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CHARACTERIZATION OF ALMOST SURELY CONTINUOUS 1-STABLE RANDOM FOURIER SERIES AND STRONGLY STATIONARY PROCESSES[1]

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We complete the results of M. Marcus and G. Pisier by showing that a strongly stationary 1-stable process $(X_t)_{t \in G}$ defined on a locally compact group has a version with sample continuous paths if (and only if) the entropy integral $\int_0^\infty \log^+ \log N(K, d_X, \varepsilon)\, d\varepsilon$ is finite, where K is a given neighborhood of the unit and d_X is the distance induced by the process.

1. Introduction. Let G be a locally compact Abelian group with dual group Γ. Following Marcus and Pisier [5], we say that a (complex) random process $(X_t)_{t \in G}$ is a strongly stationary p-stable process $(0 < p \leq 2)$ if there exists a finite positive Radon measure m on Γ such that for all $t_1, \dots, t_n \in G$ and complex numbers $\alpha_1, \dots, \alpha_n$ we have

$$E \exp i \operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j X_{t_j} \right) = \exp - \int_\Gamma \left| \sum_{j=1}^{n} \alpha_j \gamma(t_j) \right|^p dm(\gamma).$$

We associate with $(X_t)_{t \in G}$ a pseudometric d_X on G defined by

$$\forall\, s, t \in G, \qquad d_X(s, t) = \left(\int_\Gamma |\gamma(s) - \gamma(t)|^p \, dm(\gamma) \right)^{1/p}.$$

We fix a compact neighborhood K of the unit element of G. Let $N(K, d_X, \varepsilon)$ denote the smallest number of open balls of radius ε, in the pseudometric d_X, which cover K. For $p > 1$, Marcus and Pisier have shown that $(X_t)_{t \in K}$ has a version with a.s. continuous sample paths if and only if $J_q(d_X) = \int_0^\infty (\log N(K, d_X, \varepsilon))^{1/q} \, d\varepsilon < \infty$, where $1/p + 1/q = 1$. They have also shown that, when $p = 1$, a necessary condition for $(X_t)_{t \in K}$ to have a version with a.s. continuous sample paths is that

$$J(d_X) = \int_0^\infty \log^+ \log N(K, d_X, \varepsilon)\, d\varepsilon < \infty.$$

The contribution of the present paper is to show that the condition $J(d_X) < \infty$ is also sufficient for $(X_t)_{t \in K}$ to have a version with a.s. continuous sample paths. The new method we introduce dispenses with the use of nonincreasing rearrangement of the metric. It actually can be used to provide an alternative approach to sufficient conditions in the case $p > 1$ (see [1] for the details).

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THEOREM. *Let $(X_t)_{t\in G}$ be a strongly stationary 1-stable process. Then $(X_t)_{t\in K}$ has a version with a.s. continuous sample paths if and only if $J(d_X) < \infty$. Moreover, there exists a constant $L(K)$ depending only on K such that*

$$(1.1)\qquad \begin{aligned} &L(K)^{-1}(J(d_X) + m(\Gamma)) \\ &\le \sup_{c>0} cP\Big(\sup_{t\in K}|X(t)| > c\Big) \le L(K)(J(d_X) + m(\Gamma)). \end{aligned}$$

When G is compact and $K = G$, the constant $L(G)$ can be taken independent of G.

In particular (as explained in Marcus and Pisier [5]) this theorem can be applied to random Fourier series $X_t(\omega) = \sum_{\gamma\in\Gamma} a_\gamma\theta_\gamma\gamma(t)$, where $(\theta_\gamma)_{\gamma\in\Gamma}$ are i.i.d. 1-stable r.v.'s.

