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EDGEWORTH EXPANSIONS IN FUNCTIONAL LIMIT THEOREMS

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Expansions for the distribution of differentiable functionals of normalized sums of i.i.d. random vectors taking values in a separable Banach space are derived. Assuming that an $(r + 2)$th absolute moment exist, the CLT holds and the distribution of the rth derivative $r \geq 2$ of the functionals under the limiting Gaussian law admits a Lebesgue density which is sufficiently many times differentiable, expansions up to an order $O(n^{-r/2+\varepsilon})$ hold.

Applications to goodness-of-fit statistics, likelihood ratio statistics for discrete distribution families, bootstrapped confidence regions and functionals of the uniform empirical process are investigated.

1. Introduction and results. Let $X_1, \dots, X_n$ denote a sequence of i.i.d. observations taken from a measurable space $(\mathscr{X}, \mathscr{A})$. We are interested in special sequences of statistics $T_n(X_1, \dots, X_n)$ which are symmetric in $(X_1, \dots, X_n)$ and have a nonnormal limit distribution. Hence, for these statistics the "linear" term $\sum_{k=1}^{n}(E(T_n|X_j, j \neq k) - ET_n)$ of the Hoeffding expansion of T_n [see Hoeffding (1947)] with its limiting normal distribution does not dominate the higher order terms asymptotically.

Consider the following special sequence of statistics T_n. Let $(E, \|\cdot\|)$ denote a separable Banach space endowed with Borel σ-field $\mathscr{B}$. Let g denote a measurable transformation g: $(\mathscr{X}, \mathscr{A}) \to E$. For functionals F_n: $E \to \mathbb{R}$ define

$$(1.1)\qquad \begin{aligned} S_n &\triangleq n^{-1/2}(g(X_1) + \cdots + g(X_n)), \\ T_n(X_1, \dots, X_n) &\triangleq F_n(S_n). \end{aligned}$$

Assume that $Eg(X_1) = 0$ and $E\|g(X_1)\|^2 < \infty$. For examples of such statistics see Section 2.

The model (1.1) allows us to formulate rather precise moment conditions by using truncation techniques for the norm $\|g(X_1)\|$. A drawback of this approach compared to considering the general type of symmetric statistics T_n as in van Zwet (1984) and Friedrich (1989) for normal limit laws and the Berry–Esseen result is that the conditions refer to a rather arbitrary intermediate Banach space E and not directly to T_n itself. Asymptotic expansions up to the order $o(n^{-1})$ for U-statistics of degree 2 are proved in Bickel, Götze and van Zwet (1987).

For the nonnormal limit and finite order von Mises functionals (i.e., Hoeffding expansions of T_n of finite order) approximations by expansions have been obtained in Götze (1979, 1984) for the second and higher order functionals.

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According the formulation in (1.1) let $X_1, X_2, \dots$ denote from *now on* a sequence of i.i.d. random vectors taking values in a separable Banach space $(E, \|\cdot\|)$. Assume that X_j are measurable with respect to the Borel sets $\mathscr{B}$ of E. Furthermore, assume that for some integer $s \in \mathbb{N}$, $s \geq 3$ the moment condition

$$(M_s) \qquad E\|X_1\|^s < \infty, \qquad EX_1 = 0$$

holds.

Define $S_n \triangleq (X_1 + \cdots + X_n)n^{-1/2}$ and assume that the CLT holds in E, i.e.,

$$(E) \qquad S_n \Rightarrow G \quad \text{weakly},$$

where G denotes a random vector in E with Gaussian distribution.

The condition (E) always holds for Banach spaces of type 2 [see, e.g., Hoffman-Jørgensen and Pisier (1976)]. This includes for example the L^p function spaces with $2 \leq p < \infty$.

Let F: $E \to \mathbb{R}$ denote a Frechét differentiable functional *independent of n*. The rate of convergence has been investigated for $F(x) = \|x + a\|^2$ in Hilbert space in Yurinskii (1982), Zalesskii (1982), Nagaev (1985) and Sazonov and Zalesskii (1985) who relaxed the moment and variance conditions of the Berry–Esseen result of Götze (1979) and extended it to the non-i.i.d. case. For a counterexample concerning the necessary minimal number of eigenvalues see Senatov (1986).

Expansions for this special functional have been investigated in Götze (1979), Bentkus (1984a) and Bentkus and Zalesskii (1985). For general differentiable functionals there are Berry–Esseen results of order $O(n^{-1/2})$ in Götze (1981a) (using Fourier inversion) and Götze (1986) (improved results without using Fourier inversion).

Let us call F: $E \to \mathbb{R}$ a *polynomial* function of degree $r \geq 1$ if $F = F_0 + \cdots + F_r$, where $F_j(x) = F_j[x, \dots, x]$ (j arguments) and $(x_1, \dots, x_j) \to F_j[x_1, \dots, x_j]$ denotes a j-linear symmetric *continuous* functional on E^j. For such functionals expansions have been derived in Götze (1984).

The formal approximations of distribution functions of $F(S_n)$ as well as expansions of expectations of the type $Ef(S_n)$ have been studied in a general framework in Götze (1985) and in more special contexts in Götze (1981b) and Bentkus (1984b, 1986).

We shall assume that F is Fréchet differentiable in the following sense. There exist constants $c_F > 0$ and $K > 0$ such that

$$(D_{3(s-2)}) \qquad \|D^jF(x)\| \leq c_F(1 + \|x\|^K)$$

holds for every $x \in E$ and $0 \leq j \leq 3(s-2)$, $s \geq 4$.

Here $D^jF(x)$ denotes a j-linear continuous symmetric functional $D^jF(x)$: $E^j \to \mathbb{R}$ written as $(v_1, \dots, v_j) \to D^jF(x)[v_1, \dots, v_j]$ and the supremum norm of $D^jF(x)$ is defined as usual by $\|D^jF(x)\| \triangleq \sup\{|D^jF(x)[v_1, \dots, v_j]|: \|v_l\| \leq 1, l = 1, \dots, j\}$. Furthermore, let E^* denote the dual space of E endowed with the supremum norm for continuous linear functionals.

The condition $(D_{3(s-2)})$ does not guarantee a limit distribution of $F(S_n)$ with differentiable densities and a Berry–Esseen result. See Rhee and Talagrand

(1984). The essential condition which achieves this and also determines the order of approximation for discrete distributions of X_1 is the following one.

Let $G_0, G_1, \dots, G_r$, $r \geq 2$, denote i.i.d. Gaussian random vectors in E with covariance operator $(f, g) \to Ef(G_j)g(G_j)$, $f, g \in E^*$ proportional to that of X_1. Define the conditional variance

$$(1.2) \qquad \sigma_r^2(G_0; G_1, \dots, G_{r-1}) \triangleq E\Big(D^rF(G_0)[G_1, \dots, G_r]^2 \Big| G_0, \dots, G_{r-1}\Big).$$

Assume that the condition

$$(V_{r,k}) \qquad P\big(\sigma_r^2(G_0; G_1, \dots, G_{r-1}) \leq \delta\big) = O(\delta^k), \qquad \delta \downarrow 0,$$

holds.

These conditions were first used in Götze (1981a) for $r = 1$. Later Vinogradova (1985) proved under such conditions for $r = 1, 2$ a Berry–Esseen result of order $O(n^{-1+\varepsilon})$ for the distribution of symmetric differentiable functions F.

The following theorem is the main result of this paper.

THEOREM 1.3. *Assume conditions* (E), (M_s), $(D_{3(s-2)})$ *and* $(V_{r,k})$ *hold where* $r \geq 2$, $s = r + 2$, $k = c_r 2^r/\varepsilon$, $0 < \varepsilon < \frac{1}{4}$, *with* $c_r \triangleq 4.5r^2$, $r \geq 7$, *and* $c_r/r^2 \triangleq 2.1, 2.18, 2.55, 2.85, 3.1$ *for* $2 \leq r \leq 6$, *respectively. Then there exist* $3(r - j) - 2$ *times differentiable functions* $\chi_j(a)$, $0 \leq j \leq s - 3$, *depending on* F *and the moments of* X_1 *only such that*

$$(1.4) \qquad \sup_a \left| P(F(S_n) \leq a) - \sum_{j=0}^{r-1} \chi_j(a) n^{-j/2} \right| = O(n^{-r/2+\varepsilon}).$$

REMARK 1.5. The functions $\chi_j(a)$ are determined by means of Fourier inversion of derivatives of the functions

$$(\varepsilon_1, \dots, \varepsilon_l) \to E\exp[itF(G_0 + \varepsilon_1 X_1 + \cdots + \varepsilon_l \chi_l)].$$

Compare (3.9). Unfortunately explicit formulae for $\chi_j(a)$ are available in very special cases only. Therefore the bootstrap approach is studied in Section 2.

REMARK 1.6. Assuming that conditions (E), $(M_{3+\delta})$, $(D_{3+\delta})$ and condition $(V_{1,k})$ is satisfied with k sufficiently large, the Berry–Esseen result $|P(F(S_n) \leq a) - \chi_0(a)| = O(n^{-1/2})$ holds. [See Götze (1986).]

REMARK 1.7. For *smooth* distributions of X_1 in $E = \mathbb{R}^k$ in the sense of Cramér (i.e., $|E\exp[it \cdot X_1]| < \delta_1 < 1$, $\|t\| > \delta_2 > 0$, $t \in E^*$) Edgeworth approximations hold up to the error $o(n^{-(s-2)/2})$. [See Bhattacharya and Ranga Rao (1986).]

REMARK 1.8. Concerning the continuity of the distribution of X_1, the lattice distribution of X_1 in $E = R^k$, say, seems to be the "worst" case. For a homogeneous polynomial F of degree r, the rate of approximation given by Theorem 1.3 is given by at most $O(n^{-r/2+\varepsilon})$. This result is certainly not best possible but it cannot be substantially improved, since in this case $P(F(S_n) = a) = p(a)n^{-r/2} + o(n^{-r/2-\delta})$, where $p(a)$ is number theoretic function (independent of n) such that in typical examples $p(a) > 0$ for arbitrary many a [Götze (1987)].

REMARK 1.9. Whenever $F(x) = F(-x)$ for every $x \in E$ it follows from the fact that $P_l(D)$ is a differential operator of odd order if l is odd and

$$P\big(F(G + \varepsilon_1 X_1 + \cdots + \varepsilon_l X_l) \le a\big) \equiv P\big(F(-G - \varepsilon_1 X_1 - \cdots - \varepsilon_l X_l) \le a\big)$$

that $\chi_l(a) \equiv 0$ for odd integers l.

REMARK 1.10. When the moments of $\|X_1\|$ of order $s > r + 2$ exist then the condition on the covariance structure in condition $(V_{r,k})$ can be relaxed. In this case one can replace $c(r)/\varepsilon$ by a smaller constant. Furthermore, for small r the constant $c(r)$ is *not* optimal and can certainly be improved.

REMARK 1.11. When $F(x) = \langle Ax, x\rangle$ is a quadratic polynomial on a Hilbert space E condition $(V_{2,k})$ holds provided that $\mathrm{Cov}(G_0)A$ has at least k nonzero eigenvalues. [See Götze (1979).]

In statistical applications the functional F in (1.1) often does depend on n in the following way:

$$\tilde{F}_n(S_n) \triangleq F_\nu(S_n) + n^{-1/2}F_{\nu+1}(S_n) + \cdots + n^{-r/2}F_r(S_n) + \Delta_{r,n}n^{-(r+1)/2} \tag{1.12}$$

where $\Delta_{r,n}$ denotes a stochastically bounded r.v. and $F_j(x)$ denotes a homogeneous polynomial of degree j in x defined on E. Here S_n is defined as in (1.1). In many cases the stochastic expansion arises as the expansion of $n^{\nu/2}(F(n^{-1/2}S_n) - F(0))$, provided that $D^lF(0) = 0$ for $l = 1,\ldots,\nu-1$. Here we shall consider the case of nonnormal limit distributions, i.e., $\nu \ge 2$ only.

For functionals of the type (1.12), we prove the following result.

THEOREM 1.13. *Assume that conditions* (E), (M_s) *and* (V_{j,k_j}) *for* F_j, $j = \nu,\ldots,\nu+h$, *with* $\nu \ge 2$, $h \ge 1$ *and* $s \ge r+2$ *hold, where* $k_j \triangleq c_{j,r}2^j/\varepsilon$, $c_{\nu,r} \triangleq c_r$ *(as in Theorem* 1.3*),* $c_{j,r} \triangleq r^2$ *for* $j > \nu$ *and* $0 < \varepsilon < \frac{1}{4}$. *Furthermore assume*

$$P\big(|\Delta_{r,n}| > n^{1/2+\varepsilon}\big) = O\big(n^{-r/2+\varepsilon}\big).$$

Then there exist differentiable functions $\chi_j(a) \in C^{3(r-j)-2}$ *different from those in Theorem* 1.3 *such that*

$$\sup_a \left| P\big(\tilde{F}_n(S_n) \le a\big) - \sum_{j=0}^{r-1} \chi_j(a)n^{-j/2} \right| = O\big(n^{-r/2+\varepsilon}\big), \tag{1.14}$$

where $r = \nu + 2h$.

REMARK 1.15. The functions χ_j are obtained by means of the expansion (1.4) for $F(G_0) = F_\nu(G_0) + n^{-1/2}F_{\nu+1}(G_0) + \cdots + n^{-r/2}F_r(G_0)$ and an *additional* asymptotic expansion of these terms in powers of $n^{-1/2}$.

REMARK 1.16. The conditions allow for arbitrary discrete distributions of X_1. For $\nu = 2$ the limiting distributions is of weighted χ^2-type and the result of Theorem 1.13 extends the results of Chandra and Ghosh (1979) for $E = \mathbb{R}^k$ and "smooth" distributions of X_1 to *discrete* distributions of X_1 in *infinite* dimensional spaces E. For the classical χ^2-limit distribution the paper of Chandra and Ghosh provides rather explicit formulae for the expansion terms $\chi_j(\alpha)$.

Expansions for stochastic expansions with normal limit, i.e., $\nu = 1$ have been studied by Bhattacharya and Ghosh (1978).

REMARK 1.17. Assume that the conditions of Theorem 1.3 (resp. those of Theorem 1.13) hold. Define $q_\alpha \triangleq \chi_0^{-1}(\alpha)$ and assume $\chi_0'(q_\alpha) > 0$ for $0 < \alpha < 1$. Then we have for $T_n = F(S_n)$, respectively $T_n = \tilde{F}_n(S_n)$, and $\delta > 0$,

$$\sup_{\delta \le \alpha \le 1-\delta} \left| P\left(T_n \le q_\alpha + \sum_{j=1}^{r-1} n^{-j/2} \psi_j(\alpha) \right) - \alpha \right| = O(n^{-r/2+\varepsilon}), \tag{1.18}$$

where

$$\psi_1(\alpha) \triangleq -(\chi_1 \chi_0'^{-1})(q_\alpha), \qquad \psi_2(\alpha) \triangleq (\chi_1' \chi_1 \chi_0'^{-2} - \chi_2 \chi_0'^{-1} - \chi_1^2 \chi_0'' \chi_0'^{-3})(q_\alpha)$$

and the other ψ functions are defined similarly by inverting the expansion (1.4) [resp. (1.14)]. Notice that for symmetric F we have similar as in Remark 1.9: $\psi_{2l+1} = 0$, $l = 0, 1, \ldots$, and $\psi_2(\alpha) = -\chi_2 \chi_0'^{-1}$.

