THE MINIMAL EIGENFUNCTIONS CHARACTERIZE THE 
ORNSTEIN–UHLENBECK PROCESS

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A process \( X_t \) is equivalent to an Ornstein–Uhlenbeck process if and only if \( e^{-\lambda t} f(X_t) \) is a martingale for every \( f \geq 0 \) on \( \mathbb{R}^d \) such that \( \Delta f(x) - \langle x, \nabla f(x) \rangle = \lambda f(x) \).

Introduction. In [6] it was shown that the minimal solutions of the parabolic equation \( \Delta u(x, t) - \langle x, \nabla u(x, t) \rangle = u_t(x, t) \) on \( \mathbb{R}^d \times \mathbb{R} \) determine martingales that characterize the Ornstein–Uhlenbeck process on \( \mathbb{R}^d \). This is a property that the Ornstein–Uhlenbeck process on \( \mathbb{R}^d \) shares with Brownian motion on a noncompact symmetric space [5] and several other examples [6].

The Ornstein–Uhlenbeck operator is the basic example of an operator \( L \) for which the minimal solutions of the corresponding “heat” equation do not factor into the product of a nonnegative eigenfunction of \( L \) times an exponential in \( t \), which is the case for uniformly elliptic operators on \( \mathbb{R}^d \) and Brownian motion on a homogeneous space [4].

However, if \( u \geq 0 \) is a solution of \( Lu(x) = \Delta u(x) - \langle x, \nabla u(x) \rangle = \lambda u(x) \), then \( e^{-\lambda t} u(x) = v(x, t) \) is a solution of \( Lu + u_t = 0 \). In addition, if \( (X_t)_{t \geq 0} \) is an Ornstein–Uhlenbeck process with initial position \( x_0 \), then \( (v(X_t, t))_{t \geq 0} \) is a martingale with expectation \( u(x_0) \). In this note the minimal eigenfunctions (minimal nonnegative solutions \( u \) of \( Lu = \lambda u \)) are determined and it is shown that the corresponding martingales characterize the Ornstein–Uhlenbeck process.

For \( d = 1 \), it is shown that \( (e^{nt} H_n(X_t))_{t \geq 0} \) is a martingale for each Hermite polynomial \( H_n \) if the minimal eigenfunctions determine martingales. These “Hermite” martingales are used to characterize the process.

For any dimension, the minimal eigenfunctions are given by the formula

\[
\int_0^\infty r^{\lambda-1} \exp\{-r^2 + \sqrt{2r} \langle x, b \rangle\} \, dr = K(\lambda, b; x) \quad \text{for} \ \lambda > 0 \ \text{and} \ b \in S^{d-1}.
\]

Hence, the projection onto a line through the origin of a process \( (X_t)_{t \geq 0} \) for which \( K(\lambda, b; X_t) \) is always a martingale is necessarily equivalent to a real-valued Ornstein–Uhlenbeck process. From this it follows easily that \( (X_t)_{t \geq 0} \) is equivalent to an Ornstein–Uhlenbeck process.

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It follows that the Ornstein–Uhlenbeck process on Weiner space \( W = \mathcal{C}([0, 1], \mathbb{R}) \) may be characterized by the martingales of the form

\[
\int_0^\infty r^{\gamma - 1} \exp \left\{ -r^2 + \sqrt{2r} \langle \gamma, X_t \rangle \right\} \, dr = K(\lambda; \gamma; X_t),
\]

for \( \gamma \) a continuous linear functional on Weiner space of norm 1. This leads to the open question of characterizing the functions \( \int_0^\infty r^{\gamma - 1} \exp \left\{ -r^2 + \sqrt{2r} \langle \gamma, \cdot \rangle \right\} \, dr \) in terms of the generator of the process. Are they the minimal eigenfunctions?

