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TIME-AVERAGE CONTROL OF MARTINGALE PROBLEMS: A LINEAR PROGRAMMING FORMULATION[1]

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This paper studies the average cost for controlled systems given as solutions of the martingale problem for their generator. The control problem is reformulated as a linear programming problem and conditions are given for the existence of an optimal solution. It is further shown that the optimal control can be taken to depend only on the history of the system and that this cost remains optimal for systems with different initial distributions.

1. Introduction. This paper continues the study of the long-run average cost for controlled martingale problems begun in Stockbridge (1990). In this paper, the time-average control problem is shown to be equivalent to a linear programming problem in which one seeks to optimize the integral of the cost function against the stationary distributions for the system. This approach originated with Manne (1960) for discrete time with finite state and control spaces and has been studied by Wagner (1960), Derman (1962), Wolfe and Dantzig (1962), Denardo (1970) and Pittel (1971). Each of these require either time or the state and control spaces to be discrete. The present work allows for continuous time and continuous state and control spaces.

In order that this paper be as self-contained as possible, we begin in Section 2 by defining the control problem under study. It is then reformulated as a linear programming problem and existence of an optimal solution is shown in Section 3. There are two limitations to this existence result which we address in the rest of the paper. First, the optimal control depends on too much information; on the histories of both the state and control processes. In Section 4, we show existence of an optimal control which depends only on the history of the state process. Second, the optimal control and state processes are stationary, which therefore prescribes the initial state of the process. Frequently the initial distribution will not be the one given by the stationary distribution. In Section 5, we show existence of a solution with the same optimal cost as the optimal stationary solution provided the initial distribution of the state of the system is absolutely continuous (with bounded derivative) with respect to the state marginal of the optimal stationary distribution.

2. Formulation of the model. $\hat{C}(E)$ denotes the space of continuous functions on E which vanish at infinity. $\overline{C}(E \times U)$ denotes the space of bounded, continuous functions on $E \times U$ and $\|\cdot\|$ denotes the supremum norm on the appropriate space.

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Dynamics. Let the state space E and the control space U be locally compact separable metric spaces and $E^{\Delta} = E \cup \{\Delta\}$ the one-point compactification of E. Let $A: \mathscr{D}(A) \to \overline{C}(E \times U)$, $\mathscr{D}(A) \subset \hat{C}(E)$, satisfy:

(i) $\mathscr{D}(A)$ is dense in $\hat{C}(E)$.
(ii) For each $f \in \mathscr{D}(A)$ and $u \in U$, $Af(\cdot, u) \in \hat{C}(E)$.
(iii) For each $f \in \mathscr{D}(A)$ and compact $K \subset U$,

$$\lim_{x \to \Delta} \sup_{u \in K} Af(x, u) = 0.$$

(iv) For each $u \in U$, $A_u f = Af(\cdot, u)$ satisfies the positive maximum principle [i.e., if $f(x) = \sup_z f(z) > 0$, then $A_u f(x) \leq 0$].

2.1 Definition. An $E \times U$-valued process $(X(\cdot), u(\cdot))$ is a solution of the controlled martingale problem for A if there exists a filtration $\{\mathscr{F}_t\}$ such that:

(a) $(X(\cdot), u(\cdot))$ is $(\mathscr{F}_t\}$-progressive.
(b) For every $f \in \mathscr{D}(A)$, $f(X(t)) - \int_0^t Af(X(s), u(s))\, ds$ is an $\{\mathscr{F}_t\}$-martingale.

We will need to specify the initial distribution of the state. Thus for $\nu \in \mathscr{P}(E)$, $(X(\cdot), u(\cdot))$ is a solution of the controlled martingale problem for (A, ν) if, in addition to the above, $X(0)$ has distribution ν.

2.2 Remark. Conditions (i)–(iv) on A ensure the existence of solutions of the controlled martingale problem if we set $u(\cdot) = u$ for some fixed $u \in U$ and allow values in E^{Δ} [Ethier and Kurtz (1986), Chapter 4, Theorem 5.4]; however, to ensure the existence of optimal solutions in this more general setting we need to allow relaxed controls.

2.3 Definition. An $E \times \mathscr{P}(U)$-valued process $(X(\cdot), \Lambda_.)$ is a relaxed solution of the controlled martingale problem for A if there exists a filtration $\{\mathscr{F}_t\}$ such that:

(a) $(X(\cdot), \Lambda_.)$ is $\{\mathscr{F}_t\}$-progressive.
(b) For every $f \in \mathscr{D}(A)$, $f(X(t)) - \int_0^t \int_U Af(X(s), u)\Lambda_s(du)\, ds$ is an $\{F_t\}$-martingale.

