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THE PROBABILITY OF A LARGE FINITE CLUSTER IN SUPERCRITICAL BERNOULLI PERCOLATION

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We consider standard (Bernoulli) site percolation on $\mathbb{Z}^d$ with probability p for each site to be occupied. C denotes the occupied cluster of the origin and $|C|$ its cardinality. We show that for $p >$ (critical probability of the halfspace $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$) one has $P_p\{|C| = n\} \le \exp\{-C_1(p)n^{(d-1)/d}\}$ for some constant $C_1(p) > 0$. This improves a recent result of Chayes, Chayes and Newman. The proof is based on a Peierls argument which shows exponential decay of the distribution of the size of an "exterior boundary" of C.

1. Introduction and statement of results. We consider standard (Bernoulli) site percolation on $\mathbb{Z}^d$, $d \ge 2$, in which all sites are independently occupied with probability p and vacant with probability $1 - p$. The corresponding probability measure on the configurations of occupied and vacant sites is denoted by P_p. The *cluster* of the vertex x, $C(x)$, consists of all vertices which are connected to x by an occupied path. (An *occupied path* is a nearest-neighbor path on $\mathbb{Z}^d$, all of whose vertices are occupied.) By convention we always include x in $C(x)$, even if x is vacant (in the latter case $C(x) = \{x\}$). For brevity we write C for the cluster of the origin. For any collection A of vertices, $|A|$ denotes the cardinality of A. The *percolation probability* is

$$\theta(p) = P_p\{|C| = \infty\}$$

and the critical probability is

$$p_c = p_c(\mathbb{Z}^d) = \sup\{p : \theta(p) = 0\}.$$

It is well known that $0 < p_c < 1$. Instead of site percolation on $\mathbb{Z}^d$ one can also consider site percolation on the graphs $(\mathbb{Z}_+)^2 \times \{0, \dots, k\}^{d-2}$ or on $H^d := \mathbb{Z}^{d-1} \times \mathbb{Z}_+$ ($\mathbb{Z}_+ = \{0, 1, 2, \dots\}$). In obvious notation we write $p_c(\mathbb{Z}_+^2 \times \{0, \dots, k\}^{d-2})$ and $p_c(H^d)$ for the critical probabilities on these graphs. Finally, [1] introduced

$$\hat{p}_{c,2}^{\infty} = \lim_{k \to \infty} p_c\left(\mathbb{Z}_+^2 \times \{0, \dots, k\}^{d-2}\right).$$

It was recently proved ([3]) that in fact $\hat{p}_{c,2}^{\infty} = p_c(H^d)$.

Aizenman, Delyon and Souillard proved in [2] that for $p > p_c$ there exists a constant $C_1 = C_1(p) < \infty$ such that

$$P_p\{|C| = n\} \ge \exp\{-C_1 n^{(d-1)/d}\}. \tag{1.1}$$

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In [8] Kunz and Souillard proved that an inequality in the other direction holds for p close to 1, while Chayes, Chayes and Newman [5] proved that

$$P_p\{|C| = n\} \le \exp\{-C_2(\log n)^{-1} n^{(d-1)/d}\}, \tag{1.2}$$

for some $C_2 = C_2(p) > 0$ and $p > \hat{p}^{\infty}_{c,2}$. Our main result is that (1.2) can be improved to

$$P_p\{|C| = n\} \le \exp\{-C_2 n^{(d-1)/d}\}, \quad \text{for } p > p_c(H^d). \tag{1.3}$$

We remark that its proof, in Section 3, is almost independent of the other sections.

Of course (1.1) and (1.3) together say that for $p > p_c(H^d)$,

$$n^{-(d-1)/d} \log P_p\{|C| = n\} \text{ is of order } 1. \tag{1.4}$$

It is *conjectured* that

$$\lim_{n\to\infty} n^{-(d-1)/d} \log P_p\{|C| = n\} \text{ exists and lies in } (0,\infty) \quad \text{for } p > p_c(\mathbb{Z}^d).$$

However, this is not proven in general, although Alexander, Chayes and Chayes (private communication) now have a proof when $d = 2$.

While the proof of (1.2) in [5] relies on invasion percolation, our proof of (1.3) is a standard Peierls argument applied to a variant of the exterior boundary of C. Unfortunately, we have to introduce another adjacency relation to define this boundary. We remind the reader that two vertices $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ of $\mathbb{Z}^d$ are $\mathbb{Z}^d$-adjacent if and only if

$$\sum_i |u_i - v_i| = 1.$$

We also consider the graph $\mathbb{L}$ with the same vertex set as $\mathbb{Z}^d$, but with u and v $\mathbb{L}$-adjacent if and only if

$$\max_{1\le i\le d} |u_i - v_i| = 1.$$

Next we define, for integral $k \ge 1$ and $u \in \mathbb{Z}^d$, the cube

$$B_k(u) = \prod_{i=1}^{d} [ku_i, ku_i + k)$$

with "lower left-hand corner" at ku and the "fattened" clusters

$$\tilde{C}(x) = \tilde{C}_k(x) = \{u \in \mathbb{Z}^d : B_k(u) \cap C(x) \ne \varnothing\}.$$

Note that the $B_k(u)$ are disjoint for distinct u; $\tilde{C}_k(x)$ represents in some way the collection of such cubes which intersect $C(x)$. For $k = 1$, $B_k(u)$ consists of the singleton $\{u\}$ only, and $\tilde{C}_1(x)$ can be identified with $C(x)$ itself.

It is useful to think of $\tilde{C}_k(x)$ as a collection of sites of $\mathbb{L}$. Indeed, it is easily seen that $\tilde{C}_k(x)$ is $\mathbb{Z}^d$-connected, and a fortiori $\mathbb{L}$-connected, from the fact that if $u', u'' \in \tilde{C}_k(x)$, then there exist $x' \in B_k(u') \cap C(x)$ and $x'' \in B_k(u'') \cap C(x)$, and these x' and x'' are connected by a path on $\mathbb{Z}^d$ inside $C(x)$. Finally, we define $\Delta_k C(x)$ as the "exterior boundary" on $\mathbb{L}$ of $\tilde{C}_k(x)$. More precisely,

$$\Delta_k C(x) = \left\{ \begin{array}{l} u \in \mathbb{L}: u \notin \tilde{C}_k(x) \text{ but } u \text{ is } \mathbb{L}\text{-adjacent to a vertex } v \in \tilde{C}_k(x) \\ \text{and } u \text{ is connected to } \infty \text{ by a } \mathbb{Z}^d\text{-path outside } \tilde{C}_k(x) \end{array} \right\}.$$

Thus we view $\Delta_k C(x)$ as a subset of $\mathbb{L}$. Note the somewhat strange combination of $\mathbb{L}$-adjacency and $\mathbb{Z}^d$-connectedness in this definition (this is meaningful because $\mathbb{Z}^d$ is obviously a subgraph of $\mathbb{L}$). It seems that this form of the definition is needed for Lemma 1 below. We stress that nothing is said about occupancy or vacancy of the path from u to ∞ in this definition; this would not even make sense for vertices of $\mathbb{L}$. The existence of the path from u to ∞ merely excludes from $\Delta_k C(x)$ points which are completely "surrounded" by $\tilde{C}_k(x)$. Note again the special case $k = 1$. In this case, $\Delta_1 C(x)$ coincides with the exterior boundary on $\mathbb{L}$ of $C(x)$.

Here, finally, are our results.

THEOREM 1. *For all* $0 < p < 1$ *and* $k \geq 1$,

$$\lim_{n\to\infty} -\frac{1}{n}\log P_p\{|\Delta_k C| = n\} = \sigma_k(p)$$

exists and is finite.