1. Computation of the minimal eigenfunctions. Let \( Lu(x) = \Delta u(x) - \langle x, \nabla u(x) \rangle \) denote the generator of the Ornstein–Uhlenbeck process on \( \mathbb{R}^d \). Using a scaling in \( x \) by \( \sqrt{2} \) and the computations in [3] the minimal solutions \( K(y; x, t) \) of the equation \( Lu = u_t \) on \( \mathbb{R}^d \times \mathbb{R} \) are easily computed. One may also simply calculate by Martin’s method as in [3] starting from the formula for the fundamental solution \( G(x, t; y, s) = P_t \delta(x, y) \) if \( s < t \) and \( = 0 \) otherwise, where

\[
P_t(x, y) = \left[ 1/2\pi(1 - e^{-2t}) \right]^{d/2} \exp \left\{ -1/2(1 - e^{-2t}) \| e^{-x} - y \|^2 \right\}
\]

These solutions are the functions

\[
K(y; x, t) = \exp \left\{ -(e^{-2t} - 1) \| y \|^2 + \sqrt{2} e^{-t} \langle y, x \rangle \right\},
\]

where \( y \in \mathbb{R}^d \). One immediate consequence of the strict positivity of these functions is the following lemma.

**Lemma 1.1.** A positive solution \( u \) of \( Lu + u_t = 0 \) on \( \mathbb{R}^d \times (a, b) \), \( -\infty \leq a < b \leq +\infty \), which vanishes continuously on \( \mathbb{R}^d \times \{b\} \) is identically 0. Consequently, if \( u \) is a positive solution of \( Lu + u_t = 0 \) on \( \mathbb{R}^d \times \mathbb{R} \), then \( P_t(x, dy)u(y, s) = u(x, s - t) \), where \( (P_t)_{t \geq 0} \) is the transition semigroup for the Ornstein–Uhlenbeck process on \( \mathbb{R}^d \).

**Proof.** To prove the first statement, it suffices to prove the analogous statement for the operator \( Lu - u_t \). For this the proof of Theorem 3 in [4] applies without change. The second statement is an immediate consequence as \( w(x, t) = u(x, s - t) - \int P_t(x, dy)u(y, s) \) is a positive solution on \( \mathbb{R}^d \times (-\infty, s) \) that vanishes on \( \mathbb{R}^d \times \{s\} \).

Let \( u \) be a positive solution of the equation \( Lu = \lambda u \) on \( \mathbb{R}^d \). Assume \( u(0) = 1 \). As \( v(x, t) = e^{\lambda t}u(x) \) is a solution of \( Lv = v_t \), there is a unique probability \( \mu \) on \( \mathbb{R}^d \) such that \( v(x, t) = \int K(y; x, t)\mu(dy) \). Now

\[
e^{\lambda a}v(x, t) = v(x, t + a) = \int K(y; x, t + a)\mu(dy).
\]

The minimal functions for the heat equation are normalized so that \( K(\gamma; 0, 0) = 1 \). The shift in time determines an isomorphism of the cone of positive solutions
so \( K(y; x, t + a)/K(y; 0, a) = K(e^{-a}y; x, t) \). As a result the measure \( \mu \) satisfies
\[
(\ast) \quad e^{\lambda a} \mu(dy) = K(e^{a}y; 0, a) \mu_a(dy),
\]
where
\[
\int f(y) \mu_a(dy) = \int f(e^{-a}y) \mu(dy), \quad a \in \mathbb{R}.
\]

Let \( \nu + m e_0 = \mu \), where \( m = \mu((0)) \). Then \( \nu_a + m e_0 = \mu_a \) and so \( \lambda = 0 \) if \( m \neq 0 \). Assume that \( \lambda \neq 0 \).

For any Borel set \( A \subset \mathbb{R}^d \) and \( b \in S^{d-1} \) let \( A(b) = \{ s > 0 | sb \in A \} \). As \( \mu((0)) = 0 \), there is a regular conditional probability \( \pi \) such that \( \mu(A) = \int_{S^{d-1}} \pi(b, A(b)) \eta(db) \), where \( \eta \) is the projection onto \( S^{d-1} \) of \( \mu \). Condition (\ast) implies that for every nonnegative measurable function \( f \) on \((0, \infty)\) and \( a \in \mathbb{R} \),
\[
e^{\lambda a} \int_{(0, \infty)} \pi(b, ds) f(s) = \int_{(0, \infty)} \pi(b, ds) f(e^{-a}s) \exp \left( (1 - e^{-2a})s^2 \right),
\]
from which it follows that for each \( b \in S^{d-1} \), \( \pi(b, ds) \) is absolutely continuous w.r.t. the Lebesgue measure on \((0, \infty)\) and, its density \( \varphi(b, s) \) satisfies the equation
\[
\varphi(b, s) \exp \left\{ \lambda a - (e^{2a} - 1)s^2 \right\} = e^{a} \varphi(b, se^{a}).
\]

