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DIFFUSION IN A SINGULAR RANDOM ENVIRONMENT[1]

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A class of diffusions on the line is considered whose drifts are generalized functions (Schwartzian distributions). A probability measure is put on this space of drifts, giving a diffusion in a random environment. An invariance principle is then proven for the rescaled diffusion, generalizing a result of De Masi, Ferrari, Goldstein and the author.

1. Introduction. Consider the stochastic differential equation

$$dx(t) = -U'(x(t))\,dt + dw(t). \tag{1.1}$$

Here $w(\cdot)$ is a standard one-dimensional Wiener process, $w(0) = 0$ a.s., U is a function on $\mathbb{R}$, a prime denotes d/dx and $x(t) \in \mathbb{R}$ for $t \geq 0$. If U is, say, once-differentiable with a bounded, Lipschitz-continuous derivative, then (strong) solutions to (1.1) exist a.s. for any initial condition x [$x(0) = x$ a.s.] [see, e.g., McKean (1969)]. But if U is only assumed to be bounded, with perhaps some mild regularity conditions (which permit U to have discontinuities), is it still possible to associate a unique diffusion process (a process with continuous sample paths a.s.) with (1.1), in such a way that this process is the limit of suitably regularized processes? In this case (1.1) would be purely formal, as the drift could only be interpreted as a generalized function (Schwartzian distribution), and so the process would have to be characterized in some other way.

That the answer to this question is yes has been discovered by several authors. The differential equation (1.1) is replaced by the Dirichlet form

$$\tfrac{1}{2}\int dx\, e^{-2U(x)}(f'(x))^2. \tag{1.2}$$

Under suitable regularity conditions one can then associate a diffusion process with (1.2) for a.e. initial condition. This process is time-reversible with respect to the (in general infinite) measure $\exp(-2U(x))\,dx$. No smoothness condition is required on the function U [see Silverstein (1974) and Fukushima (1980), and references therein].

In this paper, I consider diffusions with singular drifts associated with quadratic forms of the type (1.2). Sufficient conditions are given (see Section 4) so that the diffusion is the weak limit of regularized processes [solutions of (1.1) in the ordinary sense], and does not explode (reach infinity in a finite time). I then consider diffusions of this type in a random environment; that is, I put a probability measure on the space of "environments" U. An invariance principle

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for the rescaled diffusion can then be proved along the lines of the theorems in Kipnis and Varadhan (1985) and De Masi, Ferrari, Goldstein and Wick (1985, 1987).

It will be more convenient to replace (1.1) and (1.2) by

$$dx(t) = V'(x(t))\,dt + \sqrt{2V(x(t))}\,dw(t) \tag{1.3}$$

and

$$\int dx\, V(x)(f'(x))^2, \tag{1.4}$$

respectively. The reason for this is that solutions of (1.3) have Lebesgue measure dx as reversible measure, and the Dirichlet form can be considered to be defined on a suitable domain in $L^2(dx)$. Thus if we take a family of "environments" $V_k(x) \to V(x)$ suitably, the corresponding family of Dirichlet forms, transition operators, infinitesimal generators, etc., are all defined on the same space. A transformation of the state space ($\mathbb{R}$) will carry the process associated with (1.3)–(1.4) to the process associated with (1.1)–(1.2) (see Section 4).

The organization of this paper is as follows. In Section 2 we construct the diffusions associated with (1.4). We also characterize the corresponding transition function as a weak solution of the backward equation (see the remarks at the end of the section). In Section 3 we put a probability measure on the space of V's, construct the "environment process as seen from the diffusing particle" and prove an invariance principle. In Section 4 we construct several examples of singular random environments. In Section 5 we prove the corresponding theorem for processes associated with (1.2). In the Appendix a scheme is given for constructing regularized environments tending in a suitable way to a given environment (a result needed in Sections 2 and 3).

Since the diffusions we treat in this paper are one-dimensional, one can also construct them (at least if U is continuous) by time-change and scale-transformation from a standard Brownian motion [see Itô and McKean (1965) or Freedman (1971)]. (Of course, this approach is not available for higher-dimensional analogs of the processes constructed in this paper.) We do not pursue this approach, but rather obtain our processes directly as weak limits of regularized processes (a result needed in any case in Section 3). For a discussion of time-changes for symmetric diffusions, see Fukushima (1980), Section 5.5].

One should contrast the cases discussed herein with that considered by Brox (1986). In the latter case the random medium U was itself a Brownian motion (I consider a similar case in Section 4), with "time-parameter" x, and $U(0) = 0$ a.s. This medium has very large excursions producing deep "valleys" which trap the diffusing particle for a long time. Thus one does not expect to get a Brownian motion (with positive diffusion constant) in the scaling limit, and in fact Brox found, informally,

$$x(t) \sim \log^2 t,$$

just as in the discrete case [Sinai (1982)]. A similar result should be expected if we take $U(x)$ to be an integral (over the interval $[0, x)$) of a Poisson point process with random signs [so that U' (x) is a linear combination of delta-functions]. The assumption that U is bounded in this paper rules out these interesting cases (yielding always a positive diffusion constant for the rescaled diffusion).

2. A family of processes reversible with respect to Lebesgue measure. Consider the stochastic differential equation

$$dx_t = V'(x_t)\,dt + \sqrt{2V(x_t)}\,dw(t). \tag{2.1}$$

Here w_t is a standard Wiener process and V is a positive function. The corresponding infinitesimal generator would be

$$L_V f(x) = \nabla V(x) \nabla f(x) \tag{2.2}$$

with $\nabla = d/dx$. If we assume, for instance, that $V \in C_b^1(\mathbb{R})$ (continuously differentiable functions with a bounded first derivative), and

$$0 < c_1 \le V(x) \le c_2 < \infty \quad \text{for all } x, \tag{2.3}$$

then weak solutions of (2.1) exist for any initial condition x and do not explode (reach infinity in a finite time). The corresponding path measures P_V^x on $C([0, \infty); \mathbb{R})$ form a Markovian diffusion process with generator given in (2.2) [see Stroock and Varadhan (1979)]. However, we do not assume this but only that V is upper semicontinuous (U.S.C.), either left- or right-continuous at each x and (2.3). Having dropped the smoothness assumption on V, (2.1) and (2.2) become purely formal. Nevertheless, we can associate a unique Markovian diffusion process with V for a.e. initial condition, which is moreover a weak limit of regularized diffusions.

