The Annals of Probability
1989, Vol. 17, No. 4, 1367–1376

THE INFINITE SELF-AVOIDING WALK IN HIGH DIMENSIONS[1]

BY GREGORY F. LAWLER

Duke University

A measure on infinite self-avoiding walks is defined which is the natural limit of the uniform measure on finite self-avoiding walks. This limit is shown to exist in sufficiently large dimensions using the methods of Slade and Brydges and Spencer.

1. Introduction. A self-avoiding walk (SAW) of length T in $\mathbb{Z}^d$ is an ordered sequence of points $[\omega(0), \dots, \omega(T)]$ in $\mathbb{Z}^d$ with $\omega(0) = 0$; $|\omega(i) - \omega(i-1)| = 1$, $0 < i \le T$; and $\omega(i) \ne \omega(j)$, $0 \le i < j \le T$. Let Ω_T denote the set of SAWs of length T and c_T the cardinality of Ω_T. The study of SAWs first arose in chemical physics as a model of polymer chains; in this model, the uniform measure on Ω_T was considered, that is, the measure $P_T(\omega) = 1/c_T$, $\omega \in \Omega_T$. Many questions about P_T are still open, in particular how does the mean-square displacement $E_{P_T}(|\omega(T)|^2)$ behave as $T \to \infty$, and what is the limiting distribution of $[E_{P_T}(|\omega(T)|^2)]^{-1/2}\omega(T)$? Recently, Slade [4, 5], using the ideas of Brydges and Spencer [1] on a related model, proved that there is a d_0 such that for $d \ge d_0$,

$$E_{P_T}\left(|\omega(T)|^2\right) \sim DT \tag{1.1}$$

and the limiting distribution is Gaussian. This result is expected to be true for all $d \ge 5$. For $d = 4$, logarithmic corrections are expected in (1.1) and for $d < 4$ a different power of T is expected.

As stated, the SAW problem is really a combinatorial rather than a probabilistic problem. In particular, the measures $\{P_T\}$ do not form a consistent family. [We call measures $\{\mu_T\}$ on Ω_T consistent if for every $R < T$ and $\omega \in \Omega_R$,

$$\mu_R(\omega) = \sum_{\substack{\xi \in \Omega_T \\ \xi > \omega}} \mu_T(\xi),$$

where $\xi > \omega$ means $\xi(i) = \omega(i)$ for $0 \le i \le R$.] To see that $\{P_T\}$ are not consistent is not difficult; indeed one can find R and walks $\omega \in \Omega_R$ which are "trapped," that is, such that

$$\sum_{\substack{\xi \in \Omega_T \\ \xi > \omega}} P_T(\xi) = 0$$

for T sufficiently large. An example with $d = 2$, $R = 8$ is pictured in Figure 1.

Received March 1988; revised July 1988.
[1]Research supported by NSF Grant DMS-87-02879 and an Alfred P. Sloan Research Fellowship.
AMS 1980 *subject classifications*. 60J15, 82A51.
Key words and phrases. Self-avoiding random walk, consistent measures, kinetically growing walks.

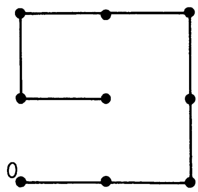

FIG. 1.

Given any consistent family of measures $\{\mu_T\}$ on Ω_T, we get a measure μ on Ω_∞, the set of infinite self-avoiding walks in the usual way. If $A \subset \Omega_T$ we define the cylinder set

$$B_A = \{\omega \in \Omega_\infty : [\omega(0), \dots, \omega(T)] \in A\}.$$

We then define $\mu(B_A) = \mu_T(A)$ and extend μ to the σ-algebra generated by the cylinder sets.

Recently a number of consistent measures have been given on SAWs. In the physics literature consistent measures are sometimes called kinetically growing walks. One which has been analyzed rigorously is the loop-erased or Laplacian self-avoiding random walk introduced by the author [2]. There are a couple of equivalent definitions for the loop-erased walk. One is to define it as the process with transition probability for $\xi > \omega$,

$$\text{(1.2)} \qquad \frac{\hat{P}_{T+1}(\xi)}{\hat{P}_T(\omega)} = \frac{P_{\xi(T+1)}\{S(j) \neq \omega(i), 0 \le i \le T, 0 \le j < \infty\}}{2dP_{\omega(T)}\{S(j) \neq \omega(i), 0 \le i \le T, 0 < j < \infty\}}.$$