Setting \( s = 1 \) and \( r = e^{a} \) gives \( \varphi(b, r) = C(b) r^{\lambda - 1}e^{-r^2} \). Therefore,
\[
e^{\lambda u(x)} = \int_{S^{d-1}} \left[ \int_{0}^{\infty} \varphi(b, s) \exp \left\{ - (e^{-2t} - 1)s^2 + \sqrt{2} e^{-t} s \langle b, x \rangle \right\} ds \right] \eta(db)
\]
\[
= \int_{S^{d-1}} C(b) \left[ \int_{0}^{\infty} s^{\lambda - 1} \exp \left\{ - s^2 e^{-2t} + \sqrt{2} e^{-t} s \langle b, x \rangle \right\} ds \right] \eta(db)
\]
\[
= \int_{S^{d-1}} e^{\lambda C(b)} \left[ \int_{0}^{\infty} r^{\lambda - 1} \exp \left\{ - r^2 + \sqrt{2} r \langle b, x \rangle \right\} dr \right] \eta(db).
\]

From this it follows that \( \lambda > 0 \). Thus there is a measure \( \eta \) on \( S^{d-1} \) such that
\[
u(x) = \int_{S^{d-1}} K(\lambda, b; x) \eta(db),
\]
where
\[
K(\lambda, b; x) = \int_{0}^{\infty} r^{\lambda - 1} \exp \left\{ - r^2 + \sqrt{2} r \langle b, x \rangle \right\} dr.
\]

On the other hand, if \( \eta \) is a measure on \( S^{d-1} \) and
\[
u(x) = \int K(\lambda, b; x) \eta(ds) \quad \text{and if} \quad v(x, t) = \int K(y; x, t) \mu(dy),
\]
where
\[
v(A) = \int_{S^{d-1}} (b) \pi(b, A(b)) \eta(db)
\]
it follows from the above that \( v(x, t) = e^{\lambda t} u(x) \).
Therefore, since \( \eta \) is the projection of \( \nu \) onto \( S^{d-1} \), it follows that there is bijection (order-preserving) between the cone of positive solutions \( u \) of \( Lu = \lambda u \) and the cone of positive measures on \( S^{d-1} \). As a result, the minimal solutions of the equation \( Lu = \lambda u \), \( \lambda > 0 \), are the functions \( K(\lambda, b; \cdot) \). This completes the proof of the following theorem.

**Theorem 1.2.** Let \( u \) be a positive solution of the equation \( Lu = \lambda u \). If \( \lambda \neq 0 \), then \( \lambda > 0 \) and

\[
u(x) = \int_{S^{d-1}} C(b) \left[ \int_0^\infty r^{\lambda-1} \exp\{ -r^2 + \sqrt{2} r \langle b, x \rangle \} \, dr \right] \eta(db).
\]

Up to a multiplicative constant, the minimal solutions of the equation \( Lu = \lambda u \), \( \lambda > 0 \), on \( \mathbb{R}^d \) are the functions \( K(\lambda, b; x) \), \( b \in S^{d-1} \), where

\[
K(\lambda, b; x) = \int_0^\infty r^{\lambda-1} \exp\{ -r^2 + \sqrt{2} r \langle b, x \rangle \} \, dr.
\]

Further, if \( \lambda = 0 \) the only nonnegative solutions are the constants.

**Remark 1.3.** When \( d = 1 \), there are two minimal functions \( \tilde{K}_\lambda^{\pm} \), where

\[
\tilde{K}_\lambda^{\pm}(x) = K(\lambda, \pm; x) = \int_0^\infty r^{\lambda-1} \exp\{ -r^2 \pm \sqrt{2} r \} \, dr.
\]

This formula appears on page 60 of Titchmarsh [7] with a difference due to scaling.