2. Applications.

2.1. *Goodness-of-fit statistics in* $\mathbb{R}^k$. Let $f\colon \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ denote a function with $3r \ge 6$ derivatives which are bounded by some polynomial functions in $|x|$ such that

$$P(|f^{(r)}(G)| \le \varepsilon) = O(\varepsilon^\alpha), \qquad \varepsilon \downarrow 0 \text{ for some } \alpha > 0 \text{ holds,} \tag{2.2}$$

where G denotes a r.v. with distribution $N(0, \lambda)$, $\lambda > 0$.

Assuming (2.2), $X_1 \in E \triangleq \mathbb{R}^k$, $EX_1 = 0$ and $E\|X_1\|^{r+2} < \infty$, it follows from Theorem 1.3 that the distribution function of

$$F(S_n) \triangleq \sum_{j=1}^{k} f(S_{nj}), \tag{2.3}$$

where $S_n \triangleq n^{-1/2}(X_1 + \ldots + X_n) \triangleq (S_{n1}, \ldots, S_{nk})$, admits an Edgeworth expansion up to an error of $O(n^{-r/2+\varepsilon})$ provided that

$$k \ge (\alpha^{-1} + r - 1) c_r 2^r / \varepsilon \quad \text{and} \quad \mathrm{Cov}(X_1) > 0 \quad (c_r \text{ as in Theorem 1.3}). \tag{2.4}$$

For an example where (2.2) holds one might consider goodness-of-fit statistics by choosing f equal to

$$f_1(x) = x^2 \exp[-\delta x^2], \qquad \delta > 0$$

or

$$f_2(x) = x^2/(1 + \delta x^2), \qquad \delta > 0.$$

Expansions of the statistic (2.3) for a given dimension k holds up to an error of $O(n^{-R/2+\varepsilon})$, where $R \triangleq \max\{r: k \geq (\alpha^{-1} + r - 1)c_r 2^r/\varepsilon, \frac{1}{4} > \varepsilon > 0\}$. Thus $R \sim \log k$.

The first statistic based on $f_1(x)$ has been used, e.g., in robust (weighted) least square methods. The goodness-of-fit statistics based on $f(x) \neq cx^2$ are of course no longer asymptotically minimax. On the other hand it is well known for the χ^2 statistic $f(x) = x^2$ [see Esseen (1945)] that in general the error cannot be smaller than $O(n^{-1+\varepsilon})$, where $\varepsilon = 1/(k+1)$. This discontinuity of the χ^2 distribution for lattice valued random vectors X_j has been investigated by Yarnold (1972), who demonstrated the unsatisfactory level asymptotics properties for small n unless one takes the number of lattice observation points on the χ^2 ellipsoid into account.

2.5. *Bootstrapped confidence regions.* Let $X_1, \dots, X_n \in (\mathscr{X}, \mathscr{A})$ denote an i.i.d. sample from the distribution P and let $g: (\mathscr{X}, \mathscr{A}) \to \mathbb{R}^d$ denote a measurable function with $\mathrm{Cov}_P(g(X_1)) > 0$. Let $\tilde{S}_n \triangleq n^{-1/2}(g(X_1) + \cdots + g(X_n))$ and standardize $\tilde{S}_n$ like $S_n(P) \triangleq \Sigma_p(\tilde{S}_n - E_P\tilde{S}_n)$, where Σ_p is a symmetric positive definite $d \times d$ matrix such that $\Sigma_p^2 = \mathrm{Cov}_P(g(X_1))^{-1}$. Assume:

(2.6)(i) $E_P\|g(X_1)\|^s < \infty$ for $s \geq r(r+2)$.
(2.6)(ii) $\mathrm{Cov}_P(g(X_1)) > 0$.
(2.6)(iii) Condition $(V_{r,k})$ holds for $r \geq 2$ and k as in Theorem 1.3.
(2.6)(iv) $F: \mathbb{R}^d \to \mathbb{R}$ is differentiable and satisfies condition (D_{3r}).

Let $\hat{P}_n$ denote the empirical distribution for a fixed sample $X \triangleq (X_1, \dots, X_n)$. Let $X_1^*, \dots, X_n^*$ denote an i.i.d. sample drawn from X. Let $S_n^*(\hat{P}_n)$ denote the sum based on $g(X_j^*)$ and $\hat{P}_n$. An application of Theorem 1.3 yields $\hat{P}_n(F(S_n^*(\hat{P}_n)) \leq a) = \chi_0(a) + n^{-1/2}\chi_1(a|\hat{P}_n) + \cdots + O_X(n^{-r/2+\varepsilon})$, where the constant in the error bound is uniform in a and is a polynomial function of $E_{\hat{P}_n}\|g(X_1^*)\|^{r+2}$ which is bounded by $K + E_p\|g(X_1)\|^{r+2}$ with probability $1 - O(n^{-r/2}K^{-r})$. As in Remark 1.17 we may invert this expansion with a similar stochastic error term $O_X(n^{-r/2+\varepsilon})$, since $\chi_0(a)$ does not depend on $\hat{P}_n$. Then the following result holds.

COROLLARY 2.7. *Assume that conditions* (i)–(iii) *hold for* $r \geq 3$. *Let* $q_\alpha^* \triangleq \inf\{q: \hat{P}_n(F(S_n^*(\hat{P}_n)) \leq q) \geq \alpha\}$ *denote the exact* α*-quantile of the distribution of* $F(S_n(\hat{P}_n))$ *under* $\hat{P}_n$. *Then we have for every* $\delta > 0$,

$$\sup_{\delta \leq \alpha \leq 1-\delta} \left|P\left(F\left(S_n^*\left(\hat{P}_n\right)\right) \leq q_\alpha^*\right) - \alpha\right| = O(n^{-1}). \tag{2.8}$$

The error term is $O(n^{-2})$ [*resp.* $O(n^{-2+\varepsilon})$] *provided that* $r = 5$ (*resp.* $r = 4$) *and* F *is symmetric.* (*Compare Remark* 1.9 *and Remark* 1.17.)

For results on confidence regions for statistics with normal limit distributions, see Hall (1986).

Similar results like (2.8) hold for stochastic expansions of the type (1.12).

2.9. *Applications to likelihood ratio statistics.* Let $l(\theta; x)$ denote the log-likelihood function for an i.i.d. sample $x = (x_1, \dots, x_n)$ from a parametric family P_θ with $(dP_\theta/d\mu)(y) = p(\theta; y)$, $y \in \mathscr{X}$, μ being a σ-finite measure on a measurable space $(\mathscr{X}, \mathscr{A})$ and parameter $\theta \in \Theta_H$ open in $\mathbb{R}^{k+1}$. Assume that $\log p(\theta, y)$ admits at least four derivatives which have absolute moments of order s, $s \geq 5$, under P_θ such that the maximum likelihood estimator $\hat{\theta}_n$ for θ exists and the likelihood ratio statistic has a stochastic expansion of the type (1.14):

$$\begin{aligned} \omega(\theta) &\triangleq 2\{l(\hat{\theta}_n; x) - l(\theta; x)\} \\ &= F_2(S_{n1}) + n^{-1/2}F_3(S_{n1}, S_{n2}) \\ &\quad + n^{-1}F_4(S_{n1}, S_{n2}, S_{n3}) + O_p(n^{-3/2}). \end{aligned} \tag{2.10}$$

Here the random vectors S_{nj} denote the components of the likelihood derivatives $(D^j l(\theta; x) - E_\theta(D^j l(\theta; x)))/\sqrt{n}$, $j = 1, 2, 3$. We identify S_n in (1.14) with $(S_{n1}, S_{n2}, S_{n3}) \in \mathbb{R}^d$, d sufficiently large. Assuming that the Fisher information matrix $I(\theta)$ is positive definite, let $\kappa^{s,t} \triangleq (I(\Theta)^{-1})_{s,t}$. Using Einstein's summation convention define $\kappa^{stu} \triangleq \kappa^{s,p}\kappa^{t,q}\kappa^{uv}(\partial^3/\partial\theta_p\,\partial\theta_q\,\partial\theta_v)\log p(\theta)$. Then

$$F_2(x) \triangleq \kappa^{s,t}x_s x_t, \qquad F_3(x, y) \triangleq \tfrac{1}{3}\kappa^{stu}x_s x_t x_u + \kappa^{s,u}\kappa^{t,v}x_s x_t y_{uv} \tag{2.11}$$

and F_4 denotes a fourth order polynomial. [See Lawley (1956) and Hayakawa (1977).] From $E\omega(\theta) = k(1 + b(\theta)n^{-1} + O(n^{-3/2}))$ one obtains the Bartlett correction $\omega'(\theta) \triangleq \omega(\theta)/(1 + b(\theta)n^{-1})$ [see Bartlett (1937)]. Chandra and Ghosh (1979), Barndorff-Nielsen and Cox (1984) and Jensen (1987) proved that $\omega'(\theta)$ has χ^2_k distribution with error $O(n^{-3/2})$ for continuous and partly continuous families of distributions. Extending these results Barndorff-Nielsen and Hall (1988) and Bickel and Ghosh (1987) proved an error $O(n^{-2})$ for continuous families. Applying Theorem 1.13 with $\nu = 2$ and $h = 1$ the following results hold for discrete families as well.

COROLLARY 2.12. *Under the condition mentioned above we have for* $0 < \varepsilon \leq \frac{1}{4}$,

$$\sup_a |P(\omega'(\theta) \leq a) - \chi(a; k)| = O(n^{-2+\varepsilon})$$

provided that $k \geq 40.8/\varepsilon$ *and that* $F_3(x, y)$ *satisfies condition* (V_{3,k_3}), *where* $k_3 \geq 128/\varepsilon$ *and* $\chi(a; k)$ *denotes the* χ^2*-distribution function with* k *degrees of freedom.*

The number of dimensions k required is rather large for practical applications, but although it may be reduced somewhat for this particular case, finding the minimum number of dimensions k in Theorem 1.3 and 1.13 (which certainly has to be smaller) is connected to unsolved problems about the asymptotic number of solutions of diophantine equations [Götze (1987)].

EXAMPLE 2.13 (Multinomial families). Let $l(\theta; x) \triangleq \log(n!\, n_1!^{-1} \cdots n_{k+1}!^{-1}\theta_1^{n_1} \cdots \theta_{k+1}^{n_{k+1}})$, where $\theta_j > 0$, $\theta_1 + \cdots + \theta_{k+1} = 1$ and $n = n_1 + \cdots + n_{k+1}$. Define $\bar{x} \triangleq (n_1, \dots, n_{k+1})n^{-1}$ and $\tilde{x} \triangleq (\bar{x} - \theta)\sqrt{n}$. Then we have by (2.10)

$$\omega(\theta) = \sum_{l=2}^{r-1} F_l(\tilde{x})n^{-(l-2)/2} + O_P(n^{-r/2}),$$

where

$$F_l(\tilde{x}) \triangleq \frac{2}{(l-1)l}\sum_{j=1}^{k+1} \tilde{x}_j^l \theta_j^{-(l-1)}$$

and the Bartlett correction is $\omega'(\theta) = \omega(\theta)/(1 + n^{-1}b(\theta))$,

$$kb(\theta) \triangleq E\big(n^{1/2}F_3(\tilde{x}) + F_4(\tilde{x})\big).$$

Similar as in Example 2.1, condition (V_{l,k_l}) can be checked for F_l. (See section 4.) Hence, $\omega'(\theta)$ admits an asymptotic expansion of the type

$$P(\omega'(\theta) \le a) = \chi^2(a; k) + \chi_4(a)n^{-2} + \cdots + O(n^{-r/2+\varepsilon})$$

provided that $k \ge c(r-1)r^2 2^r$, i.e., $r \sim \log_2 k$.

2.14. *Applications to empirical processes.* Let $x_n(t)$ denote the uniform empirical process on $[0,1]$ pertaining to i.i.d. observations $x_1, \dots, x_n$. Let $V(t,x)$ denote a function defined on $[0,1] \times \mathbb{R}$ such that for some $K > 0$:

(2.15)(i) $|(\partial^l/\partial x^l)V(t,x)| \le c(1 + |x|^K)$ for every t, x and $l \le 3r$, $r \ge 2$.
(2.15)(ii) $|V(t,x) - V(t',x)| \le c|t - t'|^{1/2}(1 + |x|^K)$.
(2.15)(iii) $P(\int_0^1 |(\partial^r/\partial x^r)V(t, w_0(t))w_1(t) \cdots w_{r-1}(t)|^2\, dt \le \delta) = O(\delta^k)$,

where $\delta \downarrow 0$, $k = 5.1c_r 2^r/\varepsilon$ and $w_j(t)$, $0 \le j \le r-1$, denote i.i.d. Brownian bridges.

REMARK 2.16. Condition (2.15)(iii) holds provided that

$$\frac{\partial^r}{\partial x^r} V(a,x)|_{x=a} \ne 0, \quad \text{for } a = 0 \text{ or } a = 1.$$

COROLLARY 2.17. *Assume that conditions* (2.15)(i)–(iii) *are satisfied. Define* $F(x(\cdot)) \triangleq \int_0^1 V(t, x(t))\, dt$. *Then* $P(F(x_n(\cdot)) \le a)$ *has an expansion of type* (1.4) *up to an error* $O(n^{-r/2+\varepsilon})$ *uniformly in* a.

3. Lemmas. Let us introduce some notation which is frequently used throughout this paper.

NOTATION. We shall use the convention that c denotes a generic constant, eventually depending on the absolute moments of the r.v. X_j but *not on* n. Furthermore, let $\eta > 0$ (resp. $L > 0$) denote arbitrarily small (resp. large) fixed constants.