Of most interest for us are the situations in which $\sigma_k(p) > 0$. The next theorem shows that this is the case for certain p, k.

THEOREM 2. *For* $d \geq 3$, $p > \hat{p}^{\infty}_{c,2} = p_c(H^d)$ *and for* $d = 2$, $p > p_c(\mathbb{Z}^2)$ *there exists a* $K = K(p, d)$ *such that*

$$\sigma_k(p) > 0 \quad \textit{for } k \geq K.$$

COROLLARY 3. (1.3) *holds for* $d \geq 3$, $p > \hat{p}^{\infty}_{c,2} = p_c(H^d)$ *and for* $d = 2$, $p > p_c(\mathbb{Z}^2)$.

The final result shows that it is necessary to introduce $\tilde{C}_k$ for $k > 1$. It is not enough to work with C and its exterior boundary alone (corresponding to $k = 1$).

THEOREM 4. *If* $d \geq 3$ *and* $p_c(\mathbb{Z}^d) < p < 1 - p_c(\mathbb{Z}^d)$, *then* $\sigma_1(p) = 0$.

NOTE. Throughout this paper C_i are strictly positive finite numbers, which may depend on p, k and d, but not on n. In the sequel we use x, y, z to denote vertices of the original graph $\mathbb{Z}^d$ and we use u, v, w to denote vertices which represent a whole box B_k of $\mathbb{Z}^d$.

2. A subadditivity argument to prove Theorem 1. We shall need the following lemma, which is contained in Lemma (2.23) of [7].

LEMMA 1. *If $C(x)$ is finite, then $\Delta_k C(x)$ is $\mathbb{Z}^d$-connected.*

The main part of the argument is the following subadditivity lemma.

LEMMA 2. *There exists a constant $L = L(k, d)$ and, for all $0 < p < 1$, constants $0 < C_i(p, k, d) < \infty$ such that*

$$P_p\{|\Delta_k C| = m + n + L\} \geq \frac{C_3}{m^d} P_p\{|\Delta_k C| = n\} P_p\{|\Delta_k C| = m\}, \qquad m, n \geq 1, \tag{2.1}$$

$$P_p\{|\Delta_k C| = n + l\} \geq C_4 P_p\{|\Delta_k C| = n\} \quad \text{for } n \geq 1,\ L \leq l \leq L + (2d + 5)3^d. \tag{2.2}$$

Lemma 1 will not be reproven here. It is used in the proof of Lemma 2 (and again in the proof of Theorem 2). We point out here that Theorem 1 follows from (2.1) and (2.2). This implication requires only trivial modifications of standard manipulations with subadditive sequences, as was already pointed out in the proof of Lemma 2 of [4].

The rest of this section is devoted to the proof of Lemma 2. This proof is very similar to that of Lemma 1 in [4]. We construct a configuration with $|\Delta_k C|$ approximately equal to $n + m$ from a configuration with $|\Delta_k C| = n$ and a translate of a configuration with $|\Delta_k C| = m$. We advise the reader to look at [4] for some of the details which we shall skip.

PROOF OF LEMMA 2.

STEP (i). The purpose of this step is to prove (2.9) below. Order the vertices of $\mathbb{Z}^d$ (or $\mathbb{L}$) lexicographically, i.e., set

$$x \succ y \quad \text{if for some } i,\ x_1 = y_1, \ldots, x_i = y_i,\ x_{i+1} > y_{i+1}. \tag{2.3}$$

Next, introduce the event

$$T^-(m, x) = T^-(m, x, k) := \{|\Delta_k C| = m \text{ and the minimal point of } C \text{ is } x\}.$$

Minimality in this definition refers to the ordering (2.3). We need a few simple observations about $T^-(m, x)$. First, if $\Delta C_k \neq \varnothing$, but $|C| = \infty$, then also $|\Delta_k C| = \infty$. [Indeed, if $u \in \Delta_k C$, then there exist infinitely many distinct u^i outside $\tilde{C}_k$ (on a path from u to ∞). If, in addition, there are infinitely many v^i in $\tilde{C}_k$, then each path from u^i to v^i must contain a point of $\Delta_k C$.] In view of this, $T^-(m, x)$ with $1 \leq m < \infty$ implies that $|C|$ is finite and $\Delta_k C$ is connected (cf. Lemma 1). Also $\Delta_k C$ must contain points on the positive and negative jth coordinate axis, $1 \le j \le d$, so that one must have

$$\Delta_k C \subset [-m, m]^d \quad \text{on } T^-(m, x). \tag{2.4}$$

If $u \in \tilde{C}_k$ and C is finite, then among all points $(v_1, u_2, \dots, u_d)$ of $\tilde{C}_k$ consider the one for which v_1 is maximal. Thus $v_1 \ge u_1$ and $(v_1 + 1, v_2, \dots, v_d) \in \Delta_k C$. This argument, applied to each of the positive and negative coordinate directions, together with (2.4), shows that also

$$\tilde{C}_k \subset (-m, m)^d \quad \text{and} \quad C \subset (-km, km + k)^d \quad \text{on } T^-(m, x).$$

In particular, $T^-(m, x)$ can occur only for x's in $(-km, km + k)^d$, and there exists for each $m \ge 1$ an $x(m) = x(m, p, k) \in (-km, km + k)^d$ for which

$$P_p\{T^-(m, x(m))\} \ge (2km + k)^{-d} P_p\{|\Delta_k C| = m\}. \tag{2.5}$$

We fix such an $x(m)$ for the remainder of this proof.

We further note that on $T^-(m, x)$ we even have

$$C \subset [x_1, km + k) \times (-km, km + k)^{d-1}, \tag{2.6}$$

by the minimality of x in C. For any fixed connected set A on $\mathbb{Z}^d$, the event $\{C = A\}$ depends only on the vertices in A or adjacent to A. Since C determines $\tilde{C}_k$ and $\Delta_k C$ this, together with (2.6), shows that $T^-(m, x)$ depends only on the vertices in

$$[x_1 - 1, km + k] \times [-km, km + k]^{d-1}. \tag{2.7}$$

Analogously to $T^-(m, x)$ we define

$$T^+(m, y) = T^+(m, y, k) := \{|\Delta_k C| = m \text{ and the maximal point of } C \text{ is } y\}.$$

This event depends only on the vertices in

$$[-km, y_1 + 1] \times [-km, km + k]^{d-1}. \tag{2.8}$$

Now define $\lceil y - x(m) \rceil_k = (k\lceil (y_1 - x_1(m))/k \rceil, k\lceil (y_2 - x_2(m))/k \rceil, \dots, k\lceil (y_d - x_d(m))/k \rceil)$, where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a, and consider the following two events:

$$T^+(n, y)$$

and

$$F(y) := \{|\Delta_k C(\lceil y - x(m) \rceil_k + 4ke_1)| = m \text{ and the minimal vertex of}$$
$$C(\lceil y - x(m) \rceil_k + 4ke_1) \text{ is } \bar{x} := \lceil y - x(m) \rceil_k + 4ke_1 + x(m)\}$$

(e_1 is the ith coordinate vector). $F(y)$ is the translate by $\lceil y - x(m) \rceil_k + 4ke_1$ of the event $T^-(m, x(m))$. Thus $P_p\{F(y)\} = P_p\{T^-(m, x(m))\}$ and $F(y)$ depends only on the vertices in $[y_1 + 4k - 1, \infty) \times \mathbb{Z}^{d-1}$. This set is disjoint from (2.8) so that $F(y)$ and $T^+(n, y)$ are independent. Therefore, for any $y \in \mathbb{Z}^d$,

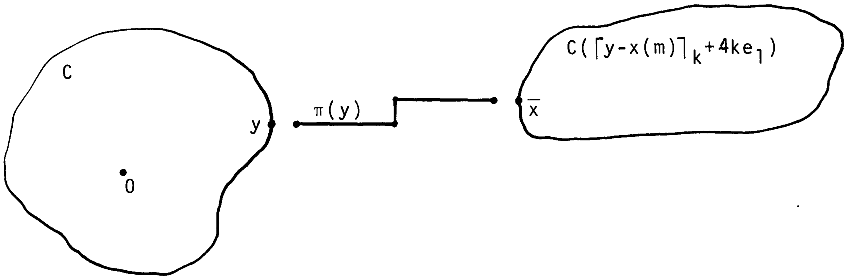

FIG. 1. *Illustration of* $\pi(y)$ *and the construction of* ω^y. $\bar{x}$ *is the point* $\lceil (y - x(m))_k \rceil + x(m) + 4ke_1$.

