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CONTINUITY OF l^2-VALUED ORNSTEIN–UHLENBECK PROCESSES

BY I. ISCOE,[1] M. B. MARCUS,[2] D. MCDONALD,[3]
M. TALAGRAND[2] AND J. ZINN[2]

Case Western Reserve University, The City College of CUNY, University of Ottawa, University of Paris VI and The Ohio State University, and Texas A & M University

A stationary l^2-valued Ornstein–Uhlenbeck process is considered which is given formally by $dX_t = -AX_t\,dt + \sqrt{2a}\,dB_t$, where A is a positive self-adjoint operator on l^2, B_t is a cylindrical Brownian motion on l^2 and a is a positive diagonal operator on l^2. A simple criterion is given for the almost-sure continuity of X_t in l^2 which is shown to be quite sharp. Furthermore, in certain special cases, we obtain simple necessary and sufficient conditions for the almost-sure continuity of X_t in l^2.

In Dawson (1972), the following semilinear stochastic differential equation on a real, separable Hilbert space H was considered

$$dX_t = -AX_t\,dt + \beta(X_t)\,dt + \alpha(X_t)\,dB_t. \tag{0}$$

Here α: $H \to L(H) = \{$the set of all bounded linear operators on $H\}$, β: $H \to H$, B_t is a cylindrical Brownian motion on H and A is a constant self-adjoint, positive definite operator on H. A is assumed to have a complete orthonormal family of eigenvectors ϕ_k corresponding to its set of positive eigenvalues λ_k, $k = 1, 2, \dots$.

Dawson (1972) gives sufficient conditions for the existence of a mild solution of (0); that is, an H-valued process X_t which is a.s. continuous in H and adapted to $\sigma(X_0, B_s; s \le t)$, and which satisfies

$$X_t = U(t)X_0 + \int_0^t U(t-s)\alpha(X_s)\,dB_s + \int_0^t U(t-s)\beta(X_s)\,ds,$$

where $U(t)$ is the semigroup on H generated by $-A$. In particular, Dawson (1972) assumed that β is a globally Lipschitz mapping, and

(1)(i) $$\lambda_k \sim ck^{1+\delta}, \qquad c, \delta > 0, \quad \text{as } k \to \infty,$$

(1)(ii) $\alpha^*(\cdot)\phi_k$ is a continuous mapping from H into H for each k, and

(1)(iii) $$\|\alpha^*(X)\phi_k\| \le K(1 + \|X\|),$$
$$\|(\alpha^*(X_1) - \alpha^*(X_2))\phi_k\| \le K\|X_1 - X_2\|$$

for each $X, X_1, X_2 \in H$, $k = 1, 2, \dots,$ where α^* is the adjoint of α.

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For more recent results on the continuity of such processes, consult Kotelenez (1984a, b) and Antoniadis and Carmona (1987).

Without loss of generality we take $H = l^2$ and we consider a special case of (0),

$$dX_t = -AX_t\,dt + \sqrt{2a}\,dB_t, \tag{2}$$

where a is a constant, positive operator such that $(\phi_k, \sqrt{a}\,\phi_k) = \sqrt{a_k}$, $(\phi_i, \sqrt{a}\,\phi_j) = 0$, $i \neq j$. The diagonal system

$$dx_k(t) = -\lambda_k x_k(t)\,dt + \sqrt{2a_k}\,dB_k(t), \qquad k = 1, 2, \dots, \tag{3}$$

where $x_k(t) = (X_t, \phi_k)$, $B_k(t) = (B_t, \phi_k)$, $x_k(0) \sim N(0, a_k/\lambda_k)$, enables us to write $x_t = \{x_k(t)\}_{k=1}^{\infty}$ as a vector of independent Ornstein–Uhlenbeck processes, i.e., mean zero Gaussian processes defined by

$$Ex_k(t)x_k(s) = \frac{a_k}{\lambda_k}\exp(-\lambda_k|t-s|). \tag{4}$$

If $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$, then for each fixed t, $x_t \in l^2$ a.s. [i.e., $\sum_{k=1}^{\infty}|x_k(t)|^2 < \infty$ a.s.]. In this paper we are concerned with the continuity of x_t in l^2 and we obtain conditions that are weaker and more general than those in (1)(i), (1)(ii) and (1)(iii). Our main result is the following simple criterion for the continuity of x_t in l^2.

THEOREM 1. *Let $f(x)$ be a positive function on $[x_1, \infty)$ such that $f(x)/x$ is nondecreasing for $x \geq x_1 > 0$ and such that*

$$\int_{x_1}^{\infty} \frac{dx}{f(x)} < \infty. \tag{5}$$

Suppose also that

$$\sum_k \frac{a_k}{\lambda_k} < \infty \tag{6}$$

and

$$\sup_k \frac{f(a_k \vee x_1)}{\lambda_k \vee 1} < \infty. \tag{7}$$

Then x_t is continuous in l^2 a.s. Moreover, this result is best possible in the sense that it is false for any function $f(x)$, which satisfies all the above hypotheses, with the exception that the integral in (5) *is infinite.*

The following simple corollaries are immediate consequences of Theorem 1.

COROLLARY 2. *x_t is continuous in l^2 a.s. if* (6) *holds and*

$$\sup_k a_k \frac{(\log^+ a_k)^r}{\lambda_k \vee 1} < \infty \tag{8}$$

for $r > 1$, where $\log^+ x = (\log x) \vee 0$.

COROLLARY 3. *Suppose that a_k/λ_k is nonincreasing and that λ_k is nondecreasing with $\lim_{k\to\infty}\lambda_k = \infty$. Then if* (6) *holds and*

$$\sum_k \frac{a_{k+1}}{\lambda_{k+1}}\log^+\frac{a_{k+1}}{a_k} < \infty, \tag{9}$$

x_t is continuous in l^2 a.s.