Fundamental to the construction will be the Dirichlet form associated to V, which we now introduce. Let $H = L^2(dx)$ and define

$$Q_V(f) = \int dx\, V(x)(f'(x))^2 \tag{2.4}$$

(the prime denoting d/dx), with domain

$$D \equiv D(Q_V) = \{ f \in H\colon f \text{ is absolutely continuous with } f' \in H \}. \tag{2.5}$$

Q_V is a closed quadratic form. [If $f_n \to f$ in H and $Q_V(f_n - f_m) \to 0$ as $n, m \to \infty$, then $f_n' \to g$ in L^2, from which it follows easily that $g = f'$ a.e. and $Q_V(f_n - f) \to 0$.] Since Q_V is also nonnegative, by a standard result [Reed and Simon (1972), Theorem 8.15] there is a self-adjoint, nonpositive operator L_v with $D(Q_V) = D((-L_v)^{1/2})$ such that

$$Q_V(f) = \|(-L_V)^{1/2} f\|_2^2 \tag{2.6}$$

($\|\cdot\|_2$ is the L^2-norm). Additionally, Q_V is Dirichlet: If $f \in D(Q_V)$ and $f^{(N)} = (fV(-N)) \wedge N$,

$$Q_V(f^{(N)}) \le Q_V(f). \tag{2.7}$$

This implies [Fukushima (1980) and Silverstein (1974)] that the resolvents $(\lambda - L_V)^{-1}$, $\lambda > 0$, and the semigroup $T_V^t = \exp(L_V t)$ are positivity-preserving.

Thus we can construct a Markov process with transition semigroup T_V^t and any initial measure ρ of form $\tilde{\rho}(x)\,dx$, $\tilde{\rho} \in L^1_+(dx)$, on $\prod_{t \in [0,\infty)} \mathbb{R}^*$ ($\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$, the one-point compactification of $\mathbb{R}$) by a standard construction [Nelson (1959)].

The Dirichlet form Q_V has several additional relevant properties, which we list below [see Fukushima (1980)].

1. It is regular: $C_0^\infty(\mathbb{R})$ is a core of $D(Q_V)$.
2. It is local: If $f, g \in D(Q_V)$ and $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$, then $Q_V(f, g) = 0$. [$Q_V(\cdot,\cdot)$ is the bilinear form obtained from Q_V by polarization.]
3. A slightly stronger condition than 2 holds: If $f, g \in D(Q_V)$ and f is constant on $\operatorname{supp} g$, then $Q_V(f, g) = 0$. [This condition is relevant to part (d) of Theorem 2.1.]

Dirichlet forms satisfying 1 and 2 are associated with diffusion processes (processes with continuous sample paths), and in fact the following theorem is a special case of a more general result on local Dirichlet forms [see Fukushima (1980), especially Section 4.5, and Silverstein (1974)].

THEOREM 2.1. *There is a null set $N \subset \mathbb{R}$ such that, for $x \notin N$, there are measures P_V^x on $C([0,\infty); \mathbb{R}^*) \equiv \Omega^*$ satisfying*:

(a) $x \to P_V^x(F)$ *is measurable for any bounded continuous F on Ω^*.*
(b) $\{P_V^x\}$ *has the (strong) Markov property.*
(c) $P_V^x f(\omega_t) = T_V^t f(x)$ *a.s. for $f \in C(\mathbb{R}^*)$ ($t \to \omega_t$ denotes a path in Ω^*).*
(d) $\omega_t \in \mathbb{R}$ *for $t \le \zeta$, where ζ (the explosion time) is a measurable stopping time, $\omega_t = \infty$ for $t \ge \zeta$ and $\lim_{t \uparrow \zeta} u_t = \infty$ a.s.*
(e) dx *is a reversible measure for this process.*

Note that $P_V^x[\zeta < \infty]$ iff $T_V^t I_{\mathbb{R}}(x) < 1$ for some $t > 0$. In this case the diffusing particle reaches infinity in a finite time, but in fact our hypotheses on V rule out this possibility. This will follow once we have recovered this process as a limit of suitably regularized processes, which we now introduce.

Let V_k, $k = 1, 2, \dots$, be a family of functions satisfying $V_k \in C_b^\infty(\mathbb{R})$ (infinitely differentiable functions with bounded derivatives);

$$V_k(x) \ge V(x), \qquad V_k(x) \to V(x) \quad \text{for all } x, \tag{2.8}$$
$$V_k(x) \le c_2 \quad \text{for all } x.$$

If V is U.S.C., it is not difficult to construct a family satisfying (2.8); an explicit construction is given in the Appendix. Let $P_k^x = P_{V_k}^x$ be the law of $x_k(\cdot)$, the solution of (2.1) (V replaced by V_k) with initial condition x [$x_k(0) = x$ a.s.]. The P_k^x form a Feller process on $C([0,\infty); \mathbb{R}) \equiv \Omega$. We regard Ω as a subset (which is measurable) of Ω^*. If $\rho = \tilde{\rho}(x)\,dx$ and $\tilde{\rho} \in L^1_+(dx)$, write P^ρ and P_k^ρ for $\int \rho(dx) P_V^x$ and $\int \rho(dx) P_{V_k}^x$, respectively.

THEOREM 2.2. (i) $P_k^\rho \to P^\rho$, *weakly as $k \to \infty$.*
(ii) *For all $T \ge 0$,*

$$\lim_{M \to \infty} \inf_k P_k^\rho\big[|\omega_t| \le M, 0 \le t \le T\big] = 1. \tag{2.9}$$

The conclusion in (i) is that, for all bounded continuous F on Ω^*, $P_k^\rho(F) \to P^\rho(F)$. Consequently,

$$\limsup_{k\to\infty} P_k^\rho(C) \le P^\rho(C) \tag{2.10}$$

for all closed $C \subset \Omega^*$. Applying this fact and conclusion (ii) to $C = [|\omega_t| \le M, 0 \le t \le T]$ for M, T fixed we conclude that

$$\lim_{M\to\infty} P^\rho[|\omega_t| \le M, 0 \le t \le T] = 1. \tag{2.11}$$

Therefore $P^\rho(\Omega) = 1$ and no explosion occurs a.s.

