The Annals of Probability
1989, Vol. 17, No. 2, 433–443

SURVIVAL OF NEAREST-PARTICLE SYSTEMS WITH LOW BIRTH RATE[1]

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Nearest-particle systems form a class of continuous-time interacting particle systems on $\mathbb{Z}$. The birth rate $\beta(l, r)$ at a given site depends on the distances l and r to the nearest occupied sites on the left and right; deaths occur at rate 1. Assume that $b(n) = \sum_{l+r=n}\beta(l, r)$, $2 \le n < \infty$, $b(\infty) = \sum_{l=1}^{\infty}\beta(l, \infty) + \sum_{r=1}^{\infty}\beta(\infty, r)$, is constant. In Liggett [6] the question was posed whether for $b(n) \equiv 1 + \varepsilon$, $2 \le n \le \infty$, with $0 < \varepsilon \le 1$, there are such systems which survive for all t. Here, we answer affirmatively for all such ε and construct a class of examples.

1. Introduction. Nearest-particle systems form a familiar class of continuous-time interacting particle systems on $\mathbb{Z}$. As time evolves, particles are born and die. At most one particle is permitted at each site at a given time; states can therefore be identified with occupied subsets $A \subset \mathbb{Z}$. The process is Markov with birth rates $\beta(l, r)$ and death rate 1:

(1a) $$A \to A \cup \{x\} \quad \text{at rate } \beta(l_A(x), r_A(x)) \text{ for each } x \notin A$$

and

(1b) $$A \to A \setminus \{x\} \quad \text{at rate 1 for each } x \in A.$$

Here

$$l_A(x) = x - \max\{y\colon\ y \le x \text{ and } y \in A\},$$
$$r_A(x) = \min\{y\colon\ y \ge x \text{ and } y \in A\} - x$$

and $l_A(x)$ or $r_A(x)$ is $+\infty$ if the maximum or minimum is not defined. It is also typically assumed that

(2) $$\beta(\infty, \infty) = 0$$

(so that $\varnothing$ is a trap), and

(3) $$\beta(l, r) = \beta(r, l) \quad \text{for all } 1 \le l, r \le \infty, \qquad \sum_{l=1}^{\infty}\beta(l, \infty) < \infty.$$

Here, ξ_t will denote such a process. $\xi_t(x) = 0, 1$ will denote the state at x, and ξ_t^A the process with initial state A. (The superscript will often be suppressed.) As is typically the case for interacting particle systems, one is interested in

Received July 1987; revised June 1988.
[1]Research supported in part by NSF Grant DMS-83-01080.
AMS 1980 *subject classification*. 60K35.
Key words and phrases. Nearest-particle system, low birth rate, survival.

formulating conditions on $\beta(l, r)$ under which the process survives, that is,

$$P\left[\xi_t^A \neq \varnothing \text{ for all } t\right] > 0.$$

One is also frequently interested in the stronger statement

$$\liminf_{t\to\infty} P\left[x \in \xi_t^A\right] > 0.$$

If the process does not survive, we will say that it dies out, or becomes extinct.

Various classes of nearest-particle systems have been studied; see Liggett [7] for a general survey. Notable examples include reversible nearest-particle systems, first studied by Spitzer [8], and the contact process. The latter has rates

$$\begin{aligned} \beta(l, r) &= \lambda && \text{if } l = r = 1, \\ &= \lambda/2 && \text{if } l = 1,\ r > 1 \text{ or } l > 1,\ r = 1, \\ &= 0 && \text{otherwise.} \end{aligned} \tag{4}$$

This process has been the subject of a considerable amount of study. It was shown in Holley and Liggett [5] that the contact process survives for $\lambda \geq 4$; one can show that for $\lambda \leq 2.36$, the process dies out (Harris [4]).

Set

$$b(n) = \sum_{l+r=n} \beta(l, r), \qquad 2 \leq n < \infty \tag{5a}$$

and

$$b(\infty) = \sum_{l=1}^{\infty} \beta(l, \infty) + \sum_{r=1}^{\infty} \beta(\infty, r). \tag{5b}$$

$b(n)$ is the total birth rate on an interval of length n between two occupied sites and $b(\infty)$ the birth rate on the union of the two infinite intervals. One can ask how survival depends on $\{b(n)\}$. By using an argument based on that of [5], it was shown in [6] that for $0 < |A| < \infty$, ξ_t^A survives irrespective of the specific rule $\beta(l, r)$ as long as $b(n) \geq 4$ for $2 \leq n \leq \infty$. On the other hand, it is easy to show that if $b(n) \leq 1$, then the process dies out. (One can dominate $|\xi_t^A|$ by the critical binary branching process.) Reversible nearest-particle systems provide examples for survival with $b(n) \equiv b$, $b > 2$ (Griffeath and Liggett [3] and Liggett [6]). In [6], the question was asked whether there are examples of nearest-particle systems with $b \leq 2$ and which survive.