$n, m \geq 1,$

$$P_p\{T^+(n,y) \text{ and } F(y)\} = P_p\{T^+(n,y)\}P_p\{T^-(m,x(m))\} \tag{2.9}$$
$$\geq \frac{1}{(2km+k)^d} P_p\{T^+(n,y)\}P_p\{|\Delta_k C| = m\}.$$

STEP (ii). We now imitate [4]. Let ω be a configuration for which $T^+(n,y) \cap F(y)$ occurs. We shall modify ω in a "neighborhood of y" to obtain a configuration ω^y for which $|\Delta_k C|$ equals approximately $n + m$. To construct ω^y we first define the path $\pi(y)$ from $y + e_1$ to $\lceil y - x(m) \rceil_k + (4k - 1)e_1 + x(m) = \bar{x} - e_1$: $\pi(y)$ goes first from $y + e_1$ to $y + 2ke_1$ by $(2k - 1)$ successive steps e_1, then successively takes $k\lceil (y_i - x_i(m))/k \rceil - (y_i - x_i(m))$ steps e_i, $1 \leq i \leq d$, and finally $(2k - 1)$ steps e_1 (see Figure 1). ω^y is now obtained from ω by

(2.10) making occupied all the vertices of $\pi(y)$

and

(2.11) making vacant all the vertices which are $\mathbb{Z}^d$-adjacent to a vertex of $\pi(y)$, but do not belong to $C \cup C(\lceil y - x(m) \rceil_k + 4ke_1) \cup \pi(y)$

(see Figure 1).

We shall write $C(\omega)$ and $C(x, \omega)$ for the clusters of the origin and of x, respectively, in the configuration ω. Similarly $C(\omega^y)$ is the cluster of the origin in the new configuration ω^y, $\Delta_k C(\omega)$ is the boundary of $\tilde{C}_k(\omega)$, etc. We conclude this step by proving

$$\tilde{C}_k(\omega^y) = \tilde{C}_k(\omega) \cup \tilde{C}_k(\lceil y - x(m) \rceil_k + 4ke_1, \omega) \tag{2.12}$$
$$\cup \{u : B_k(u) \text{ intersects } \pi(y)\}.$$

(2.12) will be immediate from

$$C(\omega^y) = C(\omega) \cup C(\lceil y - x(m) \rceil_k + 4ke_1, \omega) \cup (\text{vertices in } \pi(y)). \tag{2.13}$$

In turn (2.13) follows from the fact that in ω^y the vertices of $\pi(y)$ are occupied, as are the vertices of $C(\omega)$ (including y) and the vertices of $C(\lceil y - x(m)\rceil_k + 4ke_1, \omega)$ (including $\bar{x}$). Thus, $C(\omega^y)$ certainly contains the right-hand side of (2.13). It is also not difficult to see that $C(\omega^y)$ cannot contain more than the right-hand side of (2.13), by virtue of (2.11). Indeed, if z were a point of $C(\omega^y)$, but not contained in the right-hand side of (2.13), then consider a $\mathbb{Z}^d$-path $z_0 = z, z_1, \dots, z_r = 0$ from z to the origin, inside $C(\omega^y)$. There must be a first index s with

$$(2.14)\quad z_s \in C(\omega) \cup C(\lceil y - x(m)\rceil_k + 4ke_1, \omega) \cup (\text{vertices in } \pi(y)).$$

Since (2.14) does not hold when s is replaced by 0, but does hold when s is replaced by r, we have $0 < s \le r$. But then z_{s-1} is $\mathbb{Z}^d$-adjacent to $C(\omega) \cup C(\lceil y - x(m)\rceil_k + 4ke_1, \omega) \cup \pi(y)$, but does not belong to this set. If z_{s-1} is adjacent to $C(\omega) \cup C(\lceil y - x(m)\rceil_k + 4ke_1, \omega)$ and not in $\pi(y)$, then z_{s-1} is already vacant in ω and remains so in ω^y. Otherwise (2.11) forces z_{s-1} to become vacant in ω^y, which contradicts $z_{s-1} \in C(\omega^y)$. Thus (2.13) and (2.12) hold.

STEP (iii). In this step we show that

$$(2.15)\qquad n + m - 2 \le |\Delta_k C(\omega^y)| \le n + m + (2d + 5)3^d.$$

We begin with the left-hand inequality. Consider a vertex $u \in \Delta_k C(\omega)$ but such that

$$(2.16)\qquad B_k(u) \text{ does not intersect } \pi(y).$$

We claim that necessarily also

$$(2.17)\qquad u \in \Delta_k C(\omega^y).$$

Indeed, u is $\mathbb{L}$-adjacent to some point of $\tilde{C}_k(\omega)$ and hence also to some point of $\tilde{C}_k(\omega^y) \supset \tilde{C}_k(\omega)$. Moreover, since $C(\omega)$ is contained in $(-\infty, y_1] \times \mathbb{Z}^{d-1}$ [cf. (2.8)], we must have $ku_1 \le y_1 + k$, but $u \notin \tilde{C}_k(\omega)$ and u $\mathbb{L}$-adjacent to $\tilde{C}_k(\omega)$. By (2.12) and (2.13) this means that also $B_k(u) \cap C(\omega^y) = \varnothing$ so that $u \notin \tilde{C}_k(\omega^y)$, unless $B_k(u)$ intersects $\pi(y)$. But the latter class of vertices u has been excluded in (2.16), so that still $u \notin \tilde{C}_k(\omega^y)$. Finally, there exists a $\mathbb{Z}^d$-path $u^0 = u, u^1, u^2, \dots$ from u to ∞ outside $\tilde{C}_k(\omega)$. We show that there also must exist such a path outside $\tilde{C}_k(\omega^y)$. This will then prove that $u \in \Delta_k C(\omega^y)$. To find the required path, note that if $ku_1^i + k - 1 \le y_1$ and $u^i \notin \tilde{C}_k(\omega)$, then also $u^i \notin \tilde{C}_k(\omega^y)$, since $\tilde{C}_k(\omega^y) \setminus \tilde{C}_k(\omega)$ contains only vertices v with $kv_1 + k - 1 > y_1$. Thus, if $ku_1^i + k - 1 \le y_1$ for all i, then the path $u^0, u^1, \dots$ itself lies outside $\tilde{C}_k(\omega^y)$. We therefore only have to investigate the case for which there exists a smallest index j such that $ku_1^j + k - 1 > y_1$. First assume $j = 0$. Since $u^0 = u$ is still adjacent to $\tilde{C}_k(\omega)$ we also have $ku_1^0 \le y_1 + k$. Furthermore, (2.16) implies that for some $2 \le i \le d$ we have $y_i < ku_i = ku_i^0$ or $y_i \ge ku_i^0 + k$. For the sake of argument, let $y_2 \ge ku_2^0 + k$ and $ku_1^0 \le y_1 \le ku_1^0 + k - 1$ (see Figure 2). Then the path $u^0 = u$, $u^1 = u + e_1$, $u^i = u + e_1 - (i-1)e_2$, $i \ge 2$, lies outside $\tilde{C}_k(\omega^y)$. Similar paths exist whenever $j = 0$. If $j \ge 1$, then by the minimality of j we must have $u^j - u^{j-1} = e_1$ and either $ku_1^{j-1} \le y_1 \le ku_1^{j-1} + k - 1$ or $ku_1^j \le y_1 \le ku_1^j + k - 1$. Since

