra of X. For every $\gamma_k\in N_k^d$, let F_{γ_k} be the corresponding functional of k independent copies of W^i. Then by Theorem 4.3 and remarks following it

$$\sum_{k=1}^{\infty}\psi_\lambda(\gamma_k) \equiv \sum_{k=1}^{\infty}\lambda^{-k/2}\sum_{i_1<\cdots<i_k}\!\!\cdots\sum\sigma_{i_1}\cdots\sigma_{i_k}F_{\gamma_k}(W^{i_1},\dots,W^{i_k})$$

converges in distribution to $\sum_{k=1}^{\infty}(1/k!)I_k(F_{\gamma_k})$, provided $\sum_{k=1}^{\infty}(1/k!)\langle\gamma_k,\gamma_k\rangle<\infty$. The Gaussian family $I_1(F_{\gamma_1})$, $\gamma_1\in N_1^d$, is mean zero and

$$E\Big\{\big(I_1(F_{\gamma_1})\big)^2\Big\} = \langle\gamma_1,\gamma_1\rangle = E\big\{(\phi(\gamma_1))^2\big\},$$

so that in the Gaussian case $I_1(F_{\gamma_1}) =_{\mathscr{L}} \phi(\gamma_1)$. Then the functionals $I_k(F_{\gamma_k})$ of the family $\{I_1(F_{\gamma_1})\}$ have the same distribution as the k-multiple Wiener integral, associated with the Gaussian family $\{\phi(\gamma_1)\}$, that we denoted by $\psi(\gamma_k)$. This proves the theorem. □

Proof of Theorem 4.1. By the polarization identity

$$\prod_{i=1}^{m} a_i = \frac{1}{m!} \sum_{r=1}^{m} (-1)^{m-r} \sum_{1 \le i_1 < \cdots < i_r \le m} (a_{i_1} + \cdots + a_{i_r})^m$$

it is enough to consider $E\{(\phi_\lambda(\gamma))^n\}$, $n = 1, 2, \ldots$. But the calculations here are virtually identical to those of the proof of Theorem 4.3 in Section 7 of Adler and Epstein (1987) with $m = n$, $F_\gamma(W^i)$ instead of $F_\gamma(X_i)$ and $r_\alpha = \langle \gamma, \gamma \rangle$ in (7.14). Therefore,

$$E\{(\phi_\lambda(\gamma))^n\} = 0, \quad \text{for } n \text{ odd}, \tag{4.11}$$

$$E\{(\phi_\lambda(\gamma))^n\} \to (n-1)(n-3)\cdots 1(\langle \gamma, \gamma \rangle)^n, \quad \text{as } \lambda \to \infty \text{ for } n \text{ even}. \tag{4.12}$$

Since the moments in the right side of (4.11)–(4.12) are exactly those of $\phi(\gamma)$ and determine the Gaussian distribution, we are done. □

Theorem 4.2 can be proved in exactly the same way.

5. The properties of the limit fields. In this final section, we look at some properties of the Gaussian field $\{\phi(f),\ f \in S_d\}$ and its functionals $\{\psi(\gamma_k),\ \gamma_k \in N_k^d\}$.

Stationarity. From the field's covariance representation

$$E\{\phi(f)\phi(h)\} = \langle f, h \rangle = \iint_{R^{2d}} f(\mathbf{x}) g(\mathbf{x} - \mathbf{y}) h(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$$

$$= \int_{R^d} \hat{f}(\mathbf{k}) \hat{h}^*(\mathbf{k}) \hat{g}(\mathbf{k})\, d\mathbf{k},$$

(* means complex conjugation) we conclude that $\{\phi(f),\ f \in S_d\}$ is stationary with spectral measure

$$G(d\mathbf{k}) = \hat{g}(\mathbf{k})\, d\mathbf{k} = \frac{\ln(1 + |\mathbf{k}|^2/2)}{(|\mathbf{k}|^2/2)(1 + |\mathbf{k}|^2/2)}\, d\mathbf{k}, \qquad \mathbf{k} \in R^d. \tag{5.1}$$

Markovianess. We observe that the function $1/\hat{g}(\lambda)$, where $\hat{g}(\lambda)$ is the spectral density of the field, is not a polynomial. The conditions of the theorem in Rosanov [(1982), page 120] are violated and so $\{\phi(f),\ f \in S_d\}$ is not Markov [in the sense of the definition of Rosanov (1982), page 112].

Self-similarity and renormalizability. Adler and Epstein (1987) discussed the notion of self-similarity of random fields and introduced a new notion of renormalizability.