**2. Two corresponding entire families of martingales.** Assume that \( d = 1 \). Then, for each \( \lambda > 0 \), \( K(\lambda, +; x) \) and \( K(\lambda, -; x) \) form a fundamental set of solutions of the equation \( y'' - xy' - \lambda y = 0 \). They are analytic in \( \lambda \) if \( \text{Re} \lambda > 0 \). Another set of fundamental solutions \( y_0(\lambda, x) \) and \( y_i(\lambda, x) \) are given by the series solutions of the equation \( y'' - xy' - \lambda y = 0 \). They are

\[
y_0 = 1 + \lambda x^2 / 2! + (\lambda + 2)\lambda x^4 / 4! + \cdots
\]

\[
+ (\lambda + 2n - 2)(\lambda + 2n - 4) \cdots (\lambda + 2)\lambda x^{2n} / (2n)! + \cdots
\]

and

\[
y_i = x + (\lambda + 1)x^3 / 3! + (\lambda + 3)(\lambda + 1)x^5 / 5! + \cdots
\]

\[
+ (\lambda + 2n - 1)(\lambda + 2n - 3) \cdots (\lambda + 1)x^{2n+1} / (2n + 1)! + \cdots
\]

(cf. [1], page 157, with \( \sqrt{2} x \) replaced by \( x \) and \( -a/2 \) by \( \lambda \)).

It is not hard to see that \( y_0(\lambda, x) = y_0(\lambda, x) \) and \( y_i(x) = y_i(\lambda, x) \) are entire functions of \( \lambda \) for \( x \) fixed.

**Proposition 2.1.** Let \((\Omega, \mathcal{F}, P)\) be a probability space with an increasing filtration \((\mathcal{F}_t)_{t \geq 0}\) and let \((X_t)_{t \geq 0}\) be a process on \( \mathbb{R} \) adapted to the filtration. If for all \( \lambda > 0 \), \( e^{-\lambda t} K(\lambda, X_t) = M_t^\lambda \) is a martingale with respect to the filtration, then \( \forall z \in \mathbb{C} \), \( e^{-\lambda z} y_0(z, X_t) \) and \( e^{-\lambda z} y_i(z, X_t) \) are also martingales.
PROOF. Since all the terms in the series for \( \gamma_t(\lambda, x) \) are positive, the fact that \( e^{-\lambda t} \gamma_t(\lambda, X_t) \) is a martingale implies that the series

\[
1 + zE \left[ X_t^2 \right]/2! + (z + 2)zE \left[ X_t^4 \right]/4! + \cdots \\
+ (z + 2n - 2)(z + 2n - 4) \cdots (z + 2)zE \left[ X_t^{2n} \right]/(2n)! + \cdots 
\]

is an entire function of \( z \). Consequently, for any set \( B \in \mathcal{F} \), \( z \to \int_B \gamma_0(z, X_t) \, dP \) is an entire function and so \( e^{-\lambda t} \gamma_t(z, X_t) \) is a martingale.

Let

\[
Y_t(z) = \gamma_t(z, X_t) = X_t + (z + 1)X_t^2/3! + (z + 3)(z + 1)X_t^3/5! + \cdots \\
+ (z + 2n - 1)(z + 2n - 3) \cdots \\
\times (z + 3)(z + 1)X_t^{2n+1}/(2n)! + \cdots .
\]

First consider the integral of \( Y_t \) over the sets \( A(\pm) = \{ \pm X_t > 1 \} \). By hypothesis, \( K(\lambda, \pm; X_t) \in L^1 \) and since \( 2K(\lambda, +; X_t) = C_0 \gamma_0(\lambda, X_t) + C_1 \gamma_1(\lambda, X_t) \), this implies that \( Y_t = \gamma_t(\lambda, X_t) \in L^1 \). Consequently,

\[
\int_{A(\pm)} |Y_t(z)| \, dP \leq \int_{A(\pm)} |Y_t(|z|, X_t)| \, dP < \infty.
\]

Consequently, for any set \( B \in \mathcal{F} \), \( B \subset \{ |X_t| > 1 \} \), \( z \to \int_B Y_t(z) \, dP \) is an entire function.

To complete the proof, it suffices to consider the integral of \( Y_t(z) \) over \( \{ |X_t| \leq 1 \} \). Since \( |X_t| \leq 1 \), the series \( \int_B Y_t(z) \, dP \), \( B \in \mathcal{F} \), \( B \subset \{ |X_t| \leq 1 \} \) converges for all \( z \) by comparison with the series

\[
1 + (|z| + 1)/3! + (|z| + 3)(|z| + 1)/5! + \cdots \\
+ (|z| + 2n - 1)(|z| + 2n - 3) \cdots (|z| + 3)(|z| + 1)/(2n)! + \cdots . \quad \square
\]

COROLLARY 2.2. If for all \( \lambda > 0 \), \( e^{-\lambda t}K(\lambda, X_t) = M^\lambda_t \) is a martingale with respect to the filtration, then \( \forall \, n \geq 0 \), \( e^{nt}H_n(X_t) \) is a martingale, where \( H_n(x) \) is the nth Hermite polynomial.