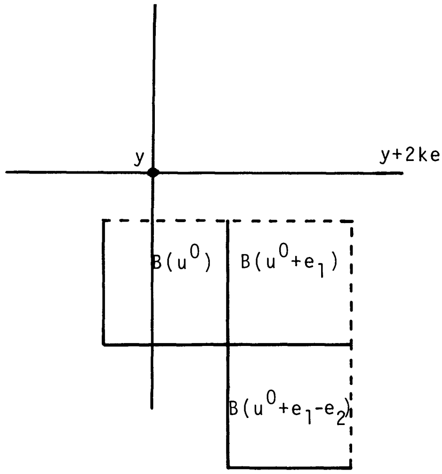

FIG. 2. *Construction of a path from u to ∞ outside* $\tilde{C}_k(\omega^y)$.

$u^{j-1} \notin \tilde{C}_k(\omega)$ and $u^j \notin \tilde{C}_k(\omega)$, we have $y \notin B_k(u^{j-1}) \cup B_k(u^j)$. We therefore have for $l = j - 1$ or j and some $2 \le i \le d$,

$$ku_1^l \le y_1 \le ku_1^l + k - 1 \quad \text{and} \quad \big(y_i < ku_i^l \text{ or } y_i \ge ku_i^l + k\big).$$

We can now continue the path from u^l in the same way as we did in the case $j = 0$. This completes the proof of (2.17) whenever $u \in \Delta_k C(\omega)$ and (2.16) holds.

In the same way we prove that (2.17) holds for all u in $\Delta_k C(\lceil y - x(m)\rceil_k + 4ke_1, \omega)$ which satisfy (2.16). Thus $\Delta_k C(\omega^y)$ contains the sets

$$\Delta_k C(\omega) \quad \text{and} \quad \Delta_k C(y - x(m) + 4ke_1, \omega) \tag{2.18}$$

with the possible exception of points u for which $B_k(u)$ does intersect $\pi(y)$. Since $C(\omega) \subset (-\infty, y_1] \times \mathbb{Z}^{d-1}$ and $C(\lceil y - x(m)\rceil_k + 4ke_1, \omega) \subset [y_1 + 4k, \infty) \times \mathbb{Z}^{d-1}$ the boundaries in (2.18) are disjoint and of size n and m, respectively. [Note that $|\Delta_k C(\lceil y - x(m)\rceil_k + 4ke_1, \omega)| = m$ on $F(y)$, because $F(y)$ is the translate of $T^-(m, x(m))$ by a vector whose coordinates are integral multiples of k.] The left-hand inequality of (2.15) is now obvious.

The right-hand inequality in (2.15) follows similarly from the fact that $u \in \Delta_k C(\omega^y)$ implies $u \in \Delta_k C(\omega)$ or $u \in \Delta_k C(\lceil y - x(m)\rceil_k + 4ke_1, \omega)$ or u is $\mathbb{L}$-adjacent to a vertex v with $B(v) \cap \pi(y) \ne \emptyset$. We leave the details to the reader.

STEP (iv). Still in imitation of [4], we try to reconstruct ω form ω^y. It is not clear that y can be uniquely determined from the configuration ω^y, but we shall show that given ω^y (and m) there are at most $4k$ possible choices for y.

Write $H(r)$ for the half space $[r, \infty) \times \mathbb{Z}^{d-1}$. Now recall that $\Delta_k C(\omega^y)$ and $\Delta_k C(\omega) \cup \Delta_k C([y - x(m)]_k + 4ke_1, \omega)$ differ at most by vertices u which are equal to or are $\mathbb{L}$-adjacent to a vertex v with $B(v) \cap \pi(y) \neq \varnothing$ [cf. Step (iii)]. In particular, the part of $\Delta_k C(\omega^y)$ in $H((y_1 + 5k)/k)$ contains at most $m - 1$ points [since $\Delta_k C([y - x(m)]_k + 4ke_1, \omega)$ has cardinality m and contains $u_0 - e_1$, where u_0 is the unique vertex with $\bar{x} \in B(u_0)$] so that $\Delta_k C(\omega)$ has at most $(m-1)$ points in $H((y_1 + 5k)/k)$. For similar reasons the part of $\Delta_k C(\omega^y)$ in $H((y_1 + k)/k)$ contains all of $\Delta_k C([y - x(m)]_k + yke_1, \omega)$ except $u_0 - e_1$, and it contains $u_0 - e_1 + e_2 \in \Delta_k C(\omega^y)$, so that its cardinality is at least m. Thus, if λ is the smallest integer for which the cardinality of $\Delta_k C(\omega^y) \cap H(\lambda)$ is at most $m - 1$, then

$$y_1 + k < k\lambda \leq y_1 + 5k.$$

Thus there are at most $4k$ choices for $y_1 + 1$. Since there is only one point of $C(\omega^y)$ in the hyperplane $\{u: u_1 = y_1 + 1\}$, namely $y + e_1$, we see that there are at most $4k$ choices for $y + e_1$, and hence also for y_1 (as claimed).

Once we have chosen y_1, there are at most C_5 possible configurations ω which could lead to the given ω^y, where C_5 depends on d and k only, since ω differs from ω^y only on vertices within distance $5k$ form y. We can now proceed more or less as in (3.7) of [4]. Fix n and m, and for any configuration $\tilde{\omega}$, let $N(\tilde{\omega})$ be the number of configurations

$$\omega \in \bigcup_y \{T^+(n, y) \cap F(y)\}$$

for which $\omega^y = \tilde{\omega}$. Then, by the above, $N(\tilde{\omega}) \leq 4kC_5$. Moreover

$$\begin{aligned} &P_p\{n + m - 2 \leq |\Delta_k C| \leq n + m + (2d+5)3^d\} \\ &\quad \geq P_p\{\tilde{\omega}: N(\tilde{\omega}) > 0\} \\ &\quad \geq (4kC_5)^{-1} E_p\{N(\tilde{\omega})\} \\ &\quad \geq (4kC_5)^{-1} \sum_y P_p\{\tilde{\omega}: \tilde{\omega} = \omega^y \text{ for some } \omega \text{ in } T^+(n,y) \cap F(y)\} \\ &\quad \geq C_6 \sum_y P_p\{T^+(n,y) \cap F(y)\}. \end{aligned} \tag{2.19}$$

Here E_p denotes expectation with respect to P_p. In the last inequality we used the fact that

$$\begin{aligned} &P_p\{\tilde{\omega}: \tilde{\omega} = \omega^y \text{ for some } \omega \text{ in } T^+(n,y) \cap F(y)\} \\ &\qquad \geq C_7(p,k,d) P_p\{T^+(n,y) \cap F(y)\} \end{aligned}$$

for some $C_7(p, k, d) > 0$, again because ω^y differs from ω on a bounded number of sites only; $C_6 = C_7(4kC_5)^{-1}$. Finally, we use (2.9) and the identity

$$\{|\Delta_k C| = n\} = \bigcup_y \{T^+(n,y)\} \quad \text{(a disjoint union!)}$$