We obtain Theorem 1 as a consequence of a recent corollary due to Fernique (1987), Theorem 3.3.3 of Talagrand's (1987) theorem on necessary and sufficient conditions for the continuity of Gaussian processes. Fernique's result gives simplified necessary and sufficient conditions for the continuity of Banach space valued Gaussian processes with stationary increments. We define

$$\sigma_k^2(u) = E\big(x_k(t+u) - x_k(t)\big)^2 = 2\frac{a_k}{\lambda_k}\big(1 - e^{-\lambda_k|u|}\big) \tag{10}$$

for the processes $\{x_k(t)\}_{k=1}^\infty$ defined in (4). For further use let us note that

$$\frac{2a_k}{e\lambda_k}(1\wedge\lambda_k|u|) \le \sigma_k^2(u) \le 2\frac{a_k}{\lambda_k}(1\wedge\lambda_k|u|). \tag{11}$$

THEOREM 4. *Let $\{x_k(t)\}_{k=1}^\infty$ be a sequence of independent Ornstein–Uhlenbeck processes as defined in* (4) *and $\sigma_k(u)$ be as defined in* (10). *There exist constants $0 < c_p \le C_p < \infty$ such that*

$$\begin{aligned} &c_p\left[E\|\{x_k(0)\}\|_p + \sup_{\{\alpha_k\}:\,\|\{\alpha_k\}\|_q\le 1}\int_0^{1/2}\frac{\left(\sum_{k=1}^\infty \alpha_k^2\sigma_k^2(u)\right)^{1/2}}{u(\log 1/u)^{1/2}}\,du\right]\\ &\quad\le E\sup_{t\in[0,1]}\|\{x_k(t)\}\|_p\\ &\quad\le C_p\left[E\|\{x_k(0)\}\|_p + \sup_{\{\alpha_k\}:\,\|\{\alpha_k\}\|_q\le 1}\int_0^{1/2}\frac{\left(\sum_{k=1}^\infty \alpha_k^2\sigma_k^2(u)\right)^{1/2}}{u(\log 1/u)^{1/2}}\,du\right], \end{aligned} \tag{12}$$

where $1/p + 1/q = 1$, $p \ge 1$, and $\|\{\alpha_k\}\|_q$ is the l^q norm of $\{\alpha_k\}_{k=1}^\infty$.

This theorem follows from Fernique (1987), Theorems 3.3.3 and 3.3.4, and well-known arguments [see, e.g., Marcus and Pisier (1981), Chapter 2, Theorem 3.4 and Lemma 3.6]. We present it for all l^p, not just l^2, since it requires no additional effort. Later in the paper we will make some remarks about the continuity of $\{x_k(t)\}_{k=1}^\infty$ in l^p for $p < 2$.

The following lemma, which generalizes some interesting inequalities of Boas (1960), enables us to use Theorem 4 to obtain Theorem 1. We will say more on the relationship of this lemma to the work of Boas at the end of this paper.

LEMMA 5. *Let $\{c_k\}_{k=1}^{\infty}$ be nonnegative real numbers such that c_k is nonincreasing in k and $c_1 \le 1$. Let $1 \le p \le 2$ and $1/p + 1/q = 1$. Then for all $\beta > 0$, there exists a constant C_β such that*

$$\sup_{\{b_k\}:\,\|\{b_k\}\|_q \le 1} \sum_{j=1}^{\infty} j^{\beta/2-1}\left(\sum_{k=j}^{\infty} b_k^2 c_k^2\right)^{1/2} \le C_\beta\left(\sum_{k=1}^{\infty} k^{\beta-1}c_k^p\right)^{1/2}\left(\sum_{k=1}^{\infty} c_k^p\right)^{(q-2)/(2q)}, \tag{13}$$

where we use the convention $(q-2)/(2q) = 1/2$ when $q = \infty$. In particular, there exist a universal constant $D > 0$ such that

$$\frac{1}{2}\left(\sum_{k=1}^{\infty} c_k^p\right)^{1/p} \le \sup_{\{b_k\}:\,\|\{b_k\}\|_q \le 1} \sum_{j=1}^{\infty} j^{-1/2}\left(\sum_{k=j}^{\infty} b_k^2 c_k^2\right)^{1/2} \le D\left(\sum_{k=1}^{\infty} c_k^p\right)^{1/p} \tag{14}$$

and constant $D_\beta > 0$ such that

$$\frac{1}{2^{\beta\vee 2}}\left(\sum_{k=1}^{\infty} k^{\beta-1}c_k^2\right)^{1/2} \le \sup_{\{b_k\}:\,\Sigma|b_k|^2 \le 1} \sum_{j=1}^{\infty} j^{\beta/2-1}\left(\sum_{k=j}^{\infty} b_k^2 c_k^2\right)^{1/2} \le D_\beta\left(\sum_{k=1}^{\infty} k^{\beta-1}c_k^2\right)^{1/2}. \tag{15}$$

PROOF. Assume $0 < \sum_{k=1}^{\infty} c_k^p < \infty$ and let $m(n) = \#\{k\colon c_k^p > 2^{-n}\}$. Observe that

$$\begin{aligned}\sum_{j=1}^{\infty} j^{\beta/2-1}\left(\sum_{k=j}^{\infty} b_k^2 c_k^2\right)^{1/2} &\le \sum_{n=1}^{\infty}\sum_{j=m(n-1)+1}^{m(n)} j^{\beta/2-1}\left(\sum_{k=m(n-1)+1}^{\infty} b_k^2 c_k^2\right)^{1/2}\\ &\le \sum_{n=1}^{\infty} a(n)\left(\sum_{h=n}^{\infty}\sum_{k=m(h-1)+1}^{m(h)} \frac{b_k^2 c_k^{2-p}}{2^{h-1}}\right)^{1/2}\\ &\le \sum_{n=1}^{\infty} \frac{a(n)}{2^{(n-1)/2}}\left(\sum_{h=n}^{\infty} \frac{B(h)}{2^{h-n}}\right)^{1/2},\end{aligned} \tag{16}$$

where

$$a(n) = \sum_{j=m(n-1)+1}^{m(n)} j^{\beta/2-1}$$

and

$$B(h) = \sum_{k=m(h-1)+1}^{m(h)} b_k^2 c_k^{2-p}.$$

By the Schwarz inequality, the last line in (16)

$$\leq \left(\sum_{n=1}^{\infty} \frac{a(n)^2}{2^{n-1}} \right)^{1/2} \left(\sum_{n=1}^{\infty} \sum_{h=n}^{\infty} \frac{B(h)}{2^{h-n}} \right)^{1/2} \equiv I_1^{1/2} I_2^{1/2}. \tag{17}$$

Since $a(n) \leq C'_\beta(m(n)^{\beta/2} - m(n-1)^{\beta/2})$ we have

$$I_1 \leq (C'_\beta)^2 \sum_{n=1}^{\infty} \frac{m(n)^\beta - (m(n-1))^\beta}{2^{n-1}} \leq C''_\beta \sum_{k=1}^{\infty} k^{\beta-1} c_k^p. \tag{18}$$