In the above, $S(j)$ denotes a simple random walk and P_x denotes probabilities starting at $x \in \mathbb{Z}^d$. We note that to give any consistent family of measures, we need only give the transition probabilities as in (1.2). In [2] it was suggested that a consistent family of measures which would correspond to the usual SAW problem could be defined as follows: If $R < T$, let $P_{R,T}$ be the measure on Ω_R,

$$P_{R,T}(\omega) = (c_T)^{-1} |\{\eta \in \Omega_T : \eta > \omega\}|$$

and

$$\text{(1.3)} \qquad \tilde{P}_R(\omega) = \lim_{T \to \infty} P_{R,T}(\omega).$$

This would correspond to the transitions

$$\frac{\tilde{P}_{T+1}(\xi)}{\tilde{P}_T(\omega)} = \lim_{k \to \infty} \frac{|\{\eta \in \Omega_k : \eta(i) = \xi(i), 0 \le i \le T+1\}|}{|\{\eta \in \Omega_k : \eta(i) = \omega(i), 0 \le i \le T\}|}.$$

The problem comes in showing that the limit in (1.3) exists. In fact, it is nontrivial even to prove that

$$\liminf_{T \to \infty} P_{R,T}(\omega) > 0$$

if ω is not "trapped." (Madras [3] has given a proof of this for all $d \ge 2$.)

In this article, we use the methods of Slade to show that for sufficiently large d, the limit exists, that is,

THEOREM 1. *There exists a d_0 such that for $d \geq d_0$, $\omega \in \Omega_R$,*

$$\lim_{T \to \infty} P_{R,T}(\omega) = \tilde{P}_R(\omega)$$

exists.

The d_0 is the same d_0 as in [4]. Our method is to develop an expansion for the characteristic function of $P_{R,T}$. While this expansion differs somewhat from that in [4], when absolute values are taken the same expression is gotten, so that results of [4] may be quoted here.

It is easy to check that $\tilde{P}_R$ gives a consistent family of measures; hence we have $\tilde{P}_\infty$ which we define as the *infinite self-avoiding walk*. We expect Theorem 1 to hold for all dimensions; however, our methods will not be applicable to low dimensions.

Section 2 of this article develops the expansion, Lemma 3. The section is self-contained although it uses the methods of [1, 4 and 5]. The third section proves the theorem and is not self-contained; it relies heavily on the estimates derived in [4]. Any reader who wishes to follow this argument will need to read [4] along with this article.

2. Expansion for the characteristic function. Let $S(n)$ denote a simple random walk starting at the origin in $\mathbb{Z}^d$ and let P and E denote probabilities and expectations with respect to this walk. If $R \leq T$, $\omega \in \Omega_R$, $\tau > 0$, let

$$N_\tau(\omega, R, T) = P\{[S(0), \dots, S(R)] = \omega,\ S(i) \neq S(j) \text{ for } 0 \leq i, j \leq T, 1 \leq |j - i| \leq \tau\}.$$

The Fourier transform is defined for $k \in [-\pi, \pi]^{Rd}$ by

$$\varphi_\tau(k, R, T) = \sum_{\omega \in \Omega_R} e^{ik \cdot (S(1), \dots, S(R))} N_\tau(\omega, R, T).$$

Note that

$$\varphi_\tau(0, R, T) = \sum_{\omega \in \Omega_R} N_\tau(\omega, R, T)$$
$$= P\{S(i) \neq S(j) \text{ for } 0 \leq i, j \leq T, 1 \leq |i - j| \leq \tau\}.$$

Let

$$\bar{\varphi}_\tau(k, R, T) = [\varphi_\tau(0, R, T)]^{-1}[\varphi_\tau(k, R, T)].$$

Then the characteristic function of $P_{R,T}$ as defined in Section 1 considered as a measure on $\mathbb{Z}^{dR}$ is $\bar{\varphi}_T(k, R, T)$. In order to prove Theorem 1 it suffices to show that there exists a neighborhood U_R about 0 in $[-\pi, \pi]^{Rd}$ and a function

$\bar{\varphi}(k, R)$ such that for $k \in U_R$,

$$\lim_{T \to \infty} \bar{\varphi}_T(k, R, T) = \bar{\varphi}(k, R). \tag{2.1}$$

[Since the sequence of measures $P_{R,T}$ is supported on a finite subset of $\mathbb{Z}^{dR}$, the sequence is tight and hence it suffices to prove (2.1) for $k \in U_R$.]

