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LOCAL LIMIT THEOREMS FOR SUMS OF FINITE RANGE POTENTIALS OF A GIBBSIAN RANDOM FIELD

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Local limit theorems are derived for sums of finite range $\mathbb{Z}$-valued potential functions of an iid random field. The resulting approximations turn out to be mixtures of standard normal densities for lattice distributions supported by residue classes of integers. The mixing weights are equal to the probability that the sum of potential functions lies in such a residue class and are nonasymptotic and computable. For finite range potential functions of a stationary Gibbsian random field with bounded and finite range interactions, conditions are given under which the global central limit theorem implies the classical local limit theorem.

1. Introduction and summary. In this paper we shall investigate m-dependent random fields generated by application of $\mathbb{Z}$-valued finite range potential or "window" functions to a real-valued iid random field, say, Y_j, $j \in \mathbb{Z}^d$, $d \geq 1$. With distances in the lattice $\mathbb{Z}^d$ measured by

$$\|x - y\| \triangleq \sup\{|x_i - y_i|: 1 \leq i \leq d\},$$

define for a fixed subset $\varnothing \neq V \subset \mathbb{Z}^d$ of diameter at most m, $m \geq 0$, and for a fixed measurable window $h_V: \mathbb{R}^V \to \mathbb{Z}$,

$$X_j^V \triangleq h_V(Y_k, k \in j + V), \qquad j \in \mathbb{Z}^d. \tag{1.1}$$

Obviously, X_j^V, $j \in \mathbb{Z}^d$, is a strict-sense stationary m-dependent random field. Especially for $d = 1$ and $V = \{0, \dots, m\}$ we shall consider m-dependent sequences defined by

$$X_j = h_V(Y_j, Y_{j+1}, \dots, Y_{j+m}), \qquad j \in \mathbb{Z}. \tag{1.2}$$

Recently Aaronson, Gilat, Keane and de Valk (1989) have shown that a representation of type (1.2) is not possible for an arbitrary m-dependent and strict-sense stationary sequence of 0–1 variables $X_1, X_2, \dots$. We do not know whether our results remain valid in the general m-dependent case, i.e., when the representation (1.2) is not possible.

Let $D_+ = \{0, \dots, m\}^d$ and $\mathscr{V}_0$ denote a nonempty collection of essentially different "windows" V such that

$$\text{(1.3)} \quad \begin{array}{lll} \text{(i)} & \varnothing \neq V \subset \mathbb{Z}^d, & V \subset D_+, \\ \text{(ii)} & V, V' \in \mathscr{V}_0, & V \neq V' \Rightarrow j + V \neq V' \quad \text{for every } j \in \mathbb{Z}^d. \end{array}$$

In order to simplify the notation let $\mathscr{V}$ denote the collection of translates of $\mathscr{V}_0$, i.e., $\mathscr{V} = \{V + j: V \in \mathscr{V}_0, j \in \mathbb{Z}^d\}$. Define accordingly $h_{V+j} \equiv h_V$ for every

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$V \in \mathscr{V}_0$, $j \in \mathbb{Z}^d$. Let $A_N \triangleq \{1, \dots, N\}^d$ denote the d-dimensional cube in $\mathbb{Z}^d$ of side length $N \geq 1$. We are interested in approximations for the point probabilities of the sum S_N defined as

$$S_N \triangleq \sum \{X_0^V: V \in \mathscr{V}, V \subset A_N\}. \tag{1.4}$$

To illustrate this notation, consider the d-dimensional Ising window as an example of a one-dependent field, where

$$\mathscr{V}_0 = \{V_1, \dots, V_d\}, \qquad V_\nu = \{0, e_\nu\},$$

$e_\nu = \nu$th unit vector in $\mathbb{Z}^d$, and $h_{V_\nu}(Y_0, Y_{e_\nu}) \triangleq 1_{\{Y_0 = Y_{e_\nu}\}}$, and where

$$\mathbb{P}\{Y_0 = +1\} = \mathbb{P}\{Y_0 = -1\} = \tfrac{1}{2}.$$

Another example would be the "d-cube window" defined by

$$\mathscr{V}_0 \triangleq \{D_+\}$$

and a measurable function $h_{D_+} = h$ on $\mathbb{R}^{D_+}$.

In the independent case, where $m = 0$ and $\mathscr{V}_0$ necessarily consists of the set $\{0\}$ only, $X_j \triangleq h(Y_j)$, $j \in \mathbb{Z}^d$, is an iid random field and $S_N = \sum_{j \in A_N} X_j$. In this case standard conditions ensure that

$$\sup_p |\mathbb{P}\{S_N = p\} - \psi_N(p)| = o(|A_N|^{-1/2}), \tag{1.5}$$

where $|A_N| = N^d$ denotes the number of elements of A_N and where

$$\psi_N(p) = \mathrm{Var}(S_N)^{-1/2} \varphi(x(N, p)),$$

$$x(N, p) = \mathrm{Var}(S_N)^{-1/2}(p - ES_N),$$

and φ denotes the standard normal density. These conditions are

$$EX_0^2 < \infty \tag{1.6}$$

and

$$X_0 \text{ has span } 1 \tag{1.7}$$

[see Bhattacharya and Ranga Rao (1986), page 231, Theorem 22.1].

Let B denote the (discrete) support of X_0. The span of X_0 is defined as the smallest positive element of the additive group generated by $B - B \triangleq \{b_1 - b_2: b_1, b_2 \in B\}$. In order to simplify the notation assume that

$$\mathbb{P}\{X_0 = 0\} > 0. \tag{1.8}$$

Then (1.7) is satisfied iff $\gcd\{a: \mathbb{P}\{X_0 = a\} > 0\} = 1$.

Relation (1.5) is the standard local limit theorem which, under certain assumptions, can be derived also for weakly dependent random fields [see Heinrich (1988) and Riauba (1986)]. This relation, however, may be violated even in the case of an m-dependent sequence. To illustrate this, we emphasize two obvious consequences of (1.5) which may not hold in the m-dependent case with $m \geq 1$: The point probabilities $\mathbb{P}\{S_N = p\}$ are of order $|A_N|^{-1/2}$ and they are approximable by a continuous function of $x(N, p)$, i.e.,

$$\limsup_{N} \sup_{p} |A_N|^{1/2} |\mathbb{P}\{S_N = p\} - \mathbb{P}\{S_N = p + k\}| = 0, \tag{1.9}$$

for every integer k. Furthermore, closely connected with the continuity is the fact that S_N does not have a preference for odd or even integers p; more generally, for any integer $k \geq 2$,

$$\lim_{N} \mathbb{P}\{S_N \equiv r \bmod k\} = 1/k, \tag{1.10}$$

for every $0 \leq r \leq k - 1$. In the m-dependent case ($m \geq 1$) the residue classes in (1.10) may have unequal weights. We shall prove (1.5) with $\psi_N(p)$ replaced by

$$\bar{\psi}_N(p) \triangleq k\mathbb{P}\{S_N \equiv p \bmod k\}\psi_N(p), \tag{1.11}$$

where k is a positive integer determined as follows. For notational convenience write Y_V for $(Y_j: j \in V)$, $V \in \mathbb{Z}^d$.