2. Tools. The representation of p-stable processes as a suitable mixture that was essential in the work of Marcus and Pisier will also be essential here. We describe this representation in the case of strongly stationary processes. By homogeneity, we can assume that m is a probability. Let W be a real positive random variable satisfying $P(W > \lambda) = e^{-\lambda}$. Let (W_i) be i.i.d. copies of W and define $\Gamma_j = \sum_{i\le j} W_i$. Let $\{Y_j\}$ be an i.i.d. sequence of Γ-valued r.v.'s distributed according to m. Let (ε_j) be a Bernoulli sequence. Let (ω_j) be a Steinhauss sequence (i.e., i.i.d. uniformly distributed on $\{z\in\mathbb{C};\ |z| = 1\}$). We assume that the sequences $\{Y_j\}, \{\Gamma_j\}, \{\varepsilon_j\}, \{\omega_j\}$ are independent of the others. Then for some constant $a(p)$, the process $(V_t)_{t\in G}$ given by

$$(2.1)\qquad \forall\, t\in G,\qquad V_t = a(p)\sum_{j\ge 1}(\Gamma_j)^{-1/p}\varepsilon_j\omega_j Y_j(t)$$

is equal in distribution to $(X_t)_{t\in G}$. It will be convenient to assume that the basic probability space is a product $\Omega\times\Omega'\times\Omega''$ and that

$$V_t(\omega,\omega',\omega'') = a(p)\sum_{j\ge 1}\big(\Gamma_j(\omega'')\big)^{-1/p}\varepsilon_j(\omega')\omega_j(\omega)Y_j(\omega)(t).$$

For a nonnegative sequence $(x_i)_{i\ge 1}$, we denote by x_i^* its nonincreasing rearrangement, i.e., $x_i^* = \sup\{u\ge 0;\ \operatorname{card}\{j:\ x_j\ge u\}\ge i\}$. Marcus and Pisier make essential use of the random distance

$$d_\omega(s,t) = \Big\|\big(j^{-1/p}\big(Y_j(\omega)(s) - Y_j(\omega)(t)\big)\big)_{j\ge 1}\Big\|_{p,\infty},$$

where

$$\|(a_j)_{j\ge 1}\|_{p,\infty} = \Big(\sup_{t>0} t^p\operatorname{card}\{j;\ |a_j| > t\}\Big)^{1/p} = \sup_{j\ge 1} j^{1/p}|a_j|^*.$$

They show in particular that for $p > 1$, we have, for some constant $b(p)$,

$$(2.2)\qquad Ed_\omega(s,t)\le b(p)d_X(s,t).$$

This unfortunately fails for $p = 1$. However, J. Zinn has given a transparent proof of (2.2) and his argument (to be given shortly) shows that in the case

$p = 1$, the failure of (2.2) is due only to the largest term of the sequence $(j^{-1/p}|Y_j(\omega)(s) - Y_j(\omega)(t)|)_{j \geq 1}$. So, the obvious thing to do will be to control a few of the largest terms of this sequence.

LEMMA 1 (J. Zinn; see Pisier [6], page 37). *Let $(X_i)_{i \geq 1}$ be independent nonnegative r.v.'s. Then for $u \geq 0$, $P(X_k^* \geq u) \leq (ea/k)^k$, where*

$$a = \sum_{i \geq 1} P(X_i \geq u).$$

PROOF. For $\lambda > 0$, we have

$$\begin{aligned} E \exp \lambda \sum_{i \geq 1} 1_{\{X_i \geq u\}} &= \prod_{i \geq 1} E \exp \lambda 1_{\{X_i \geq u\}} \\ &= \prod_{i \geq 1} \big(1 + (e^{\lambda} - 1) P(X_i \geq u)\big) \\ &\leq \exp a(e^{\lambda} - 1). \end{aligned}$$

We note that $X_k^* \geq u$ is equivalent to $\Sigma 1_{\{X_i \geq u\}} \geq k$, so that

$$P(X_k^* \geq u) \leq \exp\big(-\lambda k + a(e^{\lambda} - 1)\big).$$

The result follows with $\lambda = \log(k/a)$ if $k > a$, and is obvious otherwise. □