Also,

$$\begin{aligned} I_2^{1/2} &= \left(\sum_{h=1}^{\infty} \frac{B(h)}{2^h} \sum_{n=1}^{h} 2^n \right)^{1/2} \\ &\leq \left(2 \sum_{h=1}^{\infty} B(h) \right)^{1/2} \\ &= \sqrt{2} \left(\sum_{k=1}^{\infty} b_k^2 c_k^{2-p} \right)^{1/2} \\ &\leq \sqrt{2} \left(\sum_{k=1}^{\infty} b_k^q \right)^{1/q} \left(\sum_{k=1}^{\infty} c_k^p \right)^{(q-2)/(2q)} \end{aligned} \tag{19}$$

since $((2-p)q)/(q-2) = p$. Using (16), (17), (18) and (19), we see that the left-hand side of (13)

$$\leq C_\beta \left(\sum_{n=1}^{\infty} n^{\beta-1} c_n^p \right)^{1/2} \left(\sum_{n=1}^{\infty} c_n^p \right)^{(q-2)/(2q)}.$$

This gives us the right-hand side inequalities in (13), (14) and (15).

We now obtain the left-hand side inequality in (14). Set

$$b_k^q = c_k^p \Big/ \sum_{k=1}^{\infty} c_k^p, \qquad \forall\, k \geq 1.$$

For this choice of $\{b_k\}$ the term in the center of (14)

$$\begin{aligned} &= \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{k=j}^{\infty} c_k^{2p} \right)^{1/2} \right) \left(\sum_{k=1}^{\infty} c_k^p \right)^{-1/q} \\ &\geq \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{k=j}^{2j-1} c_k^{2p} \right)^{1/2} \right) \left(\sum_{k=1}^{\infty} c_k^p \right)^{-1/q} \\ &\geq \left(\sum_{j=1}^{\infty} c_{2j-1}^p \right) \left(\sum_{k=1}^{\infty} c_k^p \right)^{-1/q} \geq \frac{1}{2} \left(\sum_{k=1}^{\infty} c_k^p \right)^{1/p}. \end{aligned}$$

It is clear that the left-hand-side inequality in (14) also holds when $\sum_{k=1}^{\infty} c_k^p = \infty$.

We obtain the left-hand-side of (15) in a similar fashion. Assume that $0 < \sum_{k=1}^{\infty} k^{\beta-1}c_k^2 < \infty$ and set

$$b_k^2 = \frac{k^{\beta-1}c_k^2}{\sum_{k=1}^{\infty} k^{\beta-1}c_k^2}, \qquad \forall\, k \geq 1.$$

For this choice of $\{b_k\}_{k=1}^{\infty}$ the term in the center in (15)

$$= \left(\sum_{j=1}^{\infty} \left(j^{\beta-2} \sum_{k=j}^{\infty} k^{\beta-1}c_k^4 \right)^{1/2} \right) \left(\sum_{k=1}^{\infty} k^{\beta-1}c_k^2 \right)^{-1/2}$$

$$\geq \frac{1}{2^{(1-\beta)\vee 0}} \left(\sum_{j=1}^{\infty} \left(j^{2\beta-3} \sum_{k=j}^{2j-1} c_k^4 \right)^{1/2} \right) \left(\sum_{k=1}^{\infty} k^{\beta-1}c_k^2 \right)^{-1/2}$$

$$\geq \frac{1}{2^{(1-\beta)\vee 0}} \left(\sum_{j=1}^{\infty} j^{\beta-1}c_{2j-1}^2 \right) \left(\sum_{k=1}^{\infty} k^{\beta-1}c_k^2 \right)^{-1/2}$$

$$\geq \frac{1}{2^{\beta\vee 2}} \left(\sum_{j=1}^{\infty} j^{\beta-1}c_j^2 \right)^{1/2}.$$

The result follows. As above, the inequality under discussion also holds when $\sum_{k=1}^{\infty} k^{\beta-1}c_k^2 = \infty$. □

LEMMA 6. *Let $x_t = \{x_k(t)\}_{k=1}^{\infty}$ be a sequence of independent Ornstein–Uhlenbeck processes as defined in* (4). *Define the sets of integers*

(20) $\quad I_0 = \{k\colon \lambda_k \leq 1\}$ *and* $I_p = \{k\colon 2^{p-1} < \lambda_k \leq 2^p\}, \qquad \forall\, p \geq 1.$

Let

(21) $$b_p = \sup_{k\in I_p} a_k \qquad (b_p = 0 \text{ if } I_p \text{ is empty})$$

and

(22) $$c_p^2 = \sup_{r\geq p} \frac{b_r}{2^r}.$$

Then

(23) $$E \sup_{t\in[0,1]} \|\{x_k(t)\}\|_2 \leq C\left[\left(\sum_{k=1}^{\infty} \frac{a_k}{\lambda_k} \right)^{1/2} + \left(\sum_{p=1}^{\infty} c_p^2 \right)^{1/2} \right],$$

where C is a universal constant. Moreover, when the sums in (23) *are finite, x_t is a.s. a continuous function in l^2.*

PROOF. In what follows C stands for a universal constant but not necessarily the same one at each step. Let $\sum_{k=1}^{\infty}\alpha_k^2 \leq 1$, where $\{\alpha_k\}_{k=1}^{\infty}$ is a sequence of real numbers and set $\beta_p^2 = \sum_{k \in I_p} \alpha_k^2$. Let

$$S(\{\alpha_k\}) = \int_0^{1/2} \frac{\left(\sum_{k=1}^{\infty} \alpha_k^2 \sigma_k^2(u)\right)^{1/2}}{u(\log 1/u)^{1/2}}\, du.$$

