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ASYMPTOTIC EXPANSIONS FOR THE EXPECTED VOLUME OF A STABLE SAUSAGE

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Let X_t be a transient stable process on $\mathfrak{R}^d$ and let $T_B = \inf\{t > 0: X_t \in B\}$ be the hitting time of B. Set $E_B(t) = \int P_x(T_B \le t)\,dx$. Asymptotic expansions, as $t \to \infty$, to order 3 are obtained for all stable processes on $\mathfrak{R}$ that are not completely asymmetric and for all strictly stable processes on $\mathfrak{R}^d$, $d \ge 2$, whose transition density at time 1 is not zero at the origin. For those processes that are strongly transient, nontrivial O estimates of the error are also obtained. Expansions to order 2 together with O estimates of the error are given for the completely asymmetric processes on $\mathfrak{R}$, the strictly stable processes on $\mathfrak{R}^d$ whose transition density vanishes at 0 at time 1 and for linear Brownian motion with nonzero mean. Asymptotic expansions to order 3 together with O estimates of the error are given for stable processes with drift on $\mathfrak{R}^d$ having exponent $\alpha < 1$. Expansions to order 3 are also given for stable processes with drift on $\mathfrak{R}^d$ having exponent $\alpha > 1$ when the associated drift free process is isotropic, and expansions to order 2 with O estimate of the error are obtained for the other stable processes with drift on $\mathfrak{R}^d$ having exponent $\alpha > 1$.

1. Introduction. Throughout this paper, X_t will be a stable process on $\mathfrak{R}^d$ that is transient. Recall that the log of the characteristic function of $X_t - X_0$ is of the form $t\psi(\theta)$, where

$$\psi(\theta) = -i\theta \cdot b - \lambda|\theta|^\alpha \int W_\alpha(\theta, \xi)\mu(d\xi), \tag{1.1}$$

with $\lambda > 0$, μ a probability measure on the unit sphere and with

$$\begin{aligned} W_\alpha(\theta, \xi) &= \left[1 - i\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sgn}(\theta\cdot\xi)\right]\left|\frac{\theta}{|\theta|}\cdot\xi\right|^\alpha, \qquad \alpha \ne 1,\\ W_1(\theta, \xi) &= \left(\frac{\theta}{|\theta|}\xi\right) + i\frac{2}{\pi}\left(\frac{\theta}{|\theta|}\right)\ln|\theta\cdot\xi|, \qquad \alpha = 1. \end{aligned} \tag{1.2}$$

We will always assume that μ is not supported on a great circle. In this case, X_t has a bounded continuous density $p(t, x)$ that has bounded continuous derivatives of all orders. Let $\dot p(1, 0)$ be the derivative of $p(1, z)$ at $z = 0$. Then $\dot p(1, 0)z = \sum_{i=1}^d \partial p(1, 0)z_i/\partial x_i$.

If $d = 1$, then μ puts mass p at 1 and mass $q = 1 - p$ at -1. Let

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$\beta = p - q$. We can then write (1.1) as

$$-\psi(\theta) = \lambda|\theta|^{\alpha}(1 - ih\,\mathrm{sgn}(\theta)) + i\theta b \qquad (\alpha \neq 1)$$

$$= \lambda|\theta|\left(1 + i\beta\frac{2}{\pi}\,\mathrm{sgn}(\theta)\ln|\theta|\right) + i\theta b \qquad (\alpha = 1), \tag{1.3}$$

where $h = \beta \tan\left(\frac{\pi\alpha}{2}\right)$.

The constant b is called the *drift term*. If $b = 0$ and $\alpha \neq 1$, then $p(t, x)$ has the following property:

$$p(t,x) = t^{-d/\alpha}p(1, t^{-1/\alpha}x). \tag{1.4}$$

If $\alpha = 1$, $b = 0$ and $\int \xi\mu(d\xi) = 0$, then $p(t, x)$ also satisfies (1.4). Equation (1.4) is called the *scaling property*. A stable process having the scaling property is called *strictly stable*.

Let B be a bounded Borel set and let $T_B = \inf\{t > 0: X_t \in B\}$ be the hitting time of B. There has long been interest in the asymptotic behavior as $t \to \infty$ of the quantity

$$E_B(t) = \int P_x(T_B \leq t)\,dx.$$

There are two distinct interpretations of $E_B(t)$. The process $\{B + X_t\}$ is called a *stable sausage*. The volume of the sausage at time t is the Lebesgue measure of $\bigcup_{s \leq t}[B + X_s]$. The first interpretation of $E_B(t)$ is that it is the expected volume of the sausage by time t.

A second interpretation of $E_B(t)$ is as follows. At time 0, distribute particles on $\mathfrak{R}^d$ as a point process having Lebesgue measure as its intensity measure. Thereafter, let the particles move independently as processes equivalent to X_t. Then $E_B(t)$ is the expected number of distinct particles to hit B by time t.