Here, we construct such examples. Specifically, let ${}^M\xi_t$ denote the nearest-particle system with birth rates given by

$$\begin{aligned} {}^M\beta(l, r) &= (1 + \varepsilon)/((n - 1) \wedge 2M) && \text{for } l \wedge r \leq M, \\ &= 0 && \text{for } l \wedge r > M, \end{aligned} \tag{6}$$

for $2 \leq n \leq \infty$ where $n = l + r$. If $n \leq 2M + 1$, then the birth rate is evenly distributed over the interval between occupied sites, whereas if $n \geq 2M + 2$, then the birth rate is $(1 + \varepsilon)/2M$ at sites within distance M of an endpoint and 0 otherwise. Clearly, $b(n) \equiv 1 + \varepsilon$ for $2 \leq n \leq \infty$. We will show

THEOREM 1. *For given $\varepsilon > 0$ and large enough M (depending on ε),*

$$\liminf_{t\to\infty} P\left[x \in {}^M\xi_t^A\right] > 0 \tag{7}$$

if $A \neq \varnothing$ and $x \in \mathbb{Z}$.

One may wish to compare ${}^M\xi_t$ with the uniform long-range contact process in Bramson and Gray [1]. The birth rate there is given by the first line of (6) but with the denominator not truncated by $2M$; the initial state is $\mathbb{Z}$. The analog of (7) is shown for $\varepsilon > 3$.

2. Proof of Theorem 1.

Basic ideas. We begin by introducing several auxiliary processes of ${}^M\xi_t$. Let ${}^M\tilde{\xi}_t$ denote the nearest-particle system on $\mathbb{Z}$ with birth rates

$$\begin{aligned} {}^M\tilde{\beta}(l,r) &= {}^M\beta(l,r) \quad \text{for } l, r < \infty, \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned} \tag{8}$$

The process ${}^M\tilde{\xi}_t$ has the same birth (and death) rates as ${}^M\xi_t$, except that births are now only permitted between particles, that is, $b(\infty) = 0$. The leftmost occupied site is therefore an increasing function of t and the rightmost site a decreasing function.

Set

$$ {}^MI(z) = [2Mz, 2M(z+1)), \tag{9}$$

where $z \in \mathbb{Z}$. We define ${}^M\hat{\xi}_t$ so that

$$ {}^M\hat{\xi}_t = \bigcup_{z=-\infty}^{\infty} {}^M_z\hat{\xi}_t, \tag{10}$$

where ${}^M_z\hat{\xi}_t$, $t \in [Tn, T(n+1))$, is the nearest-particle system with

$$\begin{aligned} {}^M_z\hat{\xi}_{Tn} &= {}^M\xi_{Tn} \cap {}^MI(z) \quad \text{for } z = n \bmod 2, \\ &= \varnothing \qquad\qquad\qquad \text{for } z \neq n \bmod 2, \end{aligned} \tag{11}$$

and which has birth rates ${}^M\tilde{\beta}(l, r)$. ($T > 0$ is large and fixed, and will be specified later.) ${}^M_z\hat{\xi}_t$ has the same birth and death rates as ${}^M\tilde{\xi}_t$, but is periodically restarted at times Tn on alternating intervals ${}^MI(z)$ with distributions ${}^M\xi_{Tn} \cap {}^MI(z)$. It is easy to see that

$$ {}^M_z\hat{\xi}_t \subset {}^MI(z)$$

and that for $Tn \leq t_1 \leq t_2 < T(n+1)$,

$$ {}^M_z\hat{\xi}_{t_1} = \varnothing \Rightarrow {}^M_z\hat{\xi}_{t_2} = \varnothing .$$

If

$$ {}^M\xi_0 \cap {}^MI(0) \neq \varnothing, \tag{12}$$

then of course ${}^M\hat{\xi}_0 \cap {}^MI(0) \neq \varnothing$. Since the transition mechanism for ${}^M\xi_t$ is translation invariant, we can assume wlog in proving (7) that (12) holds. We also point out that ${}^M\xi_t$ and ${}^M\hat{\xi}_t$ are attractive. (See [7] for background.) This becomes obvious if one notes that ${}^M\beta(l, r)$ is increasing in l and decreasing in r. On account of (8), (10) and (11), if ${}^M\hat{\xi}_t = {}^M\xi_t$ at given t, then the birth rate for the first process at this time is at most that of the second process. Together with the attractiveness of ${}^M\xi_t$ (or of ${}^M\hat{\xi}_t$), this implies one can couple ${}^M\xi_t$ and ${}^M\hat{\xi}_t$ with

$$ {}^M\hat{\xi}_t \subset {}^M\xi_t. \tag{13} $$

(Refer to [7], page 127, for more detail.)