Proof of Theorem 2.2. Let T_k^t be the Markov semigroup on $C_b(\mathbb{R})$ associated to the process $x_k(\cdot)$. T_k^t has infinitesimal generator L_k given in (2.2) (V replaced by V_k), with domain $C_b^2(\mathbb{R})$. Integration by parts shows that L_k, on the restricted domain $C_0^2(\mathbb{R})$ (C^2-functions with compact support), is symmetric and nonpositive as an operator on $L^2(dx)$. For $\lambda > 0$, $(\lambda - L_k)(C_0^2)$ is dense in $L^2(dx)$ (by elliptic regularity), so L_k is essentially self-adjoint on C_0^2. The corresponding closed operator, which we continue to denote by L_k, has associated Dirichlet form

$$Q_k(f) \equiv \|(-L_k)^{1/2} f\|_2^2 = \int dx\, V_k(x)(f'(x))^2 \tag{2.12}$$

with $D(Q_k) = D((-L_k)^{1/2}) = \{f \in H\colon f \text{ is a.c. with } f' \in H\}$. (To see this, note that C_0^2 is a core for L_k. Hence

$$\left(-i + \sqrt{-L_k}\right)(C_0^2) = \left(i + \sqrt{-L_k}\right)^{-1}(I - L_k)C_0^2$$

is dense in H, so C_0^2 is a core of the quadratic form. On C_0^2, Q_k has the indicated form, and the result now follows by a density argument.)

By the assumed properties of the family $\{V_k\}$ and the remarks above we have

$$\begin{aligned} &D(Q_k) = D = D(Q_v),\\ &Q(f) \le Q_k(f), \qquad f \in D, \quad \text{for all } k,\\ &Q_k(f) \to Q(f) \quad \text{for } f \in D. \end{aligned} \tag{2.13}$$

Hence by a theorem of Faris [Faris (1970), Theorem 7.9], $T_k^t \to T^t$ strongly in H for each $t \ge 0$. From this we conclude the convergence of the finite-dimensional marginals of P_k^ρ to those of P^ρ. So we are reduced to proving tightness of the family $\{P_k^\rho\}$ on Ω and (2.9).

For each k define

$$h_k(x) = \begin{cases} \int_0^x V_k(y)^{-1}\,dy, & x \ge 0,\\ -\int_{-x}^0 V_k(y)^{-1}\,dy, & x < 0. \end{cases} \tag{2.14}$$

It is each to check that

$$(2.15)\qquad h_k(x_k(t)) \equiv h_k(t), \qquad h_k^2(t) - \int_0^t [2/V_k(x_k(\tau))]\, d\tau$$

are martingales relative to $\Sigma_t = \sigma(x_\tau, \tau \le t)$. [These claims follow from the computations

$$L_k h_k(x) \equiv 0, \qquad L_k h_k^2(x) = 2/V_k(x).]$$

It is also clear that, for all x, y and k,

$$(2.16)\qquad \begin{aligned} C_2^{-1}|x| &\le |h_k(x)| \le C_1^{-1}|x|, \\ C_2^{-1}|x-y| &\le |h_k(x) - h_k(y)|. \end{aligned}$$

By Doob's martingale inequality, (2.15) and (2.16), for all $T \ge 0$, $M < \infty$,

$$(2.17)\qquad \begin{aligned} &P^x\Big[\sup_{0 \le t \le T} |x_k(t)| > M\Big] \\ &\quad \le P^x\Big[\sup_{0 \le t \le T} |h_k(t)| > C_2^{-1}M\Big] \\ &\quad \le C_2 M^{-2} E^x h_k^2(T) \le C_2^2 M^{-2} 2 C_1^{-1} T. \end{aligned}$$

The same result holds with initial measure ρ. This gives (2.9).

We establish tightness of $\{P_k^x\}$, $x \in \mathbb{R}$, fixed; this implies (i) of the theorem. To do so we use the martingale $h_k(\cdot)$ and Censov's criterion [cf. Billingsley (1968), Theorem 15.1, and subsequent discussion]. We follow a line of argument that appeared in Holley and Stroock (1978), Lemma 4.6 and its proof; see also Stroock and Varadhan (1979). Our goal is to prove that, for all $T > 0$, $\eta > 0$,

$$(2.18)\qquad \lim_{\delta \to 0} \sup_k P^x\left[\sup_{\substack{0 \le s < t \le T \\ |s-t| \le \delta}} |x_k(t) - x_k(s)| > \eta\right] = 0.$$

In light of (2.16) it suffices to prove (2.18) with $h_k(\cdot)$ replacing $x_k(\cdot)$.

For any bounded function f on $[0, T]$ let

$$(2.19)\qquad \begin{aligned} \tilde{w}_f(\delta) = &\sup_{\substack{0 \le t_1 \le t_2 \le t_3 \le T \\ (t_2 - t_2) \vee (t_2 - t_1) < \delta}} \{|f(t_3) - f(t_2)| \wedge |f(t_2) - f(t_1)|\} \\ &\vee \sup_{0 \le t \le \delta} |f(t) - f(0)| \vee \sup_{0 \le t \le \delta} |f(T) - f(T-t)| \end{aligned}$$

and

$$(2.20)\qquad w_f(\delta) = \sup_{\substack{0 \le t_1 \le t_2 \le T \\ |t_2 - t_1| \le \delta}} |f(t_2) - f(t_1)|.$$

Then $w_f(\delta) \le 2\tilde{w}_f(\delta)$ [cf. Parthasarathy (1969), Lemma 6.4]. So we are reduced to proving

$$(2.21)\qquad \sup_k P^x\big[\tilde{w}_{h_k}(\delta) \ge \eta\big] \underset{\delta \to 0}{\to} 0.$$

This in turn follows from Censov's criterion if we can show

$$\sup_k E^x\Big(\big(h_k(t_3) - h_k(t_2)\big)^2\big(h_k(t_2) - h_k(t_1)\big)^2\Big) \le B^2(t_3 - t_1)^2, \tag{2.22}$$

for some constant $B < \infty$ and all $0 \le t_1 \le t_2 \le t_3 \le T$. By (2.16), $h_k^2(t) - 2C_1^{-1}T$ is a supermartingale. Hence, with $B = 2C_1^{-1}$,

$$\begin{aligned} &E^x\Big[\big(h_k(t_3) - h_k(t_2)\big)^2\big(h_k(t_2) - h_k(t_1)\big)^2\Big] \\ &\quad = E^x\Big[\Big(E\big(h_k^2(t_2)|\mathscr{F}_{t_2}\big) - h_k^2(t_2)\Big)\big(h_k(t_2) - h_k(t_1)\big)^2\Big] \\ &\quad \le B(t_3 - t_2)E^x\big(h_k^2(t_2) - h_k^2(t_1)\big) \\ &\quad \le B^2(t_3 - t_2)(t_2 - t_1) \le B^2(t_3 - t_1)^2. \end{aligned}$$