Early work by Spitzer [7] for Brownian motion and by Getoor [1] for strictly stable processes produced asymptotic expansions of order 2 for $E_B(t)$. Later Port and Stone [4] found an asymptotic expansion of order 2 for X_t any Lévy process. Even in Spitzer's early work, interest is expressed in higher order expansions of $E_B(t)$. In a footnote in [7], Spitzer states that by formal term-by-term inversion of Laplace transforms, Kac obtained the third order term in the expansion when X_t is Brownian motion on $\mathfrak{R}^3$. Just recently, Le Gall [2] investigated the expansion of order 3 when X_t is Brownian motion on $\mathfrak{R}^d$, $d \geq 3$. Le Gall obtains the third order expansions for these Brownian motions. Additionally, for $d = 3$ he obtains a nontrivial estimate of the error. The fact that such an estimate is possible for $d = 3$ seems to depend very much on the processes being Brownian motion on $\mathfrak{R}^3$, for it uses the fact that for this process one can explicitly compute $P_x(T_D \leq t)$ for D a ball.

Our purpose in this paper is to give asymptotic expansions to order 3 for stable processes. We will accomplish this goal for all strictly stable processes on

$\mathfrak{R}^d$ with $p(1,0) > 0$, and for all stable processes on $\mathfrak{R}$ with $|\beta| \neq 1$. For the strictly stable processes with $p(1,0) > 0$ such that $\alpha < d/2$ we will also obtain an O estimate of the error in the expansions to order 3. An O estimate of the error in third order expansions will also be obtained for the stable processes with drift on $\mathfrak{R}$ when $\alpha < 1/2$. For $d \geq 2$ we will obtain third order expansions together with an O estimate of the error for stable processes with drift when $\alpha < 1$. A third order expansion for stable processes with drift on $\mathfrak{R}^d$, $d \geq 2$ will also be given for $\alpha > 1$ when the corresponding drift free process is isotropic. When the corresponding drift free process is nonisotropic, we will obtain an O estimate for the error in a second order expansion for a process with drift with $\alpha > 1$, $d \geq 2$.

For the strictly stable processes with $p(1,0) = 0$, we can only obtain an O estimate of the error in a second order expansion. We also obtain such expansions for stable processes on $\mathfrak{R}$ with drift with $|\beta| = 1$ and for completely asymmetric Cauchy processes on $\mathfrak{R}$.

The methods used here, when applied to Brownian motion, are quite different from Le Gall's.

2. Preliminaries. The potential kernel of the process X_t is $g(x) = \int_0^\infty p(t,x)\,dt$. Let $\phi_B(x) = P_x(T_B < \infty)$. There is a unique measure μ_B supported on $\overline{B}$, called the capactary measure of B, such that $\phi_B(x) = \int g(y-x)\mu_B(dy)$. The total mass of μ_B is $C(B)$. The dual process to X_t is the process $-X_t$. Quantities relating to this process are denoted by $\hat{}$. For example, $\hat{\phi}_B$ is the hitting probability of B for the dual process. Recall the basic fact that $C(B) = \hat{C}(B)$.

On the event $[T_B < \infty]$, define the last hitting time L_B by

$$L_B = \sup\{t > 0\colon X_t \in B\}. \tag{2.1}$$

Let A be a bounded Borel set having Lebesgue measure 1. Set

$$r(t) = \int_t^\infty ds \int_A P_x(X_s \in A)\,dx. \tag{2.2}$$

Our main interest will be in processes such that $r(t)$ is regularly varying at ∞. For such processes, $r(t)$ has the following properties: (i) $r(t)$ is bounded, continuous and decreasing. (ii) For any h, $r(t+h)r(t)^{-1} \to 1$, $t \to \infty$. (iii) For any $a \geq 1$, $\sup_t r(t)r(at)^{-1} < \infty$. Let

$$\hat{P}_{\phi B}(\cdot) = \int \hat{P}_x(\cdot)\phi_B(x)\,dx.$$

Theorems 11.1 and 14.3 of [4] show that whenever $r(t)$ is regularly varying, then for any h and any bounded Borel function f,

$$\begin{aligned}&\int \hat{P}_{\phi B}(t < T_B \leq t+h;\, X_{T_B} \in dz) f(z)\\ &\quad = hC(B)\langle \hat{\mu}_B, f\rangle r(t) + o(r(t)),\end{aligned} \tag{2.3}$$

where $\langle \hat{\mu}_B, f\rangle = \int f(z)\hat{\mu}_B(dz)$. The basis of our approach for obtaining asymptotic expansions to order 3 of $E_B(t)$, when $r(t)$ is regularly varying, is the following lemma that is based on (2.3).

LEMMA 2.1. *Suppose $r(t)$ is regularly varying. Let*

$$G_B(t) = \int_0^t \int \hat{P}_{\phi B}(T_B \in ds,\, X_{T_B} \in dz) \hat{P}^{t-s} \hat{\phi}_B(z).$$

If $\int_0^\infty r(t)\,dt < \infty$ then $G_B(t) = O(r(t))$. If $\int_0^\infty r(t)\,dt = \infty$, then

$$G_B(t) \sim \left(\int_0^t r(s) r(t-s)\, ds \right) C(B)^3.$$

PROOF. By Theorem 12.1 of [4], uniformly in x on compacts,

$$\hat{P}^t \hat{\phi}_B(x) \sim r(t) C(B). \tag{2.4}$$

The lemma now follows from (2.3), (2.4) and the properties of $r(t)$ via routine Abelian arguments. □

Using Theorem 11.2 of [4], we find

$$E_B(t) - tC(B) = \hat{P}_{\phi B}(T_B \le t). \tag{2.5}$$

If $\int_0^\infty r(t)\,dt < \infty$, then $\hat{P}_{\phi B}(T_B < \infty) < \infty$ (see Proposition A14) and we can write the right-hand side of (2.5) as

$$\hat{P}_{\phi B}(T_B < \infty) - \hat{P}_{\phi B}(t < T_B < \infty). \tag{2.6}$$