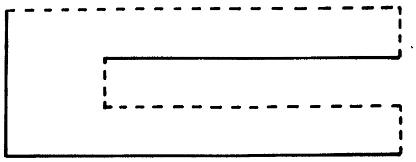

FIG. 3. *A typical $A(k,r)$. The solid lines belong to $A(k,r)$; the dashed lines are in the boundary of $A(k,r)$, but do not belong to $A(k,r)$.*

to obtain from (2.19) that

$$P_p\{n+m-2 \le |\Delta_k C| \le n+m+(2d+5)3^d\}$$
$$\ge \frac{C_8}{(m+1)^d} P_p\{|\Delta_k C| = n\} P_p\{|\Delta_k C| = m\}, \qquad n,m \ge 1,$$

for some $C_8 > 0$, independent of n, m. Finally, this proves that there exists some $j_0 = j_0(n,m,k,p) \in [-2,(2d+5)3^d]$ and a $C_9 = C_9(k,p) > 0$ such that

$$P_p\{|\Delta_k C| = n+m+j_0\} \ge \frac{C_9}{(m+1)^d} P_p\{|\Delta_k C| = n\} P_p\{|\Delta_k C| = m\}. \tag{2.20}$$

STEP (v). We now simultaneously prove (2.2) and complete the proof of (2.1). (2.2) will be proven by attaching [very much as in Steps (ii)–(iv)] to a cluster C with $|\Delta_k C| = n$ some special clusters in such a way that we have good control over the size of the boundary of the combined cluster. The control of this boundary size will rest on (2.21).

Consider the following subsets of $\mathbb{Z}^d$:

$$A(k,r) := [0,kM) \times [-k,2k)^{d-1} \setminus [k(M-r),kM) \times [0,k)^{d-1}$$

for $r = 0,1,\dots,M-1$ with $M = (6d+15)3^d + 3$. (See Figure 3.)

The set of all $u \in \mathbb{Z}^k$ such that $B(u)$ intersects $A(k,r)$ is

$$\tilde{A}(k,r) := [0,M) \times [-1,1]^{d-1} \setminus [M-r,M) \times \{0\}^{d-1}.$$

Its boundary $\Delta_k \tilde{A}(k,r)$ is defined as the set of vertices $\mathbb{L}$-adjacent to $\tilde{A}(k,r)$ but not in $\tilde{A}(k,r)$. Thus

$$\Delta_k \tilde{A}(k,r) = [0,M) \times \left([-2,2]^{d-1} \setminus [-1,1]^{d-1}\right)$$
$$\cup \{-1,M\} \times [-2,2]^{d-1} \cup [M-r,M) \times \{0\}^{d-1}.$$

The cardinality of $\Delta_k \tilde{A}(k,r)$ is written as λ_r. The only fact of importance about these cardinalities is that

$$\lambda_r = \lambda_0 + r, \qquad 0 \le r < M. \tag{2.21}$$

We now define the events

$$F(y,r) = \{C(\lceil y \rceil_k + 4ke_1) = \lceil y \rceil_k + 4ke_1 + A(k,r)\}.$$

This is the translate by $\lceil y \rceil_k + 4ke_1$ of the event $\{C = A(k,r)\}$, and therefore, there exists some constant $C_{10} = C_{10}(p) > 0$ such that

$$P_p\{F(y,r)\} \geq C_{10}, \qquad 0 \leq r < M.$$

A repetition of the argument in Steps (ii)–(iv) with $F(y)$ replaced by $F(y,r)$ and $x(m)$ by 0 shows that there exists some $C_{11} = C_{11}(p) > 0$ such that for each $r < M$ we can find a $j_1 \in [-2, (2d+5)3^d]$ with

$$P_p\{|\Delta_k C| = n + \lambda_r + j_1\} \geq C_{11} C_{10} P_p\{|\Delta_k C| = n\}. \tag{2.22}$$

In principle, j_1 may depend on n and r. However, the construction of steps (ii) and (iii) yields for each ω in $T^+(n,y) \cap F(y,r)$ a configuration ω^y for which $\Delta_k \tilde{C}(\omega^y)$ and

$$\begin{aligned} &\Delta_k C(\omega) \cup \Delta_k C(\lceil y \rceil_k + 4ke_1, \omega) \\ &\quad = \Delta_k C(\omega) \cup (\lceil y_1/k \rceil + 4, \lceil y_2/k \rceil, \dots, \lceil y_d/k \rceil) + \Delta_k \tilde{A}(k,r) \end{aligned} \tag{2.23}$$

differ only by a subset of those vertices u which are $\mathbb{L}$-adjacent to a v with $B(v) \cap \pi(y) \neq \varnothing$, or for which $B(u) \cap \pi(y) \neq \varnothing$. The path $\pi(y)$ runs from $y + e_1$ to $\lceil y \rceil_k + (4k-1)e_1$ and it does not depend on r. The difference between $\Delta_k \tilde{C}(\omega^y)$ and the set (2.23) depends only on $\pi(y)$ and the shape of $\lceil y \rceil_k + 4ke_1 + A(k,r)$ near $\lceil y \rceil_k + 4ke_1$ and is therefore also independent of r. Consequently also $|\Delta_k C(\omega^y)| - n - \lambda_r$ depends on $C(\omega)$, y and n only, but not on r. A small modification of Step (iv) now yields (2.22) for some j_1 independent of r (but possibly dependent on n).

Since $\lambda_r + j_1 = \lambda_0 + r + j_1$ varies from $\lambda_0 + j_1$ to $\lambda_0 + j_1 + M - 1 = \lambda_0 + j_1 + (6d+15)3^d + 2$ as r varies from 0 to $M - 1$, we immediately obtain (2.2) with $L = \lambda_0 + 2(2d+5)3^d$. Finally, (2.1) follows from (2.20) and

$$P_p\{|\Delta_k C| = n + m + L\} \geq C_{11} C_{10} P_p\{|\Delta_k C| = n + m + j_0\},$$

which is just a special case of (2.22) with n replaced by $n + m + j_0$ and $r = 2(2d+5)3^d - j_0 - j_1$. □

3. A Peierls argument to prove Theorem 2. Theorem 2 and Corollary 3 are known for $d = 2$ (cf. [6], Theorems 5.1 and 5.3), so we restrict ourselves to $d \geq 3$ in this section. Recall the definition

$$B_k(u) = \prod_1^d [ku_i, ku_i + k).$$

We now introduce two more boxes, whose linear dimensions are $3k$ and $5k$,

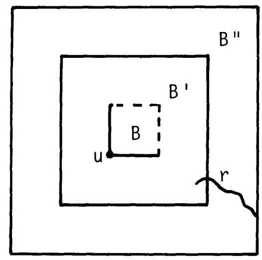

FIG. 4. *The boxes $B = B_k(u)$, $B' = B'_k(u)$ and $B'' = B''_k(u)$ and a path r for the property π_k.*

respectively, and have the same center as $B_k(u)$ (see Figure 4):

$$B'_k(u) = \prod_1^d [ku_i - k, ku_i + 2k],$$

$$B''_k(u) = \prod_1^d [ku_i - 2k, ku_i + 3k].$$

Note that B' and B'' include their full topological boundaries. We shall say that the vertex $u \in \mathbb{L}$ has *property* π_k if there exists an occupied $\mathbb{Z}^d$-path r from $B'_k(u)$ to the topological boundary of $B''_k(u)$ which is not connected by an occupied path in $B''_k(u)$ to $B_k(u)$. Clearly, whether u has property π_k or not depends only on the vertices in $B''_k(u)$.