COROLLARY 2. *Let Z_i be an i.i.d. sequence of nonnegative r.v.'s. Let $X_i = Z_i/i$. Then $P(kX_k^* \geq euEZ) \leq u^{-k}$.*

PROOF. From Lemma 1 and since $\Sigma_{i \geq 1} P(X_i \geq u) = \Sigma_{i \geq 1} P(Z_i > iu) \leq EZ/u$. □

In particular, $P(\|(X_k)\|_{1,\infty} \geq 2euEZ) \leq u^{-1}$ for $u \geq 1$. If $EZ^p < \infty$, we have $P(\|(X_i)\|_{p,\infty} \geq (2e)^{1/p} u (EZ^p)^{1/p}) \leq u^{-p}$ by applying the preceding inequality to Z^p, which proves (2.2).

We will use the following immediate consequence of Corollary 2.

COROLLARY 3. *With the notation of Corollary 2, we have*

$$P\big(X_1^* \leq 2^n eEZ;\ \forall\, i > 2^n,\ iX_i^* \leq 2eEZ\big) \geq 1 - 2^{-n+1}.$$

3. Proof of the theorem. We denote by 0 the unit of G. For $s \in G$, $\omega \in \Omega$, we set

$$X_i(\omega, s) = i^{-1}|Y_i(\omega)(s) - 1| = i^{-1}|Y_i(\omega)(s) - Y_i(\omega)(0)|.$$

Since $d_X(s, 0) = E|Y_1(s) - 1|$, it follows from Corollary 3 that for any s we have

$$\begin{aligned} P\big(X_1^*(\omega, s) \leq 2^{n+2} d_X(s, 0);\ \forall\, i > 2^n,\ iX_i^*(\omega, s) \leq 2^3 d_X(s, 0)\big) \\ \geq 1 - 2^{-n+1}. \end{aligned} \tag{3.1}$$

We set $K' = K - K = \{s - t;\ s, t \in K\}$. We fix a Haar measure of G and we denote by $|A|$ the Haar measure of a set A. For $k \geq 0$, we set $B_k = \{s \in K';\ d_X(s,0) \leq 2^{-3k}\}$. For $k, n \geq 0$, we set

$$B_{k,n,\omega} = \left\{s \in B_k;\ X_1^*(\omega, s) \leq 2^{n+2-3k};\ \forall\, i > 2^n,\ iX_i^*(\omega, s) \leq 2^{3-3k}\right\}.$$

It follows from (3.1) and Fubini's theorem that if we set

$$\Omega_{k,n} = \{\omega;\ |B_{k,n,\omega}| \geq |B_k|/2\},$$

we have $P(\Omega_{k,n}) \geq 1 - 2^{-n+2}$. We set

$$\Omega_n = \bigcap_{k<n} \Omega_{k,n} \cap \bigcap_{k \geq n} \Omega_{k,k}.$$

We have $P(\Omega_n) \geq 1 - (n+1)2^{-n+2}$. The main part of the proof is to show that a.s. conditionally on $\omega \in \Omega_n$ and on ω'' the process V_t has a.s. continuous sample paths, and that

$$E_{\omega'}\Big(\sup_{t \in K} V_t(\omega, \omega', \omega'')\Big) \leq L(K)U(\omega'')\big(n^2 + J(d_X)\big), \tag{3.2}$$

where $L(K)$ is a constant depending on K only and where $U(\omega'') = \sup_{j \geq 1} j/\Gamma_j(\omega'')$. It then follows from Fubini's theorem that $(X_t)_{t \in K}$ has a version with a.s. continuous sample paths. Also

$$E_{\omega'}\Big(\sup_{t \in K} V_t(\omega, \omega', \omega'')\Big) \leq U(\omega'')W(\omega), \tag{3.3}$$

where $(EW^2)^{1/2} < L(K)(1 + J(d_X))$ [for a new constant $L(K)$]. As pointed out by Marcus and Pisier, it is easy to see that $\sup_{u>0} uP(U > u) < \infty$, so that (3.3) easily implies the right-hand side inequality of (1.1). (The left-hand side is due to Marcus and Pisier.)