□

REMARKS. One can show that the transition function of the limiting process satisfies a weak form of the backward equation. Given $f \in D$, define

$$u_k(x, t) = T_k^t f(x), \qquad u(x, t) = T^t f(x). \tag{2.23}$$

Then for $t \ge 0$,

$$\begin{aligned} \|\nabla u_k\|_2^2 &= \int dx\big(\nabla u_k(x, t)\big)^2 \\ &\le C_1^{-1}\int ds\, V_k(x)\big(\nabla u_k(x, t)\big)^2 \\ &= C_1^{-1}\|(-L_k)^{1/2}e^{L_k t}f\,\|_2^2 \\ &\le C_1^{-1}Q_k(f) \\ &\le C_1^{-1}C_2\|f'\|_2^2, \end{aligned} \tag{2.24}$$

and if $g \in C_0^1$,

$$\big(g, \nabla u_k(\cdot, t)\big)_2 = -\big(\nabla g, T_k^t f\,\big)_2 \underset{k\to\infty}{\to} -\big(\nabla g, T^t f\,\big)_2. \tag{2.25}$$

Since C_0^1 is dense in $L^2(dx)$, we conclude that

$$\nabla u_k(\cdot, t) \to \nabla u(\cdot, t) \tag{2.26}$$

weakly in L^2. Let $h \in C_0^\infty(\mathbb{R} \times (0, \infty))$. Then

$$\iint \frac{\partial h}{\partial t} u_k\, dx\, dt = \iint \frac{\partial h}{\partial x} V_k(x) \frac{\partial u_k}{\partial x}\, dx\, dt \tag{2.27}$$

so passing to the limit $k \to \infty$ [note that $(\partial h/\partial x)V_k \to (\partial h/\partial x)V$ strongly], we obtain

$$\iint \frac{\partial h}{\partial t} u\, dx\, dt = \iint \frac{\partial h}{\partial x} V(x) \frac{\partial u}{\partial x} dx\, dt, \tag{2.28}$$

which is a weak form of the backward equation.

3. Diffusion in a singular random environment: The "environment process" and an invariance principle. In this section we put a translation-invariant probability measure on the space of environments introduced in Section 2, and prove an invariance principle for the resulting diffusion in a random environment. Let $\mathscr{E}$ be the space of environments V satisfying our regularity conditions. Give $\mathscr{E}$ the topology of convergence in measure on each finite interval and the corresponding Borel σ-algebra. Let μ be a probability measure on $\mathscr{E}$ which is translation-invariant and ergodic under translations. By this we mean that, if G is a bounded measurable function on $\mathscr{E}$ and the spatial shift operator S_y, $y \in \mathbb{R}$, is defined by

$$(3.1) \qquad S_yV(x) = V(x-y), \qquad S_yG(V) = G(S_{-y}V)$$

(note that S_y acts continuously on $\mathscr{E}$), then

$$(3.2) \qquad \mu(S_yG) = \mu(G)$$

for all $y \in \mathbb{R}$, and

$$(3.3) \qquad A \subset \mathscr{E} \text{ measurable with } S_x(A) \subset A \quad \text{for all } x \in \mathbb{R} \text{ implies } \mu(A) = 0 \text{ or } 1.$$

Note that if (3.3) fails, we may always restrict μ to an ergodic component.

We next construct the "environment process," the process of the environments seen from the diffusing particle [see Kipnis and Varadhan (1985) and De Masi, Ferrari, Goldstein and Wick (1987)]. We first consider a regularized process. For $k = 1, 2, \dots$ we construct (in the Appendix) a map $V \to V_k$: $\mathscr{E} \to C_b^\infty$, $k = 1, 2, \dots$, such that (2.8) is satisfied (note that $V_k \to V$ pointwise implies $V_k \to V$ in measure), and such that this map is measurable for each k if C_b^∞ is given its natural topology (uniform convergence of all derivatives on compact intervals) and the corresponding Borel σ-algebra. Let $x_k(\cdot)$ be the diffusion associated to V_k by solving (2.1), with $x_k(0) = \bar{x}$ a.s. Define a process $V_k(\cdot)$ by setting

$$(3.4) \qquad [V_k(t)](x) = S_{-x_k(t)}V_k(x) = V_k(x - x_k(t)).$$

We call $V_k(\cdot)$ the "environment process" [for the regularized diffusion $x_k(\cdot)$]. In a similar fashion, let $x(\cdot)$ be the diffusion associated to V with initial measure ρ, and let $V(\cdot)$ be the process given by

$$(3.5) \qquad [V(t)](x) = S_{-x(t)}V = V(x - x(t)).$$

In this way we have constructed measures $P^{\bar{x}, V_k}$ and $P^{\rho, V}$ on $C([0,\infty); \mathscr{E})$. Defining

$$(3.6) \qquad P^{\rho, V_k} = \int \rho(d\bar{x}) P^{\bar{x}, V_k},$$

it follows immediately from the results of Section 2 that

$$(3.7) \qquad P^{\rho, V_k} \to P^{\rho, V}$$

weakly on $C([0,\infty); \mathscr{E})$. Since the regularized diffusions $x_k(\cdot)$ depend smoothly on the drift and diffusion function appearing in the stochastic differential equation, $V \to P^{\rho, V_k}(G)$ is measurable for G bounded and continuous on $\mathscr{E}$.

Hence $V \to P^{\rho, V}(G)$ is measurable, and we may define

$$(3.8)\qquad \begin{aligned} P_k^{\rho,\mu} &\equiv \int \mu(dV) P^{\rho, V_k}, \\ P^{\rho,\mu} &\equiv \int \mu(dV) P^{\rho, V}. \end{aligned}$$

Since

$$P^{x, V_k} = P^{0, S_{-x} V_k}$$

for every x, it follows from (3.7), (3.8) and the translation invariance of μ that the measures $P_k^{\rho,\mu}$ and $P^{\rho,\mu}$ are actually independent of ρ. Henceforth we denote them simply by p_k^{μ} and P^{μ}, respectively. We have

$$(3.9)\qquad P_k^{\mu} \to P^{\mu}$$

weakly on $C([0, \infty); \mathscr{E})$.

In terms of the regularized processes $V_k(\cdot)$ we can recover the diffusing particle's position from the formula

$$(3.10)\qquad x_k(t) = \int_0^t [V_k'(\tau)](0)\, d\tau + \int_0^t \sqrt{2V_k(\tau)}\,(0)\, dw_\tau$$

(where the second integral is an Itô integral) holding a.s. on the probability space of the Brownian process w. The following facts are proven in an analogous fashion to the corresponding statements in De Masi, Ferrari, Goldstein and Wick (1987), Section 6.