We define the generating function

$$\begin{aligned} Z_\tau(z, k, R) &= \sum_{T=R}^{\infty} z^T \varphi_\tau(k, R, T) \\ &= \sum_{T=R}^{\infty} z^T E\left(\exp\left\{i \sum_{j=1}^{R} k_j \cdot S(j)\right\} \psi_\tau[0, T]\right) \end{aligned} \tag{2.2}$$

[we write $k = (k_1, \dots, k_R)$, $k_j \in [-\pi, \pi]^d$], where $\psi_\tau[a, b]$ is the indicator function of the event $\{S(i) \neq S(j)\colon a \le i, j \le b, 1 \le |i - j| \le \tau\}$, that is,

$$\psi_\tau[a, b] = \prod_{\substack{a \le i < j \le b \\ |i-j| \le \tau}} (1 - \delta(S(i) - S(j))).$$

Let r_τ be the radius of convergence of $Z_\tau(z, 0) = Z_\tau(z, 0, 0)$, that is, of

$$\sum_{T=0}^{\infty} z^T E(\psi_\tau[0, T]).$$

Clearly r_τ is nondecreasing in τ.

LEMMA 2. *For each R, there exists a neighborhood U_R of 0 such that for every $\tau > 0$, $k \in U_R$, the radius of convergence of $Z_\tau(z, k, R)$ is r_τ.*

PROOF. By symmetry, $E(\exp\{i\sum_{j=1}^{R} k_j \cdot S(j)\}\psi_\tau[0, T])$ is real. Hence it suffices to show for $k \in U_R$,

$$\tfrac{1}{2}E(\psi_\tau[0, T]) \le E\left(\exp\left\{i \sum_{j=1}^{R} k_j \cdot S(j)\right\} \psi_\tau[0, T]\right) \le E(\psi_\tau[0, T]).$$

But a simple calculation, using $|S(j)| \le R$, shows that the k-derivatives of the middle expression are bounded by $RE(\psi_\tau[0, T])$ which allows us to make the estimate.

Given an interval $[0, T]$, and τ, let G_T be the graph whose vertices are $\{0, 1, \dots, T\}$ and whose edges are $\{s, t\}$, $1 \le |t - s| \le \tau$. We will use graph to mean a subgraph of G_T and we let $\mathscr{G}_\tau$ be the collection of all subgraphs. If $\Gamma \in \mathscr{G}_\tau$, we write $st \in \Gamma$ to mean that $\{s, t\}$ is an edge of Γ.

A time $\sigma > 0$ is a cut-point for Γ if there do not exist $s < \sigma < t$ with $st \in \Gamma$. We call 0 a cut-point for Γ if $0t \notin \Gamma$ for each $t > 0$. Every graph Γ has a minimum cut-point, $s(\Gamma)$,

$$s(\Gamma) = \inf\{\sigma\colon \sigma \text{ cut-point of } \Gamma\}.$$

We call a graph Γ is connected if $s(\Gamma) = T$. If we let

$$U_{st} = \begin{cases} 0, & S(s) \neq S(t), \\ -1, & S(s) = S(t), \end{cases}$$

then

$$\begin{aligned} \psi_\tau[0,T] &= \prod_{st \in G_\tau} (1 + U_{st}) \\ &= \sum_{\Gamma \in \mathscr{G}_\tau} \prod_{st \in \Gamma} U_{st} \\ &= \sum_{\sigma=0}^{T} \sum_{\substack{\Gamma \in \mathscr{G}_\tau \\ s(\Gamma)=\sigma}} \prod_{st \in \Gamma} U_{st}. \end{aligned}$$

By resumming it is easy to see that

$$\sum_{\substack{\Gamma \in \mathscr{G}_\tau \\ s(\Gamma)=0}} \prod_{st \in \Gamma} U_{st} = \psi_\tau[1,T].$$

Define

$$\bar{\psi}_\tau[0,T] = \sum_{\substack{\Gamma \in \mathscr{G}_\tau \\ s(\Gamma)=T}} \prod_{st \in \Gamma} U_{st}.$$

Then for $\sigma > 0$, again by resumming we get

$$\sum_{\substack{\Gamma \in \mathscr{G}_\tau \\ s(\Gamma)=\sigma}} \prod_{st \in \Gamma} U_{st} = \bar{\psi}_\tau[0,\sigma]\psi_\tau[\sigma,T].$$