Define the discrete time process ${}^M\zeta_n$ on $\mathbb{Z}$, with ${}^M\zeta_n(z) = 0, 1$, so that ${}^M\zeta_n(z) = 1$ if

$$ {}^M_z\hat{\xi}_{T(n+1)-} \neq \varnothing. \tag{14} $$

That is, ${}^M\zeta_n(z) = 1$ if descendents under ${}^M\hat{\xi}_t$ of particles at ${}^M\xi_{Tn} \cap {}^M I(z)$ are still alive by time $T(n+1)-$. Note that on account of (11), ${}^M\zeta_n(z) = 0$ for $z \neq n \bmod 2$.

The basic idea behind the proof of Theorem 1 is to compare ${}^M\zeta_n$ with a supercritical discrete-time contact process ${}^p\eta_n$ on (n, z) with $z = n \bmod 2$. ${}^p\eta_n$ is defined so the probability that ${}^p\eta_n(z) = 1$ is $1 - (1-p)^L$, where $L = 0, 1, 2$, is the number of occupied sites of ${}^p\eta_{n-1}$ among $\{z-1, z+1\}$. (${}^p\eta_n$ can also be formulated in terms of oriented percolation.) For $p > p_0$, $p_0 < 1$ chosen appropriately,

$$ \liminf_{n\to\infty} P\left[z \in {}^p\eta^A_{2n}\right] > 0 \tag{15} $$

if $z = n \bmod 2$ and $A \cap 2\mathbb{Z} \neq \varnothing$ (Durrett [2]).

The remainder of this article is devoted to showing that for fixed $p < 1$, M may be chosen large enough (with appropriate T) so that ${}^p\eta_n$ and ${}^M\zeta_n$ can be coupled together with

$$ {}^p\eta_n \subset {}^M\zeta_n. \tag{16} $$

Theorem 1 is then an easy consequence of (a) (15) and of (13), (14) and (16), which compare ${}^M\xi_t$ to ${}^M\hat{\xi}_t$, ${}^M\hat{\xi}_t$ to ${}^M\zeta_n$ and ${}^M\zeta_n$ to ${}^p\eta_n$, and (b) the observation that

$$ \inf P\left[x \in {}^M\xi_t \middle| {}^M\xi_{T(n+1)-} = B\right] > 0, $$

where inf is taken over all B satisfying $B \cap {}^M I(z) \neq \varnothing$, all $t \in [T(n+2), T(n+4))$, $x \in {}^M I(z) \cup {}^M I(z+1)$ and all n. Part (b) follows from the attractiveness of ${}^M\xi_t$, and implies that if $z \in {}^p\eta_n$ (and hence ${}^M\xi_{T(n+1)-} \cap {}^M I(z) \neq \varnothing$), then $x \in {}^M I(z) \cup {}^M I(z+1)$ is likely to be occupied after allowing time T for the process ${}^M\xi_t$ to spread.

Notation. We justified the coupling ${}^M\hat{\xi}_t \subset {}^M\xi_t$ in (13) by using the attractiveness of ${}^M\xi_t$ and the lower birth rate of ${}^M\hat{\xi}_t$. One may also explicitly construct ${}^M\hat{\xi}_t$ from ${}^M\xi_t$ as follows: At each site $(t, x) \in \mathbb{R}^+ \times \mathbb{Z}$ where a birth for ${}^M\xi_t$ occurs, introduce an independent $[0, 1]$ uniformly distributed random variable $W_{t,x}$. If the birth rate for ${}^M\hat{\xi}_t$ at (t, x) is greater than $W_{t,x}$, we say that a birth for ${}^M\hat{\xi}_t$ occurs at (t, x); otherwise, no birth occurs there. Deaths occur at rate 1 for each process, so we say that deaths occur at occupied sites of ${}^M\hat{\xi}_t$ whenever they occur for ${}^M\xi_t$. This procedure may be carried out inductively since $|{}^M_z\xi_t| < \infty$. In this setting, ${}^M\hat{\xi}_t \subset {}^M\xi_t$ clearly holds. We denote by ${}^M\mathscr{F}_t$ the σ-algebra generated by ${}^M\xi_s$ and $W_{s,x}$ up to time t and by the independent procedure used to construct ${}_vY_i$ in (22). The construction has the advantage of allowing us to construct simultaneously other subprocesses of ${}^M\xi_t$ [notably ${}_v\tilde{\xi}_t$ in (22)] by employing $W_{t,x}$. They will all be measurable with respect to ${}^M\mathscr{F}_t$ and satisfy couplings analogous to (13). Let L_t and R_t be the leftmost and rightmost particles of the process ${}^M\tilde{\xi}_t$ from (8) constructed in this manner. On account of (8), ${}^M\tilde{\xi}_t$ evolves independently of the births and deaths for ${}^M\xi_t$ and of values of $W_{t,x}$ which occur outside of $[L_t, R_t]$.