Since

$$\hat{P}_{\phi B}(t < T_B < \infty) = \int \hat{P}^t \hat{\phi}_B(x) \phi_B(x)\, dx - G_B(t)$$

and

$$\int \hat{P}^t \hat{\phi}_B(x) \phi_B(x)\, dx = \int\int \hat{\mu}_B(da) \mu_B(db) \int_t^\infty (u-t) p(u, b-a)\, du,$$

we can write (2.5) as

$$E_B(t) - tC(B) = \hat{P}_{\phi B}(T_B < \infty) - \int\int \hat{\mu}_B(da)\mu_B(db) \times \int_t^\infty (u-t) p(u, b-a)\, du + G_B(t). \tag{2.7}$$

If $\int_0^\infty r(t)\,dt = \infty$, we write (2.5) as

$$E_B(t) - tC(B) = \hat{P}_{\phi B}(L_B \le t) + \int \hat{P}_{\phi B}(T_B \le t,\, X_t \in dy) \hat{\phi}_B(y).$$

Now

$$\hat{P}_{\phi B}(L_B \le t) = \int\int \hat{\mu}_B(da) \mu_B(db) \int_0^\infty (u \wedge t) p(u, b-a)\, du.$$

Thus, in the case when $\int_0^\infty r(t)\,dt = \infty$, we can write (2.5) as

$$E_B(t) - tC(B) = \int\int \hat{\mu}_B(da) \mu_B(db) \times \int_0^\infty (u \wedge t) p(u, b-a)\, du + G_B(t). \tag{2.8}$$

Now the first order term in the expansion of $E_B(t)$ is $tC(B)$. If $\int_0^\infty r(t)\,dt < \infty$, the second order term is $\hat{P}_{\phi B}(T_B < \infty)$. Appropriate expansions of $p(u, b - a)$ will yield the third order term together with a term of the order $r(t)$. The error term in this case is $O(r(t))$. If $\int_0^\infty r(t)\,dt = \infty$, the third order term is of the order $\int_0^t r(s)r(t-s)\,ds$. In this case, appropriate expansions of $p(u, b-a)$ and (2.8) will yield the second order term and part of the third order term. The remaining part of the third order term comes from the $C(B)^3 \int_0^t r(s)r(t-s)\,ds$ term.

All strictly stable processes on $\mathfrak{R}^d$ with $p(1,0) > 0$ have $r(t)$ regularly varying. As shown by Taylor [8], this is the case for all strictly stable processes except those with $\alpha < 1$ and with μ supported on a closed hemisphere. In particular, on $\mathfrak{R}$, $p(1,0) > 0$ except for the completely asymmetric processes with $\alpha < 1$. Also, on $\mathfrak{R}$ only the completely asymmetric stable processes with drift with $\alpha < 2$, the completely asymmetric Cauchy processes and linear Brownian motion with drift fail to have $r(t)$ regularly varying. All of these exceptional processes have $\int_0^\infty r(t)\,dt < \infty$. For these processes we use (2.7) to show

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + \varepsilon(t),$$

where an O estimate of $\varepsilon(t)$ is obtained.

3. Expansions for strictly stable processes. Throughout this section, X_t will be a strictly stable process. We first consider those processes with $p(1,0) > 0$. Then

$$p(t,x) \sim p(1,0)t^{-d/\alpha} \qquad t \to \infty.$$

If $\alpha < d/2$, $\int_0^\infty r(t)\,dt < \infty$. If $\alpha = d/2$, $r(t) \sim p(1,0)t^{-1}$, and for $\alpha > d/2$, $r(t) \sim ((d/\alpha) - 1)^{-1} p(1,0) t^{1-d/\alpha}$. We need to consider the cases $\alpha < d/2$, $\alpha = d/2$ and $\alpha > d/2$ separately. In Theorems 3.1–3.3, we assume the processes are such that $p(1,0) > 0$.

THEOREM 3.1. *For $\alpha < d/2$ and $p(1,0) > 0$,*

$$\begin{aligned} E_B(t) &= tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx \\ &\quad - C(B)^2 p(1,0)\left[\left(\frac{d}{\alpha} - 2\right)\left(\frac{d}{\alpha} - 1\right)\right]^{-1} t^{2-d/\alpha} \\ &\quad - \left[\left(\frac{d+1}{\alpha} - 2\right)\left(\frac{d+1}{\alpha} - 1\right)\right]^{-1} \\ &\quad \times \int\int \hat{\mu}_B(dx)\mu_B(dy)\dot{p}(1,0)(y-x)t^{2-(d+1)/\alpha} \\ &\quad + O(t^{1-d/\alpha}). \end{aligned}$$

Note that if $\alpha < 1$, $2 - (d+1)/\alpha < 1 - d/\alpha$. In this case, the above shows that

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx$$
$$- C(B)^2 p(1,0)\left[\left(\frac{d}{\alpha} - 2\right)\left(\frac{d}{\alpha} - 1\right)\right]^{-1} t^{2-d/\alpha} + O(t^{1-d/\alpha}).$$

[This expansion is always valid when the process is isotropic since $\dot{p}(1,0) = 0$.]