PROOF. \( C_0(-2n, x) = H_{2n}(x) \) and \( C_1(-2n - 1, x) = H_{2n+1}(x) \). \( \square \)

REMARK 2.3. If for all \( \lambda > 0 \), \( e^{-\lambda t}K(\lambda, X_t) = M^\lambda_t \) is a martingale with respect to the filtration, then \( e^{-t-\lambda^2/2} \) is a martingale.

PROOF. If \( u(x) = e^{x^2/2} \), then \( u''(x) - xu'(x) = u(x) \). Hence, \( u(x) = a_0 \gamma_0(1, x) + a_1 \gamma_1(1, x) \). \( \square \)

3. A characterization of the Ornstein–Uhlenbeck process on \( \mathbb{R} \). Corollary 2.2 and Remark 2.3 make it possible to use “Hermite” martingales to analyze the process in \( L^2(\mathbb{R}, e^{-x^2/2}) \).
If \( \varphi \in C^\infty_c(\mathbb{R}) \) and \( \varphi(x) = \phi(x)e^{-x^2/2} \), then by Cramér [2]

\[
\phi(x) = \sum_{n=0}^{\infty} \{ a_n/n! \} H_n(x)e^{-x^2/2},
\]

where the series converges uniformly and absolutely to \( \phi \). The coefficients \( a_n = \langle \varphi, H_n \rangle \), where \( \langle f, g \rangle = (1/2\pi)^{1/2}\int f(x)g(x)e^{-x^2/2} \, dx \). In addition, if \( (P_t)_{t \geq 0} \) is the transition semigroup of the Ornstein–Uhlenbeck process, then

\[
(*) \quad P_t \varphi(x) = \sum_{n=0}^{\infty} \{ a_n/n! \} P_t H_n(x) = \sum_{n=0}^{\infty} \{ a_n/n! \} e^{-nt} H_n(x).
\]

**Theorem 3.1.** Let \((\Omega, \mathcal{F}, P)\) be a probability space with an increasing filtration \((\mathcal{F}_t)_{t \geq 0}\) and let \((X_t)_{t \geq 0}\) be a process on \(\mathbb{R}\) adapted to the filtration.

For \( \lambda > 0 \) let \( M_t^{\pm \lambda} = e^{-\lambda t} K(\lambda, \pm X_t) \). The following conditions are equivalent:

1. The process is equivalent to the Ornstein–Uhlenbeck process with initial position \( x_0 \).
2. \( \forall \lambda > 0, (M_t^{\pm \lambda})_{t \geq 0} \) is a martingale with expectation

\[
\int_0^\infty r^{\lambda - 1} \exp\{ -r^2 + \sqrt{2} rx_0 \} \, dr
\]

and \((M_t^{-\lambda})_{t \geq 0}\) is a martingale with expectation

\[
\int_0^\infty r^{\lambda - 1} \exp\{ -r^2 - \sqrt{2} rx_0 \} \, dr.
\]

**Proof.** Lemma 1.1 shows that (1) \(\Rightarrow\) (2). To show the converse, note that for any \( \varphi \in C_c^\infty(\mathbb{R}) \),

\[
\left| E[\varphi(X_t)|\mathcal{F}_s] - \sum_{k=0}^{n} \{ a_k/k! \} E[H_k(X_t)|\mathcal{F}_s] \right| < \varepsilon E[e^{X_t^2/2}|\mathcal{F}_s] = \varepsilon e^{(t-s)+X_s^2/2}
\]

if

\[
\varphi(x) - \sum_{k=0}^{n} \{ a_k/k! \} H_k(x) < \varepsilon e^{x^2/2}.
\]

Hence, by (*)

\[
E[\varphi(X_t)] = \sum_{n=0}^{\infty} \{ a_n/n! \} E[H_n(X_t)]
\]

and

\[
(1) \quad E[\varphi(X_t)|\mathcal{F}_s] = P_t \varphi(x_0).
\]

From (2) it follows that the process is Markov and by (1) it has the correct distributions. \(\square\)
**Remark 3.2.** The theorem remains true if the time interval is restricted to say \([0, T]\).