We require the use of several σ-algebras. (To simplify the notation a little, we omit here and later on the superscript M, which will be implicit.) Set

$$\mathscr{G}_n = \sigma(\zeta_m, 0 \le m \le n), \tag{17}$$

and for $t \in [Tn, T(n+1))$, set

$$\begin{aligned} \hat{\mathscr{F}}_t &= \sigma(\xi_s, s \le Tn) \vee \sigma\big(\hat{\xi}_s, Tn \le s \le t\big), \\ \hat{\mathscr{F}}^t &= \sigma\big(\hat{\xi}_s, t \le s < T(n+1)\big). \end{aligned} \tag{18}$$

It is easy to check that

$$\mathscr{G}_n \subset \hat{\mathscr{F}}_t \vee \hat{\mathscr{F}}^t$$

for $t \in [Tn, T(n+1))$.

We will also use the following notation. Set $v = (k, n, z, i)$, where $k = 0, 1, \dots, K-1$, $z = n \bmod 2$ and $i = \pm 1$; the symbol v will henceforth be reserved for this 4-tuple. Introduce the times

$$\begin{aligned} S_{k,n} &= Tn + (1+t_h)k + 1, \\ T_{k,n} &= Tn + (1+t_h)(k+1); \end{aligned} \tag{19}$$

$t_h > 0$ will be specified in (36), K will be specified after (49) and T will satisfy $T = K(1+t_h)$. Note that

$$Tn < S_{0,n} < T_{0,n} < S_{1,n} < \cdots < T_{K-1,n} = T(n+1).$$

We define

$${}_vD = \big\{\omega\colon \exists\, y_1, y_2 \text{ with } y_j \in \xi_{S_{k,n}} \cap I(z+i),\ y_2 - y_1 \ge M/2\big\}. \tag{20}$$

${}_vD$ consists of those realizations which at $t = S_{k,n}$ have a pair of particles at sites y_1, y_2 in $I(z+i)$ which are "far apart." Consider tuples $v_j = (k_j, n, z_j, i_j)$, where we equate $(k_j, n, z_j, 1)$ and $(k_j, n, z_j + 2, -1)$. One can for appropriate $\varepsilon_1 > 0$ choose subsets ${}_vE \subset {}_vD$ so that for distinct v_j, $j = 1, \dots, J$,

$$P\left[\bigcap_{j=1}^{J} v_jE \,|\, \mathscr{G}_n\right](\omega) = \varepsilon_1^J \tag{21}$$

if $\zeta_n(z_j) = 1$. That is, ${}_{v_1}E, \dots, {}_{v_J}E$ are independent under $\zeta_n(z_1) = \cdots = \zeta_n(z_J) = 1$ and $\mathscr{G}_n$. For $\omega \in {}_vE$ denote by ${}_vY_1, {}_vY_2$ occupied positions y_1, y_2 such as in (20). ${}_vY_1, {}_vY_2$ may be chosen so that ${}_vY_2 - {}_vY_1$ is uniformly distributed over $[M/2, M]$ and is independent of everything else. We will justify this and (21) in Proposition 1. Let ${}_v\tilde{\xi}_t$, $t \in [S_{k,n}, T(n+2))$, denote the process defined on ${}_vE$ with

$$(22) \qquad {}_v\tilde{\xi}_{S_{k,n}} = \{{}_vY_1, {}_vY_2\}$$

and birth rate given by (8). Recall that $b(\infty) = 0$ here. Using the construction given before (17), ${}_v\tilde{\xi}_t$ is defined on the same space as ξ_t with

$$(23) \qquad {}_v\tilde{\xi}_{T(n+1)} \subset \xi_{T(n+1)} \cap I(z+i) = {}_{z+i}\hat{\xi}_{T(n+1)}.$$

Since the birth rates of both ${}_v\tilde{\xi}_t$ and ${}_{z+i}\hat{\xi}_t$ are defined by (8), one has

$$(24) \qquad {}_v\tilde{\xi}_t \subset {}_{z+i}\hat{\xi}_t \quad \text{for } t \in [T(n+1), T(n+2)).$$

[Note that $\hat{\xi}_t \cap I(z+i) = \varnothing$ for $t \in [Tn, T(n+1))$.] We will (after Lemma 1) choose M large enough so that $x, y \in {}_v\tilde{\xi}_t$ with $|x-y| = 1$ typically does not occur until $t > S_{k,n} + 2T$.