Similarly $(X(\cdot), \Lambda_.)$ is a relaxed solution of the controlled martingale problem for (A, ν), $\nu \in \mathscr{P}(E)$, if the above holds and $X(0)$ has distribution ν.

Decision criterion. Let c: $E \times U \to \mathbb{R}$ be lower semicontinuous, bounded below and satisfy $c(x, u) \to \infty$ as $x \to \Delta$. Define the long-run average cost to be

$$C(X, \Lambda) = \limsup_{t \to \infty} t^{-1} E\left[\int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds\right],$$

where $(X(\cdot), \Lambda_.)$ is an $E \times \mathscr{P}(U)$-valued process or $(X(\cdot), \Lambda_.) = (X(\cdot), \delta_{\{u(\cdot)\}})$ if $(X(\cdot), u(\cdot))$ is an $E \times U$-valued process.

The control problem is to minimize the long-run average cost $C(X, \Lambda)$ subject to the condition that $(X(\cdot), \Lambda_.)$ be a relaxed solution of the controlled martingale problem for A.

Compactness assumptions. We assume the state space E is compact. There is no loss of generality in assuming E is compact and $(1,0) \in G(A)$, the graph of A, since we can define A^{Δ}: $C(E^{\Delta}) \to \overline{C}(E \times U)$ by $(A^{\Delta}f)|_E = A(f - f(\Delta)|_E)$ and $A^{\Delta}f(\Delta, u) = 0$ for those $f \in C(E^{\Delta})$ such that $f - f(\Delta) \in \mathscr{D}(A)$.

In the case that the space U of controls is not compact, we assume

(v) There exist a strictly positive $\psi \in \hat{C}(U)$ and positive constants a and b so that

$$1/\psi(u) \leq a + b\, c(x, u) \qquad \forall\, x, u.$$

3. Linear programming formulation. We shall use the existence of a stationary relaxed solution to the controlled martingale problem to reformulate the original control problem in terms of a linear programming problem and to show the existence of an optimal solution. The linear program involves optimizing over stationary distributions and is, in fact, an extension of Manne's formulation to this more general model. In order to optimize only over stationary distributions, it is sufficient to show that for any solution of the controlled martingale problem there is a stationary solution whose cost is no greater than that of the given solution.

In the companion to this paper [Stockbridge (1990)], we showed that the stationary distributions for the controlled martingale problem are characterized by the condition

$$\int Af\, d\mu = 0 \qquad \forall\, f \in \mathscr{D}(A). \tag{3.1}$$

We begin by stating the main result from that paper. Let Γ denote the set of distributions satisfying (3.1).

3.1 THEOREM. *Let E and U be locally compact separable metric spaces. Let A: $\mathscr{D}(A) \to \overline{C}(E \times U)$, $\mathscr{D}(A) \subset \hat{C}(E)$ satisfy conditions* (i)–(iv) *and $\mathscr{D}(A)$ is an algebra. Let $\mu \in \Gamma$. Then there exists a stationary relaxed solution $(X(\cdot), \Lambda_.)$ to the controlled martingale problem for A with*

$$E\left[\chi_{\Gamma_1}(X(0))\Lambda_0(\Gamma_2)\right] = \mu(\Gamma_1 \times \Gamma_2) \qquad \forall\, \Gamma_1 \in \mathscr{B}(E), \Gamma_2 \in \mathscr{B}(U).$$

Theorem 3.1 demonstrates the existence of a stationary solution to the controlled martingale problem for any distribution $\mu \in \Gamma$. The next theorem enables the control problem to be reformulated.

3.2 THEOREM. *For each solution $(X(\cdot), \Lambda_.)$ of the controlled martingale problem for A, there exists a stationary solution $(\tilde{X}(\cdot), \tilde{\Lambda}_.)$ such that $C(\tilde{X}, \tilde{\Lambda}) \leq C(X, \Lambda)$.*

Proof. Let $(X(\cdot), \Lambda_\cdot)$ be any relaxed solution of the controlled martingale problem for A having finite long-run average cost. Define the occupation measures $\mu_t \in \mathscr{P}(E \times U)$ by

$$\mu_t(\Gamma) = t^{-1}E\left[\int_0^t \int_U \chi_\Gamma(X(s), u)\Lambda_s(du)\, ds\right]$$

for $\Gamma \in \mathscr{B}(E \times U)$. Condition (v) implies that the measures $\{\mu_t\}$ are tight and hence relatively compact.