A family of random fields $\phi^\zeta \equiv \{\phi^\zeta(f),\ f \in S_d\}$, $\zeta > 0$, is called renormalizable with renormalization parameters (α, r), if for every $\eta > 0$

$$\phi^\zeta =_{\mathscr{L}} {}^{\alpha}_{\eta}\phi^{\zeta\eta^{-r}}, \tag{5.2}$$

where

$$ {}^{\alpha}_{\eta}\phi^{\zeta\eta^{-r}}(f) := \phi^{\zeta\eta^{-r}}\left({}^{\alpha}_{\eta}f\right), \qquad {}^{\alpha}_{\eta}f(x) := \eta^{-\alpha}f(x/\eta). \tag{5.3}$$

(For $r = 0$ we get the usual definition of self-similarity.)

To define the family of Gaussian fields $\{\phi^\zeta(f),\ f \in S_d\}$, consider, for every $\zeta > 0$, the function

$$\theta = \theta(\zeta)(t, s) = se^{-s-\zeta ts}. \tag{5.4}$$

It is an easy exercise to check that, for this function, Theorems 2.7, 4.2 and 4.4 hold, and, if we define the family of Gaussian fields $\{\phi^{\theta(\zeta)}(f),\ f \in S_d\}$, $\zeta > 0$, with covariance kernels

$$g^{\theta(\zeta)}(\mathbf{x},\mathbf{y}) = \int_0^\infty \rho(\tau, \theta(\zeta))p_\tau(\mathbf{x}-\mathbf{y})\,d\tau, \tag{5.5}$$

$$\begin{aligned}\rho(\tau, \theta(\zeta)) &= \frac{1}{\zeta}e^{-\zeta\tau}\{\ln(\zeta\tau) + C + e^{\zeta\tau}E_1(\zeta\tau)\}\\ &= \frac{1}{\zeta}\rho(\zeta\tau, \theta),\end{aligned} \tag{5.6}$$

and spectral density

$$\hat{g}^{\theta(\zeta)}(\mathbf{k}) = \frac{1}{\zeta^2}\hat{g}^{\theta}\left(\mathbf{k}/\sqrt{\zeta}\right) = \frac{1}{\zeta^2}\frac{\ln\left(1 + |\mathbf{k}|^2/2\zeta\right)}{\left(|\mathbf{k}|^2/2\zeta\right)\left(1 + |\mathbf{k}|^2/2\zeta\right)}, \tag{5.7}$$

then we have

Theorem 5.1. *The family of Gaussian fields* $\{\phi^{\theta(\zeta)},\ \zeta > 0\}$ *is renormalizable with parameter* $(d/2 + 2, 2)$.

Remark. A Markov process $X(t)$, that starts at zero, i.e., $X(0) = 0$, is called self-similar if $\eta X(\eta^{-\beta}t) =_{\mathscr{L}} X(t)$ for all $\eta > 0$ [see Adler and Epstein (1987), Section 6]. We can extend this definition for the random fields, defined on R^n_+, and starting from zero on the axes, by calling X: $R^n_+ \to R^1$ self-similar with parameters β if

$$\eta^n X(\eta^{-\beta}\mathbf{t}) =_{\mathscr{L}} X(\mathbf{t}) \quad \text{for all } \eta > 0. \tag{5.8}$$

Then it is easy to see that the Brownian sheet B: $R^2_+ \to R^1$, is self-similar with parameter 2.

PROOF OF THEOREM 5.1. We deal with mean-zero Gaussian fields. Therefore, it is enough to show that the covariances match. For $\alpha = d/2 + 2$, $\eta > 0$,

$$(5.9)\quad \begin{aligned} E\{{}^{\alpha}_{\eta}\phi^{\zeta\eta^{-2}}(f)\cdot{}^{\alpha}_{\eta}\phi^{\zeta\eta^{-2}}(h)\} &= E\{\phi^{\zeta\eta^{-2}}({}^{\alpha}_{\eta}f)\cdot\phi^{\zeta\eta^{-2}}({}^{\alpha}_{\eta}h)\} \\ &= \int_{R^d}{}^{\alpha}_{\eta}\hat{f}(\mathbf{k}){}^{\alpha}_{\eta}\hat{h}^*(\mathbf{k})\hat{g}^{\theta(\zeta\eta^{-2})}(\mathbf{k})\,d\mathbf{k} \\ &= \eta^{-2\alpha+2d}\int_{R^d}\hat{f}(\eta\mathbf{k})\hat{h}^*(\eta\mathbf{k})(\zeta\eta^{-2})^{-2}\hat{g}^{\theta}\big(\mathbf{k}/(\sqrt{\zeta}\eta^{-1})\big)\,d\mathbf{k} \\ &= \eta^{-2\alpha+2d+4-d}\int_{R^d}\hat{f}(\mathbf{k})\hat{h}^*(\mathbf{k})(\zeta^{-2})\hat{g}^{\theta}(\mathbf{k}/\sqrt{\zeta})\,d\mathbf{k} \\ &= \int_{R^d}\hat{f}(\mathbf{k})\hat{h}^*(\mathbf{k})\hat{g}^{\theta(\zeta)}(\mathbf{k})\,d\mathbf{k} \\ &= E\{\phi^{\theta(\zeta)}(f)\phi^{\theta(\zeta)}(h)\}. \end{aligned}$$