Define

$$(25) \qquad {}_vF = \left\{\omega\colon {}_v\tilde{\xi}_{T_{k,n}} \neq \varnothing\right\}$$

and

$$(26) \qquad {}_uF = \bigcup_{k=0}^{K-1} {}_vF,$$

where $u = (n, z, i)$. (The symbol u will henceforth be reserved for this 3-tuple.) ${}_vF$ is the subset of ${}_vE$ over which the process ${}_v\tilde{\xi}_t$ started at time $S_{k,n}$ is still alive by $T_{k,n}$. ${}_uF$ may be thought of as the set of those realizations where the process ξ_t has managed to "take hold" in $I(z+i)$ over one of its "trial periods" $[S_{k,n}, T_{k,n}]$. (Lemma 2 will make this more precise.) It will follow from Proposition 2 that $P[{}_uF|\mathscr{G}_n] \sim 1$ for $\zeta_n(z) = 1$ if K is chosen appropriately large.

Finally, let

$$(27) \qquad \kappa(u) = \min\{k\colon \omega \in {}_vF\}$$

$[\kappa(u) = \infty$ if $\omega \notin {}_uF]$ and

$$(28) \qquad v_\kappa = (\kappa(n, z, i), n, z, i).$$

Then define

$$(29) \qquad {}_uG = \left\{\omega\colon {}_{v_\kappa}\tilde{\xi}_{T(n+2)-} \neq \varnothing\right\}.$$

$[S_{\kappa,n}, T_{\kappa,n}]$ is the interval over which ξ_t first "takes hold" and ${}_uG$ is the subset of ${}_uF$ over which ${}_{v_\kappa}\tilde{\xi}_t$ is still alive by time $T(n+2)-$. [Recall that $Tn < T_{k,n} \le T(n+1)$.] It will follow from Proposition 3 that $P[{}_uG|\mathscr{G}_n \vee \sigma({}_uF)] \sim 1$ for $\omega \in {}_uF$.

Note that the processes ${}_{v_j}\tilde{\xi}_t$ used in (25) to construct ${}_{v_j}F$ are defined over disjoint sets

$$I(z_j + i_j) \times \left[S_{k_j,n}, T_{k_j,n}\right]$$

in space-time for distinct v_j, $j = 1, \dots, J$. On account of (21) and the independence of ${}_{v_j}Y_2 - {}_{v_j}Y_1$, this implies that distinct ${}_{v_j}F$, $j = 1, \dots, J$, are independent under $\zeta_n(z_j) = 1$ and $\mathscr{G}_n$. Similarly, ${}_{u_j}F$ and ${}_{u_j}G$ are defined over disjoint sets $I(z_j + i_j)$ for distinct u_j, $u_j = (n, z_j, i_j)$, and so ${}_{u_j}F$, resp. ${}_{u_j}G$, $j = 1, \dots, J$, are also independent under $\zeta_n(z_j) = 1$ and $\mathscr{G}_n$.

Demonstration of (16). Our approach will be to show that for fixed p, if M is large enough (and T is chosen appropriately), then

$$P[{}_uG|\mathscr{G}_n](\omega) \geq 1 - (1 - p)^2 \tag{30}$$

for $z \in \zeta_n$, where $u = (n, z, i)$. By (24) and (14),

$${}_uG \subset \left\{\omega\colon {}_{z+i}\hat{\xi}_{T(n+2)-} \neq \varnothing\right\} = \{\omega\colon z + i \in \zeta_{n+1}\}. \tag{31}$$

Recall that $\mathscr{G}_n$ gives the entire history of ζ_m up to time n. Note also that the process ${}^p\eta_n$ defined above (15) is attractive. Using (30)–(31) and the independence of ${}_{u_j}G$, $j = 1, \dots, J$, under $\zeta_n(z_j) = 1$ and $\mathscr{G}_n$, one can therefore couple ${}^p\eta_n$ and ζ_n as in (16) with ${}^p\eta_n \subset \zeta_n$.

To obtain (30), we compute the probabilities of ${}_vE$ (Proposition 1), ${}_vF|{}_vE$ and ${}_uF|{}_vE$ (Proposition 2) and ${}_uG|{}_uF$ (Proposition 3) under appropriate conditioning. In order to ensure independence where needed, some care must be exercised in the choice of the σ-algebras.