4. **A characterization of the Ornstein–Uhlenbeck process on \(\mathbb{R}^d\).**

**Theorem 4.1.** Let \((\Omega, \mathcal{F}, P)\) be a probability space with an increasing filtration \((\mathcal{F}_t)_{t \geq 0}\) and let \((X_t)_{t \geq 0}\) be a process on \(\mathbb{R}^d\) adapted to the filtration.

For \(\lambda > 0\) and \(b \in S^{d-1}\), let \(M_t^{\lambda, b} = e^{-\lambda K(\lambda, b; X_t)}\). The following conditions are equivalent:

1. The process is equivalent to the Ornstein–Uhlenbeck process with initial position \(x_0\).
2. \(\forall \lambda > 0\) and \(b \in S^{d-1}\), \((M_t^{\lambda, b})_{t \geq 0}\) is a martingale with expectation
   \[
   \int_0^\infty r^{\lambda-1} \exp\{-r^2 + \sqrt{2} r \langle b, x_0 \rangle\} \, dr.
   \]

**Proof.** (1) \(\Rightarrow\) (2) by Lemma 1.1.

(2) \(\Rightarrow\) (1). For every \(b \in S^{d-1}\), \((\langle b, X_t \rangle)_{t \geq 0}\) is equivalent to a one-dimensional Ornstein–Uhlenbeck process. Consequently, the paths of \((X_t)_{t \geq 0}\) are almost surely continuous. From this (1) follows immediately because: (i) \((X_t)_{t \geq 0}\) is an Ornstein–Uhlenbeck process on \(\mathbb{R}^d\) if and only if \((B_s)_{s \geq 0}\) is a standard Brownian motion, where \(e^t X_t + X_0 = B_t, s = e^{2t} - 1\); and (ii) a standard Brownian motion is characterized by having its projection on any unit vector a one-dimensional Brownian motion. \(\Box\)

**Remark 4.2.** Also, by an obvious change of coordinates, the above result may be used to characterize the Ornstein–Uhlenbeck process on \(\mathbb{R}^n\) with generator

\[
Lu(x) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \langle x, \nabla u(x) \rangle,
\]

where \((c_{ij})\) is positive definite.

5. **The Ornstein–Uhlenbeck process on Weiner space.** The Ornstein–Uhlenbeck process on Weiner space is a process \((X_t)_{t \geq 0}\) on the Banach space \(\mathcal{W} = \mathcal{C}([0, T], \mathbb{R}^d)\) with the following property. For any finite number of continuous linear functionals \(\gamma_i, 1 \leq i \leq n\), on \(\mathcal{W}\) the process \((X_t)_{t \geq 0}\), where

\[
X_t(\omega) = (\langle \gamma_1, X_t(\omega) \rangle, \langle \gamma_2, X_t(\omega) \rangle, \ldots, \langle \gamma_n, X_t(\omega) \rangle)
\]

is an Ornstein–Uhlenbeck process on \(\mathbb{R}^n\) with generator

\[
Lu(x) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \langle x, \nabla u(x) \rangle,
\]

and \(c_{ij} = \langle \gamma_i, \gamma_j \rangle_H\) (see [8]). In particular, for any \(\gamma \in \mathcal{W}^*, (\langle \gamma, X_t \rangle)_{t \geq 0}\) is an Ornstein–Uhlenbeck process on \(\mathbb{R}\) with generator \(Lu(x) = \|\gamma\|_H^2 \frac{\partial u}{\partial x}(x) - xu'(x)\).
Assume the process on $W$ starts from $w_0 = X_0$. Then $(X_t)_{t \geq 0}$ is equivalent to the Ornstein–Uhlenbeck process on $W$ started from $w_0$ if and only if for all $\gamma \in W'$ and $\lambda > 0$, $e^{-\lambda t}K(\lambda, (1/\|\gamma\|_H)\langle \gamma, X_t \rangle)$ is a martingale with expectation $K(\lambda, (1/\|\gamma\|_H)\langle \gamma, w_0 \rangle)$.

This raises the question as to whether the functions $K(\lambda, \langle \gamma, \cdot \rangle)$ on $W$, with $\|\gamma\|_H = 1$ and fixed $\lambda > 0$, are the minimal solutions of the equation $Lu = \lambda u$ in some sense, where $L$ is the generator of the Ornstein–Uhlenbeck process on $W$.

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