This proves the theorem. □

Renormalization of functionals of the fields. Let, for $\zeta > 0$,

$$\psi^{\theta(\zeta)} \equiv \{\psi^{\theta(\zeta)}(\gamma_k),\ \gamma_k \in N_k^{\theta(\zeta),d}\}$$

be the functional of the Gaussian field $\{\phi^{\theta(\zeta)}(\gamma),\ \gamma \in N_1^{\theta(\zeta),d}\}$ defined by Theorem 4.4, with $\theta = \theta(\zeta)$ from (5.4) [to define $N_k^{\theta(\zeta)}$ change g to $g^{\theta(\zeta)}$ in (2.12)–(2.13)].

THEOREM 5.2. *The family of fields $\{\psi^{\theta(\zeta)},\ \zeta > 0\}$ is renormalizable for the pair $([k(d+4)]/2, 2)$ in the sense that for every $\gamma_k \in N_k^{\theta(\zeta),d}$ and for all $\eta > 0$,*

$$(5.10)\qquad \psi^{\theta(\zeta)}(\gamma_k) =_{\mathscr{L}} \psi^{\theta(\zeta\eta^{-r})}({}^{\alpha}_{\eta}\gamma_k),$$

with

$$(5.11)\qquad {}^{\alpha}_{\eta}\gamma_k(A) := \eta^{-\alpha+dk}\gamma(\eta^{-1}A) \quad \text{and} \quad \alpha = \frac{k(d+4)}{2}, \qquad r = 2.$$

PROOF. Note that for $\gamma_k(d\mathbf{x}_1,\dots,d\mathbf{x}_k) = q(\mathbf{x}_1,\dots,\mathbf{x}_k)\,d\mathbf{x}_1\cdots d\mathbf{x}_k$ on R^{dk}, the density function of ${}^{\alpha}_{\eta}\gamma_k$ is

$$(5.12)\qquad {}^{\alpha}_{\eta}q(\mathbf{x}_1,\dots,\mathbf{x}_k) = \eta^{-\alpha}q(\eta^{-1}\mathbf{x}_1,\dots,\eta^{-1}\mathbf{x}_k),$$

so that the definition of renormalizability of the fields, defined on measures, matches the definition (5.2)–(5.3). For the proof, we shall assume γ is absolutely continuous on R^{dk}, with density q, since the general case follows by passage to the limit. First we check that $\psi^{\theta(\zeta\eta^{-2})}({}^{\alpha}_{\eta}\gamma_k)$ is well-defined, that is, ${}^{\alpha}_{\eta}\gamma_k \in N_k^{\theta(\zeta\eta^{-2}),d}$. Since $p_\tau(\mathbf{x}-\mathbf{y})$ in (5.5) is the transition density of the Brownian motion, i.e., of the self-similar Markov process with parameter 2, it satisfies [cf. Adler and Epstein (1987), Section 6]

$$(5.13)\qquad p_{\eta^{-2}\tau}(\mathbf{x},\mathbf{y}) = \eta^d p_\tau(\eta\mathbf{x},\eta\mathbf{y}), \qquad \eta > 0.$$

Then, for the covariance kernel $g^{\theta(\zeta)}$ we have

$$(5.14)\qquad \begin{aligned} g^{\theta(\zeta)}(\mathbf{x},\mathbf{y}) &= \int_0^\infty \rho(\tau,\theta(\zeta))p_\tau(\mathbf{x}-\mathbf{y})\,d\tau \quad [\text{by (5.6)}] \\ &= \int_0^\infty \frac{1}{\zeta}\rho(\zeta\tau,\theta)p_\tau(\mathbf{x}-\mathbf{y})\,d\tau \quad (\zeta\tau = t) \\ &= \int_0^\infty \frac{1}{\zeta^2}\rho(t,\theta)p_{\zeta^{-1}t}(\mathbf{x}-\mathbf{y})\,dt \quad [\text{by (5.13)}] \\ &= \int_0^\infty \frac{1}{\zeta^2}\rho(t,\theta)\zeta^{d/2}p_t\big(\sqrt{\zeta}\,\mathbf{x}-\sqrt{\zeta}\,\mathbf{y}\big)\,dt \\ &= \zeta^{d/2-2}g^\theta\big(\sqrt{\zeta}\,\mathbf{x},\sqrt{\zeta}\,\mathbf{y}\big). \end{aligned}$$