In order to estimate various remainder terms of our expansion under acceptable moment conditions we have to introduce a truncation scheme for the random vectors X_j. In the following assume that $m \geq n$ denote integers. Let $\varphi\colon \mathbb{R} \to [0,1]$ denote a strictly monotone C^∞ function such that $\varphi(x) \equiv 1$ when $|x| \leq \frac{1}{2}$ and $\varphi(x) \equiv 0$ when $|x| \geq 1$. Let $Z_j^{\varepsilon_j}$, $\varepsilon_j \geq 0$, denote a random vector in E with distribution given by

$$P\big(Z_j^{\varepsilon_j} \in A\big) = E\varphi_j\big(\|X_j\|\varepsilon_j n^{\sigma/2}\big)1_{\{X_j\varepsilon_j \in A\}}, \tag{3.1}$$

where $\sigma \triangleq 2\varepsilon/s$ with $0 < \varepsilon < \frac{1}{4}$ as defined in (1.4), $\varphi_j(x) \triangleq \varphi(x)/E\varphi(\|X_1\|\varepsilon_j n^{\sigma/2})$ and Z_j^0 denotes the random vector which is identically zero a.s.

Define

$$Z_j \triangleq Z_j^{m^{-1/2}}. \tag{3.2}$$

This r.v. is truncated such that

$$\|Z_j\| \leq n^{-\sigma/2}. \tag{3.3}$$

In estimating some error terms we even need a further truncation of Z_j which we denote by V_j and which is defined by

$$P\big(V_j \in A\big) \triangleq P\big(Z_j \in A \,|\, \|Z_j\| \leq \tau\big), \tag{3.4}$$

where

$$\tau \triangleq \big(Nm^{-1}\big)^{1/2} \quad \text{for } 0 < N \leq m. \tag{3.5}$$

Hence,

$$\|V_j\| \leq \tau \quad \text{a.s.}$$

Definition of the expansion terms. Let $P_l(D)$, $l = 0,\dots,r$, denote differential operators with respect to variables $\varepsilon_1,\dots,\varepsilon_l$, $2l \leq 3(s-3)$ at $\varepsilon_1 = \cdots = \varepsilon_l = 0$ which are defined as follows. Let $D^P \triangleq \partial^p/\partial\varepsilon^p$. Define formal cumulant operators κ_p by means of the formal power series

$$\sum_{p=2}^{\infty} \kappa_p p!^{-1}u^p \triangleq \log\left(1 + \sum_{p=2}^{\infty} D^p p!^{-1}u^p\right) \tag{3.6}$$

using the following *convention*: $D^{p_1}\cdots D^{p_k}$ always denotes partial derivatives with respect to *k different* variables $\varepsilon_1,\dots,\varepsilon_k$ at $\varepsilon_1 = \cdots = \varepsilon_k = 0$. This convention is unambiguous when applied to *symmetric* functions of $\varepsilon_1,\dots,\varepsilon_k$. We have $\kappa_0 = \kappa_1 = 0$, $\kappa_2 = D^2$, $\kappa_3 = D^3$, $\kappa_4 = D^4 - 3D^2D^2$, etc. Finally define $P_p(D)$ by

$$\sum_{p=0}^{\infty} P_p(D)u^p \triangleq \exp\left(\sum_{p=3}^{\infty} \kappa_p p!^{-1}u^{p-2}\right). \tag{3.7}$$

In particular, $P_0(D) = 1$, $P_1(D) = \kappa_3/6$ and $P_2(D) = \kappa_4/24 + \kappa_3^2/72$. Let

$$\chi(a;\varepsilon_1,\dots,\varepsilon_l) \triangleq P\big(F(G + \varepsilon_1X_1 + \cdots + \varepsilon_lX_l) \leq a\big) \tag{3.8}$$

and define

$$\chi_j(a) \triangleq P_j(D)\chi(a;0,\dots,0), \qquad j = 0,\dots,s-3, \tag{3.9}$$

where, e.g.,

$$\chi_1(a) = \frac{1}{6}\frac{\partial^3}{\partial\varepsilon^3}\chi(a;\varepsilon)\bigg|_{\varepsilon=0}$$

and

$$\chi_2(a) \triangleq \left(\frac{1}{24}\left(\frac{\partial^4}{\partial\varepsilon_1^4} - 3\frac{\partial^2}{\partial\varepsilon_1^2}\frac{\partial^2}{\partial\varepsilon_2^2}\right) + \frac{1}{72}\frac{\partial^3}{\partial\varepsilon_1^3}\frac{\partial^3}{\partial\varepsilon_2^3}\right)\chi(a;\varepsilon_1,\varepsilon_2)\bigg|_{\varepsilon_1=\varepsilon_2=0}.$$

Define

$$T_m \triangleq Z_1 + \cdots + Z_m, \tag{3.10}$$

where $m \in \mathbb{N}$ and Z_j is defined in (3.2). In order to expand the characteristic function of $F(T_m)$ define for $\varepsilon_1,\dots,\varepsilon_m \in [-m^{-1/2}, m^{-1/2}]$,

$$h_m(t;\varepsilon_1,\dots,\varepsilon_l) \triangleq E\exp[itF(Z_1^{\varepsilon_1} + \cdots + Z_l^{\varepsilon_l} + Z_{l+1} + \cdots + Z_m)]. \tag{3.11}$$

Then we have the following expansion result.

LEMMA 3.12. (i) *Let G denote the Gaussian r.v. in condition (E) and let ε be as in Theorem* 1.3. *Let $|\alpha| = \alpha_1 + \cdots + \alpha_l$, $\alpha_j \in \mathbb{N}_0$ and $l = \lceil 3(s-3)/2 \rceil$. Then*

$$\left| E\exp[itF(T_n)] - \sum_{j=0}^{s-3} n^{-j/2}P_j(D)E\exp[itF(G + Z_1^{\varepsilon_1} + \cdots + Z_l^{\varepsilon_l})]|_{\varepsilon=0} \right|$$

$$\le n^{-(s-2)/2}\sup\left\{\left|\frac{\partial^{|\alpha|}}{\partial\varepsilon_1^{\alpha_1}\cdots\partial\varepsilon_l^{\alpha_l}}h_m(t;\varepsilon_1,\dots,\varepsilon_l)\right| : m \ge n,\ |\alpha| = s,\ |\varepsilon_j| \le m^{-1/2},\ j = 1,\dots,l\right\}$$

$$+ \sup\left\{\left|\frac{\partial^{|\alpha|}}{\partial\varepsilon_1^{\alpha_1}\cdots\partial\varepsilon_l^{\alpha_l}}E\exp[itF(G + Z_1^{\varepsilon_1} + \cdots + Z_l^{\varepsilon_l})]\right|_{\varepsilon=0} : \Sigma_j(\alpha_j - 2) \le s-3,\ \alpha_j \ge 2\right\}.$$

Notice that the terms in the expansion are just the Fourier transforms $\hat{\chi}_j(t)$ of $\chi_j(a)$ defined in (3.9).

(ii) *Let* $h(\varepsilon_1,\ldots,\varepsilon_m) \triangleq Eh(Z_1^{\varepsilon_1} + \cdots + Z_m^{\varepsilon_m})$, $h \in C^3(E)$. *Then*

$$|Eh(Z_1 + \cdots + Z_m) - Eh(G)|$$

$$\leq m^{-1/2} \sup\left\{\left|\frac{\partial^3}{\partial\varepsilon_1^{\alpha_1}\cdots\partial\varepsilon_3^{\alpha_3}} h\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, m^{-1/2},\ldots, m^{-1/2}\right)\right|: m \geq n, |\varepsilon_j| \leq m^{-1/2}, \alpha_1 + \alpha_2 + \alpha_3 = 3\right\}.$$

PROOF. (i) We have $(\partial/\partial\varepsilon_j)h_m(t; \varepsilon_1,\ldots,\varepsilon_m)|_{\varepsilon_j=0} = 0$ which follows from $EX_1 = 0$ and the fact that

$$\frac{\partial^p}{\partial\varepsilon^p} E\left(1 + \|X_1\|^l\right)\varphi\left(\varepsilon\|X_1\|n^{\sigma/2}\right) = O\left(\varepsilon^{(s-l-p)}n^{p\sigma/2}\right) \quad \text{for } p \leq s - l. \tag{3.13}$$

Furthermore, $h_m(t; 0, \varepsilon_2,\ldots,\varepsilon_m) = h_{m-1}(t; \varepsilon_2,\ldots,\varepsilon_m)$. Therefore Theorem 2.11 of Götze (1985), page 5 applies to this sequence of symmetric functions h_m, $m \geq n$, (n fixed!) and proves (i). The proof of (ii) follows similarly from an application of Proposition 2.1 in Götze (1985), page 3. □

LEMMA 3.14. (i) *Assume that* X_j, $j = 1,\ldots,m$, *satisfy conditions* (E) *and* (M_2). *Let* V_j *denote the truncation of* Z_j *at the norm* $\tau = (N/m)^{1/2}$ *as defined in* (3.5). *Then we have for every* $N \leq m$ *and* $q \geq 2$,

$$E\|V_1 + \cdots + V_N\|^q \leq C\tau^q, \tag{3.15}$$

where C *depends on* $E\|G\|^2$.

(ii) *Assume that* G *has a Gaussian distribution on* $(E, \mathscr{B})$. *Then*

$$E\exp[\alpha\|G\|] < \infty \quad \textit{for every } \alpha > 0.$$

PROOF. (i) Let $U \triangleq \|V_1 + \cdots + V_N\|$ and decompose U into martingale differences $U_l \triangleq E(U - E(U|X_j, j \neq l)|X_j, j \leq l)$ with respect to the σ-fields $\sigma(X_1,\ldots,X_l)$, $1 \leq l \leq N$, such that

$$U - EU = \sum_{l=1}^{N} U_l.$$

From $(a + b)^q \leq 2^{q-1}(a^q + b^q)$ for $a, b \geq 0$, $q \geq 2$ and well-known estimates for martingale moments by Dharmadhikari, Fabian and Jogdeo (1968) we have

$$EU^q \leq C_q'\left[(EU)^q + \frac{1}{N}\sum_{l=1}^{N} E|U_l|^q N^{q/2}\right].$$

Since by the triangle inequality and definition

$$|U_l| \leq \|V_l\| + E\|V_l\| \quad \text{and} \quad \|V_l\| \leq \tau \quad \text{a.s.},$$

we have

$$EU^q \leq C_q'\left[(EU)^q + \left(E\|X_1\|^2\right)\tau^{q-2}\right] \tag{3.16}$$

for some constant $C_q' > 0$. According to definitions (3.2) and (3.4) it follows that

$$\begin{aligned}\|EV_l\| &= \|EZ_j 1_{\{\|Z_j\| \le \tau\}}\| P(\|Z_j\| \le \tau)^{-1} \\ &= \|E\varphi(\|X_j\| m^{-1/2} n^{\sigma/2}) X_j 1_{\{\|X_j\| \le N^{1/2}\}}\| N(m^{-1/2})^{-1} P(\|Z_j\| \le \tau)^{-1}.\end{aligned} \tag{3.17}$$

Since for some constant $c > 0$,

$$\|Z_j\| \le c\|X_j\| m^{-1/2} \quad \text{a.s.,} \tag{3.18}$$

Chebyshev's inequality shows

$$P(\|Z_j\| > \tau) = O(m^{-s/2}\tau^{-s}) = O(N^{-s/2}), \tag{3.19}$$

$$N(m^{-1/2}) = 1 + O(m^{-s/2} n^{s\sigma/2}). \tag{3.20}$$

Hence by definition of φ, Chebyshev's inequality and $EX_j = 0$ we obtain from (3.17)–(3.20)

$$\begin{aligned}\|EV_l\| &= O(N^{-(s-1)/s}), \\ \|EZ_l\| &= O(m^{-(s-1)/2} n^{(s-1)\sigma/2}).\end{aligned} \tag{3.21}$$

Furthermore, by virtue of (3.20) we have

$$\begin{aligned}P((Z_1,\dots,Z_m) \in A_m) &= P((X_1,\dots,X_m) \in A_m) \\ &\qquad \times (1 + o(m^{-(s-2)/2} n^{s\sigma/2}) + o(m^{-(s-2)/2} n^{s\sigma/2})), \\ P((V_1,\dots,V_N) \in A_m) &= o(N^{-(s-2)/2}).\end{aligned} \tag{3.22}$$

Hence $\tau^{-1}Z_j$, $j = 1,\dots,N$, are infinitesimal and $W_N \triangleq \tau^{-1}(V_1 + \cdots + V_N)$ converges to the same Gaussian limit distribution as $S_n = n^{-1/2}(X_1 + \cdots + X_n)$ by condition (E). By a result of de Acosta and Giné (1979) we therefore obtain

$$\lim_{N\to\infty} E\|W_N\| = E\|G\| \le E^{1/2}\|G\|^2 < \infty.$$

Hence, part (ii) is a consequence of (3.16).

(ii) Part (ii) of the lemma follows from a well-known result of Fernique (1970). □

The key inequality which enables us to prove the rate of approximation in Theorem 1.3 is the following result.

LEMMA 3.23. *Let $S = U_1 + \cdots + U_p$, where $U_1,\dots,U_p$ denote independent random vectors in E. Let g: $E \to \mathbb{C}$ denote a measurable function such that $E|g(S)|^{2p} < \infty$. Define for $\alpha_j \in \{0,1\}$, $j = 1,\dots,p$,*

$$S_\alpha \triangleq U_1 + \alpha_2 U_2 + (1-\alpha_2)\overline{U}_2 + \cdots + \alpha_p U_p + (1-\alpha_p)\overline{U}_p,$$

where $\overline{U}_j$, $j = 1,\dots,p$, denote independent copies of U_j. Then

$$|Eg(S)|^{2^{p-1}} \le E\left|E\left(\prod_{\alpha\in\{0,1\}^{p-1}} g_\alpha(S_\alpha) | U_2, \overline{U}_2, \dots, U_p, \overline{U}_p\right)\right|, \tag{3.24}$$

where $g_\alpha \triangleq \bar{g}$ whenever $\sum_{j=2}^p \alpha_j$ is odd and $g_\alpha = g$ otherwise.

Proof. Write $Eg(S) = EE(g(S)|\mathscr{A}_2)$, where $\mathscr{A}_2 \triangleq \sigma(U_1, U_3, \dots, U_p)$. Let $\beta = (1,1,1,\dots,1)$ and $\bar{\beta} = (1,0,1,\dots,1)$. Then

$$
\begin{aligned}
|Eg(S)|^2 &\le E|E(g(s)|\mathscr{A}_2)|^2 \\
&= EE\big(g(S_\beta)\bar{g}(S_{\bar{\beta}})|\mathscr{A}_2\big) \qquad (3.25)\\
&= Eg(S_\beta)\bar{g}(S_{\bar{\beta}}).
\end{aligned}
$$

Iteration of this argument with (3.25) and

$$\mathscr{A}_j \triangleq \sigma\big(U_1, U_2, \bar{U}_2, \dots, U_{j-1}, \bar{U}_{j-1}, U_{j+1}, \bar{U}_{j+1}, \dots, U_p, \bar{U}_p\big), \qquad j = 3, \dots, p$$

yields the result (3.24). □

Corollary 3.26. *Let $F: E \to \mathbb{R}$ denote a pth order polynomial as defined in the introduction with $p \ge 1$. Then we have with the notation of Lemma* 3.23

$$|E\exp[itF(S)]|^{2^{p-1}} \le E\Big|E\Big(\exp\big(itD^pF\big[U_1, \tilde{U}_2, \dots, \tilde{U}_p\big]\big)\Big|\tilde{U}_2, \dots, \tilde{U}_p\Big)\Big|, \qquad (3.27)$$

where $\tilde{U}_j \triangleq U_j - \bar{U}_j$ and D^pF is constant and a p-linear functional.