Consider a vertex u of $\Delta_k C$ and assume $0 \notin B''_k(u)$. Then, by definition there exists a vertex v of $\tilde{C}_k$ which is $\mathbb{L}$-adjacent to u. In particular, $B_k(v) \subset B'_k(u)$, and there exists a vertex x of C in $B_k(v) \subset B'_k(u)$ which is connected by an occupied path r in C to the origin. Since $0 \notin B''_k(u)$, r connects $B'_k(u)$ to the boundary of $B''_k(u)$. On the other hand $B_k(u) \cap C = \varnothing$ (since $u \in \Delta_k C$), so that r cannot be connected to $B_k(u)$. Thus u has property π_k. In other words every $u \in \Delta_k C$ with $0 \notin B''_k(u)$ has property π_k and (by virtue of Lemma 1)

(3.1) $\{|\Delta_k C| = n\} \subset \{\exists\, \mathbb{Z}^d$-connected set of n vertices which intersects the positive and negative first coordinate axes and contains at least $n - 6^d$ vertices with property $\pi_k\}$.

Since the events $\{u$ has property $\pi_k\}$ and $\{v$ has property $\pi_k\}$ are independent as soon as $\max_i |u_i - v_i| \geq 6$, a standard Peierls argument (compare for instance [6], proof of Lemma 5.3) now shows that there exists an $\varepsilon = \varepsilon(d)$ such that

$$P_p\{0 \text{ has property } \pi_k\} < \varepsilon \tag{3.2}$$

will imply

$$P_p\{\text{right-hand side of (3.1) occurs}\} \leq C_1 \exp(-C_2 n) \tag{3.3}$$

for some constants $0 < C_1, C_2 < \infty$. Therefore, Theorem 2 is implied by the following proposition.

PROPOSITION 1. *If $p > \hat{p}^{\infty}_{c,2}$, then (3.2) holds for all sufficiently large k.*

The principal step in the proof is Lemma 3 below, which is basically already in [1]. We shall first show how Proposition 1 follows from this lemma and then indicate the proof of Lemma 3. From now on we have no need of the graph $\mathbb{L}$. "Connectedness" and "path" always refer to $\mathbb{Z}^d$ in the sequel. The notation $A \leftrightarrow B$ ($A \leftrightarrow B$ in C) is used for the event that there exists an occupied path from some vertex in A to some vertex in B (from some vertex in A to some vertex in B and lying in C). ∂B denotes the topological boundary of B.

LEMMA 3. *Let $p > \hat{p}^{\infty}_{c,2}$. Then there exist an integer L and a $\delta > 0$ such that for all $n \geq 2L$ and $x, y \in [0, n]^{d-1} \times [0, L]$,*

$$P_p\{x \leftrightarrow y \text{ in } [0,n]^{d-1} \times [0,L]\} \geq \delta.$$

PROOF OF PROPOSITION 1 FROM LEMMA 3. First we choose K_0 such that for all $k \geq K_0$,

$$P_p\{B_k(0) \text{ is not connected to } \infty\} \leq \varepsilon/2.$$

Such a K_0 exists because $p > \hat{p}^{\infty}_{c,2} \geq p_c(\mathbb{Z}^d)$. If there exists an occupied path from $B_k(0)$ to ∞, then it has a piece r_1, say, in $B''_k(0)$ which connects $B_k(0)$ to $\partial B''_k(0)$. If r is any occupied path from $B'_k(0)$ to $\partial B''_k(0)$, and if $r \leftrightarrow r_1$ in $B''_k(0)$, then $r \leftrightarrow B_k(0)$ in $B''_k(0)$. Therefore, for $k \geq K_0$

$$\begin{aligned} &P_p\{0 \text{ has property } \pi_k\} \\ &\quad \leq \frac{\varepsilon}{2} + P_p\{\exists \text{ occupied paths } r_1 \text{ from } B_k(0) \text{ to } \partial B''_k(0) \\ &\qquad \text{and } r \text{ from } B'_k(0) \text{ to } \partial B''_k(0) \text{ such that } r \\ &\qquad \text{and } r_1 \text{ are not connected in } B''_k(0)\}. \end{aligned} \tag{3.4}$$

We decompose the event in the second term in the right-hand side with respect to the initial points $x_1 \in B_k(0)$ and $x \in B'_k(0)$ of r_1 and r, respectively. There are at most $|B''_k(0)|^2$ possible choices for (x_1, x). The last term in (3.4) is bounded by

$$\sum_{x_1, x} P_p\{x \leftrightarrow \partial B''_k(0),\, x_1 \leftrightarrow \partial B''_k(0) \text{ but } x \text{ is not connected to } x_1 \text{ by an occupied path in } B''_k(0)\}. \tag{3.5}$$

To estimate the summands in (3.5) we introduce some more notation:

$$D_j = [-k - j(L+1), 2k + j(L+1)]^d.$$

The D_j are boxes surrounding $B'_k(0)$. We also define

$$\mathscr{F}_j = \sigma\text{-field generated by the vertices in } D_j$$

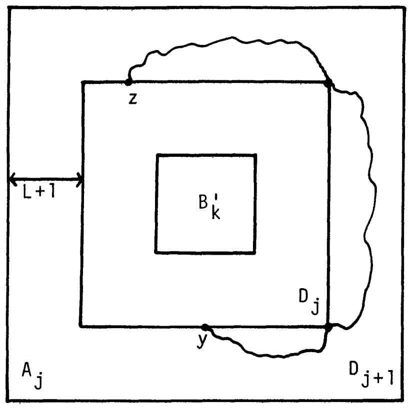

FIG. 5. *The "annulus" A_j and a connection illustrating* (3.6).

and the "annuli"

$$A_j = D_{j+1} \setminus \text{interior of } D_j$$

(see Figure 5). We claim that for all $y, z \in \partial D_j$

$$P_p\{y \leftrightarrow z \text{ in } A_j | \mathscr{F}_j\} \geq \delta^3 \quad \text{on } \{y \text{ and } z \text{ are occupied}\}. \tag{3.6}$$

To prove (3.6) consider first the case where y and z both lie on the same face of D_j, say

$$y_d = z_d = 2k + j(L+1). \tag{3.7}$$

Take $y' = y + e_d$, $z' = z + e_d$ so that

$$y'_d = z'_d = 2k + j(L+1) + 1.$$

Consider the "slab" of thickness L,

$$S := [-k - j(L+1), 2k + j(L+1)]^{d-1} \times [2k + j(L+1) + 1, 2k + (j+1)(L+1)].$$

S is contained in $D_{j+1} \setminus D_j \subset A_j$ and contains y', z' in one of its faces. By Lemma 3

$$P_p\{y' \leftrightarrow z' \text{ in } S\} \geq \delta.$$

Since $S \cap D_j = \varnothing$ we even have

$$P_p\{y' \leftrightarrow z' \text{ in } S | \mathscr{F}_j\} \geq \delta.$$

Clearly $\{y$ and z occupied$\}$ and $\{y' \leftrightarrow z'$ in $S\}$ together imply $y \leftrightarrow z$ in A_j. Consequently (3.6) holds under (3.7). If instead of (3.7)

$$y_d = 2k + j(L+1) \quad \text{and} \quad z_{d-1} = 2k + j(L+1),$$

then connect y' to $t' := (y_1, \dots, y_{d-2}, 2k + j(L+1) + 1, 2k + j(L+1) + 1)$ and t' to $z' := (z_1, \dots, z_{d-2}, 2k + j(L+1) + 1, z_d)$. The conditional probability, given $\mathscr{F}_j$, of both connections existing is at least δ^2 (by the FKG inequality). A glance at Figure 5 should convince the reader that (3.6) holds in all cases.