THEOREM 3.2. *If $\alpha > d/2$ and $p(1,0) > 0$, set*

$$L_B(t) = E_B(t) - tC(B) - C(B)^2 p(1,0)\left[\left(\frac{d}{\alpha} - 1\right)\left(2 - \frac{d}{\alpha}\right)\right]^{-1} t^{2-d/\alpha}$$
$$- C(B)^3 p(1,0)^2 \left[\frac{d}{\alpha} - 1\right]^{-2} \Gamma\left(2 - \frac{d}{\alpha}\right)^2 \left[\Gamma\left(4 - \frac{2d}{\alpha}\right)\right]^{-1} t^{3-2d/\alpha}.$$

If $\alpha > 2d/3$, then

$$L_B(t) = o(t^{3-2d/\alpha}).$$

If $\alpha \le 2d/3$, then

$$H_1(z) = \int_0^\infty u[p(u,0) - p(u,z)]\,du$$

exists,

$$\mu_B H_1 \mu_B = \int\int H_1(y - x)\hat{\mu}_B(dx)\mu_B(dy)$$

exists and, for $d \neq 3$,

$$L_B(t) = -\hat{\mu}_B H_1 \mu_B + o(t^{3-2d/\alpha}),$$

while for $d = 3$,

$$L_B(t) = -\hat{\mu}_B H_1 \mu_B + \left[\left(\frac{4}{\alpha} - 1\right)\left(2 - \frac{4}{\alpha}\right)\right]^{-1}$$
$$\times\left(\int\int \hat{\mu}_B(dx)\mu_B(dy)\dot{p}(1,0)(y - x)\right)t^{2-4/\alpha}$$
$$+ o(t^{3-6/\alpha}).$$

THEOREM 3.3. *For $\alpha = d/2$ and $p(1,0) > 0$,*

$$H_2(z) = \left[\int_1^\infty \frac{p(1,sz)}{s}\,ds + \int_0^1 \frac{p(1,sz) - p(1,0)}{s}\,ds\right]\left(\frac{d}{2}\right)$$

exists for all $z \neq 0$ and

$$\hat{\mu}_B H_2 \mu_B = \int\int \hat{\mu}_B(dx)\mu_B(dy)H_2(y - x)$$

exists. Set

$$L_B(t) = E_B(t) - tC(B) - C(B)^2 p(1,0)[1 + \ln t] - \hat{\mu}_B H_2 \mu_B - 2C(B)^3 p(1,0)^2 \frac{\ln t}{t}.$$

If $d = 1, 2$ *or* 4, $L_B(t) = o((\ln t)/t)$. *If* $d = 3$,

$$L_B(t) = \left[\left(\frac{1}{6}\right)\int\int \hat{\mu}_B(dx)\mu_B(dy)\dot{p}(1,0)(y-x)\right]t^{-2/3} + o\left(\frac{\ln t}{t}\right).$$

REMARK 1. For isotropic processes, the quantities H_1, H_2, $p(1,0)$ and $\int \phi_B(x)^2\,dx$ can be determined explicitly. This is accomplished by Propositions A3–A8 in the Appendix. By (1.4),

$$r(t) \sim \left(\frac{d}{\alpha} - 1\right)^{-1} p(1,0)t^{1-d/\alpha}. \tag{3.1}$$

This fact and Lemma 1.1 will handle the $G_B(t)$ term in (2.7) and (2.8). The strictly stable density $p(1,z)$ has a bounded second derivative. Using the scaling property (1.4) and Taylor's theorem, we can write

$$p(u,z) - p(u,0) = u^{-(d+1)/\alpha}[\dot{p}(1,0)z] + \varepsilon_1(u,z), \tag{3.2}$$

where for some numerical constant M,

$$|\varepsilon_1(u,z)| \le Mu^{-(d+2)/\alpha}|z|^2. \tag{3.3}$$

The proofs of Theorems 3.1–3.3 will be carried out in a sequence of lemmas.

LEMMA 3.1. *If* $\alpha < d/2$,

$$\int_t^\infty (u-t)p(u,z)\,du = p(1,0)\left[\left(\frac{d}{\alpha} - 2\right)\left(\frac{d}{\alpha} - 1\right)\right]^{-1} t^{2-(d/\alpha)} + \left[\left[\left(\frac{d+1}{\alpha} - 2\right)\left(\frac{d+1}{\alpha} - 1\right)\right]^{-1} \dot{p}(1,0)z\right] \times t^{2-(d+1)/\alpha} + \varepsilon_2(t,z),$$

where for a constant M_1,

$$|\varepsilon_2(t,z)| \le M_1|z|^2 t^{2-(d+2)/\alpha}.$$

PROOF. This follows from (1.4), (3.0) and (3.3). □

PROOF OF THEOREM 3.1. By Lemma 1.1, $G_B(t) = O(t^{1-d/\alpha})$. The theorem now follows from Lemma 3.1, Equation (2.7) and the fact that $2 - (d+2)/\alpha \le 1 - d/\alpha$. □

LEMMA 3.2. *If* $\alpha > d/2$,

$$\int_0^\infty (u \wedge t)p(u,0)\,du = p(1,0)\left[\left(\frac{d}{\alpha} - 1\right)\left(2 - \frac{d}{\alpha}\right)\right]^{-1} t^{2-d/\alpha}.$$