This probabilistic inequality is due to Götze (1979) for the case $p = 2$. The immediate extension to $p > 2$ has been mentioned in Yurinskii (1981). A related inequality where $U_1, \dots, U_{p-1}$ and S are uniformly distributed on the lattice points in d-dimensional rectangles is known as the generalized Weyl inequality in analytic number theory. [See Weyl (1916) for a special version of it.]

Proof. Use Lemma 3.23 with $g(x) = \exp[itF(x)]$ and the fact that the pth order difference and the pth order derivative coincide for polynomials. □

Lemma 3.28. *Let V_j, $j = 1, \dots, N$, denote the random vectors defined in* (3.4), *put $U_n \triangleq V_1 + \cdots + V_N$ and write $f \cdot x$ for $f(x)$, $f \in E^*$.*

(i) *Let $h(U_N) = h[U_N, \dots, U_N]$ denote a symmetric continuous d-linear form. For $f \in E^*$ we have for $d \ge 2$,*

$$
\begin{aligned}
&|E\exp[if \cdot U_N]h(U_N)| \\
&\qquad \le c\|h\|\tau^d(1 + \|f\tau\|^d)\big(\beta_2 + \beta_2^d\big)|E\exp[if \cdot V_1]|^{N-d},
\end{aligned} \qquad (3.29)
$$

where $\beta_l \triangleq E\|X_1\|^l$.

(ii) *Furthermore,*

$$|E\exp[if \cdot V_1]|^{N-d} \le c\exp\big[-\tau^2E(f \cdot X_1)^2/4\big] + O(N^{-(N-d)(s-2)/2}) \qquad (3.30)$$

provided that

$$E(f \cdot X_1)^2 \ge 2/3E|f \cdot X_1|^3m^{-1/2}. \qquad (3.31)$$

PROOF. Since both estimates are rather standard we will only sketch them.

(i) Centering V_j, define $Y_j \triangleq V_j - EV_j$. Then $U_N = EU_N + Y_1 + \cdots + Y_N$, where $E\|U_N\| = O(\tau N^{-(s-2)/2})$ by Chebyshev's inequality. Expanding $h(U_N)$ we obtain (using the convention $X^\nu \hat{=} X, \dots, X$ ν times)

$$h(U_N) = \sum_{\nu=0}^{d} h_\nu[U_N'^\nu], \quad \text{where } U_N' \triangleq Y_1 + \cdots + Y_N \tag{3.32}$$

and h_ν denotes the ν-linear symmetric form $h_\nu(\cdot) \triangleq c_\nu h((EU_N)^{d-\nu}, \cdot)$ such that $\|h_\nu\| \le c_\nu(\tau N^{-(s-2)/2})^{d-\nu}\|h\|$. In view of (3.32) it suffices to consider h_ν. Expanding the sum we obtain

$$\begin{aligned} I_{N\nu} &\triangleq E\exp[if \cdot U_N']h_\nu(U_N') \\ &= \sum_{k_1=1}^{N} \cdots \sum_{k_\nu=1}^{N} \prod_{j \notin I} E\exp[if \cdot Y_j] E\prod_{j\in I}\exp[if \cdot Y_j]\, h_\nu\big[Y_{k_1}, \dots, Y_{k_\nu}\big], \end{aligned} \tag{3.33}$$

where I denotes the set of different indices among $k_1, \dots, k_\nu$. We write $h[Y_{j_1}^{\alpha_1}, \dots, Y_{j_l}^{\alpha_l}]$ for $h[Y_{k_1}, \dots, Y_{k_\nu}]$, $l \le \nu$, where α_μ denotes the multiplicity of j_μ among $k_1, \dots, k_\nu$. W.l.o.g. assume in estimating a single term of (3.33) that $I = \{1, \dots, l\}$ and $\alpha_1 = \cdots = \alpha_\kappa = 1$, $\alpha_j \ge 2$ for $j > \kappa$. Using the decomposition

$$\begin{aligned} T_I &\triangleq E\prod_{j\in I}\exp[if \cdot Y_j]\, h_\nu[Y_1, \dots, Y_\kappa, Y_{\kappa+1}^{\alpha_{\kappa+1}}, \dots, Y_l^{\alpha_l}] \\ &= E\prod_{j\le\kappa}\big(\exp[if \cdot Y_j] - 1\big) \prod_{\kappa+1\le j\le l}\exp[if \cdot Y_j]\, h_\nu[Y_1, \dots, Y_\kappa, Y_{\kappa+1}^{\alpha_{\kappa+1}}, \dots, Y_l^{\alpha_l}] \end{aligned}$$

and $|\exp[if \cdot Y_1] - 1| \le \|f\|\,\|Y_1\|$, we obtain

$$|T_I| \le \|f\|^\kappa\big(E\|Y_1\|^2\big)^\kappa \prod_{j=\kappa+1}^{l} E\|Y_j\|^{\alpha_j}\|h_\nu\|. \tag{3.34}$$

This together with $E\|Y_j\|^\alpha \le E\|Y_1\|^2\tau^{\alpha-2}$ for $\alpha \ge 2$ applied to (3.34) yields, after estimating the combinatorial multiplicities of ν-tuples with exactly l indices different,

$$|I_{N\nu}| \le c\left(\sum_{l=1}^{\nu}\sum_{\kappa=0}^{l}\|f\|^\kappa\beta_2^l\tau^{\kappa+\nu}\right)\|h_\nu\|\,\big|E\exp[if \cdot Y_1]\big|^{N-d}. \tag{3.35}$$

The first factor in round brackets on the r.h.s. of (3.35) can be bounded from above by

$$c\tau^\nu\sum_{l=1}^{\nu}\beta_2^l\big[1 + (\|f\|\tau)^l\big]\|h_\nu\| \le c\|h_\nu\|\tau^\nu(\beta_2 + \beta_2^\nu)\big(1 + (\|f\|\tau)^\nu\big),$$

which immediately proves the result via (3.32) and the estimate for $\|h_\nu\|$.

(ii) Since $|E\exp[if \cdot V_1]| \le |E\exp[if \cdot X_1]| + O(N^{-s/2})$ by Chebyshev's inequality and by standard inequalities

$$|E\exp[if \cdot X_1]| \le 1 - \tfrac{1}{2}E(f \cdot X_1)^2 m^{-1} + \tfrac{1}{6}E|f \cdot X_1|^3 m^{-3/2}$$
$$\le \exp\left[-\tfrac{1}{4}E(f \cdot X_1)^2 m^{-1}\right]$$

for every f satisfying (3.31), the inequality (3.30) follows. □

LEMMA 3.36. *Let $G_0, G_1, \ldots, G_j$ denote i.i.d. Gaussian random vectors with mean zero and with the same covariance operator as X_1, which are independent of Z_j, $j = 1, \ldots, m$, as defined in* (3.2). *Define* [*see* (1.2) *and* (3.10)]

$$\Psi(t; T_m) \triangleq E\exp\left[-t^2\sigma_j^2(T_m; G_1, \ldots, G_{j-1})\right].$$

Then we have uniformly in

$$|t| \le n^{\sigma/2-\eta}, \quad \textit{for arbitrarily small } \eta > 0,$$
$$\sup_{m \ge n} \Psi(t; T_m) \le c\Psi(t; G_0) + O(n^{-L}), \quad L \textit{ arbitrarily large}. \tag{3.37}$$

PROOF. The proof is based on the Lindeberg–Feller method and a recursion argument. In order to simplify the estimates we shall use the following approach. Applying Lemma 3.12(ii) with $h(x) \triangleq E\exp[-t^2\sigma_j^2(x; G_1, \ldots, G_{j-1})]$ we obtain with $S_{M,\varepsilon} \triangleq Z_1^{\varepsilon_1} + Z_2^{\varepsilon_2} + Z_3^{\varepsilon_3} + Z_4 + \cdots + Z_M$, where m has been replaced by M in the definition (3.1) of Z_j,

$$|\psi(t; S_m) - \psi(t; G_0)|$$
$$\le m^{-1/2}\sup\left\{\left|\frac{\partial^3}{\partial\varepsilon_1^{\alpha_1}\partial\varepsilon_2^{\alpha_2}\partial\varepsilon_3^{\alpha_3}}\psi(t; S_{M,\varepsilon})\right|: \right. \tag{3.38}$$
$$\left. |\varepsilon_j| \le M^{-1/2}, M \ge m, \alpha_1 + \alpha_2 + \alpha_3 = 3\right\}.$$

The r.h.s. of (3.38) can be estimated using condition $(D_{3(s-2)})$ for F by

$$m^{-1/2}(1 + t^6)\sup\left\{E\left(1 + \left\|\sum_1^3 \varepsilon_l X_l\right\|^{3k} + \|Z_4 + \cdots + Z_m\|^{3k}\right)\right.$$
$$\times \prod_1^3 \|G_l\|^6 \prod_1^3 \|X_l\|^{\alpha_l}\left|D^\beta \prod_1^3 \varphi_j\left(M^{\sigma/2}\varepsilon_j\|X_j\|\right)\right| \tag{3.39}$$
$$\times\exp\left[-t^2\sigma_j^2(S_{M,\varepsilon}; G_1, \ldots, G_{j-1})\right]:$$
$$\left. M \ge m, |\varepsilon_j| \le M^{-1/2}, \Sigma\alpha_l + \beta_l = 3\right\},$$

where $D^\beta = D^{(\beta_1, \beta_2, \beta_3)}$ denotes the partial derivative with respect to ε_1, ε_2, ε_3.

Using Lemma 3.14(i), (ii) we obtain by Hölder's inequality the following upper bound for (3.39) in view of (3.13) for $s \geq 4$:

$$C(\eta)m^{-1/2}(1+t^6)\sup\Big\{E^{1/\eta}\exp\big[-\eta t^2\sigma_j^2(S_{M,\varepsilon}; G_1,\dots,G_{j-1})\big] \times \prod_1^3\big(1+\|Z_l\|^4\big) : |\varepsilon_j| \leq M^{-1/2},\ M \geq m\Big\}, \tag{3.40}$$

for some $\eta > 1$ sufficiently close to 1. Using Taylor expansion, condition (D_3) for F, (3.3), $\|\varepsilon_j Z_j^{\varepsilon_j}\| = O(n^{-\sigma/2})$ and the inequality $E(f(G)+\Delta(G))^2 \geq Ef(G)^2/\bar{\eta} - (\bar{\eta}/(4(\bar{\eta}-1))-1)E\Delta^2(G)$ for $\bar{\eta} > 1$ arbitrarily close to 1, we arrive at

$$\begin{aligned} t^2\sigma_j^2\big(S_{M,\varepsilon}; G_1,\dots,G_{j-1}\big) &\geq t^2\sigma_j^2\big(S_M; G_1,\dots,G_{j-1}\big)/\bar{\eta} \\ &\quad - c(\bar{\eta})\Big(t^2 n^{-\sigma}\big(1+\|S_M\|^{2k}\big)\prod_1^{j-1}\|G_l\|^2\Big) \\ &\geq t^{2/\bar{\eta}}\sigma_j^2\big(S_M; G_1,\dots,G_{j-1}\big) - c, \end{aligned} \tag{3.41}$$

where the last inequality holds on an event E_M depending on $Z_4 + \cdots + Z_M$ and $G_1,\dots,G_j$ only such that

$$P(E_M) = 1 - O(M^{-L})$$

for $L > 0$ arbitrarily large and t fulfilling (3.37). Hence, (3.41) leads to the following upper bound for (3.40):

$$O(m^{-L}) + c(\eta)m^{-1/2}(1+t^6)\sup_{M\geq m} E^{1/\eta}\exp\big[-\eta/\bar{\eta}\, t^2\sigma_j^2\big(S_M; G_1,\dots,G_{j-1}\big)\big]$$

which implies by monotonicity of ψ in $|t|$ and $\eta/\bar{\eta} > 1$,

$$\psi_m \triangleq \sup_{M\geq m}\psi(t; S_m) \leq \psi(t; G_0) + \eta_m \sup_{M\geq m}\psi^{1/\eta}(t; S_M) + O(m^{-L}),$$

where $\eta_m = O(m^{-\delta})$, $\delta > 0$, by (3.37). Hence

$$\psi_m \leq 2\psi(t; G_0) + O(m^{-L})$$

provided $\psi_m > \frac{1}{2}\eta_m^{\eta/(\eta-1)}$, since $\bar{\eta}, \eta > 1$ are arbitrarily close to 1, thus proving Lemma 3.36. □

In the following we shall use the variance condition $V_{r,k}$ of the introduction for $1 \leq l \leq r$ derivatives. Assume

$$(V_{l,R})\quad P\Big(E\big(D^lF(G_0)[G_1,\dots,G]^2\big|G_2,\dots,G_l\big) \leq \varepsilon^2\Big) = O(\varepsilon^R), \qquad \varepsilon\downarrow 0.$$

Then we have

LEMMA 3.42. *Let $\eta_l(t)$ denote the absolute value of the c.f. of $D^lF(G_0)[G_1,\dots,G_l]$. Then:*

(i) *Condition* $(V_{l,R}) \Leftrightarrow \eta_l(t) = O(|t|^{-R})$, $|t|\uparrow\infty$.

(ii) *Condition* $(V_{r,k})$ *with k as in Theorem* 1.3 *implies condition* (V_{l,R_l}), $l = 1,\dots, r-1$ *with* $R_l \triangleq \frac{2}{3}c_r 2^l/\varepsilon - \eta$ *for* $\eta > 0$ *arbitrarily small and* c_r *defined as in Theorem* 1.3.

PROOF. (i) We have for $t > 0$,

$$\begin{aligned}\eta_l(t) &= E\exp\left[-\tfrac{1}{2}t^2\sigma_l^2(G_0;G_1,\dots,G_l)\right]\\ &= \tfrac{1}{2}t^2\int_0^\infty \exp\left[-t^2x/2\right]P\left(\sigma_l^2(G_0;G_1,\dots,G_l)\le x\right)dx.\end{aligned} \tag{3.43}$$

Hence, condition $(V_{l,R})$ implies $|\eta_l(t)| = O(|t|^{-R})$, $|t|\uparrow\infty$, and

$$\eta_l(t) \ge \exp\left[-\tfrac{1}{2}\cdot 1^2\right]P\left(t^2\sigma_l^2(G_0;G_1,\dots,G_l)\le 1\right)$$

proves the equivalence $(V_{l,R}) \Leftrightarrow \eta_l(t) = O(|t|^{-R})$.