Define

$$\begin{aligned} &D \triangleq \{j \in \mathbb{Z}^d: \|j\| \leq m\}, \\ &D_0 \triangleq \{j \in D: j_\nu = 0 \text{ for some } \nu, 1 \leq \nu \leq d\} \quad \text{and} \\ &R \triangleq \sum \{h_V(Y_V): V \in \mathscr{V}, V \subset D\}. \end{aligned} \tag{1.12}$$

Conditioned on Y_j, $j \notin D_0$, the random variable R depends on Y_j, $j \in D \setminus D_0$. Let $k(Y_j, j \notin D_0)$ denote the span of the conditional distribution of R, given Y_j, $j \notin D_0$. Define the *intrinsic span* k by

$$k \triangleq \max\{l \in \mathbb{N}: k(Y_j, j \notin D_0) \in l\mathbb{Z} \text{ a.s.}\}. \tag{1.13}$$

In the above-mentioned Ising window we have in the case $d = 2$,

$$D_0 = \{0, e_1, e_2, -e_1, -e_2\}$$

and

$$\begin{aligned} R = {} & 1_{\{Y_{(0,0)} = Y_{(0,1)}\}} + 1_{\{Y_{(0,0)} = Y_{(1,0)}\}} + 1_{\{Y_{(0,0)} = Y_{(-1,0)}\}} + 1_{\{Y_{(0,0)} = Y_{(0,-1)}\}} \\ & + 1_{\{Y_{(0,1)} = Y_{(1,1)}\}} + 1_{\{Y_{(0,1)} = Y_{(-1,1)}\}} + 1_{\{Y_{(1,0)} = Y_{(1,1)}\}} + 1_{\{Y_{(1,0)} = Y_{(1,-1)}\}} \\ & + 1_{\{Y_{(-1,0)} = Y_{(-1,1)}\}} + 1_{\{Y_{(-1,0)} = Y_{(-1,-1)}\}} + 1_{\{Y_{(0,-1)} = Y_{(-1,-1)}\}} + q_{\{Y_{(0,-1)} = Y_{(1,-1)}\}} \end{aligned}$$

and finally

$$D \setminus D_0 = \{(1,1), (1,-1), (-1,1), (-1,-1)\}.$$

If for $j \in D \setminus D_0$ the values for Y_j are fixed and equal to 1, then R attains the values $\{0, 3, 4, 5, 6, 7, 8, 9, 12\}$ which yields $k = 1$ in this example.

Using the above quantities the following result generalizes (1.5) to the m-dependent case.

THEOREM 1.14. *Assume that*

(i) *for every* $V \in \mathscr{V}_0$, $\mathbb{P}\{h_V(Y_l, l \in V) \in \mathbb{Z}\} = 1$,
(ii) *for every* $V \in \mathscr{V}_0$, $E|h_V(Y_l, l \in V)|^2 < \infty$,
(iii) $\lim_N |A_N|^{-1} \operatorname{Var}(S_N) = \sigma^2 > 0$.

Then

$$\sup_p \left| \mathbb{P}\{S_N = p\} - \bar{\psi}_N(p) \right| = o(|A_N|^{-1/2}),$$

where

$$\bar{\psi}_N(p) = \omega_N(p)\psi_N(p)$$

with

$$\psi_N(p) = |A_N|^{-1/2}\sigma^{-1}\varphi(x)$$

and k is defined in (1.13) *and*

$$x = |A_N|^{-1/2}\sigma^{-1}(p - ES_N)$$

as well as

$$w_N(p) \triangleq k\mathbb{P}\{S_N \equiv p \bmod k\}.$$

It remains to demonstrate that the case of unequal weights may indeed occur for some window functions. Consider the following example.

EXAMPLE 1.15. In the one-dimensional case choose a particular window in (1.2) which leads to the one-dependent sequence

$$X_j \triangleq 3Y_j + Y_{j+1}, \qquad j \in \mathbb{Z}, Y_j \in \{0, 1\}.$$

Then

$$S_N = 4(Y_1 + \cdots + Y_{N-1}) + Y_N - Y_1.$$

Let $p \triangleq P\{Y_0 = 0\}$. Then we obtain $\mathbb{P}\{S_N \equiv k \mod 4\} = (p^2 + (1-p)^2, p(1-p), 0, p(1-p))$ for $k = 0, 1, 2, 3$, respectively. This type of example generalizes to higher-dimensional windows.

The following result describes the weights $w_N(p)$ more precisely in terms of the distribution of random variables depending on m and the window functions only. Define for $N > m$ and $j \in \mathbb{Z}^d$ the d-vector $j(N)$ via

$$j_\nu(N) \triangleq j_\nu - kN \quad \text{if } kN < j_\nu \le (k+1)N,\ k \in \mathbb{Z},$$

which induces a torus identification on A_N. Let

$$\mathscr{V}_m \triangleq \{V: V \in \mathscr{V}, V \subset A_{N+m}, \forall\, 1 \le \nu \le d\ \exists\, k, l \in V, k_\nu \le N \text{ and } l_\nu > N\}$$

and

$$H_N \triangleq \sum \left\{ h_V(Y_{l(N)}: l \in V): V \in \mathscr{V}_m \right\}.$$

Notice that the distribution of H_N does not depend on $N > m$. We shall write H for H_{m+1}.

Let us illustrate this notation in the Ising window: For $N \ge 2$ we have

$$\mathscr{V}_m = \{j_N + V_\nu, \nu = 1, \dots, d\},$$

where $j_N = (N, \dots, N)$. The random variable H_N equals

$$H_N = \sum_{\nu=1}^{d} Y_{j_N} 1_{\{Y_{j_N} = Y_{j_N - (N-1)e_\nu}\}},$$

which has the same distribution as

$$H = \sum_{\nu=1}^{d} 1_{\{Y_0 = Y_{e_\nu}\}}.$$

For $d = 2$ we have

$$\mathbb{P}\{H = p\} = \left(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\right)$$

for $p = 0, 1, 2$.

THEOREM 1.16. *Under the assumptions of Theorem* 1.14 *we have*:

(i) *For* $N > m$ *there exist numbers* $c_N \in \{0, \dots, k-1\}$ *such that* $S_N \equiv (-1)^d H_N + c_N \mod k$ *a.s.*

(ii) $N_1, N_2 > m$ *with* $N_1 \equiv N_2 \mod k$ *implies* $c_{N_1} = c_{N_2}$. *Therefore the weights* $w_N(p)$ *depend on the residue class of* N *modulo* k *only.*

(iii) *Under the assumptions* (i)–(iii) *of Theorem* 1.14 *the local central limit theorem, i.e.*,

$$\lim_{n\to\infty} |A_N|^{1/2} \sup_p |\mathbb{P}\{S_N = p\} - \psi_N(p)| = 0$$

holds if and only if either $k = 1$ *or* $k \ge 2$ *and*

$$\mathbb{P}\{H \equiv p \mod k\} = \frac{1}{k} \quad \textit{for } p = 0, \dots, k-1.$$

REMARK 1.17. For the one-dimensional case ($d = 1$) and one-dependence ($m = 1$) with $\mathscr{V}_0 = \{\{0, 1\}\}$ we have $H \triangleq h(Y_1, Y_2)$ and, mod k,

$$c_N \equiv \begin{cases} Nc_2/2, & N \text{ even}, \\ (N-3)c_2/2 + c_3, & N \text{ odd}, \end{cases}$$

where

$$c_2 \equiv h(Y_1, Y_2) + h(Y_2, Y_1) \mod k$$

and

$$c_3 \equiv h(Y_1, Y_2) + h(Y_2, Y_3) + h(Y_3, Y_1) \mod k \text{ a.s.}$$

In addition,

$$3c_2 \equiv 2c_3 \mod k.$$

Alternatively, let $a \in \mathbb{Z}/2$ be defined by

$$a \triangleq \arg(E \exp[i 2\pi h(Y_1, Y_2)/k]) \cdot k/2\pi.$$

Then $c_2 = 2a$.