1. The process $V_k(\cdot)$ is time-reversible and ergodic.
2. The function $\phi_k(V) \equiv V_k'(0)$ lies in the space $H_{-1,k}$, the dual Hilbert space to the space $H_{+1,k}$ associated with the Dirichlet form of the process. The norm in the latter is defined by

$$(3.11)\qquad \|F\|_{+1}^2 = \lim_{t \to 0^+} t^{-1}\big(F, \big(1 - \mathscr{T}_t^{k}\big) F\big)_2,$$

where $F \in L^2(\mathscr{E}, \mu)$, the inner product is in this space and $\mathscr{T}_k^t$ is the Markov semigroup acting in L^2. [The limit exists by spectral theory and is finite if and only if $F \in D((-\mathscr{L}_k)^{1/2})$, $\mathscr{L}_k$ the generator of $\mathscr{T}_t^{k}$.] $H_{+1,k}$ is then a completion in this norm.

Fact 2 is implied by general results on time-reversible Markov processes, see De Masi, Ferrari, Goldstein and Wick (1987), Section 2], but in fact we shall determine precisely the generator of the process; fact 2 is then equivalent to the statement that for some constant $C < \infty$,

$$(3.12)\qquad |(\phi_k, F)_2| \le C\sqrt{\big(F, (-\mathscr{L}_k) F\big)_2}$$

holds for F in a core of the Dirichlet form of the generator. In our context this can be checked directly.

We shall need to know rather precisely what the generator of the regularized process and the Dirichlet form are. The information we need is contained in the following proposition. Let ∇ be the operator on $L^2(\mathscr{E}; \mu)$:

$$\nabla F \equiv \lim_{h\to 0} h^{-1}(S_h F - F), \tag{3.13}$$

with domain the functions F for which the limit in (3.13) exists in L^2. Let

$$\begin{aligned} \mathscr{F}_1 &= D(\nabla) = \{F \in L^2 \colon \nabla F \in L^2\}, \\ \mathscr{F}_2 &= \{F \in L^2 \colon \nabla F \text{ and } \nabla^2 F \text{ are in } L^2\}. \end{aligned} \tag{3.14}$$

PROPOSITION 3.1. (i) *$\mathscr{F}_2$ is a core for the generator $\mathscr{L}_k$ of $\mathscr{T}_k^t$ and if $F \in \mathscr{F}_2$,*

$$\mathscr{L}_k F = \nabla V_k(0) \nabla F. \tag{3.15}$$

(ii) *$\mathscr{F}_1$ is a core for the Dirichlet form of $\mathscr{L}_k$ [defined in* (3.11)] *and if $F \in \mathscr{F}_1$,*

$$Q_k(F) \equiv \|(-\mathscr{L}_k)^{1/2} F\|_2^2 = \int d\mu\, V_k(0)(\nabla F)^2. \tag{3.16}$$

PROOF. On bounded continuous functions on $\mathscr{E}$ the semigroup of the process $V_k(\cdot)$ is given by

$$\begin{aligned} \mathscr{T}_k^t F(V) &= E^0 F(S_{-x_k(t)} V_k) \\ &= \int p_k^t(dx \mid 0) F(S_{-x} V_k), \end{aligned} \tag{3.17}$$

where $p_k^t(\cdot \mid \cdot)$ is the transition function of the diffusion $x_k(t)$ associated with V_k. Taking L^2 limits in (3.17) defines the semigroup for $F \in L^2(\mathscr{E}, \mu)$. If $F \in \mathscr{F}_2$, the derivative of (3.17) at $t = 0$ exists in L^2 and is given by (3.15) since $p_k^t(\cdot \mid \cdot)$ satisfies the forward equation associated with (2.2). On $\mathscr{F}_2$, (3.16) follows by an integration by parts (note that ∇ is skew-adjoint in L^2). Given $F \in \mathscr{F}_1 = D(\nabla)$, $\exists\, F_n \in \mathscr{F}_2$, with $F_n \to F$ and $\nabla F_n \to \nabla F$ in L^2. Hence $(\nabla F_n - \nabla F)^2 \to 0$ in L^1, which gives (3.16), since the Dirichlet form is closed. To prove that these spaces are cores for $\mathscr{L}_k$ and $(-\mathscr{L}_k)^{1/2}$, respectively, it suffices to prove that they are invariant under the semigroup [De Masi, Ferrari, Goldstein and Wick (1987), Section 3]. This follows readily from the smoothness of the transition function, since $V_k \in C_b^\infty$. □

We next introduce the rescaled diffusion. Let $\varepsilon > 0$ and

$$x^\varepsilon(t) \equiv \varepsilon x(\varepsilon^{-2} t), \qquad x_k^\varepsilon(t) \equiv \varepsilon x_k(\varepsilon^{-2} t). \tag{3.18}$$

The next theorem is the analog of Theorem 6.1 of De Masi, Ferrari, Goldstein and Wick (1987) for a process with singular drifts.

THEOREM 3.2. *Let $B(t)$ be a standard Brownian motion, $B(0) = 0$ a.s., with diffusion constant*

$$D \equiv t^{-1} E B(t)^2 = 2\langle V(0)^{-1}\rangle_\mu^{-1} \tag{3.19}$$

($\langle \cdot \rangle_\mu$ denoting the μ-integral). Then

$$x^\varepsilon(\cdot) \to B(\cdot) \tag{3.20}$$

in the weak sense of convergence of path measures on $C([0, |\infty); \mathbb{R})$.

PROOF. As in De Masi, Ferrari, Goldstein and Wick (1987) one has for the regularized processes $V_k(\cdot)$,

$$\varepsilon\int_0^{\varepsilon^{-2}t}[V_k(\tau)](0)\,d\tau = N_k^\varepsilon(t) + R_k^\varepsilon(t), \tag{3.21}$$

where $N_k^\varepsilon(t)$ is a rescaled, square-integrable martingale with stationary increments, and for all $T \geq 0$, $\delta > 0$,

$$\begin{gathered}\sup_k E_k^\mu R_k^\varepsilon(T)^2 \to 0,\\ \sup_k P_k^\mu\left[\sup_{0\leq t\leq T}|R_k^\varepsilon(t)| > \delta\right] \to 0\end{gathered} \tag{3.22}$$

as $\varepsilon \to 0$. The latter facts follow from De Masi, Ferrari, Goldstein and Wick (1987), Lemma 2.4; see also Kipnis and Varadhan (1985), and the following computation. Let $\phi_k(V) = \nabla V_k(0)$. Then by Proposition 3.1 [see also De Masi, Ferrari, Goldstein and Wick (1987), Section 2]

$$\begin{aligned}\|\phi_k\|_{-1,k}^2 &= \sup_{\substack{F\in D(\nabla)\\ Q_k(F)\neq 0}} \frac{|\int d\mu\, \phi_k F|}{\left\{d\mu\, V_k(0)(\nabla F)^2\right\}^{1/2}}\\ &= \sup_F \frac{|\int d\mu\, V_k(0)\nabla F|}{\left\{\int d\mu\, V_k(0)(\nabla F)^2\right\}^{1/2}}.\end{aligned} \tag{3.23}$$