In Proposition 1, we compute a lower bound for the conditional probability of ${}_vE$ under $z \in \zeta_n$. The basic point is that for $z \in \zeta_n$, there is a positive probability that ξ_t spreads to $I(z + i)$ over the time interval $[S_{k,n} - 1, S_{k,n}]$ as in the right-hand side of (20), and that this probability and the manner in which ξ_t spreads do not depend on $\hat{\mathscr{F}}_{T(n+1)-}$. (Recall that $T_{k,n-1} = S_{k,n} - 1$.) Since $I(z)$ has length $2M$, which is on the same scale as the birth rates for ξ_t given in (6), this probability also does not depend on M. Here

$${}_z\hat{R}_t = \max\{x\colon x \in {}_z\hat{\xi}_t\},$$
$${}_z\hat{L}_t = \min\{x\colon x \in {}_z\hat{\xi}_t\}$$

and

$${}_{n,z}B = \left\{\omega\colon {}_z\hat{\xi}_{T(n+1)-} \neq \varnothing\right\} = \{\omega\colon z \in \zeta_n\}.$$

PROPOSITION 1. *For $\omega \in {}_{n,z}B$ and $i = -1, 1$,*

$$P[{}_vD|\mathscr{G}_n](\omega) \geq \varepsilon_1 \tag{32}$$

for appropriate $\varepsilon_1 > 0$. Subsets ${}_vE \subset {}_vD$ may be chosen so that (a) *if $\omega \in {}_{n,z_j}B$ for distinct $v_j = (k_j, n, z_j, i_j)$, $j = 1, \dots, J$, then*

$$P\left[\bigcap_{j=1}^{J} {}_{v_j}E|\mathscr{G}_n\right](\omega) = \varepsilon_1^J \tag{33}$$

and (b) *on ${}_vE \cap {}_{n,z}B$, $\xi_{S_{k,n}} \cap I(z + i)$ contains occupied positions ${}_vY_1, {}_vY_2$ for which ${}_vY_2 - {}_vY_1$ is uniformly distributed on $[M/2, M]$ under $\mathscr{G}_n$, and for which* *on* $\bigcap_{j=1}^{J}({}_{v_j}E \cap {}_{n,z_j}B)$, ${}_{v_j}Y_2 - {}_{v_j}Y_1$, $j = 1,\dots, J$, *are mutually independent under* $\mathscr{G}_n$.

PROOF. By (6), the rate at which a birth of ξ_t occurs in $[{}_z\hat{R}_t + M/2, {}_z\hat{R}_t + M]$ is greater than $1/2$. (For bookkeeping purposes, allow births to occur at already occupied sites.) Such a birth, later births to the right of this point and deaths in $(\hat{R}_{T_{k-1,n}}, (2z+4)M)$ occur at rates independent of $\hat{\mathscr{F}}_{T(n+1)-}$. Also note that $\omega \in {}_{n,z}B$ implies

$$ {}_z\hat{\xi}_{T_{k-1,n}} \neq \varnothing. $$

It is therefore not difficult to check that if $\omega \in {}_{n,z}B$, then for appropriate $c_1 > 0$,

$$ \begin{aligned} P\big[\exists\, y_j,\ j = 1,\dots, J\colon\ & y_j \in \xi_{S_{k,n}},\ y_1 \in [2zM, (2z+3)M), \\ & y_{j+1} - y_j \in [M/2, M] \,|\, \hat{\mathscr{F}}_{T(n+1)-}\big] \geq (c_1/J)^J. \end{aligned} \tag{34} $$

For $J = 5$, there are at least two indices j_1, j_2, $j_2 = j_1 + 1$, for which $y_{j_1}, y_{j_2} \in I(z+1)$. Since $\mathscr{G}_n \subset \hat{\mathscr{F}}_{T(n+1)-}$, this shows (32) for $i = 1$ with $\varepsilon_1 = (c_1/5)^5$. The reasoning for $i = -1$ is analogous.

Recall that ξ_t is attractive; one can therefore choose y_{j_1}, y_{j_2} so that their difference is uniformly distributed over $[M/2, M]$ under $\hat{\mathscr{F}}_{T(n+1)-}$ (and hence $\mathscr{G}_n$). A subset ${}_vE \subset {}_vD$ can be chosen with

$$ P[{}_vE | \mathscr{G}_n](\omega) = \varepsilon_1 $$

on ${}_{n,z}B$, so that this difference remains uniformly distributed on ${}_vE \cap {}_{n,z}B$. Now assume that v_j are distinct [where we equate $(k_j, n, z_j, 1)$ and $(k_j, n, z_j + 2, -1)$]. The births as in (34) needed for ${}_{v_j}D$ occur over disjoint intervals of the form

$$ \big(R_{S_{k,n}}, (2z+4)M\big) \times [T_{k-1,n}, S_{k,n}] $$

and

$$ [(2z-2)M, L_{S_{k,n}}) \times [T_{k-1,n}, S_{k,n}] $$

in space-time. A little thought therefore shows that the subsets ${}_vE \subset {}_vD$ can be chosen so that under $\omega \in {}_{n,z_j}B$, $j = 1,\dots, J$, (33) holds and the corresponding differences ${}_{v_j}Y_2 - {}_{v_j}Y_1$ are independent under $\hat{\mathscr{F}}_{T(n+1)-}$, and hence $\mathscr{G}_n$. □