For $m = 1$, $d > 1$ and only one type of "cube window function" $h(Y_j: j \in D_+)$ we can choose $H = h(Y_j: j \in D_+)$ but the computation of c_N yields much more complex formulas.

REMARK 1.18. Although the local CLT does not hold, the CLT and a Berry–Esseen rate of convergence ($O(|A_N|^{-1/2})$) are valid: After integrating the "nonstandard" density $\bar{\psi}_N(p)$, the effect of the weights $w_N(p) \neq 1$ will be a local variation of range $k|A_N|^{-1/2}$ and size $O(|A_N|^{-1/2})$ around the standard normal d.f.

If S_N is the sum of iid lattice random variables and if U is independent of S_N and has a Fourier transform with support $[-\pi, \pi]$, then $S_N + U$ has a d.f. with valid Edgeworth expansion. When the local CLT does hold with weights different from one we obtain nonstandard terms starting at order $O(|A_N|^{-1/2})$ in addition to the usual Edgeworth expansion for $\mathbb{P}\{S_N + U \leq a\}$. Compare Götze and Hipp (1989).

REMARK 1.19. Computing the higher-order approximations in the local CLT, e.g., in the case of Example 1.15, shows that in higher orders we obtain additional interaction terms rather than a pure weighted mixture of Edgeworth expansions on residue classes: For arbitrary N, $q \geq 1$ we have

$$\mathbb{P}\{S_N = 4q\} = p^2\mathbb{P}\{Y_1 + \cdots + Y_N = q\} + (1-p)^2\mathbb{P}\{Y_2 + \cdots + Y_N = q - 1\}.$$

The next result is a local limit theorem for sums of potential functions of a Gibbsian random field. Consider a collection $\mathscr{V}_0$ of window functions satisfying (1.3) and a corresponding collection g_V, $V \in \mathscr{V}_0$, of window functions with values in $(-\infty, \infty)$. Let $\mathscr{V}$ be the corresponding set of translates. Define for finite $K \subset \mathbb{Z}^d$, $x \in \mathbb{R}^K$, $y \in \mathbb{R}^{\mathbb{Z}^d \setminus K}$,

$$\Phi_K(x, y) \triangleq \sum \{g_V(w_V): V \in \mathscr{V}, V \cap K \neq \varnothing\},$$

where

$$w_l \triangleq \begin{cases} x_l & \text{if } l \in K, \\ y_l & \text{if } l \notin K. \end{cases}$$

Let Z_j, $j \in \mathbb{Z}^d$, denote a stationary Gibbsian random field such that for finite $K \subset \mathbb{Z}^d$ and $y \in \mathbb{R}^{\mathbb{Z}^d \setminus K}$ the conditional distribution of Z_K given $Z_j = y_j$, $j \in \mathbb{Z}^d \setminus K$, has a density proportional to

$$x \to \exp[\Phi_K(x, y)],$$

with respect to a product probability measure Q^K defined on $(\mathbb{R}^K, \mathscr{B}^K)$.

Obviously Z_l, $l \in \mathbb{Z}^d$, are no longer independent. They are conditionally independent, i.e., for finite $K_1, K_2 \subset \mathbb{Z}^d$ with distance $d(K_1, K_2) > 2m$, Z_{K_1} and Z_{K_2} are stochastically independent, given $Z_{(K_1 \cup K_2)^c}$.

Furthermore, let Y_j, $j \in \mathbb{Z}^d$, denote an iid random field such that Y_j has distribution Q on $(\mathbb{R}, \mathscr{B})$. For any finite $K \subset \mathbb{Z}^d$ and $A \in \mathscr{B}^K$ the equivalence

$$\mathbb{P}\{Y_K \in A\} = 0 \quad \text{iff } \mathbb{P}\{Z_K \in A\} = 0$$

holds. We shall assume that the distribution Q of Y_j has finite support

$$T \triangleq \{a \in \mathbb{R}: \mathbb{P}\{Y_j = a\} > 0\}.$$

In this case, the equivalence reads

$$\mathbb{P}\{Z_K \in A\} = 0 \quad \text{iff } A \cap T^K = \varnothing.$$

In particular, $\mathbb{P}\{Z_j \in T\} = 1$.

Consider now a second collection of windows $\mathscr{V}_1$ satisfying (1.3), a set of translates $\mathscr{V}^*$ and a corresponding collection h_V, $V \in \mathscr{V}_1$, $h_{V+j} \equiv h_V$, of window functions. To simplify the notation, we shall identify $\mathscr{V}_0$ and $\mathscr{V}_1$ as well as the corresponding sets of translates $\mathscr{V}$ and $\mathscr{V}^*$. Define, as above,

$$S_N \triangleq \sum \{h_V(Z_V): V \in \mathscr{V}, V \subset A_N\}.$$

In this situation, we define the intrinsic span k of the problem as follows: For $y \in T^D$ let

$$R(y) = \sum \{h_V(y_V): V \in \mathscr{V}, V \subset D\}.$$

For $y_j \in T$, $j \in D \setminus D_0$, define $k(y_j,\ j \in D \setminus D_0)$ as the span of the values $R(w)$, $w \in T^D$, with $w_j = y_j$ for $j \in D \setminus D_0$ fixed. Now define

$$k \triangleq \max\{l \in \mathbb{Z}: k(y_j,\ j \in D \setminus D_0) \in l\mathbb{Z} \text{ for all } y \in T^{D \setminus D_0}\}.$$

For our discrete Q, this span coincides with the intrinsic span defined for independent random fields Y_j, $j \in \mathbb{Z}^d$.

THEOREM 1.20. *Assume that the central limit theorem holds for* (S_N), *i.e., there exist* $\mu_N \in \mathbb{R}$, $N \in \mathbb{N}$, *and* $\sigma > 0$ *such that for arbitrary* $r \in \mathbb{R}$,

$$\lim_N \mathbb{P}\{S_N - \mu_N < r\sigma N^{d/2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-x^2/2}\, dx.$$

Assume, furthermore, that $d \leq 2$ *or that for some* $y, y' \in T^D$ *with* $y_j = y'_j$, $j \neq 0$, *we have* $R(y) \neq R(y')$. *Let* k *be the intrinsic span defined above. If* $k = 1$, *then*

$$\sup_p |\mathbb{P}\{S_N = p\} - \varphi_N(p)| = o(N^{-d/2}), \tag{1.21}$$

where

$$\varphi_N(p) = \sigma^{-1}N^{-d/2}\varphi(\sigma^{-1}N^{-d/2}(p - \mu_N)).$$

This theorem improves Theorem 2 of Dobrushin and Tirozzi (1976), page 182, in which (1.21) is derived under the assumption that the central limit theorem holds for (S_N), and that the following condition B is satisfied:

There exists a finite nonvoid cube $V_0 \subset \mathbb{Z}^d$ such that for any $z \in T^{\mathbb{Z}^d}$, the conditional distribution of $\Sigma\{h_V(Z_V),\ V \in \mathscr{V},\ V \cap V_0 \neq \varnothing\}$ given $Z_l = z_l$, $l \in \mathbb{Z}^d \setminus V_0$, has span 1.