Now consider a sequence of times $\{t_k\}$, $t_k \to \infty$, such that there exists $\mu \in \mathscr{P}(E \times U)$ with $\mu_{t_k} \Rightarrow \mu$ and

$$\int c\, d\mu_{t_k} \to \limsup_{t\to\infty} t^{-1}E\left[\int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds\right].$$

Since $(X(\cdot), \Lambda_\cdot)$ is a solution of the controlled martingale problem for A, we have for each $f \in \mathscr{D}(A)$,

$$t_k^{-1}E[f(X(0))] = t_k^{-1}E[f(X(t_k))] - t_k^{-1}E\left[\int_0^{t_k} \int_U Af(X(s), u)\Lambda_s(du)\, ds\right]$$

$$= t_k^{-1}E[f(X(t_k))] - \int Af(x, u)\mu_{t_k}(dx \times du)$$

and so letting $k \to \infty$, we see $\mu \in \Gamma$. We use here the facts that Af is bounded and continuous and f is bounded in passing to the limit.

By Theorem 3.1, there exists a stationary solution $(\tilde{X}(\cdot), \tilde{\Lambda}_\cdot)$ corresponding to μ. The cost associated with $(\tilde{X}(\cdot), \tilde{\Lambda}_\cdot)$ is

$$\lim_{t\to\infty} t^{-1}E\left[\int_0^t \int_U c(\tilde{X}(s), u)\tilde{\Lambda}_s(du)\, ds\right] = \int c(x, u)\mu(dx \times du).$$

We now compare the cost of the two solutions. Since $\mu_{t_k} \Rightarrow \mu$, the lower semicontinuity of $c(x, u)$ implies

$$C(X, \Lambda) \ge \int c(x, u)\mu(dx \times du) = C(\tilde{X}, \tilde{\Lambda}). \qquad \square$$

Theorem 3.2 shows that for every solution to the controlled martingale problem, there is a stationary solution whose cost is not greater and Theorem 3.1 shows that for every stationary distribution μ, there is a stationary solution to the controlled martingale problem. Therefore, it suffices to solve the following linear programming problem:

$$\text{minimize } \int c(x, u)\mu(dx \times du)$$

$$\text{subject to } \int Af(x, u)\mu(dx \times du) = 0 \qquad \forall f \in \mathscr{D}(A).$$

Now we consider the existence of an optimal solution to this problem.

3.3 THEOREM. *There exists an optimal solution to the controlled martingale problem for* A.

PROOF. If the cost is infinite for every solution, any solution will be optimal so we assume the optimal cost is finite and let M denote this infimal cost. Then there exists a sequence $\{\mu_k\} \subset \Gamma$ for which

$$\int c\, d\mu_k \leq M + 1 \quad \text{and} \quad \lim_{k \to \infty} \int c\, d\mu_k = M.$$

Such a sequence is relatively compact due to the compactness assumptions. Let μ be a weak limit of $\{\mu_k\}$ and note that $\mu \in \Gamma$. Again the lower semicontinuity of $c(x, u)$ implies

$$\inf_{\nu \in \Gamma} \int c\, d\nu = \lim_{k \to \infty} \int c\, d\mu_k \geq \int c\, d\mu$$

and thus μ is an optimal stationary distribution. Theorem 3.1 then establishes the existence of a stationary relaxed solution corresponding to μ which is therefore optimal. □

We wish to close this section by relaxing the assumption that the state space E is compact. When E is not compact, we can solve the control problem on E^Δ, the one-point compactification of E (refer to the compactness assumptions). In this case, the optimal μ is a distribution on $E^\Delta \times U$. However, the assumption that the optimal cost is finite together with $c(\Delta, u) = \infty$ implies that $\mu(\{\Delta\} \times U) = 0$ so the optimal solution does not spend any positive fraction of time in Δ. A rather restrictive condition implies that the solution, in fact, never hits Δ. Its proof uses the following two results [Ethier and Kurtz (1986), Chapter 4, Lemma 3.2 and Chapter 2, Proposition 2.15, respectively].

3.4 LEMMA. *Let* X *be a measurable process,* $f \in B(E)$ *and* $g \in B(E \times U)$. *Then for fixed* $\lambda \in \mathbb{R}$,

$$f(X(t)) - \int_0^t \int_U g(X(s), u) \Lambda_s(du)\, ds$$

is an $\{\mathscr{F}_t\}$*-martingale if and only if*

$$e^{-\lambda t} f(X(t)) + \int_0^t \int_U e^{-\lambda s} (\lambda f(X(s)) - g(X(s), u)) \Lambda_s(du)\, ds$$

is an $\{\mathscr{F}_t\}$*-martingale.*

3.5 PROPOSITION. *Let* X *be a right continuous nonnegative* $\{\mathscr{F}_t\}$*-supermartingale and let* $\tau_c(0)$ *be the first contact time with* 0. *Then* $X(t) = 0$ *for all* $t \geq \tau_c(0)$ *with probability* 1.