Since $E\|\{x_k(0)\}\|_2 \le (\sum_{k=1}^{\infty} a_k/\lambda_k)^{1/2}$ it is clear from (12) that what we must do is find an upper bound for

$$\sup_{\{\alpha_k\}: \sum|\alpha_k|^2 \le 1} S(\{\alpha_k\}). \tag{24}$$

We have

$$S(\{\alpha_k\}) \le \sum_{j=1}^{\infty} \frac{\left(\sum_{p=0}^{\infty} \sum_{k \in I_p} \alpha_k^2 \sigma_k^2(2^{-j})\right)^{1/2}}{j^{1/2}} \tag{25}$$

which by (11), (20) and (21)

$$\le C \sum_{j=1}^{\infty} \frac{1}{j^{1/2}} \left(\sum_{p \le j} \sum_{k \in I_p} \alpha_k^2 \frac{a_k}{2^j} + \sum_{p > j} \sum_{k \in I_p} \alpha_k^2 \frac{a_k}{2^p} \right)^{1/2}$$

$$\le C \sum_{j=1}^{\infty} \frac{1}{j^{1/2}} \left(2^{-j} \sum_{p \le j} \beta_p^2 b_p + \sum_{p > j} \beta_p^2 \frac{b_p}{2^p} \right)^{1/2} \tag{26}$$

$$\le C \sum_{j=1}^{\infty} \frac{2^{-j/2}}{j^{1/2}} \left(\sum_{p \le j} \beta_p^2 b_p \right)^{1/2} + C \sum_{j=1}^{\infty} \frac{1}{j^{1/2}} \left(\sum_{p > j} \beta_p^2 \frac{b_p}{2^p} \right)^{1/2}$$

$$\equiv A + B. \tag{27}$$

By changing the order of summation we have

$$A \le C \sum_{j=1}^{\infty} \frac{2^{-j/2}}{j^{1/2}} \sum_{p \le j} \beta_p b_p^{1/2}$$

$$\le C \left(\sum_{p=1}^{\infty} \frac{2^{-p/2}}{p^{1/2}} \beta_p b_p^{1/2} + \beta_0 b_0^{1/2} \right) \tag{28}$$

$$\le C \sum_{p=0}^{\infty} 2^{-p/2} \beta_p b_p^{1/2}.$$

Therefore

$$\sup_{\{\beta_p\}: \sum \beta_p^2 \le 1} A \le C \left(\sum_{p=0}^{\infty} \frac{b_p}{2^p} \right)^{1/2} \le C \left(\sum_{p=0}^{\infty} \sum_{k \in I_p} \frac{a_k}{\lambda_k} \right)^{1/2} \le C \left(\sum_{k=1}^{\infty} \frac{a_k}{\lambda_k} \right)^{1/2}. \tag{29}$$

Also we see that

(30) $$\sup_{\{\beta_p\}:\Sigma\beta_p^2\le 1} B \le \sup_{\{\beta_p\}:\Sigma\beta_p^2\le 1} C\sum_{J=1}^{\infty}\frac{1}{j^{1/2}}\left(\sum_{p\ge j}\beta_p^2c_p^2\right)^{1/2},$$

where c_p is defined in (22). Since c_p is nonincreasing we have by (14) of Lemma 5 that the second term in (30)

(31) $$\le C\left(\sum_{p=1}^{\infty}c_p^2\right)^{1/2}.$$

Finally, we note that

(32) $$\sup_{\{\alpha_k\}:\Sigma\alpha_k^2\le 1} S(\{\alpha_k\}) \le \sup_{\{\beta_p\}:\Sigma\beta_p^2\le 1} A + \sup_{\{\beta_p\}:\Sigma\beta_p^2\le 1} B.$$

Thus (23) follows from (32), (29), (30) and (31).

Suppose that the sums in (23) are finite. Define $x'_{t,N} = \{x'_k(t)\}_{k=1}^{\infty}$, where

$$x'_k(t) = \begin{cases} 0, & 1 < k < N,\\ x_k(t), & k \ge N.\end{cases}$$

Consider $x_t = \{x_k(t)\}_{k=1}^{\infty}$ and assume that $\lim_{k\to\infty}\lambda_k = \infty$. Given an integer $M > 0$, one can always choose N such that $\lambda_k > 2^{M-1}$, $k \ge N$. Now consider $x'_{t,N}$ and the numbers b_p and c_p given in (21) and (22) for this process. Since $b_p = 0$ for $p < M$ we see that for $x'_{t,N}$,

$$c_p = c_M, \qquad p = 1,\dots,M-1,$$

and therefore

(33) $$\sum_{p=1}^{\infty}c_p^2 = Mc_M^2 + \sum_{p=M+1}^{\infty}c_p^2.$$

Since c_p is nonincreasing and, by assumption, $\Sigma_{p=1}^{\infty}c_p^2 < \infty$, we see that both terms on the right-hand-side of (33) go to 0 as $M \to \infty$. Therefore there exists an $N(\varepsilon)$ such that

(34) $$E\sup_{t\in[0,1]}\|x'_{t,N(\varepsilon)}\|_2 \le \varepsilon$$

since, of course, $\lim_{N\to\infty}\Sigma_{k=N}^{\infty}a_k/\lambda_k = 0$.

Now suppose that $\sup_k\lambda_k < \infty$. Since (6) holds we see that $\lim_{k\to\infty}a_k = 0$. Consider $x'_{t,N}$ in this case. It is obvious that

(34a) $$\sum_{p=1}^{\infty}c_p^2 \le \sup_{k\ge N}a_k.$$

This last inequality and (6) show that (34) also holds when $\sup_k\lambda_k < \infty$.

Note that for any $\rho > 0$ and $\delta > 0$,

$$P\Big(\sup_{(s-t)\le\delta:\ s,\,t\in[0,1]} \|x_s - x_t\|_2 > 3\rho\Big)$$

$$\le P\left(\sup_{(s-t)\le\delta:\ s,\,t\in[0,1]} \left(\sum_{k<N(\varepsilon)} |x_k(s) - x_k(t)|^2\right)^{1/2} > \rho\right) \tag{35}$$

$$+ 2P\left(\sup_{t\in[0,1]} \left(\sum_{k\ge N(\varepsilon)} |x_k(t)|^2\right)^{1/2} > \rho\right).$$

Given $\varepsilon > 0$, we see from (34) that the last term in (35) can be made arbitrarily small by choosing $N(\varepsilon)$ sufficiently large. The first term on the right-hand side of the inequality in (35) goes to 0 as $\delta \to 0$ because the Ornstein–Uhlenbeck processes are continuous. Thus we see that for all $\rho > 0$,

$$\lim_{\delta\to 0} P\Big(\sup_{(s-t)\le\delta:\ s,\,t\in[0,1]} \|x_s - x_t\|_2 > 3\rho\Big) = 0$$

which gives us the a.s. continuity of x_t in l^2. □

PROOF OF THEOREM 1. Suppose $\sup_k \lambda_k < \infty$. Using the notation of Lemma 6, we see, as in (34a), that

$$\sum_{p=1}^{\infty} c_p^2 \le \sup_{k\ge 1} a_k$$

which is finite since (6) holds. Thus it follows from Lemma 6 that x_t is a.s. continuous in l^2 in this case.