PROOF. This follows at once from the fact that $p(u,0) = u^{-d/\alpha}p(1,0)$. □

LEMMA 3.3. *If* $\alpha > d/2$,

$$t\int_t^\infty [p(u,z) - p(u,0)]\,du = \left[\left(\frac{d+1}{\alpha} - 1\right)^{-1} \dot{p}(1,0)z\right]t^{2-(d+1)/\alpha} + O(|z|^2t^2 - (d+2)/\alpha).$$

PROOF. This follows at once from (1.4), (3.2) and (3.3). □

LEMMA 3.4. *Suppose* $\alpha > d/2$. *If* $d = 1$ *or* 3, *or* $d = 2$ *and* $\alpha < 3/2$, *then*

$$H_1(z) = \int_0^\infty u[p(u,0) - p(u,z)]\,du \tag{3.4}$$

exists, $\hat{\mu}_B H_1 \mu_B = \int\int \hat{\mu}_B(dx)\mu_B(dy)H_1(y-x)$ *exists and*

$$\begin{aligned}\int_0^t u[p(u,0) - p(u,z)]\,du &= H_1(z) + \left[(2-(d+1)/\alpha)^{-1}\dot{p}(1,0)z\right]t^{2-(d+1)/\alpha} \\ &\quad + O\left(|z|^2t^{2-(d+2)/\alpha}\right).\end{aligned} \tag{3.5}$$

PROOF. Using (1.4), we see

$$u[p(u,0) - p(u,z)] = u^{1-d/\alpha}[p(1,0) - p(1,u^{-1/\alpha}z)].$$

Since $\sup_z p(1,z) < \infty$, it follows that, whenever $\alpha > d/2$,

$$\int_0^1 u|p(u,0) - p(u,z)|\,du \le c_1,$$

for some constant c_1. For $d = 1$, $d = 2$ and $\alpha < 3/2$, or $d = 3$ and $\alpha < 2$, $u^{1-(d+1)/\alpha}$ is integrable on $(1,\infty)$. Since $|\dot{p}(1,z)|$ is bounded, it follows from (3.2) and (3.3) that for these cases

$$\int_1^\infty u|p(u,0) - p(u,z)|\,du \le c_2|z|,$$

for some constant c_2. If $d = 3$, $\alpha = 2$, then for some constant c_3,

$$u|p(1,0) - p(1,u^{-1/2}z)| \le c_3u^{-3/2}|z|^2.$$

Since $u^{-3/2}$ is integrable on $(1,\infty)$, it follows that for this process

$$\int_1^\infty u|p(u,0) - p(u,z)|\,du \le c_4|z|^2$$

for some constant c_4. Since $\hat{\mu}_B(dx)\mu_B(dy)$ is a finite measure with compact support, it follows that

$$\int\int \hat{\mu}_B(dx)\mu_B(dy)\int_0^\infty u|p(u,0) - p(u,y-x)|\,du < \infty.$$

Equation (3.5) now follows from (3.2), (3.3) and the equation

$$\int_0^t u[p(u,0) - p(u,z)]\,du = H_1(z) - \int_t^\infty u[p(u,0) - p(u,z)]\,du. \qquad \square$$

LEMMA 3.5. *If $d = 2$ and $\alpha = 3/2$,*

$$\int_0^t u[p(u,z) - p(u,0)]\,du = [\dot{p}(1,0)z]\ln t + O(1).$$

PROOF. Write

$$\int_0^t u[p(u,z) - p(u,0)]\,du = \int_0^1 u[p(u,z) - p(u,0)]\,du + \int_1^t u[p(u,z) - p(u,0)]\,du$$

and use (3.2) and (3.3). $\square$

LEMMA 3.6. *If $d = 2$ and $\alpha > 3/2$,*

$$\int_0^t u[p(u,z) - p(u,0)]\,du = \left[\left(2 - \frac{3}{\alpha}\right)^{-1} \dot{p}(1,0)z\right] t^{2-3/\alpha} + O(1).$$

PROOF. Use (3.2) and (3.3). $\square$

LEMMA 3.7. *If $\alpha > d/2$,*

$$G_B(t) \sim C(B)^3 p(1,0)^2[(d/\alpha) - 1]^{-2}\Gamma(4 - 2(d/\alpha))^{-1} \times\Gamma(2 - (d/\alpha))^2 t^{3-2(d/\alpha)}.$$

PROOF. This follows from Lemma 2.1, (3.1) and a well known Abelian theorem on convolutions. $\square$

PROOF OF THEOREM 3.2. Note that for $d = 1$ or 2, $2 - (d+1)/\alpha < 3 - 2(d/\alpha)$, while for $d = 3$, $2 - (d+1)/\alpha > 3 - 2(d/\alpha)$. Also, if $\alpha > 2d/3$, $t^{3-2(d/\alpha)} \to \infty$ as $t \to \infty$. These observations together with Lemmas 3.3–3.6 and (2.8) suffice to establish the theorem. $\square$

LEMMA 3.8. *Let $\alpha = d/2$ and let $H_2'(z) = (d/2)\int_1^\infty [p(1, uz)/u]\,du$. Then*

$$\hat{\mu}_B H_2' \mu_B = \int\int H_2'(y - x)\hat{\mu}_B(dx)\mu_B(dy) < \infty.$$

Also, for $H_2''(z) = (d/2)\int_0^1([p(1, uz) - p(1, 0)]/u)\,du$, $H_2''(y - x)$ is $\hat{\mu}_B(dx)\mu_B(dy)$ integrable.