(ii) Let $l \ge 1$. Notice that $G_0 \overset{\mathscr{D}}{=} \alpha_0 G_0 + \alpha_1 G_{l+1} + \cdots + \alpha_1 G_r$ where $\alpha_0^2 + (r-l)\alpha_1^2 = 1$. Applying Corollary 3.26 to the function

$$x \to D^lF(\alpha_0G_0 + x)[G_1,\dots,G_l]$$

with $j \triangleq r - l$ we obtain

$$\eta_l(t) \le E^{2^{-j}}\exp[it\Delta^jD^lF],$$

where

$$\Delta^jD^lF \triangleq \sum_\beta(-1)^{|\beta|}D^lF\Big(\alpha_0G_0 + \alpha_1\big(\beta_1G_{l+1} + (1-\beta_1)\overline{G}_{l+1} + \cdots + \beta_jG_r + (1-\beta_j)\overline{G}_r\big)\Big)$$

and the sum extends over all tuples $\beta \in \{0,1\}^j$. Hence,

$$\eta_l(t) \le E^{2^{-j}}\exp\left[-t^2E\left(\Delta^jD^lF^2|\mathscr{A}_1\right)/2\right], \tag{3.44}$$

where $\mathscr{A}_1 \triangleq \sigma(G_\nu, \nu\ne 1, \overline{G}_{l+1},\dots,\overline{G}_r)$. Expanding in α_0 around 1 and in α_1 around 0 yields

$$\Delta^jD^lF = d_rF\cdot\alpha_1^j + \left(1 + \|G_0\|^L\right)\prod_1^l\|G_r\|\prod_{l+1}^r\left(\|G_r\|^2 + \|\overline{G}_r\|^2 + 1\right)O\left(\alpha_1^{j+1}\right),$$

where $d_rF \triangleq D^rF(G_0)[G_1,\dots,G_{l+1} - \overline{G}_{l+1},\dots,G_r - \overline{G}_r]$.

Hence, by Lemma 3.14(ii) we have for $\alpha_1\downarrow 0$, $\Delta^jD^lF = d_rF\cdot\alpha_1^j + O(\alpha_1^{j+1-\delta})\|G_1\|$ with probability $1 - O(\alpha_1^L)$, where $L > O$ is arbitrarily large and

$\delta > 0$ arbitrarily small. Therefore,

$$
\begin{aligned}
(3.45)\qquad & P\big(E\big(\Delta^j D^l F^2|\mathscr{A}_1\big) \le \varepsilon^2\big) \\
& \le P\big(E\big(d_r F^2|\mathscr{A}_1\big) \le \epsilon^2/\alpha_1^{2j} - O\big(\alpha_1^{1-2\delta}\big)\big) + O\big(\alpha_1^L\big).
\end{aligned}
$$

Choosing $\alpha_1 < c\varepsilon^{2/(2j+1-2\delta)}$ condition $(V_{r,k})$ yields the upper bound $O(\varepsilon^{k(2j-2\delta)/(2j+1-2\delta)}$ for (3.45). This inequality together with (3.44) implies

$$\eta_l(t) = O\big(|t|^{-k2^{-j}(2j/(2j+1))(1-\delta)}\big) \quad \text{for some } \delta > 0,$$

thus proving assertion (ii). □

We mentioned that Lemma 3.23 and Corollary 3.26 contain the key tools to show that the d.f. of $F(S_n)$ has jumps of sufficiently small order only. The following lemma applies these inequalities in order to bound the remainder terms of the expansion of the c.f. of $F(S_n)$ by $n^{-r/2+\varepsilon}g(t)$, where $g(t)$ decreases sufficiently fast.

LEMMA 3.46. *Assume that F satisfies condition $(D_{3(s-2)})$ with some constants $c_F > 0$ and $K > 0$ and assume g satisfies condition $(D_{3(s-2)-M})$ with constants c_g and K, where $M \le s$. Then (recall $T_m = Z_1 + \cdots + Z_m$):*

(i) $\sup\{|E\exp[itF(T_m)]t^M g(T_m)|\; m \ge n\} \le cc_g(1+|t|)^{-\eta}$ *for some* $\eta > 0$ *and every t such that $|t| \le T_1 \triangleq m^{\gamma_1/2}$ [see (3.78)], where c is independent of n.*

(ii) $|E\exp[itF(T_m)]| \le O(n^{-r/2-\varepsilon-\eta})$ *for* $T_1 < |t| \le T_r \triangleq n^{r/2-\varepsilon}$.

(iii) *Let $\eta > 0$ be sufficiently small. Then*

$$\sup_{0\le l\le r-1} n^{-l/2}|\hat{\chi}_l(t)| = O(n^{-r/2+\varepsilon-\eta})$$

for every $|t| \ge T_1$ and $\chi_l \in C^{3(r-l)-2}$, i.e.,

$$|\hat{\chi}_j(t)| = O\big(|t|^{-3(r-l)+2-\eta}\big)$$

for $j = 1,\dots,r-1$ and arbitrarily small $\eta > 0$.

REMARK 3.47. Under the conditions of Lemma 3.46(i) we have

$$
\begin{aligned}
& \operatorname{ess\,sup}\Big\{\Big|E\Big(\exp\big[itF\big(Z_1^{\varepsilon_1} + \cdots + Z_l^{\varepsilon_l} + Z_{l+1} + \cdots + Z_m\big)\big] \\
(3.48)\qquad & \qquad \times t^M g(T_m)|Z_\gamma^{\varepsilon_\gamma}, \gamma = 1,\dots,l\Big)\Big| : m \ge n, |\varepsilon_\gamma| \le m^{-1/2}, Z_\gamma^{\varepsilon_\gamma}\Big\} \\
& \le cc_g\big(1+|t|\big)^{-\eta}.
\end{aligned}
$$

PROOF. We shall develop the estimation techniques in general steps which when combined or selectively used will prove Lemma 3.46(i)–(iii).

STEP 1. (Additional truncation of $Z_1,\dots,Z_N$.) Let Z_j be defined as in (3.2) and denote by A_μ^* the event that exactly $\mu = 0,1,\dots,N+\nu$ of the random vectors $Z_1,\dots,Z_{N+\nu}$ $(N,\nu \in \mathbb{N}$, to be chosen later) have norm

$\|Z_j\|^2 > \tau^2 \triangleq N/m < 1$. Define $p(\tau) \triangleq P(\|Z_1\| > \tau)$ and

$$A_\mu \triangleq \{\|Z_l\| > \tau, l = N + \nu, \dots, N + \nu - \mu + 1, \|Z_l\| \le \tau, l \le N + \nu - \mu\},$$

$$P_\mu \triangleq P(\cdot|A_\mu), \qquad E_\mu f \triangleq \int f \, dP_\mu.$$

Then we may write for integrable f,

$$Ef = E \sum_{\mu=0}^{N+\nu} \binom{N+\nu}{\mu} \mu! P(A_\mu) E_\mu f. \tag{3.49}$$

We have by Lemma 3.14(i) applied to $Z_1 + \cdots + Z_{N+\nu-\mu}$ and $Z_{N+\nu+1}, \dots, Z_m$ together with $\|Z_{N+\nu-\mu+1} + \cdots + Z_{N+\nu}\| \le \mu n^{-\sigma/2}$ and the differentiability of g,

$$E_\mu |g(T_m)| \le c c_g \big(1 + \mu^k n^{-\sigma/2} + C(k)\big). \tag{3.50}$$

Since

$$q_\mu(\tau) \triangleq \binom{N+\nu}{\mu} \mu! P(A_\mu) \le a(\tau)^\mu,$$

where $a(\tau) \triangleq (N + \nu) p(\tau)$, it follows that

$$\sum_{\mu=\nu+1}^{N+\nu} \mu^k q_\mu(\tau) \le c a(\tau)^{\nu+1}, \tag{3.51}$$

because $a(\tau) = O(N^{-(s-2)/2})$ for bounded $\nu \in \mathbb{N}$. From (3.51) and (3.50) we conclude

$$\begin{aligned} E \exp[itF(T_m)] g(T_m) &= \sum_{\mu=0}^{\nu} q_\mu(\tau) E_\mu \exp[itF(T_m)] g(T_m) \\ &\quad + O(N^{-(\nu+1)(s-2)/2}), \end{aligned} \tag{3.52}$$

which shows that for $N \ge c m^\delta$, $\delta > O$, ν may be chosen as a constant independent of m such that the error in (3.52) is $O(m^{-L})$ for arbitrarily large fixed $L > 0$.

Step 2. (Conditioning on $M - N$ random vectors in T_m.) We split the sum T_m into two parts:

$$T_m \triangleq U_N + T_{m,N}, \qquad U_N \triangleq \sum_{j=\mu-1}^{N} Z_j, \qquad 0 < N < m.$$

By Taylor expansion we have

$$\begin{aligned} g(T_m) &= \sum_{\gamma=0}^{p-1} g_\gamma + R_p(g), \quad \text{where } g_\gamma \triangleq D^\gamma g(T_{m,N})[U_N^\gamma] \gamma!^{-1}, \\ F(T_m) &= F_{\bar{p}-1,N} + R_{\bar{p}}(F), \quad \text{where } F_{l,N} \triangleq \Sigma_{\gamma=0}^{l} D^\gamma F(T_{m,N})[U_N^\gamma] \gamma!^{-1}. \end{aligned} \tag{3.53}$$

Conditioning on $T_{m,N}$ we have by Lemma 3.14(i) applied to U_N and to

$Z_{N+\nu+1} + \cdots + Z_m$ together with $\|Z_l\| \le n^{-\sigma/2}$ for $l = N+1, \dots, N+\nu$,

$$E_\mu |R_p(g)| \le c c_F \tau^p \quad \text{and} \quad E_\mu |g(T_m) R_{\bar{p}}(F)| \le c c_F \tau^{\bar{p}}. \tag{3.54}$$

Define $\Delta_{j+1,N} \triangleq F_{\bar{p},N} - F_{j,N}$. Then we have by expansion

$$\begin{aligned} &|E_\mu \exp[itF(T_m)]\, g(T_m)| \\ &\quad \le |E_\mu \exp[itF_{j,N}] \exp[it\Delta_{j+1,N}]\, g(T_m)| + O(\tau^{\bar{p}}|t|) \\ &\quad \le \sum_{\gamma=0}^{p-1} E_\mu \left| E_\mu\left(\exp[itF_{j,N}]\, g_\gamma \sum_{\gamma=0}^{q(\gamma,j)-1} \Delta^\nu_{j+1,N} t^\nu \nu!^{-1} \middle| T_{m,N} \right) \right| \\ &\qquad + O(\tau^p + |t\tau^{\bar{p}}|) + \sum_{\gamma=0}^{p-1} |t|^{q(\gamma,j)} O(\tau^q), \end{aligned} \tag{3.55}$$

for some $q > p$ and $q(\gamma, j) \triangleq \lfloor (q-\gamma)/(j+1) \rfloor$, where $\lceil x \rceil$ denotes the smallest integer larger or equal to x.

Furthermore, expand

$$\Delta^\nu_{j+1,N} = (\text{terms of order } l \le q - \gamma - 1 \text{ in } U_N) + O\big((1 + \|T_{m,N}\|^L)\|U_N\|^{q-\gamma}\big),$$

which yields by means of Lemma 3.14(i),

$$\begin{aligned} &|E_\mu \exp[itF(T_m)]\, g(T_m)| \\ &\quad \le \sum_{\gamma,l}^{*} E_\mu \left| E_\mu\left(\exp[itF_{j,N}]\, H_{\gamma,l}(T^\gamma_{m,N})[U^\gamma_N]\, t^l \middle| T_{m,N} \right) \right| \\ &\qquad + O(\tau^p + |t\tau^{\bar{p}}|) + O\big(\tau^q(1 + |t|^{\lceil q/(j+1) \rceil})\big), \end{aligned} \tag{3.56}$$

where $H_{\gamma,l}$ denotes a continuous γ-linear form, such that

$$\|H_{\gamma,l}(x)\| \le c(1 + \|x\|^L)$$

for some $L > 0$ sufficiently large and where the summation $\sum^*$ extends over l, γ such that $\gamma \le q-1$ and $l(j+1) \le q-1$.

STEP 3. [The symmetrization inequality (Lemma 3.23, Corollary 3.26).] In view of (3.56) it is sufficient to estimate

$$\varphi_N(t) \triangleq E_\mu\big(\exp[itF_{N,j}]\, h[U^k_N] \,\big|\, T_{m,N}\big), \tag{3.57}$$

where h denotes a continuous k-linear form with norm $\|h\| \le C(1 + \|T_{m,N}\|^L)$, $L > 0$ sufficiently large. Notice that $E_\mu(f | T_{m,N} = T) = E_0(f | T_{m,N} = T)$. Splitting U_N into j sums of independent summands of approximately equal length yields $U_N = U_{N1} + \cdots + U_{Nj}$. By Lemma 3.23 and Corollary 3.26 we obtain

$$|\varphi_N(t)|^{2^{j-1}} \le E\left(\left| E\left(\exp[it\Delta^j F_N] \prod_{\alpha \in \{0,1\}^{j-1}} h^{(\alpha)}[U^k_{N\alpha}] \,\middle|\, \mathscr{C} \right) \right| \,\middle|\, T_{m,N} \right), \tag{3.58}$$

where $\mathscr{C} \triangleq \sigma(T_{m,N}, U_{N2}, \overline{U}_{N2}, \dots, U_{Nj}, \overline{U}_{Nj})$, $\overline{U}_{Nj}$ denotes an independent copy of

U_{Nj}, $U_{N\alpha}$ is defined as S_α in Lemma 3.23 and

$$\Delta^j F_N \triangleq D^j F(T_{m,N})\big[U_{N1}, U_{N2} - \bar{U}_{N2}, \ldots, U_{Nj} - \bar{U}_{Nj}\big]. \tag{3.59}$$

Notice that *conditionally* on $\mathscr{C}$ the mapping $U_{N1} \to \Delta^j F_N$ is a continuous linear functional, i.e., $it\Delta^j F_N = f \cdot U_{N1}$, where $f \in E^*$ depends on $\mathscr{C}$ and t.

Furthermore, $H(U_{N1}) \triangleq \prod_\alpha h^{(\alpha)}[U_{N\alpha}^k]$ is a polynomial of degree $2^{j-1}k$ in U_1 given $\mathscr{C}$. Expanding $H(U_{N1}) = H_0(U_{N1}) + \cdots + H_{2^{j-1}k}(U_{N1})$ into homogeneous forms of degree ν for $H_\nu(U_{N1})$ we may assume w.l.g. that H is a form of degree $\nu \le k2^{j-1}$ in U_{N1}.