Notice that in this condition the span must equal 1 for *all* conditioning vectors z, while our $k = 1$ is satisfied, e.g., when the span of a similar quantity equals 1 for *one* conditioning z. Notice, however, that we consider finite potentials only.

That our condition "$k = 1$" is weaker than condition B can be seen in the following example in which condition B is not satisfied, while $k = 1$ holds.

EXAMPLE 1.22. Consider the case $m = 1$, $d = 2$, $\mathscr{V}_0 = \{\{(0,0),(0,1)\}, \{(0,0),(1,0)\}\}$, and the functions $h_V = h, V \in \mathscr{V}_0$, with $h(x, y) = 1$ if $x = y$ and $h(x, y) = 0$ elsewhere. The intrinsic span k equals 1 in this example: The values of R at the three points with $y_j = 0, j \in D/D_0$, and

$$y_{0,1} = y_{0,-1} = y_{1,0} = y_{-1,0} = 1, \qquad y_{0,0} = 0,$$
$$y_{-1,0} = y_{1,0} = y_{0,-1} = 1, \qquad y_{0,1} = y_{0,0} = 0,$$

and

$$y_{0,0} = y_{0,-1} = 1, \qquad y_{-1,0} = y_{1,0} = y_{0,1} = 0,$$

are 0, 3 and 7, respectively.

We now show that condition B is not satisfied. Let V_0 be an arbitrary finite nonvoid subset of $\mathbb{Z}^2$, and for $y \in \{0,1\}^{\mathbb{Z}^2}$ write

$$W = \sum \{h_V(y_V) : V \in \mathscr{V}_0, V \cap V_0 \neq \varnothing\}.$$

For $j \notin V_0$ let $y_j \in \{0,1\}$ be fixed. Then W, as a function of y_j, $j \in V_0$, has span 2. To see this, fix $j_0 \in V_0$, choose and fix $y_j \in \{0,1\}$, $j \in V_0 \setminus \{j_0\}$, and let w_i be the value of W when $y_{j_0} = i$, $i = 0,1$. Then $w_0 - w_1$ is even which is easily seen by inspection. Notice that $w_0 - w_1$ only depends on y_j, $j \in D + j_0$, $j \neq j_0$. Since $j_0 \in V_0$ was arbitrary, we obtain that for fixed $y_j \in \{0,1\}$, $j \notin V_0$, the function W, as a function of $y_j, j \in V_0$, has all values in $a + 2\mathbb{Z}$, where, e.g., a is the value for W when $y_j = 0$, $j \in V_0$. So W, as a function of y_j, $j \in V_0$, has span 2. This implies that condition B is not satisfied.

For related results of the type "global CLT implies local CLT" for special translation invariant two-point potential functions, see the papers of Campanino, Capocaccia and Tirozzi (1979a, b).

2. Lemmas and proofs.

For $A \subset \mathbb{Z}^d$ define

$$S_A \triangleq \sum \{h_V(Y_V) : V \in \mathscr{V}, V \subset A\}.$$

With this notation, our formerly defined R equals S_D. The following six lemmas are special cases of Lemmas 2.1 to 2.4, 2.12 and 3.1 in Götze and Hipp (1989).

LEMMA 2.1. *Under assumptions* (ii) *and* (iii) *of Theorem* 1.14, *the conditional variance of S_D, given Y_j, $j \neq 0$, cannot be* 0 *almost everywhere.*

As in Götze and Hipp (1983), consider conditional characteristic functions, given certain Y_j. Let

$$u(t) \triangleq E\left|E\left(\exp(itS_D)|Y_j\colon j \neq 0\right)\right|$$

and

$$v(t) \triangleq E\left|E\left(\exp(itS_D)|Y_j\colon j \in D \setminus D_0\right)\right|.$$

Notice that $u(t)$ and $v(t)$ coincide for $d = 1$.

LEMMA 2.2. *Under assumptions* (ii) *and* (iii) *of Theorem* 1.14 *there exist positive constants* ε, α *such that for all* $t_0 \in \mathbb{R}$ *with* $u(t_0) = 1$ *and* $t \in \mathbb{R}$ *with* $|t - t_0| < \varepsilon$, *we have*

$$u(t) \leq \exp\left(-\alpha(t - t_0)^2\right).$$

PROOF. In the proof given in Götze and Hipp (1989) the additional assumption

$$\forall\, V \in \mathscr{V}_0, \qquad E\left|h_V(Y_l, l \in V)\right|^3 < \infty$$

is used. In the following we shall give a proof for the case that only second-order moments are finite.

For $y = (y_j) \in \mathbb{R}^{D\setminus\{0\}}$ write $\sigma^2(y)$ for the conditional variance of S_D, given $Y_j = y_j$, $j \in D \setminus \{0\}$. If $\sigma^2(y) > 0$, then there exists $\varepsilon(y) > 0$ such that for $|t| < \varepsilon(y)$ we have

$$\left|E\left(\exp(itS_D)|Y_j = y_j, j \in D \setminus \{0\}\right)\right| \leq 1 - \varepsilon(y)t^2.$$

Choosing $\varepsilon(y)$ as large as possible, we can achieve that $y \to \varepsilon(y)$ is measurable and positive on $\{\sigma^2(y) > 0\}$. Then we can find a positive constant δ such that

$$\mathbb{P}\left\{\varepsilon\left(Y_j\colon j \in D \setminus \{0\}\right) \geq \delta\right\} \geq \delta.$$

Hence

$$u(t) \leq 1 - \delta + \delta(1 - \delta t^2) \leq \exp(-\delta^2 t^2).$$

Since the left-hand side remains unchanged if we substitute t by $t - t_0$ with $u(t_0) = 1$, this proves the lemma. □

LEMMA 2.3. *When* $v(t) < 1$, *then there exists* $j_0 \in D_0$ *such that*

$$E\left|E\left(\exp(itS_D)|Y_j\colon j \neq j_0\right)\right| < 1.$$

LEMMA 2.4. *There exists a positive constant* β *depending on* d *and* m *only such that for all* $t \in \mathbb{R}$ *and* $N \in \mathbb{N}$,

$$\left|E\exp(itS_N)\right| \leq u(t)^{[\beta N^d]}.$$

Here $[a]$ *denotes the integral part of the number* a.

The proof of this lemma uses the fact that for a maximal subset $A^* \subset A_N$ such that

$$j \in A^* \Rightarrow j + D \subset A_N \text{ and } j, j' \in A^*,\ j \neq j', \text{ imply } (j + D) \cap (j' + D) = \varnothing$$

the sum S_N, conditioned on Y_j, $j \in A_N \setminus A^*$, is a sum of $|A^*|$ independent random variables.

LEMMA 2.5. *$v(t) < \rho < 1$ implies $|E \exp[itS_N]| \le \delta(\rho)^N$ for some $0 < \delta(\rho) < 1$ not depending on N.*

It can be shown via Lemma 2.3 that $v(t) < 1$ implies that there is at least one segment, say S, of size $m^{d-\kappa}N^{\kappa}$, $\kappa \ge 1$, of the m-boundary skeleton of A_N on which

$$\exp\big[\,it\{\text{sum of window functions depending on } S \text{ only}\}\big]$$

is not constant and can be proven to be exponentially small by the method outlined after Lemma 2.4.