(3.6) immediately shows that

$$P_p\{x \text{ and } x_1 \text{ are both connected to } \partial D_{j+1} \text{ but not to each other in } D_{j+1} | x \text{ and } x_1 \text{ are both connected to } \partial D_j, \text{ but not to each other in } D_j\} \le (1-\delta^3). \tag{3.8}$$

Indeed, once the configuration in D_j is fixed, we merely pick any y in ∂D_j which is connected to x in D_j and any z in ∂D_j which is connected to x_1 in D_j and apply (3.6) to this y, z. We then have a conditional probability at least δ^3 of connecting y and z, and hence x and x_1 in D_{j+1}. Iteration of (3.8) shows that

$$P_p\{x \text{ and } x_1 \text{ are both connected to } \partial D_j \text{ but not to each other in } D_j\} \le (1-\delta^3)^j.$$

Thus, as long as $D_j \subset B_k''(0)$, or equivalently $j(L+1) \le k$, the summand of (3.5) is bounded by $(1-\delta^3)^j$. Consequently (3.5) is at most

$$\sum_{x_1, x} (1-\delta^3)^{\lfloor k/(L+1) \rfloor} \le (5k+1)^{2d}(1-\delta^3)^{\lfloor k/(L+1) \rfloor},$$

where $\lfloor a \rfloor$ is the largest integer less than or equal to a. Proposition 1 follows. □

PROOF OF LEMMA 3. We only indicate the proof since most ideas are already in Lemma 4.3 of [1]. Assign i.i.d. uniform random variables on $[0,1]$, $U(x)$, to the vertices x of $\mathbb{Z}^d$. Call x t-occupied if $U(x) \le t$. If $p > \hat{p}_{c,2}^{\infty}$ find an L and a $\bar{p}$ such that

$$p > \bar{p} > p_c(\mathbb{Z}_+^2 \times [0, L]^{d-2}).$$

Now let $x, y \in [0, n]^{d-1} \times [0, L]$ for some $n \ge 2L$. Without loss of generality let $0 \le x_i \le n/2$, $1 \le i \le d$, and, for simplicity, assume n even. We shall show that

$$P_p\left\{x \leftrightarrow \left(\frac{n}{2}, x_2, x_3, \dots, x_{d-1}, 0\right) \text{ in } [0,n]^{d-1} \times [0, L]\right\} \ge \theta \tag{3.9}$$

for some $\theta > 0$, independent of x, y, n. The lemma will then follow by combining a number of paths which connect vertices z and z' with $z_i = z_i'$ for all $i \ne j$ and $z_j' = n/2$ for some choice of $j \le d-1$.

To prove (3.9) we observe that there exists some $\theta_1 = \theta_1(\bar{p}) > 0$ such that

$$P_p\left\{\exists\ \bar{p}\text{-occupied path } r \text{ from } x \text{ to } \left\{\frac{n}{2}\right\} \times [x_2, n] \times \prod_3^{d-1} [x_i, x_i + L] \times [0, L] \text{ in } \left[x_1, \frac{n}{2}\right] \times [x_2, n] \times \prod_3^{d-1} [x_i, x_i + L] \times [0, L]\right\} \ge \theta_1 \tag{3.10}$$

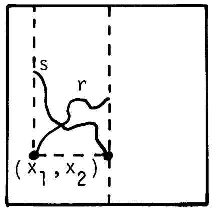

FIG. 6. *Projections of paths r and s in* (3.10) *and* (3.11) *onto the subspace* $\{x: x_i = 0, i \geq 3\}$.

[because $\bar{p} > p_c(\mathbb{Z}_+^2 \times [0, L]^{d-2})$] (see Figure 6). Similarly

$$
\begin{aligned}
&P_p\Big\{\exists\ \bar{p}\text{-occupied path } s \text{ from } \left(\frac{n}{2}, x_2, \dots, x_{d-1}, 0\right) \text{ to} \\
(3.11)\qquad &\{x_1\} \times [x_2, n] \times \prod_3^{d-1} [x_i, x_i + L] \times [0, L] \text{ in} \\
&\left[x_1, \frac{n}{2}\right] \times [x_2, n] \times \prod_3^{d-1} [x_i, x_i + L] \times [0, L]\Big\} \geq \theta_1.
\end{aligned}
$$

The projections of the paths r and s in (3.10) and (3.11) on the two-dimensional subspace $\{x: x_i = 0,\ i \geq 3\}$ must intersect in some point (z_1, z_2), say. Then r contains a point $\alpha = (z_1, z_2, a_3, \dots, a_d)$ and s a point $\beta = (z_1, z_2, b_3, \dots, b_d)$ with $x_i \leq a_i, b_i \leq x_i + L$, $3 \leq i \leq d - 1$ and $0 \leq a_d, b_d \leq L$. Now, when we know for each vertex whether it is $\bar{p}$-occupied or not, then we know whether r, s, α and β exist. If they do, then conditionally on all this information, there is still a conditional probability of at least

$$\left(\frac{p - \bar{p}}{1 - \bar{p}}\right)^{\Sigma\, |a_i - b_i|}$$

that α and β are connected by a p-occupied path in $\{z_1\} \times \{z_2\} \times \prod_3^{d-1} [x_i, x_i + L] \times [0, L]$. If such a connection exists, then x is connected to $(n/2, x_2, \dots, x_{d-1}, 0)$ in $[0, n]^{d-1} \times [0, L]$. Therefore (3.9) holds for

$$\theta = \theta_1^2 \left(\frac{p - \bar{p}}{1 - \bar{p}}\right)^{dL}. \qquad \square$$

PROOF OF COROLLARY 3. We shall restrict ourselves to $d \geq 3$. Fix k such that $\sigma_k(p) > 0$. Then

$$(3.12)\qquad \{|C| = n\} \subset \{k^{-d} n \leq |\tilde{C}_k| \leq n\} \subset \{C_3 n^{(d-1)/d} \leq |\Delta_k C| < \infty\}$$

for some $C_3 > 0$. The last inclusion follows from the fact that the projection of $\tilde{C}_k$ on some coordinate hyperplane $\{u: u_i = 0\}$ must have cardinality greater than or equal to $C_3 n^{(d-1)/d}$ (see [5], Theorem 6). If this holds for $i = 1$, say, then for each point $(0, u_2, \dots, u_d)$ in the projection of $\tilde{C}_k$ onto $\{u_1 = 0\}$ there must be some point $(\bar{u}_1, u_2, \dots, u_d)$ in ΔC_k. (3.12) and Theorems 1 and 2 show that for large n

$$P_p\{|C| = n\} \le \sum_{C_3 n^{(d-1)/d} \le m < \infty} \exp\left(-\tfrac{1}{2}\sigma_k m\right), \tag{3.13}$$

which proves Corollary 1. [Theorem 1 can be avoided in this proof: Some form of (3.13) can be obtained directly from the Peierls argument at the beginning of this section.] □

4. Proof of Theorem 4. This proof is very similar to [2]. By means of the ergodic theorem we show that there is a probability of at least $\frac{1}{2}$ that the cube $S(n) := [-n, n]^d$ contains of the order n^d vertices which are connected to $\partial S(n)$ by an occupied path and which have a neighbor which is connected to $\partial S(n)$ by a vacant path. By some modifications in the boundary layer $S(n+3) \setminus S(n)$ we can then see that

$$\begin{aligned} &P_p\{\exists \text{ finite cluster } C \text{ in } S(n+2) \text{ such that } |C| \text{ and } |\Delta_1 C| \text{ are of order } n^d\} \\ &\quad \ge C_1 \exp(-C_2 |S_{n+2} \setminus S_n|) \\ &\quad \ge C_1 \exp\left(-C_3 n^{d-1}\right). \end{aligned}$$

Replacing n^d by n immediately shows then that $P_p\{|\Delta_1 C| = n\}$ cannot decrease exponentially in n, whence Theorem 4.