Thus

$$(5.15)\qquad \begin{aligned} \langle {}^\alpha_\eta\gamma_k, {}^\alpha_\eta\gamma_k\rangle_{\theta(\zeta\eta^{-2})} &= \iint_{R^{2dk}} {}^\alpha_\eta q(\mathbf{x}_1,\dots,\mathbf{x}_k)g^{\theta(\zeta\eta^{-2})}(\mathbf{x}_1,\mathbf{y}_1)\cdots \\ &\quad\times g^{\theta(\zeta\eta^{-2})}(\mathbf{x}_k,\mathbf{y}_k)\,{}^\alpha_\eta q(\mathbf{y}_1,\dots,\mathbf{y}_k)\,d\mathbf{x}_1\cdots d\mathbf{x}_k\,d\mathbf{y}_1\cdots d\mathbf{y}_k \\ &= \iint_{R^{2dk}} \eta^{-\alpha}q\big(\eta^{-1}\mathbf{x}_1,\dots,\eta^{-1}\mathbf{x}_k\big)\big(\zeta^{d/2-2}\eta^{-d+4}\big)^k \\ &\quad\times g^\theta\big(\sqrt{\zeta}\,\eta^{-1}\mathbf{x}_1,\sqrt{\zeta}\,\eta^{-1}\mathbf{y}_1\big)\cdots \\ &\quad\times g^\theta\big(\sqrt{\zeta}\,\eta^{-1}\mathbf{x}_k,\sqrt{\zeta}\,\eta^{-1}\mathbf{y}_k\big)\eta^{-\alpha} \\ &\quad\times q\big(\eta^{-1}\mathbf{y}_1,\dots,\eta^{-1}\mathbf{y}_k\big)\,d\mathbf{x}_1\cdots d\mathbf{x}_k\,d\mathbf{y}_1\cdots d\mathbf{y}_k \\ &= \eta^{-2\alpha-dk+4k+2dk}\iint_{R^{2dk}} q(\mathbf{x}_1,\dots,\mathbf{x}_k)\big(\zeta^{d/2-2}\big)^k \\ &\qquad\times g^\theta\big(\sqrt{\zeta}\,\mathbf{x}_1,\sqrt{\zeta}\,\mathbf{y}_1\big)\cdots g^\theta\big(\sqrt{\zeta}\,\mathbf{x}_k,\sqrt{\zeta}\,\mathbf{y}_k\big) \\ &\qquad\times q(\mathbf{y}_1,\dots,\mathbf{y}_k)\,d\mathbf{x}_1\cdots d\mathbf{x}_k d\mathbf{y}_1\cdots d\mathbf{y}_k \\ &= \langle\gamma_k,\gamma_k\rangle_{\theta(\zeta)}, \quad \text{for } \alpha = [k(d+4)]/2. \end{aligned}$$

Now, by Theorem 4.4, for $\lambda \to \infty$ the sum

$$\lambda^{-k/2}\sum_{i_1<\cdots<i_k}\cdots\sum \sigma_{i_1}\cdots\sigma_{i_k}F^{\theta(\zeta\eta^{-2})}_{{}^\alpha_\eta\gamma_k}(W^{i_1},\dots,W^{i_k})$$

converges in distribution to $(1/k!)\psi^{\theta(\zeta\eta^{-2})}({}^\alpha_\eta\gamma_k)$, the k-order multiple Wiener integral, associated with the Gaussian family $\{\phi^{\theta(\zeta\eta^{-2})}({}^\alpha_\eta\gamma)\} \equiv \{\psi^{\theta(\zeta\eta^{-2})}({}^\alpha_\eta\gamma_1)\}$. But by (5.15),

$$\begin{aligned} E\Big\{\Big(\phi^{\theta(\zeta\eta^{-2})}\big({}^\alpha_\eta\gamma\big)\Big)^2\Big\} &= \langle {}^\alpha_\eta\gamma, {}^\alpha_\eta\gamma\rangle_{\theta(\zeta\eta^{-2})} \\ &= \langle\gamma,\gamma\rangle_{\theta(\zeta)} = E\Big\{\big(\phi^{\theta(\zeta)}(\gamma)\big)^2\Big\}, \end{aligned}$$

so that $\phi^{\theta(\zeta\eta^{-2})}({}^{\alpha}_{\eta}\gamma) =_{\mathscr{L}} \phi^{\theta(\zeta)}(\gamma)$, $\gamma \in N_1^{\theta(\zeta), d}$, and therefore also $\psi^{\theta(\zeta\eta^{-2})}({}^{\alpha}_{\eta}\gamma_k) =_{\mathscr{L}} \psi^{\theta(\zeta)}(\gamma_k)$, the multiple Wiener integral, associated with the Gaussian family $\{\phi^{\theta(\zeta)}(\gamma)\}$. This proves the theorem. □

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