Suppose $\inf_k \lambda_k \ge (f(x_1) \vee 1)$. By (7), $m = \sup_k f(a_k \vee x_1)\lambda_k^{-1} < \infty$. Replacing f by f/m, in case $m > 1$, we may assume, without loss of generality, that $f(a_k \vee x_1)\lambda_k^{-1} \le 1$, $\forall\ k$, i.e., that $a_k \vee x_1 \le f^{-1}(\lambda_k)$, $\forall\ k$. In the notation of Lemma 6 and using the fact that $f^{-1}(x)/x$ is nonincreasing for $x \ge f(x_1)$, we have $\forall\ r \ge p$,

$$\frac{b_r}{2^r} \le \sup_{k\in I_r} \frac{a_k \vee x_1}{2^r} \le \sup_{k\in I_r} \frac{f^{-1}(\lambda_k)}{2^r} \le \frac{f^{-1}(2^r)}{2^r} \le \frac{f^{-1}(2^p)}{2^p}.$$

Therefore

$$c_p^2 \le \frac{f^{-1}(2^p)}{2^p}$$

and so, $\sum_{p=1}^{\infty} c_p^2 < \infty$ if

$$\int_0^{1/f(x_1)} f^{-1}\left(\frac{1}{u}\right) du < \infty. \tag{36}$$

The equivalence of (36) and (5) follows from integration by parts. We now use Lemma 6 to see that x_t is a.s. continuous in l^2 when $\inf_k \lambda_k \ge (f(x_1) \vee 1)$. This completes the proof of the first part of Theorem 1 since, in general, we can prove continuity separately for those terms in which $\lambda_k < (f(x_1) \vee 1)$ and for those in which $\lambda_k \geq (f(x_1) \vee 1)$. The proof of the statement that this result cannot be improved will be given in Remark 9. □

PROOF OF COROLLARY 2. Let $f(x) = x(\log^+ x)^r$, $r > 1$, in (5). □

It is clear that $f(x)$ can be taken to be smaller than $x(\log x)^r$ for $r > 1$. However, Corollary 2 is false if r is taken to be 1. To see this, note that a necessary condition for $\{x_t(t)\}_{k=1}^{\infty}$ to be continuous in l^2 is that $\sup_{k \geq k_0} E \sup_{t \in [0,1]} |x_k(t)| \to 0$ as $k_0 \to \infty$. It follows from Fernique (1975), Theorem 7.2.2, and Marcus and Pisier (1981), Chapter 2, Lemma 3.6, or from the left-hand-side of (12) that for $\lambda_k > 4$,

$$\begin{aligned} E \sup_{t \in [0,1]} |x_k(t)| &\geq C\left(\int_0^{1/2} \frac{\sigma_k(u)}{u(\log 1/u)^{1/2}}\, du\right) \\ &\geq C\left(\frac{a_k}{\lambda_k}\right)^{1/2} \int_{1/\lambda_k}^{1/2} \frac{du}{u(\log 1/u)^{1/2}} \\ &\geq C\left(\frac{a_k}{\lambda_k}\right)^{1/2} (\log \lambda_k)^{1/2}, \end{aligned} \tag{37}$$

where $C > 0$ is a constant independent of k, not necessarily the same for each occurrence.

Now let $\{a_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ be such that (6) holds and $\inf \lambda_k \geq 4$. Then, since $a_k \leq \lambda_k$, $\forall\, k \geq k_0$ sufficiently large, we see by (37), that

$$\limsup_{k \to \infty} \left(\frac{a_k}{\lambda_k}\right)(\log a_k) > 0$$

implies that $\{x_k(t),\ t \in [0,1]\}_{k=1}^{\infty}$ is discontinuous in l^2. On the other hand, we see from (8) that for $r > 1$,

$$\sup_k \left(\frac{a_k}{\lambda_k}\right)(\log a_k)^r < \infty$$

implies that $\{x_k(t),\ t \in [0,1]\}_{k=1}^{\infty}$ is almost surely continuous in l^2.

x_t is almost surely in l^2, for t fixed, if $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$ but as seen above it may be discontinuous. The following example illustrates some curious path behavior for such an l^2-discontinuous process.

EXAMPLE 7. Let $x_t = \{x_k\}_{k=1}^{\infty}$ be as defined in (4) with

$$\frac{a_k}{\lambda_k} = \frac{1}{k^2} \quad \text{and} \quad \lambda_k = \frac{1}{k}\exp\left(\frac{k^2 x_0^2}{2}\right), \qquad \forall\, k \geq 1. \tag{38}$$

Then

$$\lim_{k \to \infty} \sup_{0 \leq t \leq T} |x_k(t)| = x_0 \quad \text{a.s.}$$

PROOF. The following inequality is due to Newell (1962): For any $x > x_0$,

$$P\Big(\sup_{0\le t\le T}|x_k(t)| > x\Big) = 1 - F\left(\left(\frac{\lambda_k}{a_k}\right)^{1/2} x, \lambda_k T\right),$$

where $F(z, s) = [\exp(-\Lambda_1(z)s)](1 + O(\varepsilon(z))) + O(\varepsilon(z)\exp(-\Lambda_2(z)s))$ is given in Dawson (1972) and where

$$\frac{1}{\Lambda_1(z)} \sim \left(\frac{\pi}{2}\right)^{1/2}\exp\left(\frac{z^2}{2}\right)\left(\frac{1}{z} + \frac{1}{z^3} + \frac{3}{z^5} + \cdots\right),$$

$$\varepsilon(z) = \left(\frac{2}{\pi}\right)^{1/2} 2z\exp\left(-\frac{z^2}{2}\right)(\log z + O(1)),$$

$$\Lambda_2(z) > 2$$

(all O terms are uniform in s as $z \to \infty$). In particular, $\varepsilon(z) = O(\exp(-z^2/4))$. For $x > x_0$,

$$\begin{aligned}
&P\Big(\sup_{0\le t\le T}|x_k(t)| > x\Big)\\
&\le 1 - \left[1 + O\left(\varepsilon\left(\left(\frac{\lambda_k}{a_k}\right)^{1/2} x\right)\right)\right]\exp\left[-K\left(\frac{\lambda_k}{a_k}\right)^{1/2} x\exp\left(-\frac{\lambda_k}{2a_k}x^2\right)\lambda_k T\right]\\
&\quad + O\left[\varepsilon\left(\left(\frac{\lambda_k}{a_k}\right)^{1/2} x\right)\exp(-2\lambda_k T)\right]\\
&\le 1 - \exp\left[-KxT\exp\left[-\frac{1}{2}\left(\frac{\lambda_k}{a_k}(x^2 - x_0^2)\right)\right]\lambda_k\left(\frac{\lambda_k}{a_k}\right)^{1/2}\exp\left(-\frac{\lambda_k}{2a_k}x_0^2\right)\right]\\
&\quad + O\left(\exp\left(-\frac{\lambda_k}{4a_k}x^2\right)\right)
\end{aligned}$$