Now for some details. We remind the reader that $d \ge 3$ in this entire section. We begin by showing

$$\begin{aligned} &P_p\{0 \leftrightarrow \infty \text{ and some neighbor of } 0 \text{ is connected to } \infty \text{ by a vacant path}\} \\ &\quad > 0, \end{aligned} \tag{4.1}$$

whenever $p_c < p < 1 - p_c$. In fact, since $p > p_c$ we have for all large l

$$P_p\{\partial S_l \leftrightarrow \infty\} \ge \tfrac{3}{4}. \tag{4.2}$$

In the same way, $p < 1 - p_c$ or $P_p\{0 \text{ is vacant}\} = 1 - p > p_c$ implies

$$P_p\{\exists \text{ vacant path from } \partial S_l \text{ to } \infty\} \ge \tfrac{3}{4} \tag{4.3}$$

for large l. (4.2) and (4.3) together show that for some l and $x, y \in \partial S_l$,

$$P_p\{x \leftrightarrow \infty \text{ outside } S_{l-1} \text{ and } \exists \text{ vacant path from } y \text{ to } \infty \text{ outside } S_{l-1}\} \ge \tfrac{1}{2}|\partial S_l|^{-2}.$$

Since there is a strictly positive probability of an occupied path in S_{l-1} from 0 to a neighbor of x and a vacant path in S_{l-1} from a neighbor of 0 to a neighbor of y, one obtains (4.1).

(4.1) and the ergodic theorem show that there exists some $C_3 = C_3(p, d) > 0$ such that for large n,

$$\begin{aligned} &P_p\{\exists \text{ at least } C_3 n^d \text{ points } x \in S(n) \text{ such that } x \leftrightarrow \partial S(n) \text{ and} \\ &\quad \text{there exists a vacant path from a neighbor of } x \text{ to } \partial S(n)\} \ge \tfrac{1}{2}. \end{aligned} \tag{4.4}$$

The endpoints of all the occupied connections in the event in (4.4) are in the union of the $2d$ faces of $S(n)$. By symmetry we may restrict ourselves to endpoints on the "right face"

$$F_n = \{n\} \times [-n, n]^{d-1}.$$

We may even assume that all the occupied paths have an endpoint with u_2 even or with u_2 odd. More precisely, if

$$F_n^e = \{n\} \times \{2k: -n \le 2k \le n\} \times [-n, n]^{d-2}$$

and

$$F_n^0 = \{n\} \times \{2k+1: -n \le 2k+1 \le n\} \times [-n, n]^{d-2},$$

then for $* = e$ or 0

$$\begin{aligned} &P_p\{\exists \text{ at least } (4d)^{-1}C_3 n^d \text{ points } x \in S(n) \text{ such that } x \leftrightarrow F_n^* \text{ in } S(n) \\ &\quad \text{and there exists a vacant path from a neighbor of } x \text{ to } \partial S(n)\} \\ &\quad \ge (8d)^{-1}. \end{aligned} \tag{4.5}$$

For the sake of argument, assume that (4.5) holds for $* = e$. One now would like to connect all occupied points of F_n^e in $S(n+2) \setminus S(n)$ and also would like to connect all vacant points of $\partial S(n)$ by a vacant path to $\partial S(n+3)$ and to make all of $\partial S(n+3)$ vacant. On the event in (4.5) we would then have a cluster $C \subset S(n+2)$ of at least $(4d)^{-1}C_3 n^d$ points x in $S(n)$, such that each x has a neighbor which belongs to $\Delta_1 C$ [since some neighbor of such an x has a vacant path to $\partial S(n+3)$; this path can be continued to ∞ outside $S(n+3)$ and hence outside C]. Thus we would have

$$|\Delta_1 C| \ge (8d^2)^{-1} C_3 n^d.$$

To carry out this construction we need a little care, because some points of F_n^e will be vacant. We can, however, do this as follows. Fix the configuration in $S(n)$. If $(n, 2k, z_3, \dots, z_d)$ is an occupied point of F_n^e, then make also $(n+1, 2k, z_3, \dots, z_d)$ and $(n+2, 2k, z_3, \dots, z_d)$ occupied. In addition, make occupied all points of $F_n^e + 2e_1$ (which lies in $\{x_1 = n+2\}$ and is congruent to F_n^e) and connect these points by an occupied path in $\{n+2\} \times [-n+2, n+2]^{d-1}$ which does not go through any points of the form $\{n+2\} \times \{2l+1\} \times [-n, n]^{d-2}$ with $-n \le 2l+1 \le n$. Figure 7 shows such a path in the plane $x_1 = n+2$ when $d = 3$. Next, if $(n, z_2, \dots, z_d)$ is a vacant point in F_n, with z_2 odd, then make vacant also the points $(n+i, z_2, \dots, z_d)$ for $i = 1, 2, 3$. If z_2 is even, then make vacant the points $(n+1, z_2, \dots, z_d)$, $(n+1, z_2+\eta, z_3, \dots, z_d)$, $(n+2, z_2+\eta, z_3, \dots, z_d)$, $(n+3, z_2+\eta, z_3, \dots, z_d)$ with $\eta = \pm 1$ chosen such that $|z_2 + \eta| \le n$. Finally, make all of $\partial S(n+3)$ vacant and all points in $[-n-3, n-1] \times [-n-3, n+3]^{d-1} \setminus S(n)$ [i.e., in the part of $S(n+3)$ to the "left" of $\{x_1 = n\}$ and outside $S(n)$]. It can be checked that these assignments are consistent. The probability that all vertices in $S(n+3) \setminus S(n)$ have these assigned values is at least

$$[\min(p, 1-p)]^{|S(n+3) \setminus S(n)|} \ge [\min(p, 1-p)]^{C_4 n^{d-1}}.$$

Thus, if the event in (4.5) with $* = e$ occurs, then we have a conditional probability of at least

$$[\min(p, 1-p)]^{C_4 n^{d-1}}$$

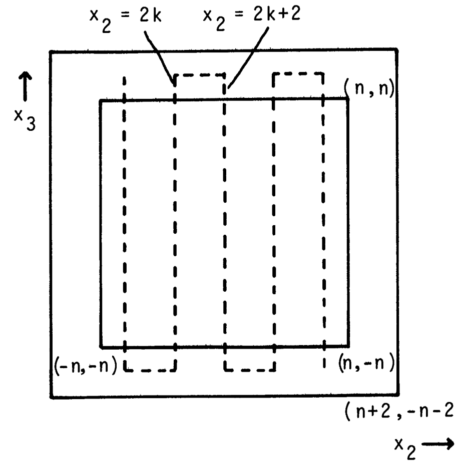

FIG. 7. *A path connecting points with even x_2.*

to have a cluster C inside $S(n+2)$ with $|C| \geq (4d)^{-1}C_3n^d$ and $|\Delta_1 C| \geq (8d^2)^{-1}C_3n^d$. In particular, for some $x \in S(n+2)$ (and hence for all x),

$$P_p\{|\Delta_1 C(x)| \geq (8d^2)^{-1}C_3n^d\} \geq (8d)^{-1}(2n+5)^{-d}[\min(p, 1-p)]^{C_4n^{d-1}}.$$

Note added in proof. Recently G. Grimmett and J. M. Marstrand have proven that all the critical probabilities $p_c(\mathbb{Z}^d)$, $p_c(H^d)$ and $\hat{p}^{\infty}_{c,2}$ are equal. Thus our result (1.3) holds for all $p > p_c(\mathbb{Z}^d)$.

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