PROOF. Since $\hat{P}_{\phi B}(L_B \le t) < \infty$ for all t, we find that

$$\int_0^t u\,du \int\int \hat{\mu}_B(dx)\mu_B(dy)p(u, y - x) < \infty.$$

Now using (1.4) and the change of variable $s = u^{-2/d}$, we find

$$\left(\frac{d}{2}\right)\int\int \hat{\mu}_B(dx)\mu_B(dy)\int_{t^{-2/d}}^{\infty}\frac{1}{s}p(u, s(y-x))\,dx < \infty.$$

Taking $t = 1$ shows $\hat{\mu}_B H_2'\mu < \infty$. Since $|\dot{p}(1,z)|$ is bounded,

$$\int_0^1 |p(1,uz) - p(1,0)|\frac{du}{u} \le \Big(\sup_z |\dot{p}(1,z)|\Big)|z|,$$

so $H_2''(y - x)$ is integrable as claimed. □

LEMMA 3.9. *If* $\alpha = d/2$,

$$t\int_t^{\infty} p(u,z)\,du = p(1,0) + \left[\left(\frac{2}{d} + 1\right)^{-1}\dot{p}(1,0)z\right]t^{-2/d} + O(|z|t^{-4/d}).$$

PROOF. This follows at once from (1.4) and (3.2). □

LEMMA 3.10. *If* $\alpha = d/2$,

$$\begin{aligned}\int\int \hat{\mu}_B(dx)\mu_B(dy)\int_0^t up(u, y - x)\,du \\ &= \hat{\mu}_B H_2\mu_B - C(B)^2 p(1,0)\ln t \\ &\quad - \left[\frac{d}{2}\int\int \hat{\mu}_B(dx)\mu_B(dy)\dot{p}(1,0)(y-x)\right]t^{-2/d} + O(t^{-4/d}).\end{aligned}$$

PROOF. Note that for $t > 1$,

$$\begin{aligned}\int_0^t up(u,z)\,du &= \left(\frac{d}{2}\right)\int_{t^{-2/d}}^{\infty} s^{-1}p(1,sz)\,ds \\ &= \left(\frac{d}{2}\right)\int_1^{\infty} s^{-1}p(1,sz)\,ds + \left(\frac{d}{2}\right)\int_0^1 s^{-1}[p(1,sz) - p(1,0)]\,ds \\ &\quad + p(1,0)\ln t - \frac{d}{2}\int_0^{t^{-2/d}}\frac{1}{s}[p(1,sz) - p(1,0)]\,ds.\end{aligned}$$

The lemma now follows from Lemma 3.8 and a Taylor series expansion of the integrand of the last integral on the right to order 2. □

LEMMA 3.11. *If* $\alpha = d/2$,

$$G_B(t) \sim 2C(B)^3 p(1,0)^2 t^{-1}\ln t.$$

PROOF. Here $r(t) \sim p(1,0)t^{-1}$. A routine Abelian argument shows that

$$\int_0^t r(s)r(t-s)\,ds \sim 2p(1,0)^2 t^{-1}\ln t.$$

The lemma now follows from this fact and Lemma 2.1. □

PROOF OF THEOREM 3.3. If $d = 1, 2$, $(\ln t)/t > t^{-2/d}$ or if $d = 3$, $t^{-2/3} > (\ln t)/t$. If $d = 4$, $\dot{p}(1,0) = 0$. The theorem follows from these observations, Lemmas 3.8–3.11 and (2.7). □

We will now consider those strictly stable processes with $p(1,0) = 0$. As shown by Taylor [8], this can happen iff $\alpha < 1$ and μ lies in some closed hemisphere. In this case, there is a closed convex cone k with vertex 0 such that $p(1,x) = 0$ for all $x \notin k$. Since $p(1,x)$ is C^∞, it follows that $p(1,x)$ and all its derivatives vanish at 0. Using (1.4) we see that, uniformly in z on compacts,

$$\lim_{t\to\infty} t^{-n}p(t,z) = 0$$

for any positive n. Consequently, by (2.7), for all of these processes, $\int_0^\infty r(t)\,dt < \infty$ and the following holds.

THEOREM 3.4. *Assume* $p(1,0) = 0$. *Then*

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + \varepsilon(t),$$

where $\varepsilon(t) = o(t^{-n})$ *for any positive* n.

4. Expansion for stable processes with drift on $\mathfrak{R}$ with $\alpha < 1$. For stable processes on $\mathfrak{R}$ with $\alpha \neq 1$ we can write

$$\psi(\theta) = -i\theta b - \lambda|\theta|^\alpha(1 - ih\,\mathrm{sgn}(\theta)), \tag{4.1}$$

where $h = \beta\tan(\pi\alpha/2)$ and $|\beta| \le 1$. In this section, we will give the expansion of $E_B(t)$ for such processes with $b \neq 0$. We need only consider the case $b > 0$. The results for $b < 0$ follow from these by replacing β with $-\beta$ and b with $|b|$ in the corresponding formulas. We assume $|\beta| < 1$. As usual, $p(t,x)$ is the transition density. We let $f(x)$ be the density of the corresponding drift free process at time $t = 1$. Then

$$p(t,x) = t^{-1/\alpha}f\big(t^{-1/\alpha}(x + bt)\big). \tag{4.2}$$

THEOREM 4.1. *Assume* $|\beta| < 1$. *If* $\alpha < 1/2$,

$$\begin{aligned} E_B(t) = {} & tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx \\ & - C(B)^2\left[\left[\left(\frac{1}{\alpha} - 2\right)\left(\frac{1}{\alpha} - 1\right)\right]^{-1} f(0)\right]t^{2-1/\alpha} \\ & + O(t^{1-1/\alpha}). \end{aligned} \tag{4.3}$$