for some constant K. Using the hypothesis and the fact that $\Sigma a_k/\lambda_k < \infty$, we see that the right-hand side of the inequality is summable in k. [Note: if $c_k > 0$ and $\Sigma 1/c_k < \infty$, then $\Sigma_{k=1}^{\infty}(1 - \exp(-e^{-c_k})) \le \Sigma_{k=1}^{\infty} e^{-c_k} < \Sigma_{k=1}^{\infty} 1/c_k < \infty$.] Therefore, by the Borel–Cantelli lemma,

$$P\Big(\sup_{0\le t\le T}|x_k(t)| > x \text{ i.o.}\Big) = 0.$$

Similarly, for $x < x_0$, we use Newell's expansion for $P(\sup_{0\le t\le T}|x_k(t)| < x)$ and show for each k that this probability is dominated by a term in a convergent series. Hence

$$P\Big(\sup_{0\le t\le T}|x_k(t)| < x \text{ i.o.}\Big) = 0.$$

Consequently, $\lim_{k\to\infty}\sup_{0\le t\le T}|x_k(t)| = x_0$ a.s. □

PROOF OF COROLLARY 3. If $\sup_k a_k < \infty$, then the corollary follows from Lemma 6. Otherwise let $a'_k = \sup_{j \le k} a_k$ and let $\{a_{k(j)}\}_{j=1}^{\infty}$ be the largest subsequence of $\{a'_k\}_{k=1}^{\infty}$ such that $a_{k(j)}$ is strictly increasing and such that, if there are two or more candidates for $a_{k(j)}$, we choose the one with the smallest subscript. Define, for $x \ge a_1$,

$$f(x) = \frac{x\lambda_{k(j+1)}}{a_{k(j+1)}}, \qquad a_{k(j)} < x \le a_{k(j+1)}, \forall\, j \ge 1, \tag{38a}$$

and note that $f(x)/x$ is nondecreasing because λ_k/a_k is nondecreasing by hypothesis. Also note that, by construction, if $a_{k(j)} < a_k \le a_{k(j+1)}$, then $k \ge k_{j+1}$ so $a_k/\lambda_k \le a_{k(j+1)}/\lambda_{k(j+1)}$. Thus the function f defined in (38a) satisfies (7).

Since $a_{k(1)} = a_1$, we have

$$\begin{aligned}\int_{a_1}^{\infty} \frac{dx}{f(x)} &= \sum_{j=1}^{\infty} \frac{a_{k(j+1)}}{\lambda_{k(j+1)}} \int_{a_{k(j)}}^{a_{k(j+1)}} \frac{dx}{x} \\ &= \sum_{j=1}^{\infty} \frac{a_{k(j+1)}}{\lambda_{k(j+1)}} \log \frac{a_{k(j+1)}}{a_{k(j)}} \le \sum_{k=1}^{\infty} \frac{a_{k+1}}{\lambda_{k+1}} \log^+ \frac{a_{k+1}}{a_k}\end{aligned} \tag{39}$$

since, for $j \ge 1$,

$$\begin{aligned}\sum_{k=k(j)+1}^{k=k(j+1)} \frac{a_k}{\lambda_k} \log^+ \frac{a_k}{a_{k-1}} &\ge \frac{a_{k(j+1)}}{\lambda_{k(j+1)}} \sum_{k=k(j)+1}^{k(j+1)} \log \frac{a_k}{a_{k-1}} \\ &= \frac{a_{k(j+1)}}{\lambda_{k(j+1)}} \log \frac{a_{k(j+1)}}{a_{k(j)}}.\end{aligned}$$

By (9) and (39) the function defined in (38a) satisfies (5) and (7). Since (6) holds by hypothesis we see from Theorem 1 that x_t is continuous in l^2 a.s. □

The upper and lower bounds in (12) are equivalent. Therefore, under sufficient smoothness conditions on $\{a_k/\lambda_k\}$ and $\{\lambda_k\}$, the right-hand side of (23) is also a lower bound. Nevertheless, we will not systematically develop necessary conditions for the boundedness of x_t in l^2 but will instead present some broad classes of examples in which we have sharp estimates for $E \sup_{t \in [0,1]} \|x_t\|_2$. We will also give some results in l^p because they follow so easily from Lemma 5.

THEOREM 8. *Let $x_t = \{x_k(t)\}_{k=1}^{\infty}$ be a sequence of Ornstein–Uhlenbeck processes, where $x_k(t)$ is as defined in* (4). *Let a_k and λ_k, $k \ge 1$, be as defined in* (4) *and assume that a_k/λ_k is nonincreasing in k.*

(i) *Let $\lambda_k \le 2^{k^\beta}$, $\forall\, k \ge 1$ (resp. $\lambda_k \ge 2^{k^\beta}$, $\forall\, k \ge 1$), $\beta \ge 0$. Then there exist constants $0 < c_\beta, C_\beta < \infty$ depending only on β such that*

$$\begin{aligned}\left[c_\beta \left(\sum_{k=1}^{\infty} k^{(\beta-1)\vee 0} \frac{a_k}{\lambda_k} \right)^{1/2} \le \right] E \sup_{t \in [0,1]} \|\{x_k(t)\}\|_2 \\ \le C_\beta \left(\sum_{k=1}^{\infty} k^{(\beta-1)\vee 0} \frac{a_k}{\lambda_k} \right)^{1/2}.\end{aligned} \tag{40}$$

(ii) *Let* $\lambda_k \le 2^k, \forall\, k \ge 1$ (*resp.* $\lambda_k \ge 2^k, \forall\, k \ge 1$), $1 \le p \le 2$. *Then there exist constants* $0 < c_p, C_p < \infty$ *depending only on p such that*

$$\left[c_p\left(\sum_{k=1}^{\infty}\left(\frac{a_k}{\lambda_k}\right)^{p/2}\right)^{1/p} \le \right] E \sup_{t\in[0,1]} \|\{x_k(t)\}\|_p \le C_p\left(\sum_{k=1}^{\infty}\left(\frac{a_k}{\lambda_k}\right)^{p/2}\right)^{1/p}. \tag{41}$$