Given two subsets A, B of K, we denote by $N(A, B)$ the smallest number of translates of B by elements of A that can cover A. We set $K'' = K + K'$. The following is classical (see Marcus and Pisier [4], page 16).

Lemma 4.

(i) $$N(K, A) \geq \frac{|K|}{|A \cap K'|}.$$

(ii) $$N(K, A - A) \leq \frac{|K''|}{|A \cap K'|}.$$

We now fix n and $\omega \in \Omega_n$. For convenience, we set $C_k = B_{k,n,\omega}$ for $k < n$ and $C_k = B_{k,k,\omega}$ for $k > n$, so that we have $|C_k| \geq |B_k|/2$ for $k \geq 0$. It follows from Lemma 4 that we have

$$N(K, C_k - C_k) \leq \frac{2|K''|}{|K|} N\big(K, d_X, 2^{-3k}\big),$$

so that

$$\sum_{k\geq 0} 2^{-3k}\log^{+}\log N(K, C_k - C_k) \leq L(K)(1 + J(d_X)), \tag{3.4}$$

where $L(K)$ depends only on K. Actually, if $K = G$, we have $|K''|/|K| = 1$, so that in this case $L(K)$ is a universal constant (this is the only place where the constants involved in the proof depend on K).

The relation (3.2) will follow from (3.4) by the standard chaining argument and a proper deviation inequality. The following, again, is standard.

LEMMA 5. *Consider a sequence (a_i) such that $|a_i|^* \leq a/i$ for $i > l$. Let (ε_i) be a Bernoulli sequence. Then for $u > 4a$,*

$$P\left(\left|\sum_{i\geq 1}\varepsilon_i a_i\right| \geq b + u\right) \leq \exp\left(-\exp\frac{u}{4a}\right),$$

where $b = \sum_{i\leq l}|a_i|^*$.

PROOF. There is no loss of generality to assume that the sequence $(|a_i|)_{i\geq 1}$ is nonincreasing. Let $\tau = \exp(u/2a)$. Let $\rho = \max(l, \tau)$. We have $\sum_{l<i\leq\rho}|a_i| \leq a\log\tau \leq u/2$. Thus

$$P\left(\left|\sum_{i\geq 1}\varepsilon_i a_i\right| \geq b + u\right) \leq P\left(\left|\sum_{i>\rho}\varepsilon_i a_i\right| \geq u/2\right) \leq \exp\left(-u^2/8\sum_{i>\rho}|a_i|^2\right)$$

by the standard sub-Gaussian inequality [let us recall that this inequality is a consequence of Chebyshev's inequality and the elementary fact that $E\exp\varepsilon_i\lambda \leq \exp(\lambda^2/2)$]. Now, $\sum_{i>\rho}|a_i|^2 \leq a^2\sum_{i>\rho}1/i^2 \leq a^2/(\tau - 1)$. Since $u/2a \geq 2$, we have $u^2/ia^2(\tau - 1) \geq \exp(u/4a)$. This completes the proof. □

COROLLARY 6. *If $s \in t + C_k - C_k$, then for all $u > 0$,*

$$\begin{aligned} &P_{\omega'}\left(\left|\sum_{j\geq 1}\Gamma_j^{-1}(\omega'')\varepsilon_j(\omega')\omega_j(\omega)\big(Y_j(\omega)(s) - Y_j(\omega)(t)\big)\right| \geq U(\omega'')(b_k + u)\right) \\ &\quad\leq 2\exp\left(-\exp 2^{3k-6}u\right), \end{aligned} \tag{3.5}$$

where $b_k = \min(4(1 + n), 2^{2n+3-3k})$.