LEMMA 2.6. *$v(t_0) = 1$ implies $\exp[it_0 S_N] = \exp[it_0(-1)^d H_N + c_N)]$ a.s., where c_N is a constant.*

PROOF. This result is Lemma 3.1 of Götze and Hipp (1989). Therefore we shall give only a sketch of the proof. Also some of the proofs of our results will use the following notions for cyclic "boundaries." Define for $I \subset \{1, \dots, d\}$, $1 \le i \le d$, and $\alpha = 0, 1$,

$$B_{i,\alpha} \triangleq \{j \in A_{N+m}\colon N - (1 - \alpha)m < j_i \le N + \alpha m\}$$

and

$$\mathscr{T}_I \triangleq \{V \in \mathscr{V}\colon V \cap A_N \neq \varnothing, V \subset A_{N+m}, V \cap B_{i,\alpha} \neq \varnothing \text{ for all } \alpha = 0, 1, i \in I\}.$$

Notice that by definition

$$\mathscr{T}_I \cap \mathscr{T}_J = \mathscr{T}_{I \cup J}.$$

Define

$$H(\mathscr{A}) \triangleq \sum \big\{h_V(Y_{j(N)}\colon j \in V)\colon V \in \mathscr{A}\big\},$$

for any subset $\mathscr{A} \subset \mathscr{V}$. Obviously H is an additive set function. Let $T_I \triangleq H(\mathscr{T}_I)$. Then

$$\{V \in \mathscr{V}\colon V \subset A_N\} = \mathscr{T}_\varnothing \setminus \left(\bigcup_{p=1}^{d} \mathscr{T}_{\{p\}} \right)$$

implies by the inclusion–exclusion principle

$$S_N = \sum (-1)^{|I|} T_I, \tag{2.7}$$

where the sum extends over all subsets $I \subset \{1, \dots, d\}$. Notice that $H_N = T_{\{1, \dots, d\}}$.

Since T_I represents the sum of window functions over an essentially $(d - |I|)$-dimensional "torus strip" we obtain

(2.8) $\exp[it_0 T_I] = \text{const. a.s. for all } |I| < d \text{ provided } v(t_0) = 1.$

In order to prove (2.8) consider first the case $I = \varnothing$. Here we use the fact that $v(t_0) = 1$ implies $u(t_0) = 1$. Together with the stationarity and independence of the random field Y_j, $j \in \mathbb{Z}^d$, this immediately implies that the sum over this d-dimensional torus field is constant modulo $2\pi/t_0$.

For $|I| > 0$ we have to define a quantity similar to $v(t)$ for cubes of Y_j at the "I-boundary" of A_N, say, $v_I(t)$, and show as for $I = \varnothing$ that $v_I(t_0) = 1$ implies (2.8). Finally, $v_I(t_0) < 1$ for $d > |I| > 0$ would imply—similarly as in the proof of Lemma 2.4—that $|E \exp[it_0 T_\varnothing]|$ converges to 0 exponentially in N in contradiction to (2.8) for $I = \varnothing$.

Define

$$\gamma_N \triangleq \sup\Big\{E|h_V(X_V)|^2 1_{\{|h_V(X_V)| < |A_N|^{1/2}\}} \colon V \in \mathscr{V}_0\Big\}.$$

For the expansion of the characteristic functions of m-dependent summands we need the following result.

Lemma 2.9. *Suppose that conditions* (i)–(iii) *of Theorem* 1.14 *hold. Let* $\sigma_N \triangleq \operatorname{Var}(S_N)^{1/2}$ *and* f_N *be the characteristic function* (*Fourier transform*) *of* S_N. *Then we have for every* $|t| \le \varepsilon_N \gamma_N^{-1/2}$, $\varepsilon_N \downarrow 0$,

$$\begin{aligned} &\left| f_N(t\sigma_N^{-1})\exp[-it\sigma_N^{-1}\,\mathrm{E}S_N] - \exp[-t^2/2]\right| \\ &\qquad \le c\gamma_N\{\min(|t|, 1) + (|t|^3 + |t|^6)\exp[-t^2/4]\} + \rho^{N^\alpha}(|t|^2 + |t|^3), \end{aligned}$$

where $0 < \rho < 1$ *and* $c, \alpha > 0$ *are constants.*

A proof can be obtained via Tikhomirov's differential equation method for $f_N'(t\sigma_N^{-1})$ [see Tikhomirov (1980)] and is a special case of the expansion result of Heinrich (1989), Lemmas 1 and 2, adapted for the case of second moments by means of truncation of h_V at $|A_N|^{1/2}$ and using

$$|f_N(t) - f_N^*(t)| \le \gamma_N \min(1, |A_N|^{1/2}|t|),$$

where $f_N^*(t)$ denotes the c.f. of the sum over the truncated window functions.

Proof of Theorem 1.14. By definition of k in (1.13) and from (1.8) we obtain

$$\{t_0 \in \mathbb{R} \colon v(t_0) = 1\} = 2\pi\mathbb{Z}/k.$$

Define the Fourier transform

$$g_N(t) \triangleq \int \exp[itx]\psi_N(x)\,dx.$$

Furthermore, define j_t to be the integer j of *smallest* absolute value such that $|t-jh| \le h/2$, where $h \triangleq 2\pi/k$. Let

$$g_N^*(t) \triangleq (E\exp[ij_t hS_N])g_N(t-j_t h).$$

Since

$$\mathbb{P}(S_N = p) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} f_N(t)\exp[-ipt]\,dt,$$

we will split the periodicity interval $[-\pi,\pi]$ of $f_N(t)$ into the intervals

$$I_l \triangleq \{t: |t-hl| \le h/2\} \cap [-\pi,\pi],$$

for integers $l = -\lfloor k/2\rfloor,\dots,\lfloor k/2\rfloor$, where $\lfloor x\rfloor$ denotes the largest integer smaller than x. Then we have by change of variables

$$\begin{aligned}(2.10)\qquad &\frac{1}{2\pi i}\int_{-\pi}^{\pi} g_N^*(t)\exp[-itp]\,dt\\ &= \sum_l E\Theta_N^l \frac{1}{2\pi i}\int_{I_l} g_N(t-lh)\exp[-i(t-lh)p]\,dt\\ &= E\Big(\sum_l \Theta_N^l\Big)(\psi_N(p) + O(N^{-K})),\end{aligned}$$

where $K>0$ is arbitrarily large and where the random variable $\Theta_N \triangleq \exp[ih(S_N-p)]$ is a kth root of unity.