Now since ∇ generates a unitary semigroup in $L^2(\mu)$, $\{\nabla F\colon F \in D(\nabla)\}$ is dense in the orthogonal complement of one in $L^2(\mu)$. ($i\nabla$ is self-adjoint and the union of its spectral subspaces corresponding to intervals not containing 0 are dense in $1^\perp$.) Hence $\{V_k)(0)^{1/2}\nabla F\}$ is dense in the orthogonal complement of $V_k(0)^{-1/2}$. Hence, from (3.23),

$$\|\phi_k\|_{-1,k}^2 = \|(1 - P_{V_k(0)^{-1/2}})V_k(0)^{1/2}\|_{L^2(\mu)}^2$$

$$= \langle V_k(0)\rangle_\mu - \langle V_k(0)^{-1}\rangle_\mu^{-1} \tag{3.24}$$

$$\leq c_2 - c_1. \tag{3.25}$$

It follows as in De Masi, Ferrari, Goldstein and Wick (1987), Section 2, that if $\psi_k^\lambda = (\lambda - \mathscr{L}_k)^{-1}\phi_k$, then $\|\psi_k^\lambda\|_{+1,k} \to_{\lambda\to 0} \|\psi_k\|_{+1,k} \leq c_2 - c_1$ for all k, from which (3.22) follows as in De Masi, Ferrari, Goldstein and Wick (1987) and Kipnis and Varadhan (1985).

Writing

$$M_k^\varepsilon(t) \equiv \varepsilon\int_0^{\varepsilon^{-2}t}\sqrt{2[V_k(\tau)](0)}\,dw_\tau, \tag{3.26}$$

we have

$$x_k^\varepsilon(t) = N_k^\varepsilon(t) + M_k^\varepsilon(t) + R_k^\varepsilon(t). \tag{3.27}$$

We next establish that the sum of the martingales in (3.27) is tight (for $\varepsilon > 0$ fixed). This will follow from the Censov criterion, as in Section 2, if we can bound the compensators of the squares of the martingales. For $M_k^\varepsilon(t)$ we know from Itô calculus that

$$\left(M_k^\varepsilon(t)\right)^2 - \varepsilon^2 \int_0^{\varepsilon^{-2}t} 2(V_k(\tau))(0)\, d\tau \tag{3.28}$$

is another martingale. Since $V_k \leq c_2$, $(M_k^\varepsilon(t))^2 - 2c_2 t$ is a supermartingale. For $N_k^\varepsilon(t)$ we note, from its construction [De Masi, Ferrari, Goldstein and Wick (1987), Section 2],

$$\begin{aligned} N_k(t) &= L^2 - \lim_{\lambda \to 0} N_k^\lambda(t), \\ N_k^\lambda(t) &= \int_0^t \mathscr{L}_k \psi_k^\lambda(V(\tau))\, d\tau - \psi_k^\lambda(V(t)) - \psi_k^\lambda(V(0)) \end{aligned} \tag{3.29}$$

and that $\psi_k^\lambda \in \mathscr{F}_2$ [follows from the smoothness of the kernel of the resolvent $(\lambda - \mathscr{L}_k)^{-1}$]. Hence by Proposition 3.1 and an easy computation [see De Masi, Ianiro, Pellegrinotti and Presutti (1984), Chapter 4]

$$N_k^\lambda(t)^2 - \int_0^t 2(V_k(\tau))(0)\left(\nabla\psi_k^\lambda\right)^2(V(\tau))\, d\tau \tag{3.30}$$

is a martingale. Since

$$\lambda\psi_k^\lambda - \mathscr{L}_k\psi_k^\lambda = \nabla V_k(0), \tag{3.31}$$

given $F \in D(\nabla)$, we have, after integrating by parts,

$$\lambda\left(F, \psi_k^\lambda\right) + \left(V_k(0)\nabla F, \nabla\psi_k^\lambda\right) = \left(1, V_k(0)\nabla F\right). \tag{3.32}$$

Since $\{V_k(0)\nabla F\}$ is dense in the orthogonal complement of $V_k(0)^{-1}$ and from results in De Masi, Ferrari, Goldstein and Wick (1987)

$$\begin{aligned} \psi_k^\lambda &\to \psi_k \quad \text{in } H_{+1,k}, \\ \lambda\psi_k^\lambda &\to 0 \quad \text{in } L^2, \qquad \text{as } \lambda \to 0, \end{aligned} \tag{3.33}$$

we conclude that

$$\nabla\psi_k = 1 + cV_k(0)^{-1} \tag{3.34}$$

for some constant c, which (since $\langle \nabla\psi_k \rangle_\mu = 0$) must have the value

$$c = -\langle V_k(0)^{-1} \rangle_\mu^{-1}. \tag{3.35}$$

Thus $N_k(t)^2 - 8c_2c_1^{-4}t$ is a supermartingale. The same holds for $N_k^\varepsilon(t)$, and this gives the sought-after tightness of the martingales.

Now let F be bounded continuous on $C([0, T]; \mathbb{R})$. Then for $\delta > 0$ there is ε_0 such that if $\varepsilon < \varepsilon_0$,

$$\sup_k |EF(x_k^\varepsilon(\cdot)) - EF(M_k^\varepsilon(\cdot) + N_k^\varepsilon(\cdot))| < \delta \tag{3.36}$$

for all k. By the tightness, along a subsequence $\{k_j\}$,

$$M_{k_j}(\cdot) + N_{k_j}(\cdot) \to_d \tilde{M}(\cdot) \tag{3.37}$$

weakly. Then $\tilde{M}(\cdot)$ is another square-integrable martingale with stationary increments. Hence by the martingale form of the invariance principle [Helland (1982); see also the discussion in De Masi, Ferrari, Goldstein and Wick (1987), Section 2]

$$\tilde{M}^\varepsilon(\cdot) \to \tilde{B}(\cdot) \tag{3.38}$$

weakly where $\tilde{B}(\cdot)$ is a Brownian motion with variance

$$E\tilde{B}(t)^2 = \lim_{\varepsilon \to 0} E\tilde{M}^\varepsilon(t)^2 = \tilde{D}t. \tag{3.39}$$

Furthermore, from (3.37) there follows

$$\tilde{D} \le \limsup_{k \to \infty} D_k, \tag{3.40}$$

and, by results in De Masi, Ferrari, Goldstein and Wick (1987), Section 2, and (3.24),

$$\begin{aligned} D_k &= 2\langle V_k(0)\rangle_\mu - 2\|\phi_k\|_{-1,k}^2 \\ &= 2\langle V_k(0)^{-1}\rangle_\mu^{-1} \\ &\underset{k\to\infty}{\to} 2\langle V(0)^{-1}\rangle_\mu^{-1}. \end{aligned} \tag{3.41}$$

Thus taking $j \to \infty$ we obtain from (3.37), (3.38) and the results of Section 1, for $\varepsilon < \varepsilon_0$,

$$|EF(x^\varepsilon(t)) - EF(\tilde{M}^\varepsilon(\cdot))| < \delta. \tag{3.42}$$

Taking $\varepsilon \to 0$, we obtain the invariance principle with diffusion constant $\tilde{D}$, which is bounded above by the expression in (3.19).