We now give a sufficient condition to assure that the optimal solution on $E^\Delta \times U$ is, in fact, an optimal solution on $E \times U$.

3.6 PROPOSITION. *Suppose there exist $f \in \hat{C}(E)$, $g \in C(E \times U)$ and $\lambda > 0$ such that f is strictly positive:*

(i) *$f(X(t)) - \int_0^t \int_U g(X(s), u)\Lambda_s(du)\,ds$ is a martingale.*
(ii) *$\lambda f(x) - g(x, u) \geq 0$ for all x and u.*

Then $P\{X(t) \in E$ for all $t \geq 0\} = 1$.

PROOF. Condition (i) implies

$$e^{-\lambda t} f(X(t)) + \int_0^t \int_U e^{-\lambda s}(\lambda f(X(s)) - g(X(s), u))\Lambda_s(du)\,ds$$

is a martingale by Lemma 3.4. Therefore by condition (ii), $e^{-\lambda t} f(X(t))$ is a nonnegative, right continuous, supermartingale. The result follows by Proposition 3.5. □

EXAMPLE. Let $E = \mathbb{R}$, $U = \mathbb{R}$ and consider the stochastic differential equation

$$dX(t) = -X(t)/\left(u^2(t) + 1\right) dt + \left[2/\left(u^2(t) + 1\right)\right]^{1/2} dW(t).$$

The corresponding generator of the martingale problem is

$$Af(x, u) = 1/(u^2 + 1)f''(x) - x/(u^2 + 1)f'(x)$$

for $f \in C_c^\infty(\mathbb{R})$. Note that the control only affects the system by determining the rate at which the diffusion runs. Taking $f(x) = 1/(x^2 + 1)$, $g(x, u) = 1/(u^2 + 1)[f''(x) - xf'(x)]$ and $\lambda > 4$, the conditions of the proposition are satisfied for any solution of the martingale problem for A.

4. Control adapted to the state of the process. The goal of this section is to show that the optimal control can depend only on the history of the state process. This is accomplished by showing the existence of a "predictable projection" $\eta_\cdot$ of $\Lambda_\cdot$ which inherits its stationarity from $\Lambda_\cdot$. It follows that $(X(\cdot), \eta_\cdot)$ is a relaxed solution of the controlled martingale problem having the same cost as $(X(\cdot), \Lambda_\cdot)$. We use the following theorem [Métivier (1982), Theorem 14.2] in the construction of $\eta_\cdot$.

4.1 THEOREM. *For every bounded, real measurable process $X(\cdot)$, there exists (up to indistinguishability) a unique predictable process $\tilde{X}(\cdot)$ such that, for every predictable stopping time τ,*

$$E\left[X(\tau)\chi_{\{\tau < \infty\}}\right] = E\left[\tilde{X}(\tau)\chi_{\{\tau < \infty\}}\right].$$

We now construct the predictable projection $\eta_\cdot$ of a relaxed control $\Lambda_\cdot$. For convenience, we denote the processes by $\eta(\cdot, \cdot, \cdot)$ and $\Lambda(\cdot, \cdot, \cdot)$, respectively, where the first argument is a set from $\mathscr{B}(U)$, the second argument is $s \in [0, \infty)$ and the third argument is $\omega \in \Omega$.

4.2 PROPOSITION. *Let* $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ *be given. Let* U *be a locally compact, separable, metric space. Let* $\Lambda(\cdot, \cdot, \cdot)$ *be a* $\mathscr{P}(U)$*-valued measurable process. Then there exists a predictable projection* $\eta(\cdot, \cdot, \cdot)$ *of* $\Lambda(\cdot, \cdot, \cdot)$ *in the sense that* η *is a* $\mathscr{P}(U)$*-valued measurable process and for each* $B \in \mathscr{B}(U)$, $\eta(B, \cdot, \cdot)$ *as a real-valued process is the predictable projection of* $\Lambda(B, \cdot, \cdot)$.

PROOF. Let $\{u_i\}$ be a countable dense subset of U and let $\{B_i\}$ be an enumeration of the balls $\{B(u_j, 1/k)\colon\ j, k = 1, 2, 3, \dots\}$. Define $G_n = \sigma(B_1, \dots, B_n\}$ and $\Gamma_n = \{C_1 \cap \cdots \cap C_n\colon C_i = B_i \text{ or } B_i^c\}$.