STEP 4. (Estimate of the c.f. of sums of i.i.d. random variables conditionally on $\mathscr{C}$.) Applying Lemma 3.28(i) and (ii) with f defined above, we obtain for $\nu \le k2^{j-1}$, some $\lambda > 0$ and $M = (N/j - \nu)(s-2)/2$,

$$\begin{aligned}|\varphi_N(t)|^{2^{j-1}} \le cE_0\big(\|H(\mathscr{C})\|\{\tau^\nu(1 + \|f\tau\|^\nu)\exp\big[-\lambda\tau^2\sigma^2(\mathscr{C})\big] \\ + 1_{\{\sigma^2(\mathscr{C}) < 2/3|t|\beta_3(\mathscr{C})m^{-1/2}\}}\}\big|T_{m,N}\big) \\ + O(N^{-M}),\end{aligned} \tag{3.60}$$

where $\bar{f} \triangleq D^j F(T_{m,N})[X_1, U_{N2} - \bar{U}_{N2}, \ldots, U_{Nj} - \bar{U}_{Nj}]$, $\sigma^2(\mathscr{C}) \triangleq E_0(\bar{f}^2|\mathscr{C})$, $\beta_3(\mathscr{C}) \triangleq E_0(|\bar{f}|^3|\mathscr{C})$,

$$\|H(\mathscr{C})\| \triangleq \|h\|^{2j-1}\big(1 + \|T_{m,N}\|^L\big)\big(\|U_{N2}\|^\kappa + \|\bar{U}_{N2}\|^\kappa + \cdots + \|\bar{U}_{Nj}\|^\kappa\big) \tag{3.61}$$

for some $L > 0$ and $\kappa \triangleq k2^{j-1} - \nu$. Notice that

$$\|f\| \le c_F\big(1 + \|T_{m,N}\|^K\big)\|U_{N2} - \bar{U}_{N2}\| \cdots \|U_{Nj} - \bar{U}_{Nj}\|\,|t|. \tag{3.62}$$

By Lemma 3.14(i) we have for $\delta > 0$ arbitrarily small, $l = 1, \ldots, j$,

$$P_\mu\big(\|U_{Nl}\| > \tau N^\delta\big) = O(m^{-L}), \qquad L > 0 \text{ arbitrarily large}, \tag{3.63}$$

provided $N \ge m^\delta$. Furthermore, again by Lemma 3.14(i) with $N = m$ and $\|Z_j\| \le n^{-\sigma/2}$ for $j = N+1, \ldots, N+\nu$, we have

$$P_\mu\big(\|T_{m,N}\| > m^\delta\big) = O(m^{-L}). \tag{3.64}$$

Hence, by $E_\mu\|H(\mathscr{C})\|1_B \le E_0^{1/L}\|H(\mathscr{C})\|^L P(B)^{1-1/L}$ we conclude by (3.61)–(3.64),

$$\begin{aligned}E_0\big(\|H(\mathscr{C})\|1_{\{\sigma^2(\mathscr{C}) < 2/3|t|\beta_3(\mathscr{C})m^{-1/2}\}}\big|T_{m,N}\big) \\ \le c\tau^{k2^{j-1}}E_0^{1-1/L}\big(\exp\big[-\lambda\sigma^2(\mathscr{C})m^{1/2}|t|^{-1-3(j-1)}\varepsilon_N\big]\big) \\ + O(m^{-L}),\end{aligned} \tag{3.65}$$

where $\varepsilon_N = m^{-\delta}$, $\delta > 0$ small.

Using Hölder's inequality and the last result we obtain from Lemma 3.14(i) and (3.60) for some $\lambda_1, \lambda_2 > 0$,

$$\begin{aligned}|\varphi_N(t)| \le c\|h\|\tau^k E_0\big((|t\tau^{2j}|^k + 1)\psi_N\big(\lambda_1\tau^{2j}t\big) + \psi_N\big(\lambda_2 m_t|T_{m,N}\big)\big) \\ + O(m^{-L}),\end{aligned} \tag{3.66}$$

where

$$\psi_N(a) \triangleq E_0^{2^{-(j-1)}(1-1/L)}\Big(\exp\big[-a^2\sigma^2(\mathscr{C})/\tau^2(j-1)\big]\big|T_{m,N}\Big),$$

$$m_t^2 \triangleq N^{1/2}|t\tau^j|^{-1}\varepsilon_N.$$

Furthermore, we would like to get rid of the r.v. Z_l, $N+\nu-\mu+1 \le l \le N+\nu$, which satisfy $\tau \le \|Z_l\| \le n^{-\sigma/2}$ and the expectations E_μ. Notice that in (3.52) we have to estimate $q_\mu(\tau)E_\mu(\cdots)$. By definition of A_μ and (3.51) from the first step it follows that

$$\begin{aligned} q_\mu(\tau)E_\mu(1+\|T_{m,N}\|^L)\psi_N(a) &\le q_\mu(\tau)P(A_\mu)^{-1}E1_{A_\mu}(1+\|T_{m,N}\|^L)\psi_N(a) \\ &\le C\binom{N+\nu}{\mu}\mu!P(A_\mu)^\alpha E^\beta\psi_N(a)^{1/\beta} \quad \text{(Hölder)} \\ &\le cE^\beta\psi_N(a)^{1/\beta} \end{aligned} \tag{3.67}$$

for every $\mu \le \nu$ and $\alpha+\beta<1$, $\alpha>0$. Hence, choosing an appropriate constant α, it follows from (3.52), (3.56), (3.57), (3.66) and (3.67) that

$$\begin{aligned} &|E\exp[itF(T_m)]g(T_m)t^M| \\ &\quad\le c(1+\tau^{q-1})|t|^M(1+|\tau^{j+1}t|^{[q/(j+1)]}) \\ &\qquad\times\Big[(1+|t\tau^j|^{q-1})E^{1/4}\psi_N\big(\lambda_1(t\tau)^2\tau^{2(j-1)}\big)^4 \\ &\qquad\qquad+E^{1/4}\psi_N(\lambda_2 m_t)^4+O(m^{-L})\Big] \\ &\qquad+O\big((\tau^p+|t|\tau^p)t^M\big)+O\big(t^M(1+t^{[q/(j+1)]})\tau^q\big). \end{aligned} \tag{3.68}$$

STEP 5. [Replace $U_{lN}-\overline{U}_{lN}$, $l=2,\dots,j$, by Gaussian r.v.'s in $\psi_N(a)$.] Rewriting $\psi_n(a)$ as a c.f. we have for $\gamma \triangleq 2^{-(j-1)}(1-L^{-1})$,

$$\psi_N(a) = E_0^\gamma\Big(\exp\big(iaD^jF(T_{m,N})\big)\big[G_1,U'_{2N},\dots,U'_{jN}\big]\Big)\Big|T_{m,N}\Big),$$

where G_1 denotes a Gaussian r.v. with the same covariance functional as X_1 and $U'_{pN} \triangleq (U_{pN}-\overline{U}_{pN})m^{1/2}N^{-1/2}$. Thus $E_0U'_{pN}=0$ and $U'_{pN} \Rightarrow G_p$ weakly, where $G_0,\dots,G_p$ denote i.i.d. Gaussian r.v.'s. Since

$$U'_{2N} \mapsto D^jF(T_{m,N})\big[G_1,U'_{2N},U'_{3N},\dots,U'_{jN}\big]$$

is linear given $\mathscr{C}_2 \triangleq \sigma(T_{m,N},G_1,U'_{3N},\dots,U'_{jN})$, Lemma 3.28(i) and (ii) imply [using arguments similar to (3.60)–(3.66)]

$$\begin{aligned} \psi_N(a) &\le E_0^\gamma\Big(\exp\big[-\lambda_3\sigma^2(\mathscr{C}_2)a^2\big]\big|T_{m,N}\Big) \\ &\quad+cE_0^\gamma\Big(\exp\big[-\lambda_4N^{1/2}|a|^{-1}\varepsilon_{N1}\sigma^2(\mathscr{C}_2)\big]\big|T_{m,N}\Big) \\ &\quad+O(m^{-L}), \qquad L>0 \text{ arbitrarily large,} \end{aligned} \tag{3.69}$$

where $\lambda_3,\lambda_4>0$ denote positive constants $\varepsilon_{N1}=O(m^{-\delta})$ and

$$\sigma_j^2(\mathscr{C}_2) \triangleq E_0\Big(D^jF(T_{m,N})\big[G_1,X_0,U'_{3N},\dots,U'_{jN}\big]^2\Big|\mathscr{C}_2\Big)$$

and X_j, $j = 0, 1, 2, \dots$, are i.i.d. r.v. independent of $G_1, \dots, G_q$. Iterating the estimate (3.69) above, we obtain by the monotonicity of ψ_N a sum of ψ_N function values for arguments like $a = a_0 \le N^{-\kappa+1/2}m^{-\eta}$ ($\eta, \kappa > 0$ small) and $a_{l+1} \triangleq N^{1/2}a_l^{-1}m^{-\eta}$. Hence for $l \ge 1$, $N \ge m^{\eta}$, $a_0 \ge 1$ we have $N^{\kappa}m^{-\eta} \le a_l \le N^{-\kappa+1/2}m^{-\eta}$. This implies for $|a| \le N^{-\kappa+1/2}m^{-\eta}$,

$$\begin{aligned}&\psi_N(a) \le \psi(a) + C\psi(cN^{\kappa}m^{-\eta}) + O(m^{-L}), \quad \text{where}\\ &\psi(a) \triangleq E^{\gamma}\Big(\exp\big[-a^2\lambda_r\sigma_j^2(T_{m,N}; G_1, \dots, G_j)\big]\big|T_{m,N}\Big)\end{aligned} \tag{3.70}$$

and $\lambda_r > 0$.

STEP 6. (Replace $T_{m,N}$ by a Gaussian r.v. G_0.) In view of (3.68) and (3.70) we have for $c > 0$ by Hölder's inequality

$$E^{1/4}\psi_N(a)^4 \le c\psi(ca; T_{m,N})^{\gamma} + C\psi(cN^{\kappa}m^{-\eta}; T_{m,N})^{\gamma} + O(m^{-L}),$$

where

$$\psi(a; T_{m,N}) \triangleq E\exp\big[-a^2\sigma_j^2(T_{m,N}; G_1, \dots, G_{j-1})\big]$$

for every $|a| \le N^{-\kappa+1/2}m^{-\eta}$. Using Lemma 3.36 we obtain with $a_n = n^{\sigma(1-\eta)/2}$ for arbitrarily small $\eta > 0$ by monotonicity of ψ,

$$\psi(a; T_{m,N}) \le \bar{\psi}(a; G_0) \triangleq \psi(\min(|a|, a_n); G_0). \tag{3.71}$$

Using the inequality (3.41) in the proof of Lemma 3.36 we may replace $F(Z_1^{\varepsilon_l} + \cdots + Z_l^{\varepsilon_l} + x)$ by $F(x)$ uniformly in $Z_r^{\varepsilon_r}$ [since $\|Z_r^{\varepsilon_r}\| = O(n^{-\sigma/2})$] in the definiton of the function ψ. Summarizing, we have by (3.68), (3.70) and the last inequality,

$$\begin{aligned}&|E\exp[itF(T_m)]g(T_m)t^M|\\ &\quad\le c|t|^M(1 + |\tau^q t^{\lceil q/(j+1)\rceil}|)\\ &\qquad\times\big[\big(\bar{\psi}(ct\tau^j; G_0) + \bar{\psi}(cN^{\kappa}m^{-\eta}; G_0)\big)(1 + |t\tau^j|^{q-1}) + O(m^{-L})\big]\\ &\qquad\quad+O\big((\tau^p + \tau^{\bar{p}}|t|)t^M\big) + O\big(t^{M+\lceil q/(j+1)\rceil}\tau^q\big)\end{aligned} \tag{3.72}$$

for every t such that

$$|t\tau^{2j}| \le N^{-\kappa+1/2}m^{-\eta}. \tag{3.73}$$

STEP 7. [Proof of (i)–(iii) by appropriate choice of N, κ, p, q, M and g.] Choose for $0 < \varepsilon < \frac{1}{2}$ defined in (1.4) and $j \ge 1$,

$$\begin{aligned}&\kappa \triangleq \tfrac{1}{4}, \qquad \delta_j \triangleq \alpha\varepsilon/(j+1), \quad \text{where}\\ &\alpha \triangleq 2(r+1)(r-2\varepsilon)^{-1}(1-2\kappa)(r-1+2\kappa)^{-1},\\ &N \triangleq c\big\lceil m|t|^{-2(1-\delta_j)/j}\big\rceil, \qquad |t| \ge 1.\end{aligned} \tag{3.74}$$

Notice that

$$\alpha \sim 1/r, \qquad r \uparrow \infty, \qquad \tau \sim |t|^{-(1-\delta_j)/j}, \qquad |\tau^j t| \sim |t|^{\delta_j}, \tag{3.75}$$
$$\alpha = 1.34, 0.64, 0.41, 0.3, 0.232, 0.19, 0.16 \quad \text{for } r = 2, \dots, 8.$$

Defining T_j as the maximum frequency such that (3.73) holds for every $m \geq n$ we obtain after some computations,

$$T_j \triangleq cn^{j\gamma_j/2}, \quad \text{where for } \eta > 0 \text{ arbitrarily small,} \tag{3.76}$$
$$\gamma_j \triangleq 1 - \eta - (j - 1 + 2\kappa)\delta_j / \big(1 - 2\kappa + (j - 1 + 2\kappa)\delta_j\big) \triangleq 1 - \nu_j.$$

Notice that $T_r = O(n^{-r/2+\varepsilon})$ by the choice of α.

Using these choices of parameters we obtain by Lemma 3.42 the following upper bound for (3.72):

$$O\Big(|t|^{M+1-q(1-\varepsilon\alpha)/j(j+1)} + |t|^{M+\delta_j(q-1)} \min\big(|t|^{\delta_j}, N^{\kappa}, a_n\big)^{-R_j 2^{-(j-1)}(1-\eta)} \tag{3.77}$$
$$+ |t|^{M+1-\bar{p}(1-\delta_j)/j} + |t|^{M-p(1-\delta_j)/j} I(g \not\equiv \text{const.}) + m^{-L}\Big),$$

where

$$R_j \triangleq c_r^* 2^{j-1}/\varepsilon, \qquad \eta > 0 \text{ small, } L > 0 \text{ arbitrarily large,}$$
$$a_n \triangleq n^{\varepsilon/(r+2)-\eta} \quad \text{and} \quad c_r^* > \tfrac{4}{3}c_r.$$

This means $c_r^* = 6r^2$ for $r \geq 7$ and $c_r^*/r^2 = 2.8, 2.9, 3.4, 3.8, 4.1$ for $2 \leq r \leq 6$.