PROOF. We observe that, since $Y_i(\omega')$ is a character, the sets C_k are symmetric, i.e., $s \in C_k$ if and only if $-s \in C_k$. It follows that $C_k - C_k = C_k + C_k$. Hence, it suffices to prove that if $s \in t + C_k$, we have

$$\begin{aligned} &P_{\omega'}\left(\left|\sum_{j\geq 1}\Gamma_j^{-1}(\omega'')\varepsilon_j(\omega')\omega_j(\omega)\big(Y_j(\omega)(s) - Y_j(\omega)(t)\big)\right| \geq \tfrac{1}{2}U(\omega'')(b_k + u)\right) \\ &\quad\leq \exp\left(-\exp 2^{3k-6}u\right). \end{aligned}$$

We observe that the left-hand side is bounded by

$$P_{\omega'}\left(\left|\sum_{j\geq 1} j^{-1}\varepsilon_j(\omega')\omega_j(\omega)\big(Y_j(\omega)(s) - Y_j(\omega)(t)\big)\right| \geq \tfrac{1}{2}(b_k + u)\right)$$

and that $|\omega_j(\omega)| = 1$. We observe that $|Y_j(\omega)(s) - Y_j(\omega)(t)| = |Y_j(\omega)(s - t) - 1|$ and that, since $s - t \in C_k$, the sum of the first $2^{\min(n,k)}$ largest terms of the sequence $|Y_j(\omega)(s - t) - 1|/j$ is less than or equal to $b_k/2$ [there we use the fact that $|Y_j(\omega)(s - t) - 1|/j \leq 2/j$]. The result then follows from Lemma 5, used with $l = 2^{\min(n,k)}$, and $u/2$ instead of u. □

We note that the series Σb_k has a sum less than or equal to Ln^2, for some number L independent of n. Then (3.2) follows from (3.4) and (3.5) by the standard chaining argument, that we sketch now.

We fix ω'' and we set, for $t \in T$,

$$Z_t = U(\omega'')^{-1} \sum_{j\geq 1} \Gamma_j^{-1}(\omega'')\varepsilon_j(\omega')\omega_j(\omega)Y_j(\omega)(t),$$

so that by (3.5) we have, if $s \in t + C_k - C_k$,

$$P_{\omega'}(|Z_s - Z_t| \geq b_k + u) \leq 2\exp(-\exp 2^{3k-6}u). \tag{3.6}$$

For $k \geq 0$, consider a finite set $N_k \subset K$ such that card $N_k = N(K, C_k - C_k)$ and $K \subset N_k + C_k - C_k$. For each $t \in K$, we pick $\phi_k(t) \in N_k$ such that $t - \phi_k(t) \in C_k - C_k$. Fix $m \geq 0$. For $t \in N_m$, we set $t_m = t$ and by decreasing induction we define $t_k = \phi_k(t_{k+1})$ for $0 \leq k < m$. Since

$$t - t_0 = \sum_{0<k\leq m} t_k - t_{k-1} = \sum_{0<k\leq m} t_k - \phi_{k-1}(t_k)$$

we have, letting $z_k = \log^+ \log \operatorname{card} N_k$,

$$P\left(\exists\, t \in N_m;\ |Z_t - Z_{t_0}| \geq \sum_{k\geq 1} b_{k-1} + u\left(\sum_{k\geq 1} 2^{-3k+9}z_k\right)\right)$$

$$\leq \sum_{k\geq 1} P\big(\exists\, t \in N_k;\ |Z_t - Z_{\phi_{k-1}(t)}| \geq b_{k-1} + u2^{-3k+9}z_k\big)$$

$$\leq \sum_{k\geq 1} 2 \operatorname{card} N_k \exp(-\exp uz_k)$$

by (3.6). Using (3.4) this implies (3.2) by a simple computation, letting $m \to \infty$. □

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