PROOF. Define

$$\Phi(u) = \left(\sum_{j=1}^{\infty} \alpha_j^2\sigma_j^2(u)\right)^{1/2} \tag{42}$$

and observe that $\Phi(u)$ is increasing in u. We see that for $\beta > 0$,

$$I(\Phi) \equiv \int_0^{1/2} \frac{\Phi(u)}{u(\log 1/u)^{1/2}}\,du = \sqrt{\log 2}\,\beta\int_1^{\infty}\Phi(2^{-v^{\beta}})v^{\beta/2-1}\,dv. \tag{43}$$

We can find constants $0 < g_\beta, G_\beta < \infty$ such that $g_\beta k^{\beta/2-1} \le (k-1)^{\beta/2-1}$ for $k \ge 2$ and $(k+1)^{\beta/2-1} \le G_\beta k^{\beta/2-1}$ for $k \ge 1$. Thus there exist constants $0 < d_\beta$, $D_\beta < \infty$ such that

$$d_\beta \sum_{k=2}^{\infty} k^{\beta/2-1}\Phi(2^{-k^{\beta}}) \le I(\Phi) \le D_\beta \sum_{k=1}^{\infty} k^{\beta/2-1}\Phi(2^{-k^{\beta}}). \tag{44}$$

Note that for $\lambda_k \le 2^{k^{\beta}}$,

$$\sigma_k^2(u) \le 2\frac{a_k}{\lambda_k}2^{k^{\beta}}u \quad \text{for } u2^{k^{\beta}} < 1,$$

and

$$\sigma_k^2(u) \le 2\frac{a_k}{\lambda_k} \qquad \text{for } u2^{k^{\beta}} \ge 1.$$

Under the alternative hypothesis, if $\lambda_k \ge 2^{k^{\beta}}$ then, by (11)

$$\sigma_k^2(u) \ge \left(\frac{2}{e}\right)\frac{a_k}{\lambda_k} \quad \text{for } u2^{k^{\beta}} \ge 1.$$

Let $c_k^2 = a_k/\lambda_k$. We see that

$$\Phi(2^{-k^{\beta}}) \le \left(2\sum_{j=1}^{k-1}\alpha_j^2c_j^2 2^{j^{\beta}}2^{-k^{\beta}}\right)^{1/2} + \left(2\sum_{j=k}^{\infty}\alpha_j^2c_j^2\right)^{1/2} \tag{45}$$

and, under the alternative hypothesis, that

$$\Phi(2^{-k^{\beta}}) \ge \left(\frac{2}{e}\sum_{j=k}^{\infty}\alpha_j^2c_j^2\right)^{1/2}. \tag{46}$$

It follows from (44), (45) and (46) that there exist constants $0 < d_\beta'$, $D_\beta' < \infty$

such that

$$\begin{aligned}\left[d'_\beta \sum_{k=1}^\infty k^{\beta/2-1}\left(\sum_{j=k}^\infty \alpha_j^2 c_j^2\right)^{1/2} \leq \right] I(\Phi) \\ \leq D'_\beta\left(\sum_{k=1}^\infty k^{\beta/2-1}\left(\sum_{j=1}^{k-1} \alpha_j^2 c_j^2 2^{j^\beta} 2^{-k^\beta}\right)^{1/2} + \sum_{k=1}^\infty k^{\beta/2-1}\left(\sum_{j=k}^\infty \alpha_j^2 c_j^2\right)^{1/2}\right).\end{aligned} \tag{47}$$

Note that

$$\begin{aligned}\sum_{k=1}^\infty k^{\beta/2-1}\left(\sum_{j=1}^{k-1} \alpha_j^2 c_j^2 2^{j^\beta} 2^{-k^\beta}\right)^{1/2} &\leq \sum_{k=1}^\infty k^{\beta/2-1} \sum_{j=1}^{k-1} \alpha_j c_j 2^{j^\beta/2} 2^{-k^\beta/2} \\ &= \sum_{j=1}^\infty \alpha_j c_j 2^{j^\beta/2} \sum_{k=j+1}^\infty k^{\beta/2-1} 2^{-k^\beta/2} \\ &\leq K_\beta \sum_{j=1}^\infty \alpha_j c_j j^{-\beta/2} \\ &\leq K_\beta\left(\sum_{j=1}^\infty |\alpha_j|^q\right)^{1/q}\left(\sum_{j=1}^\infty c_j^p j^{-\beta p/2}\right)^{1/p},\end{aligned} \tag{48}$$

where $1/p + 1/q = 1$ and K_β is a constant depending only on β. This follows since $k^{\beta/2-1}2^{-k^\beta/2}$ is decreasing for $k \geq j$ for j big enough, so we may use an integral comparison. We can now obtain upper bounds for

$$\sup_{\{\alpha_k\}:\, \|\{\alpha_k\}\|_q \leq 1} I(\Phi)$$

by using (47), (48) and Lemma 5; and, under the alternative hypothesis, lower bounds by (47) and Lemma 5. Using these bounds and the fact that

$$c'_p\left(\sum_{k=1}^\infty c_k^p\right)^{1/p} \leq E\|\{x_k(0)\}\|_p \leq C'_p\left(\sum_{k=1}^\infty c_k^p\right)^{1/p}$$

for constants, $0 < c'_p \leq C'_p < \infty$, in (12), we get Theorem 8. □

As an application of Theorem 8 consider the sequence $\{x_k(t)\}_{k=1}^\infty$ given in Example 7. Although

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} |x_k(t)| = x_0 \quad \text{a.s.}, \tag{49}$$

it is not clear from the methods used in the example whether, in the case when $x_0 \neq 0$, we have

$$\sup_{0\leq t\leq T} \|\{x_k(t)\}\|_2 = \infty \quad \text{a.s.} \tag{50}$$

That (50) is true follows from Theorem 8. Note that in Example 7

$$\lambda_{nk} = \frac{1}{nk}\exp\left(\frac{(nkx_0)^2}{2}\right) \geq 2^{(k-1)^2}$$

for $k \geq k_0$ sufficiently large and for $x_0 \geq \sqrt{2\log 2}/n$, $\forall\ n \geq 1$. Thus for $x_0 \geq \sqrt{2\log 2}/n$ we see from the left-hand side of (40) that

$$E \sup_{0 \leq t \leq T} \|\{x_k(t)\}\|_2 = \infty. \tag{51}$$

(Note that the value of $0 < T < \infty$ is immaterial so we can take it to be 1.) It follows from (51) and Jain and Marcus (1978), Chapter 2, Corollary 4.7, that (50) holds for $x_0 \geq \sqrt{2\log 2}/n$ and, since n can be taken as large as we want, it holds for all x_0.