In the last relation of (2.10) we employed Theorem 1.14(iii) using the inequality

$$(2.11)\qquad |g_N(t)| \le \exp\big[-c\sigma^2|A_N|t^2\big],$$

for some absolute constant $c>0$. Finally, $E\sum_l \Theta_N^l = kE1_{\{\Theta_N=1\}} = w_N(p)$. Notice that the summation $\sum_l$ used here and in (2.10) extends over $l = -\lfloor k/2\rfloor,\dots,\lfloor k/2\rfloor$ for k odd and over $l = -\lfloor k/2\rfloor,\dots,\lfloor k/2\rfloor - 1$ for k even. Hence ψ_N is approximately the Fourier inversion of g_N^* and we obtain by means of (2.10) and (2.11)

$$\begin{aligned}(2.12)\qquad \sup_p|\mathbb{P}(S_N=p) - \bar\psi_N(p)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_N(t) - g_N^*(t)|\,dt + O(N^{-K})\\ &= \sum_l J_l + O(N^{-K}),\end{aligned}$$

say, where J_l denotes the integral over one of the k intervals I_l on the r.h.s. of (2.10). (For k even combine the intervals $I_{-k/2}$ and $I_{k/2}$ of length $h/2$ by periodicity into one interval.) Furthermore, let for some $\varepsilon>0$ and $\mu_N \triangleq (\varepsilon_N\gamma_N)^{-1/2}\uparrow\infty$ as defined in Lemma 2.9

$$I_l = I_{l1}\cup I_{l2}\cup I_{l3},$$

where

$$I_{l1} \triangleq \{t: |t - lh| \le |A_N|^{-1/2}\mu_N\},$$
$$I_{l2} \triangleq \{t: |A_N|^{-1/2}\mu_N < |t - lh| \le \varepsilon\},$$
$$I_{l3} \triangleq \{t: \varepsilon < |t - lh| \le h/2\}$$

and let

$$J_{lq}, q = 1, 2, 3,$$

denote the corresponding integrals over these intervals such that

$$J_l = J_{l1} + J_{l2} + J_{l3}. \tag{2.13}$$

In order to estimate J_{l1} we decompose $f_N(t)$ as follows:

$$f_N(t) = E\exp[i(t - lh)S_N]\exp[ilhS_N]. \tag{2.14}$$

The relation $v(lh) = 1$ implies that $\exp[ilhS_N]$ is equivalent to a $\sigma(Y_j,\ j \notin D_0 + j_0)$-measurable function for *every* j_0 such that $j_0 + D \subset A_N$. Since Y_j are iid, this implies that $\exp[ilhS_N]$ is even equivalent to a $\sigma(Y_j,\ j \in C_{m,N})$-measurable function, where

$$C_{m,N} \triangleq \{j \in A_N: \exists\ v = (v_1, \dots, v_d) \in A_N,$$
$$v_i = 1 \text{ or } v_i = N,\ i = 1, \dots, d,\ \|v - j\| < m\}$$

denotes an m-neighborhood of the vertices of the cube A_N. Let

$$\Delta_N \triangleq \sum \{h_V(Y_V: V \in \mathscr{V}, V \subset A_N, V \cap C_{m,N} \ne \varnothing\}.$$

Then $S_N - \Delta_N$ and $\exp[ilhS_N]$ are independent. For $t \in I_{l1}$ we have by (2.14), $|\exp(x)| \le 1 + O(|x|)$ and $E|\Delta_N| \le c_m$ for some constant c_m not depending on N,

$$f_N(t) = E\exp[i(t - lh)(S_N - \Delta_N)]E\exp[ilhS_N] + O\big(|A_N|^{-1/2}\mu_N\big).$$

Hence

$$\begin{aligned} J_{l1} &\le \int_{I_{l1}} |E\exp[i(t - lh)S_N] - g_N(t - lh)|\,dt + O\big(|A_N|^{-1}\mu_N^2\big) \\ &= o\big(|A_N|^{-1/2}\big), \end{aligned} \tag{2.15}$$

by change of variables $t - lh \to t'$ and application of Lemma 2.9.

Notice that $v(lh) = 1$ implies $u(lh) = 1$. From Lemmas 2.2 and 2.4, (2.11) and (2.14), we obtain for some constants $c_1, c_2, c_3 > 0$,

$$J_{l2} \le c_1 \int_{I_{l2}} \exp\big[-\alpha(t - lh)^2\big] \le c_2\exp\big[-c_3|A_N|^{-2}\mu_N^2N^d\big].$$

Hence

$$\int_{I_{l2}} J_{l2}\,dt = O\big(\mu_N|A_N|^{-1/2}\exp[-c_4\mu_N^2]\big) = o\big(|A_N|^{-1/2}\big). \tag{2.16}$$

Finally, we have by

$$\rho \triangleq \sup\{|v(t)|: \varepsilon < |t - lh| \le h/2\} < 1$$

and Lemma 2.5

$$J_{l3} \le \int_{I_{l3}} \delta^N(\rho)\, dt = o\big(|A_N|^{-1/2}\big).$$

This together with (2.16), (2.15), (2.13) and (2.12) proves the approximation result of Theorem 1.14. □

PROOF OF THEOREM 1.16. (i) Since t_0 with $v(t_0) = 1$ satisfies $t_0 = 2\pi j/k = jh$ for some $j \in \mathbb{Z}$ we obtain from (2.8) for every "torus" sum T_I: $T_I \equiv c_I \bmod k$ a.s. if $|I| < d$, where c_I denotes a constant. From (2.7) we obtain

$$S_N \equiv \sum_{|I|<d} (-1)^{|I|} c_I + (-1)^d H_N \mod k \text{ a.s.,} \tag{2.17}$$

which proves part (i).

(ii) Let us compute c_I inductively for an arbitrary rectangle

$$R_{N_1,\dots,N_d} \triangleq \{1,\dots,N_1\} \times \{1,\dots,N_2\} \times \cdots \times \{1,\dots,N_d\}$$

in $\mathbb{Z}^d$ with $N_j \ge 2m + 1$ instead of the cube A_N. For $|I| < d$ assume w.l.o.g. that $1 \notin I$. We shall consider the two rectangles $R_1 = R_{N_1,\dots,N_d}$ and $R_0 = R_{N_1-1,\dots,N_d}$. Define the "torus field" Y_j^* on R_1 by $Y_j^* \triangleq Y_{(j_1(N_1),\dots,j_d(N_d))}$, $j \in \mathbb{Z}^d$, where

$$j_i(N_i) = \begin{cases} j_i, & \text{if } 0 < j_i \le N_i, \\ j_i - N_i, & \text{elsewhere,} \end{cases}$$

and write Y_j^{**} for the corresponding torus field on R_0. Let $T_I(R_1)$, $T_I(R_0)$, respectively, $c_I(R_1), c_I(R_0)$, denote the corresponding torus sums and constants for these rectangles. We claim that $c_I(R_1) - c_I(R_0)$ is independent of N_1 provided that $N_1 \ge 2m + 2$. This follows from the fact that

$$c_I(R_1) - c_I(R_0) \equiv \sum h_V(Y_V^*) - \sum h_V(Y_V^{**}) \mod k,$$

where in the first sum the summation extends over $V \in \mathscr{V}$ intersecting R_1 and satisfying

$$V \subset R_1' \triangleq R_{N_1+m,\dots,N_d+m}$$

and

$$\forall\, i \in I\ \exists\, l^{(1)}, l^{(2)} \in V\colon N_i - m < l_i^{(1)} \le N_i \text{ and } N_i < l_i^{(2)} \le N_i + m; \tag{2.18}$$

in the second sum the summation extends over $V \in \mathscr{V}$ intersecting R_0 and satisfying $V \subset R_0' \triangleq R_{N_1-1+m,\dots,N_d+m}$ and relation (2.18) with N_1 replaced by $N_1 - 1$. All the terms corresponding to V with $V \subset R_0$ cancel out, and therefore

$$
\begin{aligned}
&c_I(R_1) - c_I(R_0) \\
(2.19)\qquad &\equiv \sum \{h_V(Y_V^*): V \subset R_1' \text{ and } V \cap B_{N_1} \neq \varnothing\} \\
&\quad - \sum \{h_V(Y_V^{**}): V \subset R_0' \text{ and } V \cap B_{N_1 - 1} \neq \varnothing\} \mod k,
\end{aligned}
$$

where B_a denotes the subspace $\{j_1 \geq a\}$ in $\mathbb{Z}^d$. It is easy to see that the number and the types of summands on the r.h.s. of (2.19) do not depend on N_1 provided we exclude "overlapping effects" by assuming $N_1 \geq 2m + 2$. From the iid assumption on Y_j we obtain that the r.h.s. of (2.19) must have identical distribution for $N_1 \geq 2m + 1$ which is constant mod k thus proving our claim.