Once particles spread from $I(z)$ to neighboring blocks $I(z-1)$ and $I(z+1)$, they start to reproduce; we need to know that their descendents survive with some probability. To make this more precise, we introduce the continuous-time birth–death process X_t on $0, 1, 2, \dots,$ with transitions

$$ \begin{aligned} k &\to k+1 \quad \text{at rate } (1+\varepsilon)(k-1)^+, \\ &\to k-1 \quad \text{at rate } k, \end{aligned} \tag{35} $$

and $X_0 = 2$. Comparison with a supercritical binary branching process shows that:

LEMMA 1. *$P[X_t > 0$ for all $t] = \varepsilon_2 > 0$.*

Choose t_h so that

$$P\left[X_t > 0 \; \forall \; t | X_{t_h} > 0\right] \geq h, \tag{36}$$

where $h \geq 1/2$. We will need $h \sim 1$ to obtain (30) with $p \sim 1$. Also, introduce the stopping time

$$\tau^A = \inf\left\{t: |x - y| = 1 \text{ for some } x, y \in {}^M\tilde{\xi}_t^A\right\}, \tag{37}$$

where ${}^M\tilde{\xi}_t^A$ is the process defined by (8). It follows easily from (6) that for $A = \{0, m\}$ and fixed t,

$$\sup_{m \geq M/2} P[\tau^A \leq t] \to 0 \quad \text{as } M \to \infty. \tag{38}$$

Choose M_h so that for $2m \geq M \geq M_h$,

$$P[\tau^A \leq 2T] \leq \varepsilon_2(1 - h). \tag{39}$$

[T will be specified later on; recall that $(1 + t_h)$ divides T.]
$|\tilde{\xi}_t^A|$ can be coupled with X_t so that

$$|\tilde{\xi}_t^A| = X_t \quad \text{for } t \leq T^A.$$

One therefore obtains from Lemma 1, (36) and (39) that:

LEMMA 2. *For $A = \{0, m\}$ with $2m \geq M \geq M_h$,*

$$\text{(a)} \qquad P\left[\tilde{\xi}_t^A \neq \varnothing \text{ for } t \leq t_h\right] \geq \varepsilon_2/2,$$

$$\text{(b)} \qquad P\left[\tilde{\xi}_t^A \neq \varnothing \text{ for } t \leq 2T | \tilde{\xi}_{t_h}^A \neq \varnothing\right] \geq 2h - 1.$$

By utilizing Proposition 1 and Lemma 2, we now compute lower bounds on the conditional probability of ${}_uF$.

PROPOSITION 2. *For $z \in \zeta_n$,*

$$P[{}_uF | \mathscr{G}_n] \geq 1 - (1 - \varepsilon_1\varepsilon_2/2)^K, \tag{40}$$

where $\mathscr{G}_n$ and ${}_uF$ are as in (17) *and* (26) *and $K = T/(1 + t_h)$.*

PROOF. By Proposition 1,

$$P[{}_vE | \mathscr{G}_n](\omega) \geq \varepsilon_1 \tag{41}$$

for $\omega \in {}_{n,z}B$. Also, by the Markov property and Lemma 2(a),

$$\begin{aligned} P\left[{}_vF | \mathscr{F}_{S_{k,n}}\right](\omega) &= P\left[{}_vF | \sigma({}_vY_2 - {}_vY_1)\right](\omega) \\ &\geq \varepsilon_2/2 \end{aligned} \tag{42}$$

for $\omega \in {}_vE$. Because of (8), it is not difficult to see that the random variable on the left-hand side of (42) is independent of $\hat{\mathscr{F}}^{S_{k,n}}$. Therefore,

$$P\left[{}_vF | \mathscr{F}_{S_{k,n}} \vee \hat{\mathscr{F}}^{S_{k,n}}\right](\omega) \geq \varepsilon_2/2 \tag{43}$$

for $\omega \in {}_vE$. Since

$$\mathscr{G}_n \subset \mathscr{F}_{S_{k,n}} \vee \hat{\mathscr{F}}^{S_{k,n}},$$

it follows from (41) and (43) that

$$P[{}_vF | \mathscr{G}_n](\omega) \geq \varepsilon_1 \varepsilon_2/2 \tag{44}$$

on $\{\omega\colon z \in \zeta_n\} = {}_{n,z}B$. But, ${}_{(k,n,z,i)}F$ are independent for $k = 0, 1, \dots, K-1$. Therefore,

$$P[{}_uF | \mathscr{G}_n](\omega) \geq 1 - (1 - \varepsilon_1\varepsilon_2/2)^K \tag{45}$$

for $z \in \zeta_n$. □

Using Lemma 2, we can also compute lower bounds for ${}_uG|{}_uF$.