Define η^0 on $\bigcup_n G_n \times [0, \infty) \times \Omega$ recursively by:

(i) $\eta^0(B_1, s, \omega)$ is the predictable projection of $\Lambda(B_1, s, \omega)$ and $\eta^0(B_1^c, s, \omega) = 1 - \eta^0(B_1, s, \omega)$.

(ii) For $B \in G_n$, $\eta^0(B \cap B_{n+1}, s, \omega)$ is the predictable projection of $\Lambda(B \cap B_{n+1}, s, \omega)$ and $\eta^0(B \cap B_{n+1}^c, s, \omega) = \eta^0(B, s, \omega) - \eta^0(B \cap B_{n+1}, s, \omega)$.

For each $C \in \Gamma_n$, $C \neq \varnothing$, fix $z_C \in C$. Now define the $\mathscr{P}(U)$-valued process η^n by

$$\eta^n(B, s, \omega) = \sum_{C \in \Gamma_n} \delta_{z_C}(B) \eta^0(C, s, \omega)$$

for $B \in \mathscr{B}(U)$. Note that for $B \in \bigcup_n G_n$, $\eta^n(B, s, \omega) = \eta^0(B, s, \omega)$.

Define $\nu^n \in \mathscr{P}(U)$ by

$$\begin{aligned} \nu^n(\cdot) &= \int_0^\infty e^{-s} E[\eta^n(\cdot, s, \omega)]\, ds \\ &= \int_0^\infty e^{-s} \left\{ \sum_{C \in \Gamma_n} \delta_{z_C}(\cdot) E\left[\eta^0(C, s, \omega)\right] \right\} ds \\ &= \int_0^\infty e^{-s} E\left[\sum_{C \in \Gamma_n} \delta_{z_C}(\cdot) \Lambda(C, s, \omega) \right] ds. \end{aligned}$$

Then $\nu^n \Rightarrow \nu \in \mathscr{P}(U)$, where $\nu(\cdot) = \int_0^\infty e^{-s} E[\Lambda(\cdot, s, \omega)]\, ds$.

Since $\nu^n \Rightarrow \nu$, $\{\nu^n\}$ are tight and hence for each m, there exists a compact set $K_m \subset U$ such that $\inf_n \nu^n(K_m) \geq 1 - 2^{-m}$. Also for each m, there exist N_m and $B \in G_{N_m}$ (and thus $B \in G_n\ \forall\ n \geq N_m$) such that $K_m \subset B \subset K_m^{1/m}$, where $K_m^{1/m} = \{u\colon d(u, K_m) < 1/m\}$. Thus

$$\int_0^\infty e^{-s} E\left[\inf_{n \geq N_m} \eta^n\left(K_m^{1/m}, s, \omega\right) \right] ds \geq \int_0^\infty e^{-s} E\left[\eta^0(B, s, \omega)\right] ds \geq 1 - 2^{-m}$$

and as a result

$$2^{-m} \geq \int_0^\infty e^{-s} E\left[1 - \inf_{n \geq N_m} \eta^n\left(K_m^{1/m}, s, \omega\right) \right] ds.$$

Therefore, $m/2^{-m} \geq \lambda\{(s, \omega)\colon \inf_{n \geq N_m} \eta^n(K_m^{1/m}, s, \omega) \leq 1 - 1/m\}$, where λ denotes the measure on $[0, \infty) \times \Omega$ given by $d\lambda = e^{-s}\, ds \times dP$. Letting

$$G = \left\{(s, \omega) \colon \liminf_{n\to\infty} \eta^n\left(K_m^{1/m}, s, \omega\right) \geq 1 - 1/m \text{ for all but finitely many } m\right\},$$

Borel–Cantelli (Theorem A.1 of the Appendix) yields $\lambda(G) = 1$ and so for each $(s, \omega) \in G$, it follows that $\{\eta^n(\cdot, s, \omega)]$ is relatively compact.

Since $\lim_{n\to\infty} \eta^n(B, s, \omega) = \eta^0(B, s, \omega)$ for every $B \in \bigcup_n G_n$, it follows that for every $(s, \omega) \in G$ there exists $\eta(\cdot, s, \omega)$ such that $\eta^n(\cdot, s, \omega) \Rightarrow \eta(\cdot, s, \omega)$. Let $\eta^* = \mathscr{P}(U)$ be fixed and extend $\eta(\cdot, s, \omega)$ to all of $[0, \infty) \times \Omega$ by setting $\eta(\cdot, s, \omega) = \eta^*(\cdot)$ for $(s, \omega) \notin G$. Then for all ω, $\eta^n(\cdot, s, \omega) \Rightarrow \eta(\cdot, s, \omega)$.