If $\alpha > 1/2$, *set*

$$L_B(t) = E_B(t) - tC(B) - C(B)^2\left[\left(\frac{1}{\alpha} - 1\right)\left(2 - \frac{1}{\alpha}\right)\right]^{-1} f(0)t^{2-1/\alpha}.$$

Then for $1/2 < \alpha < 2/3$,

$$H_3(z) = \int_0^\infty u\left[p(u,z) - u^{-1/\alpha}f(0)\right] du$$

exists,

$$\hat{\mu}_B H_3 \mu_B = \int\int \hat{\mu}_B(dx)\mu_B(dy)H_3(y-x)$$

exists and

$$\begin{aligned} L_B(t) = \hat{\mu}_B H_3 \mu_B - C(B)^2&\left[\left[\left(\frac{2}{\alpha} - 2\right)\left(3 - \frac{2}{\alpha}\right)\right]^{-1} f'(0)b\right. \\ &\left. + C(B)f(0)^2\left(\frac{1}{\alpha} - 1\right)^{-2}\Gamma\left(4 - \frac{2}{\alpha}\right)^{-1}\Gamma\left(2 - \frac{1}{\alpha}\right)^2\right]t^{3-2/\alpha} \\ &+ o(t^{3-2/\alpha}), \end{aligned} \tag{4.4}$$

where

$$\hat{\mu}_B H_3 \mu_B = \int\int \hat{\mu}_B(dx)\mu_B(dy)H_3(y-x).$$

If $\alpha = 2/3$, *then*

$$\begin{aligned} H_4(z) = &\int_0^1 u\left[p(u,z) - u^{-3/2}f(0)\right] du \\ &+ \int_1^\infty u\left[p(u,z) - u^{-3/2}f(0) - u^{-3}bf'(0)\right] du \end{aligned}$$

exists and is $\hat{\mu}_B(dx)\mu_B(dy)$ *integrable and*

$$L_B(t) = C(B)^2 f'(0)b[1 + \ln t] + 4\pi C(B)^3 f(0)^2 + \hat{\mu}_B H_4 \mu_B + o(1), \tag{4.5}$$

where $\hat{\mu}_B H_4 \mu_B = \int\int \hat{\mu}_B(dx)\mu_B(dy)H_4(y-x)$. *If* $\alpha > 2/3$,

$$\begin{aligned} L_B(t) = C(B)^2&\left[\left[\left(\frac{2}{\alpha} - 2\right)\left(3 - \frac{2}{\alpha}\right)\right]^{-1} f'(0)b\right. \\ &\left. + C(B)f(0)^2\left(\frac{1}{\alpha} - 1\right)^{-2}\Gamma\left(4 - \frac{2}{\alpha}\right)^{-1}\Gamma\left(2 - \frac{1}{\alpha}\right)^2\right]t^{3-2/\alpha} + o(t^{3-2/\alpha}). \end{aligned} \tag{4.6}$$

If $\alpha = 1/2$, *then*

$$H_5(z) = \int_1^\infty s^{-1}f\left(s\left(1 + \frac{zs}{b^2}\right)\right) + \int_0^1 s^{-1}\left[f\left(\left(1 + \frac{zs}{b^2}\right)s\right) - f(0)\right] ds$$

exists and is such that

$$\hat{\mu}_B H_5 \mu_B = \int\int \hat{\mu}_B(dx)\mu_B(dy)H_5(y-x)$$

exists and

$$E_B(t) = tC(B) + \hat{\mu}_B H_5 \mu_B + C(B)^2[f(0)\ln t - f(0)\ln b + f(0)] + 2C(B)^3 p(1,0)^2 \frac{\ln t}{t} + o\left(\frac{\ln t}{t}\right).$$

REMARK 2. Propositions A11 and A12 in the Appendix explicitly determine the quantities $f(0)$, $f'(0)$ and $\int \hat{\phi}_{\{0\}}(x)\phi_{\{0\}}(x)\,dx$.

LEMMA 4.1. *For any* z *and* $u > 0$,

$$p(u,z) - f(0)u^{-1/\alpha} = u^{1-2/\alpha}f'(0)(b - z/u) + \varepsilon(z,u), \tag{4.7}$$

where

$$|\varepsilon(z,u)| \le u^{2-3/\alpha}|b + z/u|^2 \sup_x |f''(x)|. \tag{4.8}$$

PROOF. Use (4.2) and Taylor's Theorem. □

The proof of Theorem 4.1 uses Lemma 4.1 and arguments quite similar to those used to prove Theorems 3.1–3.3. For this reason we will be brief.