Note that $\bar{\psi}_\tau[0,\sigma]$ and $\psi_\tau[\sigma,T]$ are independent random variables. We then get

$$\psi_\tau[0,T] = \psi_\tau[1,T] + \sum_{s=1}^{T} \bar{\psi}_\tau[0,s]\psi_\tau[s,t]. \tag{2.3}$$

LEMMA 3. *For Z_τ as defined in* (2.2),

$$\begin{aligned} Z_\tau(z,k,R) &= zD(k)Z_\tau(z,\bar{k}_1,R-1) \\ &\quad + \sum_{s=1}^{R-1} z^s Z_\tau(z,\bar{k}_s,R-s) E\left(\exp\left\{i\sum_{j=1}^{R} k_j \cdot S(j \wedge s)\right\}\bar{\psi}_\tau[0,s]\right) \\ &\quad + Z_\tau(z,0)H_\tau(z,k,R), \end{aligned}$$

where

$$k = (k_1,\dots,k_R), \quad k_j \in [-\pi,\pi]^d; \quad k_j = (k_j^1,\dots,k_j^d);$$

$$D(k) = \frac{1}{d}\sum_{m=1}^{d} \cos\left(\sum_{j=1}^{R} k_j^m\right); \quad \bar{k}_s = (k_{s+1},\dots,k_R)$$

and

$$H_\tau(z,k,R) = \sum_{s=R}^{\infty} z^s E\left(\exp\left\{i\sum_{j=1}^{R} k_j \cdot S(j)\right\}\bar{\psi}_\tau[0,s]\right).$$

PROOF. By (2.3)

$$Z_\tau(z,k,R) = \sum_{T=R}^{\infty} z^T E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j)\right\}\left(\psi_\tau[1,T] + \sum_{s=1}^{T}\bar{\psi}_\tau[0,s]\psi_\tau[s,T]\right)\right).$$

$$\begin{aligned}
&\sum_{T=R}^{\infty} z^T E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j)\right\}\psi_\tau[1,T]\right)\\
&\quad = zE\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(1)\right\}\right)\\
&\qquad \times \sum_{T=R}^{\infty} z^{T-1}E\left(\left\{i\sum_{j=1}^{R} k_j\cdot (S(j)-S(1))\right\}\psi_\tau[1,T]\right)\\
&\quad = zD(k)Z_\tau(z,\bar{k}_1,R-1).
\end{aligned}$$

$$\begin{aligned}
&\sum_{T=R}^{\infty} z^T E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j)\right\}\sum_{s=1}^{R-1}\bar{\psi}_\tau[0,s]\psi_\tau[s,t]\right)\\
&\quad = \sum_{T=R}^{\infty} z^T \sum_{s=1}^{R-1} E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j\wedge s)\right\}\bar{\psi}_\tau[0,s]\right)\\
&\qquad \times E\left(\exp\left\{i\sum_{j=s+1}^{R} k_j\cdot (S(j)-S(s))\right\}\psi_\tau[s,T]\right)\\
&\quad = \sum_{s=1}^{R-1} z^s E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j\wedge s)\right\}\bar{\psi}_\tau[0,s]\right)\\
&\qquad \times \sum_{T=R}^{\infty} z^{T-s}E\left(\exp\left\{i\sum_{j=s+1}^{R} k_j\cdot (S(j)-S(s))\right\}\psi_\gamma t[s,T]\right)\\
&\quad = \sum_{s=1}^{R-1} z^s E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j\wedge s)\right\}\bar{\psi}_\tau[0,s]\right)Z_\tau(z,\bar{k}_s,R-s).
\end{aligned}$$

$$\begin{aligned}
&\sum_{T=R}^{\infty} z^T E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j)\right\}\sum_{s=R}^{T}\bar{\psi}_\tau[0,s]\psi_\tau[s,T]\right)\\
&\quad = \sum_{s=R}^{\infty} z^s E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j)\right\}\bar{\psi}_\tau[0,s]\right)\sum_{T=s}^{\infty} z^{T-s}E(\psi_\tau[s,T])\\
&\quad = Z_\tau(z,0)\sum_{s=R}^{\infty} z^s E\left(\exp\left\{i\sum_{j=1}^{R} k_j\cdot S(j)\right\}\bar{\psi}_\tau[0,s]\right)\\
&\quad = Z_\tau(z,0)H_\tau(z,k,R).
\end{aligned}$$