In order to verify η is the predictable projection of Λ, we need to verify that for each $B \in \mathscr{B}(U)$, $\eta(B, s, \cdot\,)$ is the predictable projection of $\Lambda(B, s, \cdot\,)$. Note we already have this for $B \in \bigcup_n G_n$. By Theorem 4.1, it suffices to show that for every $B \in \mathscr{B}(U)$, $\eta(B, \tau, \cdot\,) = E[\Lambda(B, \tau, \cdot\,) \mid \mathscr{F}_{\tau-}]$ for all predictable stopping times τ. Note this is true for each $B \in \bigcup_n G_n$ and that $\bigcup_n G_n$ is a π-system. Let

$$G = \{B \colon \eta(B, \tau, \cdot) = E[\Lambda(B, \tau, \cdot) \mid \mathscr{F}_{\tau-}] \ \forall \text{ predictable stopping times } \tau\}.$$

Then $\mathscr{G}$ is a λ-system and so by Dynkin's π-λ theorem (Theorem A.3 of the Appendix) $\mathscr{G} \supset \sigma(\bigcup_n G_n) = \mathscr{B}(U)$. □

The following lemma establishes the stationarity of η since deterministic times are predictable stopping times.

4.3 LEMMA. *Let $(X(t), Y(t))_{t\in\mathbb{R}}$ be a jointly stationary $E \times U$-valued process and let $\mathscr{F}_t = \sigma(Y(s)\colon -\infty < s \leq t)$. Let $f\colon E \to \mathbb{R}$ be measurable. Then $(X(t), Y(t), E[f(X(t)) \mid \mathscr{F}_t])$ is jointly stationary.*

PROOF. Consider $E[f(X(0)) \mid \mathscr{F}_0]$. There exist a Borel measurable function $G\colon U^\infty \to \mathbb{R}$ and $\{-t_j \leq 0\colon j = 1,2,3,\dots\}$ such that $E[f(X(0)) \mid \mathscr{F}_0] = G(Y(-t_1), Y(-t_2), \dots)$.

Fix $h \in \mathbb{R}$ and let $s_1 \leq s_2 \leq \cdots \leq s_m \leq h$. Let $\Gamma \in \mathscr{B}(U^m)$. By stationarity, we have

$$\begin{aligned} E[\chi_\Gamma(Y(s_1), \dots, Y(s_m))G(Y(h - t_1), y(h - t_2), \dots)] \\ = E[\chi_\Gamma(Y(s_1 - h), \dots, Y(s_m - h))G(Y(-t_1), Y(-t_2), \dots)] \\ = E[\chi_\Gamma(Y(s_1 - h), \dots, Y(s_m - h)) f(X(0))] \\ = E[\chi_\Gamma(Y(s_1), \dots, Y(s_m)) f(X(h))]. \end{aligned}$$

Since this is true for any m and $\Gamma \in \mathscr{B}(U^m)$, $G(Y(h - t_1), Y(h - t_2), \dots)$ is a version of $E[f(X(h)) \mid \mathscr{F}_h]$ and the result follows. □

4.4 REMARK. A similar proof with $\mathscr{F}_t$ and $X(t)$ replaced by $\mathscr{F}_{t-}$ and $X(t-)$, respectively, establishes the stationarity of $(X(t-), Y(t), E[f(X(t-)) \mid \mathscr{F}_{t-}])$.

We have shown so far that (X, η) is stationary. We conclude this section by showing that it is, in fact, a relaxed solution of the controlled martingale problem for A with respect to the filtration $\mathscr{F}_t^X = \sigma(X(s): 0 \le s \le t)$ and that the average cost does not change.

For $f \in \mathscr{D}(A)$, $h_1, \ldots, h_m \in \overline{C}(E)$ and $0 \le t_1 < \cdots < t_m < t_{m+1}$ we have by conditioning with respect to $\mathscr{F}_{s-}^X$ inside the expectation

$$E\left[\left\{f(X(t_{m+1})) - f(X(t_m)) - \int_{t_m}^{t_{m+1}} \int_U Af(X(s), u)\eta_s(du)\, ds\right\} \prod_{i=1}^m h_i(X(t_i))\right]$$

$$= E\left[\{f(X(t_{m+1})) - f(X(t_m))\} \prod_{i=1}^m h_i(X(t_i))\right]$$

$$- \int_{t_m}^{t_{m+1}} E\left[E\left[\int_U Af(X(s-), u)\eta_s(du)\, |\mathscr{F}_{s-}^X\right] ds \prod_{i=1}^m h_i(X(t_i))\right]$$