This appears to be the most that we can obtain from general theory (in this context). However, we have available (in one dimension) an additional argument, based on the construction of the martingale h_k in Section 2. Following the line of reasoning in De Masi, Ferrri, Goldstein and Wick (1987), Chapter 6 (final remarks), we can construct for each k a martingale $\tilde{h}_k$, a functional of the process $V_k(\cdot)$, which is square-integrable with stationary increments [$\tilde{h}_k$ is equal to $h_k(x_k(t))$, regarding $x_k(t)$ as a functional of $V_k(\cdot)$]. From (2.15),

$$E\tilde{h}_k^2(t) = 2t\langle V_k(0)^{-1}\rangle_\mu. \tag{3.43}$$

In addition, by explicit computation

$$L_k(h_k(x))^4 = 12h_k^2(x)V_k(x)^{-1}, \tag{3.44}$$

so that $\tilde{h}_k(t) \in L^4(P^\mu)$ with

$$E^\mu \tilde{h}_k^4(t) \le 24 c_1^{-2} t. \tag{3.45}$$

Since $\tilde{h}_k$ lies in L^4, $\tilde{h}_k^2$ is uniformly integrable; since the compensator of $\tilde{h}_k^2$ is bounded (by $2c_1^{-1}$), $\{\tilde{h}_k\}_k$ is tight. Hence

$$\tilde{h}_k(\cdot) \to_d \tilde{h}(\cdot), \tag{3.46}$$

$\tilde{h}$ is a square-integrable martingale with stationary increments, and

$$E^\mu \tilde{h}_k^2(t) \underset{k\to\infty}{\to} E^\mu \tilde{h}(t)^2 \tag{3.47}$$

so that

$$E^\mu \tilde{h}(t)^2 = \langle V(0)^{-1} \rangle_\mu. \tag{3.48}$$

Since

$$h_k(x) \to h(x) \tag{3.49}$$

[$h(x)$ given by (2.14) with V_k replaced by V] uniformly on compact sets, we have that $h(t) \sim_d h(x(t))$. Furthermore, by stationarity and ergodicity,

$$h(t)/x(t) \underset{t\to\infty}{\overset{P}{\to}} \langle V(0)^{-1} \rangle_\mu. \tag{3.50}$$

[Note that $|x(t)|$ diverges in probability, since $c_1|h(t)| \le |x_t|$ a.s.] We conclude from (3.50) by the martingale CLT that, as $t \to \infty$,

$$E^\mu \exp\left[ix(t)/\sqrt{t}\right] \sim E^\mu \exp\left[i\left(h(t)/\langle V(0)^{-1}\rangle_\mu/R\sqrt{t}\right)\right] \underset{t\to\infty}{\to} \exp\left[-\tfrac{1}{2}Dt^2\right], \tag{3.51}$$

where D is given in (3.19). This concludes the proof of Theorem 3.2. □

REMARKS. That the process lives in one dimension was used twice in the proof of Theorem 3.2: once to obtain the process seen from the moving particle as a limit of regularized processes, and again to obtain a lower bound on the diffusion constant. General theory should suffice to obtain the process V_t and prove the invariance principle in higher dimensions [our conditions on V must be modified in an appropriate way—the crucial point is to keep $\nabla V(0)$ in H_{-1}]. However, a special argument will apparently be needed to ensure that the diffusion matrix is nonsingular.

4. Some examples of singular random environments. Consider the diffusions of Sections 2 and 3. As a first example, we construct a translation-invariant measure μ on $\mathscr{E}$ supported on highly discontinuous functions. Restricting μ to an ergodic component then yields a nontrivial example of a singular random environment of the type considered in Section 3.

Let $H_{y,r}$, $y \in \mathbb{R}$, $0 < r < \infty$, be defined by

$$H_{y,r}(x) = \begin{cases} 1, & y - r \le x \le y + r, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Construct the following objects on the same probability space:

$\{y_i, -\infty < i < +\infty\}$, a locally finite, stationary point process on $\mathbb{R}$.

$\{a_i, -\infty < i < +\infty\}$, a stationary sequence of nonnegative random variables.

$\{r_i, -\infty < i < +\infty\}$, a stationary sequence of random variables taking values in $(0, R)$ for some $R > 0$ fixed.

V_0, a random variable taking values in $[c_1, +\infty)$ for some $c_1 > 0$ fixed.

Let

$$\hat{V}(x) \equiv \sum_{-\infty < i < +\infty} a_i H_{y_i, r_i}(x) + V_0 \tag{4.2}$$

and for $c_2 > c_1$ fixed let

$$V(x) = \hat{V}(x) \wedge c_2. \tag{4.3}$$

The law of V provides an example of a measure of the indicated type. Note that $V'(x)$, thought of as a distribution (generalized function), is a linear combination of delta-functions.

REMARKS. (i) Although in this example μ is supported on piecewise constant V (so $V' = 0$ a.e.), the drift still plays a role. To see this, note that the limiting Brownian motion in Theorem 3.2 has variance $2\langle V(0)^{-1}\rangle^{-1}t \neq 2\langle V(0)\rangle t$, the limiting variance of the Itô integral one would have in the absence of the drift, if $V(0)$ is not constant.

(ii) By replacing the family $\{H_{y,r}\}$ by wilder functions one can construct more exotic examples, e.g., measures supported on functions with discontinuities of the second kind (oscillatory type).

(iii) The uniform bound $V \le c_2$ in the definition of $\mathscr{E}$ can presumably be replaced by a milder restriction, e.g., $V(x) \le c_2(1 + |x|)$, without invalidating the theorems of Sections 2 and 3. The truncation step (4.3) can presumably then be dispensed with for certain point processes. Can one allow even milder restrictions, i.e., $V(x) \le c(V)(1 + |x|)$ with $c(V) < \infty$, μ a.e.? This would permit taking $\{y_i\}$ to be a Poisson process [and dropping (4.3)].