REMARK 9. Theorem 1 is best possible in the following sense. Suppose that we can find a function $f_1(x)$ that satisfies the hypotheses of Theorem 1 except that instead of (5) holding we have

$$\int_{x_1}^{\infty} \frac{dx}{f_1(x)} = \infty. \tag{52}$$

Now let us consider a sequence of independent Ornstein–Uhlenbeck processes $\{x_k(t)\}_{k=1}^{\infty}$ defined as in (4) with $\lambda_k = 2^{k^\beta}$ and $a_k = f_1^{-1}(\lambda_k) \wedge \lambda_k k^{-\delta}$, $1 < \delta < \beta$. As we remarked at the end of the proof of Theorem 1, (52) is equivalent to

$$\sum_k \frac{f_1^{-1}(2^k)}{2^k} = \infty$$

which, by a change of variables, is equivalent to

$$\sum_k k^{\beta-1} \frac{f_1^{-1}(2^{k^\beta})}{2^{k^\beta}} = \infty. \tag{53}$$

However, (53) and the definition of a_k imply by (40) that

$$E \sup_{t \in [0,1]} \|\{x_k(t)\}\|_2 = \infty$$

which, as we remarked following (51), implies that $\sup_{t\in[0,1]}\|\{x_k(t)\}\|_2$ is unbounded a.s. Therefore Theorem 1 is false if (5) does not hold.

The right-hand-side inequalities in Lemma 5 generalize some interesting inequalities of Boas (1960). One of these which plays a role in the study of Gaussian processes [see Jain and Marcus (1978), Chapter 4, Lemma 2.2] is the following: Let $\{c_k\}_{k=1}^{\infty}$ be a nonincreasing sequence of nonnegative real numbers. Then there exists an absolute constant C such that

$$\sum_{j=1}^{\infty} j^{-1/2}\left(\sum_{k=j}^{\infty} c_k^4\right)^{1/2} \leq C\left(\sum_{k=1}^{\infty} c_k^2\right).$$

This inequality is given by (14) with $p = q = 2$ and with the specific sequence $b_k^2 = c_k^2/\sum_{k=1}^{\infty} c_k^2$, $\forall\, k \geq 1$. Of course, in (14), we show that the inequality is valid for all sequences $\{b_k\}_{k=1}^{\infty}$ satisfying $\|\{b_k\}\|_2 \leq 1$. The interested reader can check that (13) generalizes several of the inequalities given by Boas (1960), Theorem 2, (4).

Finally, let us note that whereas a great deal is known about Ornstein–Uhlenbeck processes we only used the fact that they are Gaussian processes with a particular covariance function. Our methods can be equally well used to study the a.s. continuity in l^p of sequences of independent Gaussian processes with stationary increments. For example, let $\{Z_{k,\alpha}(t)\}_{k=1}^{\infty}$ be a sequence of independent mean zero stationary Gaussian processes satisfying

$$EZ_{k,\alpha}(s)Z_{k,\alpha}(t) = \frac{a_k}{\lambda_k}\exp(-\lambda_k|s-t|^{\alpha}), \qquad 0 < \alpha \leq 2. \tag{54}$$

All the results we have obtained immediately extend to these processes because of the following simple consequence of Theorem 4.

Theorem 10. *There exist constants $0 < c_{\alpha,p} \leq C_{\alpha,p} < \infty$ depending only on $0 < \alpha \leq 2$ and $1 \leq p \leq \infty$ such that*

$$c_{\alpha,p} E \sup_{t\in[0,1]} \|\{Z_{k,1}(t)\}\|_p \leq E \sup_{t\in[0,1]} \|\{Z_{k,\alpha}(t)\}\|_p \leq C_{\alpha,p} E \sup_{t\in[0,1]} \|\{Z_{k,1}(t)\}\|_p,$$

where $\{Z_{k,1}(t)\}$ is a sequence of independent Ornstein–Uhlenbeck processes.

Proof. Define

$$\sigma_{k,\alpha}^2(u) = 2\frac{a_k}{\lambda_k}(1 - e^{-\lambda_k|u|^{\alpha}})$$

and

$$\Phi_\alpha(u) = \left(\sum_{k=1}^{\infty} a_k^2 \sigma_{k,\alpha}^2(u)\right)^{1/2}.$$

For I as defined in (43), we have

$$\begin{aligned} I(\Phi_\alpha) &= \frac{1}{\sqrt{\alpha}}\int_0^{2^{-\alpha}} \frac{\Phi_\alpha(u^{1/\alpha})}{u(\log 1/u)^{1/2}}\,du \\ &= \frac{1}{\sqrt{\alpha}}\int_0^{2^{-\alpha}} \frac{\Phi_1(u)}{u(\log 1/u)^{1/2}}\,du. \end{aligned}$$

This is the I function for the Ornstein–Uhlenbeck process with a minor change in the upper limit of integration. The upper limit can be brought back to $1/2$ by altering the domain of the process, i.e., by considering $\{Z_{k,\alpha}(\beta t)\}_{k=1}^{\infty}$ where $\beta = 2^{1-1/\alpha}$, and then, by stationarity and the triangle inequality, extending the domain, if necessary, back to $[0,1]$. [Note that we use the monotonicity of $\sigma_{k,\alpha}^2(u)$ very strongly.] □

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I. ISCOE
DEPARTMENT OF MATHEMATICS
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106

M. B. MARCUS
DEPARTMENT OF MATHEMATICS
TEXAS A & M UNIVERSITY
COLLEGE STATION, TEXAS 77843

D. MCDONALD
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OTTAWA
OTTAWA, ONTARIO
CANADA K1N6N5

M. TALAGRAND
EQUIPE D'ANALYSE, TOUR 46
U.A. AU C.N.R.S. N° 754
UNIVERSITÉ PARIS VI
4 PLACE JUSSIEU
75230 PARIS CEDEX 05
FRANCE
AND
DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
231 WEST 18TH AVENUE
COLUMBUS, OHIO 43210

J. ZINN
DEPARTMENT OF MATHEMATICS
TEXAS A & M UNIVERSITY
COLLEGE STATION, TEXAS 77843