$$= E\left[\{f(X(t_{m+1})) - f(X(t_m))\} \prod_{i=1}^m h_i(X(t_i))\right]$$

$$- \int_{t_m}^{t_{m+1}} E\left[E\left[\int_U Af(X(s-), u)\Lambda_s(du)\, |\mathscr{F}_{s-}^X\right] ds \prod_{i=1}^m h_i(X(t_i))\right]$$

$$= E\left[\left\{f(X(t_{m+1})) - f(X(t_m)) - \int_{t_m}^{t_{m+1}} \int_U Af(X(s), u)\Lambda_s(du)\, ds\right\} \times \prod_{i=1}^m h_i(X(t_i))\right]$$

$$= 0.$$

The last equality follows from $(X(\cdot), \Lambda_.)$ being a solution of the controlled martingale problem for A. Similarly,

$$E\left[\int_0^t \int_U c(X(s), u)\eta_s(du)\, ds\right] = \int_0^t E\left[\int_U c(X(s-), u)\eta_s(du)\right] ds$$

$$= \int_0^t E\left[E\left[\int_U c(X(s-), u)\Lambda_s(du) | \mathscr{F}_{s-}\right]\right] ds$$

$$= E\left[\int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds\right].$$

5. Initial conditions. The initial distribution of the optimal solution $(X(\cdot), \Lambda_.)$ given in Section 3 must satisfy $P\{X(0) \in \Gamma\} = \mu\{\Gamma \times U\}\ \forall\, \Gamma \in \mathscr{B}(E)$, where μ is the optimal stationary distribution obtained via linear programming. We wish to address the situation in which the initial distribution is not the one given by the marginal of μ.

First observe that the optimal cost obtained by the linear programming approach of Section 3 provides a lower bound on the optimal cost for processes having a prescribed initial distribution. This follows directly from Theorem 3.2. Thus what we seek to obtain in this section are conditions on the initial distribution which will ensure that the optimal cost remains the same as that given by the stationary solution.

In this section, we set $\Omega = D_E[0,\infty) \times \mathscr{P}(U)^{[0,\infty)}$, $(X(\cdot), \Lambda_\cdot)$ to be the coordinate random variable, $\mathscr{F}_t = \mathscr{F}_t^X = \sigma(X(s)\colon 0 \le s \le t)$, P to be the optimal stationary solution of the controlled martingale problem, μ to be the optimal stationary distribution, $\mu_x(\cdot) = \mu(\cdot \times U)$ and α to be the optimal long-run average cost.

5.1 PROPOSITION.

$$\lim_{t\to\infty} t^{-1} \int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds = \alpha \quad a.s.\,(P).$$

PROOF. By the ergodic theorem (Theorem A.4 of the Appendix), $\lim_{t\to\infty} t^{-1}\int_0^t\int_U c(X(s), u)\Lambda_s(du)\, ds$ exists a.s. Call the limit Z and observe that $Z \ge 0$ a.s. and $E[Z] = \alpha$. We wish to show that $Z = \alpha$ a.s.

Assume $P\{Z < \alpha\} > 0$. Then there exist $\delta > 0$ and $\varepsilon > 0$ such that $P\{Z < \alpha - 4\varepsilon\} \ge \delta$. Choose δ_0 such that $P(F) < \delta_0$ implies $E[Z\chi_F] < \delta\varepsilon/2$. Let

$$\Omega_1(T) = \left\{ \left| t^{-1} \int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds - Z \right| < \varepsilon \text{ for all } t \ge T \right\}$$

and choose T large enough such that $P\{\Omega_1(T)\} \ge 1 - (\delta_0 \wedge \delta/2)$. Observe that

$$G = \left\{ T^{-1} \int_0^T \int_U c(X(s), u)\Lambda_s(du)\, ds < \alpha - 3\varepsilon \right\} \supset \Omega_1(T) \cap \{Z < \alpha - 4\varepsilon\}$$

and so $P(G) \ge P(\Omega_1(T) \cap \{Z < \alpha - 4\varepsilon\}) \ge \delta/2$. Define $Q \ll P$ by

$$dQ/dP = \chi_G/P(G)$$

and $(\tilde{X}(\cdot), \tilde{\Lambda}_\cdot) = (X(T+\cdot), \Lambda_{T+\cdot})$. Then $(\tilde{X}(\cdot), \tilde{\Lambda}_\cdot)$ is a solution of the controlled martingale problem for A under Q. We now determine the cost associated with this solution:

$$\begin{aligned} &E^Q\left[t^{-1} \int_0^t \int_U c(\tilde{X}(s), u)\tilde{\Lambda}_s(du)\, ds \right] \\ &\quad = (T+t)/tE^Q\left[1/(T+t) \int_0^{T+t} \int_U c(X(s), u)\Lambda_s(du)\, ds \right] \\ &\qquad - t^{-1}E^Q\left[\int_0^T \int_U c(X(s), u)\Lambda_s(du)\, ds \right]. \end{aligned}$$