Another approach to constructing interesting examples is to take a stationary process $V(x)$ with state space $[c_1, c_2]$ and "time-parameter" x. For instance, let $V(x)$, $-\infty < x < +\infty$, be Brownian motion on $[c_1, c_2]$ with reflecting boundary conditions and "initial" measure $dV/c_2 - c_1$. This construction gives an example of a random environment with continuous but nowhere differentiable sample paths. The limiting Brownian motion of the rescaled diffusion has diffusion constant $(c_2 - c_1)(\log c_2 - \log c_1)^{-1}$.

5. Processes associated with (1.1). We next consider processes associated (formally) with the S.D.E. (1.1). Let (for the moment) $\tilde{V} \in C_b^\infty$ and let

$$\tilde{L}_{\tilde{V}} f(x) = -\tilde{V}'(x) f'(x) + \tfrac{1}{2} f''(x). \tag{5.1}$$

Define a map $\Psi_{\tilde{V}}: \mathbb{R} \to \mathbb{R}$ by

$$(5.2) \qquad \Psi_{\tilde{V}}(x) = \begin{cases} \int_0^x e^{-2\tilde{V}(y)}\, dy, \\ -\int_x^0 e^{-2\tilde{V}(y)}\, dy, \end{cases}$$

and a corresponding operator by

$$(5.3) \qquad R_\Psi f(x) = f \circ \Psi(x).$$

Then by a straightforward computation

$$(5.4) \qquad R_{\Psi_{\tilde{V}}^{-1}} \tilde{L}_{\tilde{V}} R_{\Psi_{\tilde{V}}} f = L_V f,$$

where L_V is given in (2.2) with

$$(5.5) \qquad V(x) = (\tfrac{1}{2}) e^{-4\tilde{V}} \circ \Psi_{\tilde{V}}^{-1}(x).$$

Hence $\Psi_{\tilde{V}}$ transforms the diffusion associated with (1.1) ($\tilde{V} \equiv U$), which has reversible measure

$$(5.6) \qquad e^{-2\tilde{V}(x)}\, dx,$$

into the diffusion associated with (2.1), which has Lebesgue measure as reversible measure. If $\tilde{V}$ enjoys the regularity conditions specified in Section 2 except for the bounds, which are replaced by

$$(5.7) \qquad -c_2 \le \tilde{V}(x) \le c_2$$

for some $c_2 < +\infty$, one can then construct a diffusion $x(t)$ associated with V and set

$$(5.8) \qquad \tilde{x}(t) = \Psi_{\tilde{V}}^{-1}(x(t)), \qquad t \ge 0.$$

Since $\Psi_{\tilde{V}}^{-1}$ is continuous, $\tilde{x}(\cdot)$ is a diffusion. The associated quadratic form is

$$(5.9) \qquad \int dx\, e^{-2V(x)} (f'(x))^2,$$

defined in the space $L^2(e^{-2V(x)}\, dx)$. Furthermore, we can obtain $\tilde{x}(t)$ as a limit of regularized processes $\tilde{x}_k(t)$ by transforming the processes $x_k(t)$. The proof of Theorem 3.2 then goes through (with the necessary changes), and we obtain the following theorem.

THEOREM 5.1. *Let $\tilde{\mu}$ be a translation-invariant, ergodic probability measure on $\tilde{\mathscr{E}}$ (the space of suitably regular $\tilde{V}$), and let $\tilde{B}(\cdot)$ be a Brownian motion, $\tilde{B}(0) = 0$ a.s., with diffusion constant*

$$(5.10) \qquad \tilde{D} = \langle e^{2\tilde{V}(0)} \rangle_\mu^{-1} \langle e^{-2\tilde{V}(0)} \rangle_{\tilde{\mu}}^{-1}.$$

Then

$$(5.11) \qquad \tilde{x}^\varepsilon(\cdot) \to \tilde{B}(\cdot)$$

weakly as $\varepsilon \to 0$.

Examples of measures $\tilde{\mu}$ defining singular random environments supported on $\tilde{\mathscr{E}}$ may be constructed in a manner similar to the examples presented in this section.

APPENDIX

Construction of regularized environments. Consider the intervals, for $-\infty < j < +\infty$, $k = 1, 2, \ldots$,

$$I_{j,k} = \left((j - \tfrac{1}{4})2^{-k}, (j + 1 + 1/4)2^{-k}\right), \tag{A.1}$$

which form an open cover of $\mathbb{R}$ for each k. Let $\{h_{j,k}\}$ be a C^∞ partition of unity subordinate to this cover. This means that

$$\begin{aligned} &h_{j,k} \in C^\infty, \qquad 0 \le h_{j,k}(x) \le 1, \\ &\operatorname{supp} h_{j,k} \subset I_{j,k}, \qquad \sum_j h_{j,k}(x) = 1, \end{aligned} \tag{A.2}$$

for all $x \in \mathbb{R}$. Given V satisfying the conditions of Section 2, define

$$s_{j,k}(V) \equiv \sup_{y \in I_{j,k}} V(y), \tag{A.3}$$

and let

$$V_k(x) \equiv \sum_j s_{j,k}(V) h_{j,k}(x). \tag{A.4}$$

Then $V_k \in C_b^\infty$, $V(x) \le V_k(x) \le c_2$ and $V_k(x) \to V(x)$ for $x \in \mathbb{R}$. [The last claim follows from the U.S.C. of V. If U_k are open intervals, $x \in U_k$, $\operatorname{diam} U_k \to 0$, then

$$\sup_{y \in U_k} V(y) \to V(x). \tag{A.5}$$

Each x is contained in the support of at most two $h_{j,k}$, which sum to 1. Therefore $V_k(x)$ is a convex combination of two numbers, both greater than $V(x)$ and tending to $V(x)$ as $k \to \infty$.]

We must also show that, for each k, the map $V \to V_k$ is measurable. Since V is left-or-right-continuous at each x and U.S.C., the supremum in (A.3) can be replaced by a supremum over the rationals in $I_{j,k}$. So it suffices to prove that, for each x_0, $V \to V(x_0)$ is measurable. But again by the one-sided continuity and the semicontinuity,

$$V(x_0) = \limsup_{\delta \downarrow 0} \left(\delta^{-1} \int_{(x_0 - \delta, x_0)} V(y)\, dy\right) \vee \left(\delta^{-1} \int_{(x_0, x_0 + \delta)} V(y)\, dy\right), \tag{A.6}$$

which expresses $V(x_0)$ as a superior limit of continuous functionals on $\mathscr{E}$. Thus $V(x_0)$ is measurable.

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