Iteration of this procedure yields

$$
(2.20)\qquad c_I(R_1) \equiv (N_1 - 2m)(c_I(R_3) - c_I(R_2)) + c_I(R_2) \mod k,
$$

with $R_2 \triangleq R_{2m, N_2, \dots, N_d}$ and $R_3 \triangleq R_{2m+1, N_2, \dots, N_d}$. In order to make the r.h.s. of (2.20) independent of $N_2, \dots, N_d$ we apply the same procedure to the rectangles R_2 and R_3 and the other coordinates in $\{1, \dots, d\} \setminus I$ and obtain

$$
(2.21)\qquad c_I(A_N) \equiv \sum (N - 2m - 1)^{\alpha}(N - 2m)^{\beta} c_{\alpha\beta} \mod k,
$$

where the summation extends over $0 \leq \alpha + \beta \leq d - |I|$ and $c_{\alpha\beta}$ are integer constants depending on sums over functions of Y_j^*, $j \in \mathbb{Z}^d$, and rectangles $R_{2m+\varepsilon_1, \dots, 2m+\varepsilon_d}$, $\varepsilon_j \in \{0, 1\}$, and suitable torus identifications. Hence c_I depends mod k on the residue class of N mod k only thus proving part (ii) via relation (2.17).

(iii) Suppose that the local limit theorem holds. Let p_N denote an integer which is closest to ES_N and is congruent to p modulo k. Then the local CLT implies $\lim_{N\to\infty} w_N(p) = 1$. Since $w_N(p)$ does depend on the residue class of N mod k only this implies $w_N(p) = 1$ for every $p \in \mathbb{Z}$ or $\mathbb{P}(H \equiv p) = 1/k$ for every $p \in \mathbb{Z}$. The converse implication is obvious. □

Proof of Remark 1.17. For $m = d = 1$ and $h_{ij} \triangleq h(Y_i, Y_j)$ we obtain $H_N = h_{N1}$ and $S_N \equiv c_N - h_{N1}$ mod k, where the constant c_N equals $T_N \triangleq h_{12} + h_{23} + \cdots + h_{N1}$ mod k. From (2.20) we obtain that for $N \geq 3$,

$$
(2.22)\qquad c_N \equiv (N - 2)(h_{12} + h_{23} + h_{31}) - (N - 3)(h_{12} + h_{21}) \mod k.
$$

(Notice that $h_{12} + h_{21} = S_2 - H_2 \equiv c_2$.) Since $h_{12} + h_{23} + h_{31}$ and $h_{13} + h_{32} + h_{21}$ have the same distribution and are constant mod k they have to be equal mod k. When N is even we may write $(N - 2)/2$ sums of the $(N - 2)$ sums in (2.22) in the reversed order and obtain from the fact that $h_{ij} + h_{ji}$, $i \neq j$, has the same distribution as $h_{12} + h_{21}$,

$$
\begin{aligned}
c_N &\equiv (N - 2)/2(h_{12} + h_{21} + h_{23} + h_{32} + h_{31} + h_{13}) - (N - 3)(h_{12} + h_{21}) \\
&\equiv N/2 c_2 \mod k.
\end{aligned}
$$

Thus

$$c_N \equiv \begin{cases} Nc_2/2, & N \text{ even}, \\ (N-3)c_2/2 + c_3, & N \text{ odd}. \end{cases} \tag{2.23}$$

From (2.22) and (2.23), applied for $N = 4$, we obtain

$$3c_2 \equiv 2c_3.$$

For $t_0 = 2\pi/k$ and N even $S_N + h_{N1}$ can be split into two parts of $N/2$ summands each such that

$$E\exp\left[it_0 \sum_{j=0}^{N/2} h_{2j+1,2j}\right] = \exp[it_0 c_N] E\exp\left[-it_0\left(\sum_{j=1}^{N/2-1} h_{2j,2j+1} + h_{N1}\right)\right],$$

which implies

$$\exp[it_0 c_N] = \left(E\exp[it_0 h_{12}]/E\exp[-it_0 h_{12}]\right)^N \triangleq \exp[it_0 a]^N, \qquad a \in \{1, \dots, k-1\},$$

which implies $Na \in \mathbb{Z}$ and $c_N \equiv Na \mod k$ for N even. This means, e.g., $h_{12} + h_{21} \equiv 2a \mod k$. Hence $a \in \mathbb{Z}/2$. For N odd write $S_N = S_N + h_{N,N+1} + h_{N+1,1} - h_{N,N+1} + h_{N+1,1}$ to reduce the computation of c_N to the case of N even. This together with (2.23) proves Remark 1.17. □

Proof of Theorem 1.20. Define the Fourier transforms

$$f_N(t) \triangleq E\exp[itS_N]$$

and

$$g_N(t) \triangleq E\exp\left[it\sigma^{-1}N^{-d/2}(S_N - \mu_N)\right].$$

Then

$$\mathbb{P}\{S_N = p\} = (2\pi i\sigma N^{d/2})^{-1}\int_{-\sigma N^{d/2}\pi}^{\sigma N^{d/2}\pi} \exp\left(-it\sigma^{-1}N^{-d/2}(p - \mu_N)\right)g_N(t)\,dt.$$

Furthermore,

$$\varphi_N(p) = (2\pi i\sigma N^{d/2})^{-1}\int_{-\sigma N^{d/2}\pi}^{\sigma N^{d/2}\pi} \exp\left(-it\sigma^{-1}N^{-d/2}(p - \mu_N)\right)e^{-t^2/2}\,dt + o(N^{-d/2}).$$

The global central limit theorem implies that for all $M > 0$,

$$\lim_N \sup_{|t|\le M} \left|g_N(t) - e^{-t^2/2}\right| = 0.$$

Hence it suffices to show that

$$\lim_{M\to\infty}\lim_{N\to\infty}\int_M^{\sigma N^{d/2}\pi} |g_N(t)|\,dt = 0. \tag{2.24}$$

Let S_A, $A \subset \mathbb{Z}^d$, be the sum defined at the start of Section 2. We shall first show that there exists $y, y' \in T^D$ with $y_j = y'_j$, $j \ne 0$, and $R(y) \ne R(y')$. For $d \ge 3$ this was assumed in Theorem 1.20. For $d = 1$ this follows from $u(t) = v(t)$ and our assumption $k = 1$. Consider the case $d = 2$ now and assume that no such pair y, y' exists. Then for $N > m$ the "torus sum"

$$S_N' = \sum \{h_V(Z_{l(N)}: l \in V): V \in \mathscr{V}, V \subset A_{N+m}, V \cap A_N \neq \varnothing\}$$

equals a constant, say, c_N. (For this we use the equivalence of the random fields Z_l, $l \in A_N$, and Y_l, $l \in A_N$.) This implies that for these N the sum S_N equals c_N plus a sum of $O(N)$ "boundary" terms which are uniformly bounded. Hence for these N the random variable $N^{-1}S_N$ is restricted to an interval of the kind

$$[N^{-1}c_N - M, N^{-1}c_N + M]$$

and therefore $N^{-1}(S_N - \mu_N)$ cannot be asymptotically normal.