As $t \to \infty$, the last term converges to 0 and $(T+t)/t \to 1$ and so we consider

$$\begin{aligned} &E^Q\left[1/(T+t) \int_0^{T+t} \int_U c(X(s), u)\Lambda_s(du)\, ds \right] \\ &\quad = E^P\left[1/(T+t) \int_0^{T+t} \int_U c(X(s), u)\Lambda_s(du)\, ds\, \chi_G/P(G) \right]. \end{aligned}$$

Since $1/(T+t)\int_0^{T+t}\int_U c(X(s),u)\lambda_s(du)\,ds \to Z$ a.s. and in L^1,

$$E^P\left[1/(T+t)\int_0^{T+t}\int_U c(X(s),u)\Lambda_s(du)\,ds\,\chi_G/P(G)\right] \to E^P[Z\chi_G/P(G)]$$

and

$$\begin{aligned} E^P[Z\chi_G/P(G)] &= E^P\left[Z\chi_G/P(G)\chi_{\Omega_1}\right] + E^P\left[Z\chi_G/P(G)\chi_{\Omega_1^c}\right] \\ &\le E^P\left[(\alpha-2\varepsilon)\chi_G/P(G)\chi_{\Omega_1}\right] + 1/P(G)E^P\left[Z\chi_{\Omega_1^c}\right] \\ &\le \alpha - 2\varepsilon + 2/\delta.\delta\varepsilon/2 = \alpha - \varepsilon. \end{aligned}$$

Thus

$$\lim_{t\to\infty} E^Q\left[t^{-1}\int_0^t\int_U c(\tilde{X}(s),u)\tilde{\Lambda}_s(du)\,ds\right] = \alpha - \varepsilon,$$

which contradicts the optimality of α. Therefore $P\{Z<\alpha\}=0$ and since $E[Z]=\alpha$, it follows that

$$\lim_{t\to\infty} t^{-1}\int_0^t\int_U c(X(s),u)\Lambda_s(du)\,ds = \alpha \quad \text{a.s. } (dP). \qquad \square$$

5.2 Corollary. *Suppose $\nu_x \in \mathscr{P}(E)$ with $\nu_x \ll \mu_x$ and $d\nu_x/d\mu_x$ bounded. Define the new probability $Q \ll P$ to have*

$$dQ/dP = d\nu_x/d\mu_x(X(0)).$$

Then

$$\lim_{t\to\infty} t^{-1}E^Q\left[\int_0^t\int_U c(X(s),u)\Lambda_s(du)\,ds\right] = \alpha.$$

Proof. This follows immediately from Proposition 5.1 since

$$E^Q\left[t^{-1}\int_0^t\int_U c(X(s),u)\Lambda_s(du)\,ds\right] \to E^P[\alpha\, d\nu_x/d\mu_x(X(0))] = \alpha$$

as $t\to\infty$. $\square$

APPENDIX

For completeness, we give the statements of well-known theorems in this section to which we refer in the paper.

A.1 Theorem (Borel–Cantelli). *Let A_n be a sequence of events. If $\sum_{n=1}^{\infty} P(A_n)$ converges, then $P(\limsup_n A_n)=0$.*

A.2 Definition. (i) A class $\mathbb{P}$ of subsets of a space Ω is a π-system if $A, B \in \mathbb{P}$ implies $A \cap B \in \mathbb{P}$.

(ii) A class $\mathscr{L}$ is a λ-system if:

(a) $\Omega \in \mathscr{L}$.

(b) $A, B \in \mathscr{L}$ and $A \subset B$ imply $A - B \in \mathscr{L}$.

(c) $A_1, A_2, \ldots \in \mathscr{L}$ and $A_n \nearrow A$ imply $A \in \mathscr{L}$.

A.3 THEOREM (Dynkin's π-λ theorem). *If* $\mathbb{P}$ *is a* π-*system and* $\mathscr{L}$ *is a* λ-*system, then* $\mathbb{P} \subset \mathscr{L}$ *implies that* $\sigma(\mathbb{P}) \subset \mathscr{L}$.

A.4 THEOREM (ergodic theorem). *Let* $X(\cdot)$ *be stationary with* $E[|X(0)|] < \infty$. *Then there exists a random variable* Z *such that*

$$\lim_{t\to\infty} t^{-1} \int_0^t X(s)\, ds = Z \quad (w.p.1).$$

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