PROPOSITION 3. *For* $\omega \in {}_uF$,

$$P\left[{}_uG | \mathscr{G}_n \vee \sigma({}_uF)\right](\omega) \geq 2h - 1. \tag{46}$$

PROOF. By the Markov property and Lemma 2(b),

$$\begin{aligned} P\left[{}_uG | \mathscr{F}_{T_{k,n}} \vee \sigma(\kappa)\right](\omega) &= P\left[{}_uG \cap \{\kappa = k\} | \mathscr{F}_{T_{k,n}} \vee \sigma(\kappa)\right](\omega) \\ &\geq 2h - 1 \end{aligned} \tag{47}$$

for

$$\omega \in {}_uF \cap \{\kappa = k\} = {}_vF. \tag{48}$$

The random variable in (47) depends only on events in $I(z + i)$ and is therefore independent of $\hat{\mathscr{F}}^{T_{k,n}}$. Since

$$\mathscr{G}_n \subset \mathscr{F}_{T_{k,n}} \vee \hat{\mathscr{F}}^{T_{k,n}},$$

this implies that

$$P\left[{}_uG | \mathscr{G}_n \vee \sigma(\kappa)\right](\omega) \geq 2h - 1$$

for ω as in (48). Since the inequality does not depend on k (as long as $k < \infty$) and since ${}_uF = \bigcup_{k=0}^{K-1}\{\kappa = k\}$, (46) follows. □

By combining Propositions 2 and 3, one sees that for $z \in \zeta_n$,

$$P[{}_uG | \mathscr{G}_n](\omega) \geq (2h - 1)\left[1 - (1 - \varepsilon_1\varepsilon_2/2)^K\right]. \tag{49}$$

Choosing h close enough to 1 and K large enough, one obtains

$$P[{}_uG | \mathscr{G}_n](\omega) \geq 1 - (1 - p)^2 \tag{50}$$

for fixed $p < 1$. One can now specify t_h [using (36)] and T, with $T = K(1 + t_h)$.

(39) tells us how to choose M. (50) therefore gives us the desired inequality (30). This completes the proof of Theorem 1.

We close with a few observations. First note that the assumption $b(n) \equiv 1 + \varepsilon$ is not essential. One can instead demonstrate Theorem 1 for processes satisfying $\liminf_{n \to \infty} b(n) \geq 1 + \varepsilon$ and $b(\infty) \geq \varepsilon$. [Replacing $1 + \varepsilon$ by ε for $b(\infty)$ only changes the constant ε_1 in Proposition 1.] The basic proof goes through, although one needs to condition more carefully when the process is not attractive. Also, note that for $|A| = \infty$, the comparison we have made with the contact process shows that for each t_0,

$$P\left[x \in {}^M\xi_t^A \text{ for some } t \geq t_0\right] = 1. \tag{51}$$

One can instead condition on nonextinction rather than assuming $|A| = \infty$.

If one wishes, one can with somewhat less effort than needed for Theorem 1 exhibit processes ξ_t^A, with $b(n) \equiv 1 + \varepsilon$, for which

$$P\left[\xi_t^A \neq \varnothing \text{ for all } t\right] > 0, \tag{52}$$

$0 < |A| < \infty$. For these processes, $\beta(l, \infty) = \beta(\infty, l)$ has a "very fat tail" (with not even fractional moments) and $\beta(l, r) = 0$ except near $l = r$. The basic idea for demonstrating (52) is to use the tail behavior of $\beta(l, \infty)$ to obtain births at $R_t + m$ far to the right of the rightmost particle R_t of ξ_t, with the rate only decreasing slowly as $m \to \infty$. As in Lemma 2, for large m the number of progeny between particles at R_t and $R_t + m$ can be compared with the birth–death process X_t for long times. As m increases and this time increases, one has more time to look for births much further to the right of $R_t + m$. Iterating in this manner, one can obtain (52). These processes, while providing examples of survival, are not as convincing as ${}^M\xi_t$ since (1) survival is only shown on sets $A_n \subset {}^M\xi_{t_n}$, $\min\{x\colon x \in A_n\} \to \infty$ as $t_n \to \infty$, and (2) these processes, with their fat tails, are scarcely representative members of the class with $b(n) \equiv 1 + \varepsilon$.

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