For $t \in \mathbb{R}$, let f_t be the conditional characteristic function of $R(Z_l: l \in D)$, given Z_j, $j \neq 0$. The random variable f_t depends on Z_l, $l \in D \setminus \{0\}$, only. For $t \in \mathbb{R}$ let h_t be the conditional expectation of $|f_t|$, given Z_l, $l \notin D$. We shall show that there exists $\varepsilon > 0$ such that for $z \in T^{\mathbb{Z}^d \setminus D}$ and $|t| < \varepsilon$,

$$h_t(z) \leq 1 - \varepsilon t^2. \tag{2.25}$$

Fix the above $y \in T^D$. Then there exists $\delta > 0$ such that for $|t| < \delta$,

$$|f_t(y_l: l \in D \setminus \{0\})| \leq 1 - \delta t^2.$$

For $z \in T^{\mathbb{Z}^d \setminus D}$ the conditional probability of $\{Z_l = y_l,\ l \in D \setminus \{0\}\}$, given $Z_l = z_l$, $l \in \mathbb{Z}^d \setminus D$, is positive, and it depends on z_l, $l \in 2D$, only. Hence there is a positive lower bound, say, α, for these conditional probabilities. For arbitrary $z \in T^{\mathbb{Z}^d \setminus D}$ and $|t| < \delta$ we obtain

$$h_t(z) \leq 1 - \alpha + \alpha(1 - \delta t^2) = 1 - \alpha\delta t^2,$$

which is (2.25).

With the help of (2.25) we shall prove that for $|t| < \varepsilon$ and some positive ε_1,

$$F_N(t) \leq \exp(-\varepsilon_1 N^d t^2). \tag{2.26}$$

To this aim we choose a maximal subset A of A_N with the following properties:

(i) $j, j' \in A$, $j \neq j' \Rightarrow \|j - j'\| > 3m$;
(ii) $j \in A \Rightarrow j + D \subset A_N$.

The set A contains at least cN^d elements, where $c > 0$ depends on m and d only. For $t \in \mathbb{R}$ we have

$$\begin{aligned} |f_N(t)| &= |EE(\exp(itS_N)|Z_j: j \notin A)| \\ &\overset{\text{(ii)}}{\leq} E\left|E\left(\prod_{j \in A} \exp(itS_{j+D})|Z_l: l \notin A\right)\right| \overset{\text{c.i.}}{=} E\left|\prod_{j \in A} E(\exp(itS_{j+D})|Z_l: l \neq j)\right| \\ &= E\prod_{j \in A} |E(\exp(itS_{j+D})|Z_l: l \neq j)|. \end{aligned}$$

The relation marked with "c.i." follows from conditional independence. Notice that

$$W_j \triangleq \left| E\left(\exp(itS_{j+D}) | Z_l : l \neq j\right) \right|$$

depends on Z_l: $l \in j + D$, only. Let

$$B_N = \{l \in \mathbb{Z}^d : l \notin j + D \text{ for all } j \in A\}.$$

Given Z_l, $l \in B_N$, the random variables W_j are independent. Hence

$$E \prod_{j \in A} W_j = EE\left(\prod_{j \in A} W_j | Z_l : l \in B_N \right) = E \prod_{j \in A} E(W_j | Z_l : l \in B_N).$$

Stationarity of Z_l, $l \in \mathbb{Z}^d$, conditional independence, and (2.25) imply that

$$E(W_j | Z_l : l \in B_N) = E(W_j | Z_l : l \notin j + D) \leq 1 - \varepsilon t^2,$$

and therefore

$$| f_N(t) | \leq (1 - \varepsilon t^2)^{cN^d} \leq \exp(-c\varepsilon N^d t^2).$$

This is (2.26).

The proof of (2.24) will be complete if we show that

$$\lim_{N \to \infty} \int_{\varepsilon N^{d/2}}^{\sigma N^{d/2} \pi} | g_N(t) | \, dt = 0.$$

This will be done proving that for $|t| \geq \varepsilon_1$,

$$| f_N(t) | \leq \rho^N, \tag{2.27}$$

where $\rho < 1$ does not depend on N. Our assumption "$k = 1$" implies that for some $\rho < 1$, some $j^* \in D_0$ and some $z \in T^{D \setminus \{j^*\}}$ and for $\varepsilon_1 \leq t \leq \pi$,

$$\left| E \exp(itS_D | Z_l = z_l, l \in D \setminus \{j^*\}) \right| \leq \rho.$$

As in the above proof for (2.25) we conclude that for some $\rho_1 < 1$ and all $\varepsilon_1 \leq t \leq \pi$ and all $z \in T^{\mathbb{Z}^d \setminus D}$,

$$E\left(\left| E(\exp(itS_D) | Z_l : l \neq j^*) \right| | Z_l = z_l, l \notin 2D \right) \leq \rho_1. \tag{2.28}$$

For $j^* \neq 0$ the proof for (2.28) is similar to the one given above. For $j^* \neq 0$ we construct a maximal subset B of A_N with the following properties:

(i) $j, j' \in B$, $j \neq j' \Rightarrow \|j - j'\| > 3m$;
(ii) $j \in B \Rightarrow j + D \subset A_N$;
(iii) $j \in B$, $V \in \mathscr{V}$, $V \subset A_N$ and $j + j^* \in V \Rightarrow V \subset j + D$.

Property (iii) can be achieved by choosing $j \in B$ close to the boundary of A_N in the direction j^*. In the case $d = 3$, $m = 1$ and $j^* = (-1, 1, 0)$, we may take as elements of B the points $(2, N - 1, 2k)$, $k = 1, 2, \ldots, (N - 1)/2$. The set B contains at least cN elements, with $c > 0$ not depending on N. As above we obtain

$$| f_N(t) | \leq E \prod_{j \in B} \left| E\left(\exp(itS_{j+D}) | Z_l : l \neq j + j^*\right) \right|.$$

Again let

$$B_N = \{l \in \mathbb{Z}^d : l \notin j + 2D \text{ for all } j \in B\}.$$

Given Z_l, $l \in B$, the random variables

$$W_j' = \left| E\big(\exp(itS_{j+D}) | Z_l : l \neq j + j^*\big) \right|$$

are independent. Hence

$$|f_N(t)| \le EE\bigg(\prod_{j \in B} W_j' | Z_l : l \in B_N\bigg) = E \prod_{j \in B} E(W_j' | Z_l : l \notin j + 2D).$$

Stationarity and (2.28) imply

$$|f_N(t)| \le \rho_1^{cN},$$

which is (2.27). □

REMARK 2.29. Our condition "$k = 1$" is weaker than condition B of Dobrushin and Tirozzi. If $k \neq 1$ then for some $0 < t < 2\pi$, the random variable $\exp(itS_D)$ does not depend on Z_0. Let

$$W = \sum \{h_V(Z_V) : V \in \mathscr{V}, V \cap V_0 \neq \varnothing\}.$$

By stationarity, $\exp(itW)$ does not depend on Z_l, $l \in V_0$, and then $\exp(itW)$, conditioned on Z_l, $l \notin V_0$, is a constant. In this situation, condition B is violated.

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