By adding up the contributions above we get the lemma. □

If we define $F_\tau(z, k, R)$ by

$$Z_\tau(z, k, R) = F_\tau(z, k, R) Z_\tau(z, 0),$$

Lemma 3 becomes

$$\begin{aligned} F_\tau(z, k, R) &= zD(k) F_\tau(z, \bar{k}_1, R-1) \\ &\quad + \sum_{s=1}^{R-1} z^s F_\tau(z, \bar{k}_s, R-s) E\left(\exp\left\{ i \sum_{j=1}^{R} k_j \cdot S(j \wedge s) \right\} \bar{\psi}_\tau[0, s] \right) \\ &\quad + H_\tau(z, k, R). \end{aligned} \tag{2.4}$$

3. Proof of Theorem 1. The proof of Theorem 1 follows [4] and we omit a large amount of the hard analysis which is done in that article. The function $H_\tau(z, k, R)$ defined in Lemma 3 is analogous to $\Pi_\tau(k, z)$ in [4]; in our notation the latter is defined by

$$\Pi_\tau(k, z) = \sum_{T=1}^{\infty} z^T E\left(e^{ik \cdot S(T)} \bar{\psi}_\tau[0, T]\right), \qquad k \in [-\pi, \pi]^d.$$

One can see that the functions differ only in the form of the complex exponential term (and the fact that H_τ has fewer terms).

We will show that a number of the results for $\Pi_\tau(k, z)$ also hold for $H_\tau(z, k, R)$ with similar if not identical proofs. We first refer to the derivation of (2.11) and (2.12) of [4]. In the proof of (2.11), the complex exponential term is just estimated by one, so the same proof works verbatim for H_τ. We are only interested in (2.12) for z-derivatives of H_τ. To get (2.12) for the z-derivatives of Π_τ, the exponential term must be separated into independent exponentials, that is, for $0 = T_0 < T_1 < T_2 < \cdots < T_n = T$ we can write

$$\exp\{ik \cdot S(T)\} = \prod_{m=1}^{n} \exp\{ik \cdot (S(T_m) - S(T_{m-1}))\}.$$

After differentiation by z, these terms are again estimated by one. The exponential term for H_τ can be split similarly,

$$\exp\left\{ i \sum_{j=1}^{R} k_j \cdot S(j) \right\} = \prod_{m=1}^{n} Y_m,$$

where $Y_m = 1$ if $T_{m-1} \geq R$ and otherwise

$$Y_m = \exp\left\{ i \sum_{j=T_{m-1}}^{R} k_j \cdot (S(j \wedge T_m) - S(T_{m-1})) \right\}.$$

With this splitting of the exponential into independent parts we can then follow the proof exactly. Hence we get that

$$|H_\tau(z, k, R)| \le \text{RHS of (2.11)},$$
$$|\partial_z H_\tau(z, k, R)| \le \text{RHS of (2.12)}.$$

Similarly (2.14) holds for $\dot{H}_\tau$ for those estimates which do not involve derivatives in k.

LEMMA 4 (Theorem 4.3 of [4]). *For* $d \ge d_0$, $k \in U_R$, $H_\tau(z, k, R)$ *and* $\partial_z H_\tau(z, k, R)$ *are analytic in* $D_\tau(\frac{1}{2}) = \{z\colon\ |z| \le r_\tau(1 + \frac{1}{2}(\log \tau/\tau))\}$ *and* $|H_\tau(z, k, R)|, |\partial_z H_\tau(z, k, R)| \le c/d$, *where* c *is a constant independent of* z, R, τ.

PROOF. Refer to the proof of Theorem 4.3 in [4]. The proof uses (2.12) and estimates for the RHS—since (2.12) holds in our case the identical proof holds. □

The generating function $Z_\tau(z, k, R)$ is analogous to $N_\tau(k, z)$ of [4]; in fact

$$Z_\tau(z, 0) = N_\tau(0, z),$$
$$Z_\tau(z, 0, R) = N_\tau(0, z) - \rho_\tau(z, R),$$

where $\rho_\tau(z, R)$ is the polynomial

$$\rho_\tau(z, R) = \sum_{T=0}^{R-1} z^T E(\psi_\tau[0, T]).$$

Our r_τ is the same as $r_\tau = r_\tau(0)$ of [4] and

$$\operatorname*{Res}_{z=r_\tau} Z_\tau(z, 0, R) = \operatorname*{Res}_{z=r_\tau} N_\tau(0, z). \tag{3.1}$$