Proof of (i). Let $j = 1$. Then we have

$$\gamma_1 = 1 - 2\kappa\varepsilon\alpha/(1 - 2\kappa + \varepsilon\alpha\varepsilon) < 1 - 2\varepsilon. \tag{3.78}$$

Choose

$$q \triangleq \lceil 2(M+1)/(1-\varepsilon\alpha) \rceil,$$
$$\bar{p} \triangleq \lceil (M+1)/(1-\varepsilon\alpha/2) \rceil,$$
$$p \triangleq \lceil M/(1-\alpha\varepsilon/2) \rceil.$$

Notice that p derivatives of g exist since $p \leq 3r - (r+2) + M - 1$ and $M \leq r + 2$. This holds true for our choice of α for $r \geq 2$ provided that $\varepsilon < \frac{1}{4}$. Let $j = 1$ in (3.77). The various terms in (3.77) are bounded by $O(n^{-\eta}) + O(t|^{-\eta})$, $\eta > 0$, for $|t| \leq n^{\gamma_1/2}$ provided that $c_r^* > \max(\xi_1, \xi_2, \xi_3)$, with $\xi_1 \triangleq 2(r+2)\varepsilon/(1-\varepsilon\alpha)$, $\xi_2 \triangleq (r+2)((r+2) + (r+3)\alpha\varepsilon/(1-\alpha\varepsilon))\gamma_1/2$ and $\xi_3 \triangleq \varepsilon\xi_2/((r+2)\gamma_1\kappa(1-\alpha\varepsilon/2) - \kappa\varepsilon)$ which follows from our choice of c_r^*.

Proof of (ii). Let $M = 0$ in (3.77), let $g = 1$ and $m = n$. Fix a j with $2 \leq j \leq r$. Choose

$$b \triangleq (r-2)/\big((j-1)\gamma_{j-1}\big), \qquad \bar{p} \triangleq \big\lceil (b+1)j/(1-\delta_j) \big\rceil, \tag{3.79}$$
$$q \triangleq \lceil (1-\varepsilon\alpha)j(j+1)(b+1) \rceil \quad \text{and} \quad p \triangleq \big\lceil bj/(1-\delta_j) \big\rceil.$$

Notice that $p \leq j(r - 2\varepsilon)/(\gamma_{j-1}(1 - \alpha\varepsilon/(j+1))(j-1)) < 3r$. Thus this choice is allowed by our differentiability constraints on F provided that $\varepsilon < \frac{1}{4}$. By the choice of R_j and T_{j-1} we obtain with the choices (3.79) after elementary calculations the upper bound $O(n^{-r/2+\varepsilon-\eta})$, $\eta > 0$, for (3.77) in the intervals $T_{j-1} \le |t| \le T_j$ provided that $c_r^* > \max(\xi_4, \xi_5, \xi_6)$, where

$$\xi_4 \triangleq (j+1)b(1-(j+1)\alpha\varepsilon)/(\alpha(1-\alpha\varepsilon)),$$
$$\xi_5 \triangleq ((2r-\varepsilon)(r+2)+\varepsilon b(j+1)j/(1-\alpha\varepsilon))/2,$$
$$\xi_6 \triangleq \varepsilon\big(r-2\varepsilon+j\gamma_j bj\alpha\varepsilon/(1-\alpha\varepsilon)\big)/\big(2\kappa(\nu_j+\delta_j\gamma_j)\big).$$

This follows from the choice of c_r in condition $(V_{r,k})$ and completes the proof of part (ii).

Proof of (iii). By the definition (3.6) and (3.7) of the expansion of the c.f. $\hat{\chi}_l(t)$ of $\chi_l(a)$ we have

$$\hat{\chi}_l(t) = \sum_{M=l}^{3l} t^M E\exp[itF(G)]g_M(G), \tag{3.80}$$

where g_M consists of sums of products of derivatives of F such that g_M fulfills the differentiability condition $(D_{3r-1-(3l-M)})$ for some c_{g_M} sufficiently large. Starting with relation (3.77) we let $m = n$ tend to infinity and define N, τ, δ_j as in (3.74). Thus $N^\kappa, a_n \to \infty$. Let $d \triangleq 3r-1-3l+M$ and let

$$j=1, \qquad q \triangleq \lceil 2(d+1)/(1-\varepsilon\alpha)\rceil \quad \text{and} \quad p \triangleq \lceil (d-1)/(1-\alpha\varepsilon/2)\rceil. \tag{3.81}$$

Notice that the choice of p is possible since $g \in C^d$ provided that $\varepsilon < \frac{1}{4}$ which given the choice of α forces $p = d$. From (3.81) and (3.77) we obtain for a particular term of (3.80) the following upper bound for $j = 1$ and *arbitrary* t by choosing $c_r^* > 2(3r-2)(1-2\varepsilon/(1-\alpha\varepsilon))/\alpha$ (which follows by the choice of c_r),

$$\begin{aligned} &|t|^{M+\delta_1(q-1)}\psi\big(c|t|^{\delta_1}; G_0\big)^\gamma + O\big(|t|^{M-\bar{p}(1-\alpha\varepsilon/2)+1}\big) \\ &\quad + O\big(t^{M-p(1-\alpha\varepsilon/2)}\big) + O\big(|t|^{M-q(1-\varepsilon\alpha)/2}\big) \le c|t|^{-3(r-l)+2-\eta}. \end{aligned} \tag{3.82}$$

Hence, for $|t| \ge n^{\gamma_1/2}$ we have

$$n^{-l/2}|\hat{\chi}_l(t)| = O(n^{-\gamma_1(3(r-l)/2-1)-l/2}) = O(n^{-r/2+\varepsilon}) \quad \text{for } 0 \le l \le r-1$$

by the choice of γ_1 in (3.78). The differentiability of χ_j follows by Fourier inversion from (3.82). This proves part (iii) and completes the proof of Lemma 3.46. □

PROOF OF REMARK 3.47. The proof is immediate by the argument following inequality (3.71). □

With the notation of 1.12 we have

LEMMA 3.83. *Assume that F_j satisfies conditions (V_{j,k_j}), $j = \nu, \dots, \nu+h$, and let g denote a homogeneous polynomial of degree d. Let $R \triangleq r - 2\varepsilon$. Then*:

(i) $\sup\{(1+|t|^{r+2})|E\exp[it\tilde{F}_n(T_m)]g(T_m)|\colon m \ge n\} \le cc_g(1+|t|^\eta)^{-1}$ *for every t such that $|t| \le n^{\gamma_1/2}$.*

(ii) $\sup\{|E\exp[it\tilde{F}_n(T_n)]|\colon n^{\gamma_1/2} \le |t| \le n^{R/2}\} = O(n^{-R/2-\eta})$.

(iii) $\sup\{n^{-j/2}P_j(D)E\exp[it\tilde{F}_n(G+\varepsilon_1 X_1+\cdots+\varepsilon_l X)]|_{\varepsilon=0}\colon\ n^{\gamma_1/2}\le|t|\le n^{R/2}\}=O(n^{-R/2-\eta})$ *for some* $\eta>0$ *and* $j=0,\dots,r-1$.

(iv) *For* $|t|\le n^{\gamma_1/2}$ *we have*

$$\sum_{j=0}^{r-1} n^{-j/2}P_j(D)E\exp\big[it\tilde{F}_n(G+\varepsilon_1X_1+\cdots+\varepsilon_lX_l)\big]\big|_{\varepsilon=0}$$

$$=\sum_{j=0}^{r-1} n^{-j/2}\hat{\chi}_j(t)+O(n^{-R/2-\eta}),$$

where $\hat{\chi}_j$ *denote the Fourier transforms of differentiable functions* $\chi_j\in C^{3(r-j)-2}$ (χ_0 *nondecreasing*) *which do not depend on* n.

PROOF.

1. *The frequency range* $|t|\le n^{\nu/2-\varepsilon}$. Here the proof of (i) and (iii) is similar to the case where F does not depend on n. We use Lemma 3.42 in order to verify conditions of the type $(V_{j,k})$ for $T_{j-1}\le t<T_j$, $j=1,\dots,\nu$, as defined in (3.76). Moreover, since the derivatives of order $l>\nu$ of $\tilde{F}_n$ are of order $O(n^{-(l-\nu)_+/2})$ instead of (3.72) we obtain for $T_{j-1}<|t|<T_j$, $\mu\triangleq(j-\nu)_+$, $\bar{\mu}\triangleq(j-\nu+1)_+$ and some polynomial of degree d

$$\sup_{m\ge n}\big|E\exp\big[it\tilde{F}_n(T_m)\big]g(T_m)t^M\big|$$

$$\le c|t|^M\big(1+|\tau^{j+1}tn^{-\bar{\mu}/2}|^{q/(j+1)}\big)\Big(\bar{\psi}\big(ct\tau^j;G_0\big)^\gamma+\bar{\psi}\big(cN^\kappa n^{-\eta+\mu/2};G_0\big)^\gamma\Big)$$
$$\times\big(1+|t|^{\delta_j(q-1)}\big)+O(n^{-L})+O\big(|\tau^{j+1}tn^{-\bar{\mu}/2}|^{q/(j+1)}\big)\tag{3.84}$$

for every t such that

$$|t|^{\delta_j}\le cN^{-\kappa+1/2}.\tag{3.85}$$

This inequality is derived similarly as (3.73) but since we deal with polynomials we do not need to consider the Taylor expansion remainder terms for g and F_j. Furthermore, we adjusted all expressions for the order of the jth derivative of $\tilde{F}_n$ for $j>\nu$. As parameters we choose similarly as in (3.74) for $0<\varepsilon<\frac{1}{4}$,

$$\kappa=\tfrac{1}{4},\qquad \delta_j\triangleq\alpha\varepsilon/(j+1),\qquad N\triangleq\big\lceil cm^{1+\mu/j}|t|^{-2(1-\delta_j)/j}\big\rceil,$$
$$\tau\sim|t|^{-(1-\delta_j)/j}n^{\mu/(2j)},\qquad |t\tau^j|\sim|t|^{\delta_j}n^{\mu/2}.\tag{3.86}$$

For $1\le j\le\nu$ these are precisely the definitions of (3.74). Choosing α as in (3.74) yields the same intervals $[T_{j-1},T_j]$ as in (3.76) with T_ν proportional to $n^{-\nu/2+\varepsilon}$. Furthermore, choosing q exactly as in (3.78) and (3.79) we can prove part (i), since k_ν in the condition has been choosen large enough (depending on r) such that the error in part (ii) is $O(n^{-R/2})$ for $T_1\le|t|\le n^{\nu/2-\varepsilon}$.

2. *The case* $|t| \geq n^{\nu/2-\varepsilon}$. For $j \geq \nu + 1$, (3.86) defines a new scaling. The definition of N here is still meaningful (i.e., $1 \leq N \leq n$) provided that $n^{\mu/(2(1-\delta_j))} \leq |t| \leq n^{(j+\mu)/(2(1-\delta_j))}$. Furthermore, (3.85) yields

$$T_j \triangleq cn^{\gamma_j(j+\mu)/2}, \qquad j = \nu, \dots, \nu + h. \tag{3.87}$$

Here γ_j is defined as in (3.76) and with the choice of α in (3.74) we obtain $T_{\nu+h} > n^{R/2}$. Choosing

$$q = \left[Rj(j+1)/\left(\gamma_{j-1}(\mu + j - 2)(1 - \varepsilon\alpha) + j - \mu\right)\right]$$

and choosing in Theorem 1.13, $c_{j,r} \triangleq r^2$, we obtain after some tedious but straightforward calculations that

$$\begin{aligned} 2c_{j,r} > \max\Big(&\left\{(\alpha\gamma_{j-1})^{-1}R(j+1)(\mu + j - 2)^{-1} + (q-1)\varepsilon\right\}, \\ &\left\{(r+2)R + \gamma_j\alpha(\mu + j)(j+1)^{-1}(q-1)\varepsilon(r+2)\right\}, \\ &\left\{\left[\gamma_j j(j+1)^{-1}\varepsilon\alpha(q-1) + 1 + jR(\mu + j)^{-1}\right]\varepsilon/2\kappa\right\}\Big) \end{aligned}$$

implies that (3.84) is bounded by $O(n^{-R/2-\eta})$ in $[T_{j-1}, T_j]$. This proves part (ii) of Lemma 3.82.

Proof of (iii). Consider the limit $m \to \infty$ and fix n in the stochastic expansion $\tilde{F}_n(x)$. By the arguments of Lemma 3.46(iii) we obtain, since k_ν is chosen depending on r, that the choices (3.81) yield for the expansion $\chi_{l,n}$ of $\tilde{F}_n$,

$$|\hat{\chi}_{l,n}(t)| = O\left(|t|^{-3(r-l)+2-\eta}\right), \tag{3.88}$$

which in turn proves part (iii) as in (3.82).