By (5.11) of [4],

$$|N_\tau(0, z)| \le c|z - r_\tau|^{-1}.$$

Since $Z_\tau(z, 0, R) = N_\tau(0, z) - \rho_\tau(z, R)$ this clearly implies

$$Z_\tau(z, 0, R) \le c_R|z - r_\tau|^{-1}. \tag{3.2}$$

In fact, we could prove the above estimate with a constant independent of R but we will not need to do so. Since (2.14) holds for H_τ, the estimates above (5.15) of [4] for $\delta\Pi = \Pi_\tau - \Pi_\sigma$ can be used to show for $\sigma < \tau$,

$$|H_\sigma(r_\sigma, 0, R) - H_\tau(r_\sigma, 0, R)| \le c\sigma^{-1}. \tag{3.3}$$

Also (5.15) gives

$$r_\tau - r_\sigma \le c\sigma^{-1}. \tag{3.4}$$

In (3.3), (3.4) the constant c is independent of τ. Combining (3.4) with Lemma 4, we get

$$|H_\tau(r_\tau, 0, R) - H_\tau(r_\sigma, 0, R)| \le c\sigma^{-1}$$

and hence with (3.3) we get

$$|H_\sigma(r_\sigma, 0, R) - H_\tau(r_\tau, 0, R)| \le c\sigma^{-1}. \tag{3.5}$$

We assume the neighborhoods U_R of Section 2 have been chosen so that if $k = (k_1, \dots, k_d) \in U_R$, then $\bar{k}_s = (k_{s+1}, \dots, k_R) \in U_{R-s}$ for every $1 < s < R - 1$. Then by induction on R we see from (2.4), Lemma 4 and (3.5) that $F_\tau(z, k, R)$ is analytic in $D_\tau(\frac{1}{2})$ and

$$\begin{aligned} &|F_\tau(z, k, R)|, |\partial_z F_\tau(z, k, R)| \le c_R, \\ &|F_\sigma(r_\sigma, k, R) - F_\tau(r_\tau, k, R)| \le c_R \sigma^{-1}, \end{aligned} \tag{3.6}$$

where c_R is independent of τ, σ (but depends on R).

Following [4], we let C be the circle of radius $\frac{1}{2}$ around 0, oriented counterclockwise, and let $k \in U_R$. Then

$$\begin{aligned} \varphi_\tau(k, R, T) &= \frac{1}{2\pi i} \int_C Z_\tau(z, k, R) \frac{dz}{z^{T+1}} \\ &= -\operatorname*{Res}_{z = r_\tau} z^{-(T+1)} Z_\tau(z, k, R) + \frac{1}{2\pi i} \int_{\partial D_\tau(1/2)} Z_\tau(z, k, R) \frac{dz}{z^{T+1}} \\ &= -r_\tau^{-(T+1)} F_\tau(r_\tau, k, R) \operatorname*{Res}_{z = r_\tau} Z_\tau(z, 0) \\ &\quad + \frac{1}{2\pi i} \int_{\partial D_\tau(1/2)} Z_\tau(z, k, R) \frac{dz}{z^{T+1}}. \end{aligned}$$

From (3.2) the absolute value of the second term is bounded by $c_R r_T^{-(T+1)} T^{-1/2} \log T$ for $\tau = T$. By Corollary 4.2 of [4] [see the comment in [4] below (5.7)],

$$\operatorname*{Res}_{z = r_\tau} Z_\tau(z, 0) = -1 + O\left(\frac{1}{d}\right).$$

Hence we get

$$\begin{aligned} \bar{\varphi}_T(k, R, T) &= \frac{\varphi_T(k, R, T)}{\varphi_T(0, R, T)} \\ &= \frac{F_T(r_T, k, R)}{F_T(r_T, 0, R)} \left(1 + O_R(T^{-1/2} \log T)\right), \end{aligned}$$

where we write O_R to indicate that the term may depend on R. But the uniform estimates (3.6) show that there exists a $\bar{\varphi}(k, R)$ such that

$$\lim_{T \to \infty} \frac{F_T(r_T, k, R)}{F_T(r_T, 0, R)} = \bar{\varphi}(k, R)$$

and hence

$$\lim_{T\to\infty} \bar{\varphi}_T(k, R, T) = \bar{\varphi}(k, R),$$

which proves the theorem. □

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DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NORTH CAROLINA 27706