Proof of (iv). Expanding $\exp[it\tilde{F}_n(G)]$ in terms of $tn^{-1/2}$ and multiplying this expansion (in $n^{-1/2}$) with the expanded derivatives of $\tilde{F}_n(G)$ we obtain the desired expansion with an error term of the type

$$\begin{aligned} &\left| E\left(\exp[itF_0(G)] \sum_\alpha{}^* (tn^{-1/2})^{\alpha_1} n^{-\alpha_2/2} F_{\alpha_1,\alpha_2}(G)\right)\right| \\ &\qquad + O\left((1 + \|G\|^K)(tn^{-1/2})Q\right), \end{aligned} \tag{3.89}$$

where the sum Σ^* extends over all $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 \geq r$, $\alpha_1 \leq Q - 1$, Q is chosen such that $Q(1 - \gamma_1) \geq r + 1$ and $F_{\alpha,\beta}(G)$ denotes a polynomial function of G. Hence,

$$\hat{\chi}_j(t) \triangleq \sum_{l=j+2}^{3j} E\left(\exp[itF_\nu(G_0)]\, g_l(G_0)\right) t^l,$$

where g_l denotes a polynomial of degree at most $3l(\nu - 1)$. For $k_\nu = 2^\nu c_r/\varepsilon$ and c_r as defined in Theorem 1.3 we obtain similar as in (3.82) that $|\hat{\chi}_j(t)| = O(|t|^{-3(r-j)+2-\eta})$ for $1 \le j \le r-1$ and by the choice of γ_1 in (3.78) it follows that $|\hat{\chi}_\nu(t)n^{-j/2}| = O(n^{-R/2-\eta})$ and $\chi_j \in C^{3(r-j)-2}$, thus proving part (iv) of Lemma 3.83. □

4. Proof of the results.

PROOF OF THEOREM 1.3. By means of (3.22) we obtain with $m = n$ uniformly in a,

$$P(F(S_n) \le a) = P(F(T_n) \le a) + O(n^{-r/2+\varepsilon}). \tag{4.1}$$

Define $\hat{\psi}_N(t) \triangleq E\exp[itF(T_n)]$. Using the well-known Berry–Esseen lemma [see, e.g., Bhattacharya and Ranga Rao (1986), Lemma 12.1, page 100] we have

$$\begin{aligned}\sup_a |P(F(T_n) \le a) - \theta_n(a)| &\le c_1 \int_{|t| \le n^{r/2-\varepsilon}} |\hat{\psi}_N(t) - \hat{\theta}_n(t)| \frac{dt}{|t|} \\ &\quad + c_2 \sup_a |\theta'(a)| n^{-r/2+\varepsilon} \\ &\triangleq I_1 + I_2, \quad \text{say},\end{aligned} \tag{4.2}$$

where

$$\theta_n(a) \triangleq \sum_{l=0}^{r-1} \chi_l(a) n^{-l/2}.$$

By Lemma 3.46(iii) we have $\sup_a |\theta_n'(a)| \le c\int |\hat{\theta}(t)|\, dt < c < \infty$. Thus $I_2 = O(n^{-r/2+\varepsilon})$. In order to estimate the term I_1 we split the domain of integration into two parts,

$$\{|t| \le n^{r/2-\varepsilon}\} = J_1 \mathbin{\mathring{\cup}} J_2, \quad \text{where } J_1 \triangleq \{|t| \le n^{\gamma_1/2}\}$$

and

$$J_2 \triangleq \{n^{\gamma_1/2} < |t| \le n^{r/2-\varepsilon}\}.$$

Then

$$\begin{aligned} I_1 &\le \int_{J_1} |\hat{\psi}_N(t) - \hat{\theta}_N(t)| \frac{dt}{|t|} + \int_{J_2} |E\exp[itF(T_n)]| \frac{dt}{|t|} + \int_{J_2} |\hat{\theta}_n(t)| \frac{dt}{|t|} \\ &\triangleq I_3 + I_4 + I_5, \quad \text{say}. \end{aligned} \tag{4.3}$$

Define $T_{\varepsilon,m} \triangleq Z_1^{\varepsilon_1} + \cdots + Z_r^{\varepsilon_r} + Z_{r+1} + \cdots + Z_m$, $\alpha = (\alpha_1, \dots, \alpha_r)$, $\alpha_l \ge 1$, $\beta =$

$(\beta_1, \dots, \beta_r)$ and $j = (j_1 \dots j_r)$, $1 \le j_l \le r$. By Lemma 3.12(i) we have

$$|\hat{\psi}_N(t) - \hat{\theta}_N(t)| \le cn^{-r/2} \sup \Bigg\{ \sum_{\alpha, \beta, j} \Bigg| E \exp[itF(T_{\varepsilon, m})] t^{\nu} \prod_{\mu=1}^{\nu} D^{\alpha_\mu} F(T_{\varepsilon, m}) \big[X_{j_\mu}^{\alpha_\mu} \big]$$

$$\times n^{|\beta|\sigma/2} \prod_{\mu=1}^{\nu} \|X_\mu\|^{\beta_\mu} \varphi_0^{(\beta_\mu)} \big(\varepsilon_\mu \|X_\mu\| n^{\sigma/2} \big) + I_6 \Bigg| : \tag{4.4}$$

$$m \ge n, |\varepsilon_\mu| \le m^{-1/2} \Bigg\},$$

where the sum extends over all vectors α, β and j such that $|\alpha| + |\beta| \le r + 2$ and I_6 denotes the second term in the estimate of Lemma 3.12(i).

For $t \in J_1$ we obtain by Lemma 3.12(i), Lemma 3.46(i) and (iii) the following estimate of (4.4):

$$O\big(n^{-(r+2)\sigma/2 - r/2 - \eta}\big) \sum_{\beta, \gamma} \prod_{\mu=1}^{\nu} E \|X_\mu\|^{\gamma_\mu} \varphi^{(\beta_\mu)} \big(\varepsilon_\mu \|X_\mu\| n^{\sigma/2} \big) \tag{4.5}$$

$$+ n^{-r/2}(1 + |t|)^{-\eta} = O(n^{-r/2 + \varepsilon - \eta}), \quad \text{where } |\gamma| \le r + 2.$$

This implies $I_3 = O(n^{-r/2+\varepsilon})$. Furthermore, by Lemma 3.46(ii), $I_4 = O(n^{-r/2+\varepsilon})$ and Lemma 3.46(iii) entails $I_5 = O(n^{-r/2+\varepsilon})$. Thus (4.2)–(4.5) together complete the proof of Theorem 1.3. □

PROOF OF THEOREM 1.13. Let $\psi_n(a)$ denote the expansion. We have

$$\begin{aligned} &\sup_a |P(T_n \le a) - \psi_n(a)| \\ &\quad \le \sup_a |P(\tilde{F}_n(a) \le a) - \psi_n(a)| + P(|\Delta_n| > n^{\varepsilon + 1/2}) \\ &\qquad + \sup_a |\psi_n(a + O(n^{-R/2})) - \psi_n(a)| \\ &\quad = \sup_a |P(\tilde{F}_n(a) \le a) - \psi_n(a)| + O(n^{-R/2}), \end{aligned} \tag{4.6}$$

by assumption and the uniform differentiability of $\psi_n(a)$. The expansion scheme of Lemma 3.12 is still working for $\tilde{F}_n(T_n)$ provided that we fix the n in the expansion of $\tilde{F}_n$ and approximate the limit distribution $\lim_{m \to \infty} \tilde{F}_n(T_m) \overset{\mathscr{D}}{=} \tilde{F}_n(G)$. This together with Lemma 3.83(iv) yields the expansion terms of Theorem 1.13. Using Lemma 3.83(i)–(iii) the proof is now analogous to the proof of Theorem 1.3. □

PROOF OF EXAMPLE 2.1. Let Σ denote the covariance matrix of G_j and let λ denote its maximal eigenvalue. Then there is a constant $c(\Sigma)$ depending on Σ only, such that $\varphi_{0,\Sigma}(x) \le c(\Sigma)\varphi_{0,\lambda Id}(x)$ holds for every $x \in \mathbb{R}^k$, where $\varphi_{0,\Sigma}$ denotes the multivariate normal density with mean zero and covariance Σ. Hence in proving condition $(V_{r,k})$ we may assume w.l.g. that G_l has independent components $G_{l1}, \dots, G_{lk}$. Then we have by conditioning $b_j \triangleq G_{j2} \cdots G_{jr}$, $j =$

$1, \ldots, k$, and (2.2)

$$(4.7)\qquad \begin{aligned} P\big(\sigma_r^2(G_0; G_2, \ldots, G_r) \le \delta^2\big) &= EP\left(\sum_{j=1}^{k} f^{(r)}\big(G_{0j}\big)^2 b_j^2 \le \delta^2 \middle| b_1, \ldots, b_k \right) \\ &\le c \prod_{j=1}^{k} E\big(\delta^2/\big(\delta^2 + b_j^2\big)\big)^{\alpha/2}. \end{aligned}$$

Let $\beta = 1/(\alpha^{-1} + r - 1)$. Since $P(|b_j| \le \delta^{\beta(r-1)}) \le rP(|G_{01}| \le \delta^{\beta})$ we obtain the upper bound $O(\prod_{j=1}^{k} \delta^{\beta})$ for (4.7), thus proving (2.4). □

Proof of Corollary 2.7. Given the proof of Theorem 1.13, the proof is similar to the proof of the result of Hall (1986). Therefore we shall confine ourselves to a brief outline of the arguments.

Since the limiting d.f. $\chi_0(a)$ is independent of P by virtue of the normalization [provided that the minimum eigenvalue of the covariance matrix of $g(X_1)$ given P, say λ_P is positive], we obtain that for every constants $\delta, c_1 > 0$ there exist constants $c_2, c_2 > 0$ such that

$$(4.8)\qquad \begin{aligned} \sup\Bigg\{ \Bigg| P\Bigg(F(S_n(P)) \le q_\alpha + \sum_{j=1}^{r-1} \psi_j(P_\alpha|P) n^{-j/2} \Bigg) - \alpha \Bigg| n^{r/2-\varepsilon}: \\ P \text{ such that } \lambda_P > \delta,\ E_P\|X_1\|^{r+2} \le c_1, n \ge c_1 \Bigg\} \\ \le c_2 \le \infty. \end{aligned}$$

Let Σ (resp. Σ_n) denote the covariance matrices of $g(Z_1)$ given P (resp. $\hat{P}_n$). By Chebyshev's inequality it follows that $|x^T(\Sigma - \Sigma_n)x| \le \delta_P/2\|x\|^2$ holds on a set $\mathscr{C}_1$ of samples X such that $P(\mathscr{C}_1) = 1 - O(n^{-r/2}\sigma_P^{-r})$. Hence, $\lambda_{P_n} > \sigma_P/2 \triangleq \sigma > 0$ on $\mathscr{C}_1$ and if M denotes the (sample) expectation of $\|Z\|^l$, $l \le r + 2$, given P_n we obtain $P(M \le E_P M + c_4) = 1 - O(n^{-r/2})$. Let $\mathscr{C}_2$ denote the latter event. Since $F(x)$ and the condition $(V_{r,k})$ do not depend on P we conclude that the expansion in (4.8) holds uniformly with error $c_2 n^{-r/2+\varepsilon}$ for every sample in $\mathscr{C} \triangleq \mathscr{C}_1 \cap \mathscr{C}_2$.

Let $\chi_{n,r}(a)$ denote the expansion (1.4) of length r. Since $\chi_0'(q_\alpha) > 0$ by assumption it follows $\chi_{n,r}'(q_\alpha) > \delta\chi_0'(q_\alpha)$ for n sufficiently large and $1 > \alpha > 0$. Hence there exists a constant c_5 (independent of n) such that the exact α-quantile q_α^* can be bounded from below and above by the Cornish–Fisher expansion quantile $q_{\alpha,n}^*$ defined as in (1.18) for the bootstrap distribution $P = \hat{P}_n$ with $\psi_j = \psi_j(\cdot|\hat{P}_n)$ uniformly for all samples in $\mathscr{C}$:

$$(4.9)\qquad q_{\alpha,n}^* - c_5 n^{-r/2+\varepsilon} \le t_\alpha^* \le q_{\alpha,n}^* + c_5 n^{-r/2+\varepsilon}.$$

Let $q_{\alpha,n}$ denote the Cornish–Fisher expansion quantile defined in (1.18). Since $\psi_1(\cdot|P)$ is proportional to the third cumulant of $g(x_1)$ given P and $\psi_2(\cdot|P)$ is a polynomial in the third and fourth cumulants given P, we may expand $\psi_j(\cdot|\hat{P}_n)$

in terms of the empirical process measure $e_n \triangleq (\hat{P}_n - P)n^{1/2}$. It follows that there are functions $\tilde{\psi}_j(\cdot|P)$ such that

$$
(4.10)\qquad \begin{aligned} q^*_{\alpha,n} = q_{\alpha,n} &+ n^{-1}\int \tilde{\psi}_1(y|P)\,de_n(y) \\ &+ n^{-3/2}\int \tilde{\psi}_2(y|P)\,de_n(y) + O_P(n^{-2}). \end{aligned}
$$

Hence, (4.8), (4.9) and (4.10) together imply for $r = 3$,

$$|P\big(F\big(S_n^*(\hat{P}_n)\big) \le t^*_\alpha,\ X \in \mathscr{C}\big) - \alpha| + P(X \in \mathscr{C}) = O(n^{-1}),$$

thus proving the first part of Corollary 2.7. Choosing $r \ge 5$ and F symmetric it follows that $\tilde{\psi}_1 \equiv 0$ in (4.10). Furthermore, $\int\tilde{\psi}_2(y|P)\,de_n(y)$ converges weakly to a r.v. G^* with normal distribution which drops out of condition $(V_{r,k})$ for $F(G_0) - G^*n^{-3/2}$ by differentiation. Hence, (4.8) holds for $\tilde{F}_n \triangleq F(S_n(P)) - n^{-3/2}\int\tilde{\psi}_2(y|P)\,de_n(y)$ as well. Since in the expansion of the c.f. of $\tilde{F}_n$ the term $\tilde{\psi}_2$ does not enter linearly because of the symmetry of the region $F(x) \le q$ it follows that $|P(F(S_n^*(\hat{P}_n)) \le t_\alpha) - \alpha| = O(n^{-2})$ [resp. $= O(n^{-2+\varepsilon})$ for $r = 4$], which completes the proof of Corollary 2.7. □

Proof of Corollary 2.12. The proof follows immediately from Theorem 1.13 and the known results on the formal expansion terms $\chi_j(a)$ mentioned in Section 2.9 which imply $\chi_j(a) \equiv 0$ for $j = 1, 2, 3$. □

Proof of Example 2.13. With the notation of the proof of Example 2.1 we have

$$
(4.11)\qquad E\Big(D^lF_l[G_{1j},\dots,G_{lj}]^2\Big|G_2,\dots,G_l\Big) = c_l\sum_{j=1}^{k+1}\theta_j\Big(b_j\theta_j^{-(l-1)} - \bar{b}\Big)^2,
$$

where $c_l > 0$ and $\bar{b} \triangleq \sum_{j=1}^{k+1}\theta_j^{-(l-2)}b_j$. Since $(G_{p1},\dots,G_{p(k-2)},\sum_1^k G_{pj})$, $p = 2,\dots,l$, are Gaussian vectors with positive covariance, we may replace them as in the proof of Example 2.1 by Gaussian r.v. with independent components. Therefore (4.11) can be bounded from above by $\sigma_l^2 \triangleq c\sum_1^{k-1}\theta_j(b_j\theta_j^{-(l-1)} - \bar{b})^2$. Furthermore, $\sigma_l^2 \le \delta^2$ implies $|b_j - b_1| \le \delta c(\theta)$ for $j = 2,\dots,k-1$, which in turn implies the claim by conditioning on b_1 and using $\max_z P(|b_j - z| \le \delta) = O(\delta^{1/(l-1)})$. □

Proof of Remark 2.16. Assume $a = 0$ and let $V^{(r)}t, x)$ denote the rth derivative with respect to x. Then $|V^{(r)}(t, t^{1/2}\log t^{-1})| \ge c > 0$ for $0 < t \le \eta$, η sufficiently small which implies $|V^{(r)}(t, w_0(t))| \ge c$ with probability $1 - O(\eta^K)$ for $0 \le t \le \eta$ and K arbitrary large. Furthermore, the event $c^2\int_0^\eta w_1(t)^2 \cdots w_{r-1}(t)^2\,dt \le \delta$ has probability $O(\delta^L)$, L arbitrary large. (Use conditioning and the well-known properties of the Brownian bridge.) □

Proof of Remark 2.17. Exactly as in Remark 1.12 of Götze (1986) we conclude that (2.15)(i)–(iii) implies condition $(V_{r,k})$ for $F(x(\cdot))$. □

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