LEMMA 4.2. *If* $\alpha < 1/2$, *then*

$$\int_t^\infty (u-t)p(u,z)\,du = \left[\left(\frac{1}{\alpha} - 2\right)\left(\frac{1}{\alpha} - 1\right)\right]^{-1} f(0)t^{2-1/\alpha} + O(t^{1-1/\alpha}).$$

PROOF. Use Lemma 4.1. □

LEMMA 4.3. *If* $\alpha > 1/2$, *then*

$$\int_0^\infty (u \wedge t)u^{-1/\alpha}f(0)\,du = \left[\left(\frac{1}{\alpha} - 1\right)\left(2 - \frac{1}{\alpha}\right)\right]^{-1} f(0)t^{2-1/\alpha}$$

and

$$t\int_t^\infty [p(u,z) - u^{-1/\alpha}f(0)]\,du = \left(\frac{2}{\alpha} - 2\right)^{-1} f'(0)bt^{3-2/\alpha} + \varepsilon(z,t),$$

where

$$\sup_{z\in\overline{B}} |\varepsilon(z,t)| = O(t^{2-2/\alpha}).$$

PROOF. The first assertion is obvious. The second assertion follows from Lemma 4.1. □

LEMMA 4.4. *If* $\alpha > 2/3$,

$$\int_0^t u\left[p(u,z) - u^{-1/\alpha}f(0)\right] du = (3 - 2/\alpha)^{-1} f'(0) b t^{3-2/\alpha} + \varepsilon(z,t),$$

where

$$\sup_{z \in \overline{B}} |\varepsilon(z,t)| = O(t^{4-3/\alpha}).$$

If $\alpha = 2/3$, *then*

$$\begin{aligned}\int_0^t u\left[p(u,z) - u^{3/2}f(0)\right] du &= \int_0^1 u\left[p(u,z) - u^{-3/2}f(0)\right] du\\ &\quad + \int_1^\infty u\left[p(1,z) - u^{-3/2}f(0) - u^{-3}bf'(0)\right] du\\ &= H_4(z)\end{aligned}$$

exists and is $\hat{\mu}_B(dx)\mu_B(dy)$ *integrable and*

$$\begin{aligned}\int_0^t U\left[p(u,z) - u^{-3/2}f(0)\right] du &= H_4(z) + bf'(0)[1 + \ln t]\\ &\quad + O(t^{-1/2}|z| \vee |z|^2).\end{aligned}$$

If $1/2 < \alpha < 2/3$,

$$\int_0^\infty u\left[p(u,z) - u^{-1/\alpha}f(0)\right] du = H_3(z)$$

exists, $\hat{\mu}_B H_3 \mu_B$ *exists and*

$$\begin{aligned}&\int\int \hat{\mu}_B(dx)\mu_B(dy)\int_0^t u\left[p(u,y-x) - u^{-1/\alpha}f(0)\right] du\\ &\quad = \hat{\mu}_B H_3 \mu_B - C(B)^2\left[\left(3 - \frac{2}{\alpha}\right)\right]^{-1} f'(0) b t^{3-2/\alpha} + O(t^{2-2/\alpha}).\end{aligned} \tag{4.9}$$

PROOF. Since f is bounded,

$$\sup_{z \in B}\int_0^1 u^{1-1/\alpha}\left| f\left(u^{1-1/\alpha}(b + z/u)\right) - f(0)\right| du < \infty,$$

whenever $\alpha > 1/2$. If $\alpha < 2/3$, $\int_1^\infty u^{2(1-1/\alpha)}\,du < \infty$. Since $f'(z)$ is bounded, it follows that $H_3(z)$ exists for $1/2 < \alpha < 2/3$. Using Lemma 4.1 and the above facts, we see that (4.9) holds. The other assertions of the lemma also follow from Lemma 4.1. □

LEMMA 4.5. *Let $\alpha = 1/2$. Then*

$$H_5(z) = \int_1^\infty s^{-1} f\left(s\left(1 + \frac{zs}{b^2}\right)\right) ds + \int_0^1 s^{-1}\left[f\left(s\left(1 + \frac{zs}{b^2}\right)\right) - f(0)\right] ds$$

exists and $H_5(y - x)$ is $\hat{\mu}_B(dx)\mu_B(dy)$ integrable. Let $\hat{\mu}_B H_5 \mu_B$ be its integral. Then

$$\int \hat{\mu}_B(dx)\mu_B(dy)\int_0^t up(u, y - x)\,du$$
$$= \hat{\mu}_B H_5 \mu_B - C(B)^2 f(0)\ln(b/t) + O(1/t)$$

and

$$\int \hat{\mu}_B(dx)\mu_B(dy) t\int_t^\infty p(u, y - x)\,du = C(B)^2 f(0) + O(1/t).$$

PROOF. Note that

$$t\int_t^\infty p(u, z)\,du = t\int_t^\infty u^{-2} f(u^{-1}(b + z/u))\,du = f(0) + O(1/t). \tag{4.10}$$

Also, for $t > 1$

$$\int_0^t up(u, z)\,du = \int_0^t f\left(\frac{b}{u}\left(1 + \frac{zu}{b}\right)\right)\frac{du}{u} = \int_{b/t}^\infty f\left(s + \frac{zs}{b^2}\right)\frac{ds}{s}$$

$$= \int_1^\infty f\left(s\left(1 + \frac{zs}{b^2}\right)\right)\frac{ds}{s} + \int_0^1 \left[f\left(s\left(1 + \frac{zs}{b^2}\right)\right) - f(0)\right]\frac{ds}{s} \tag{4.11}$$
$$- f(0)\ln(b/t) + O(1/z).$$

Proceeding as in the proof of Lemma 3.8, we find H_5 is $\hat{\mu}_B(dx)\mu_B(dy)$ integrable. The lemma now follows from this fact and Equations (4.10) and (4.11). □

PROOF OF THEOREM 4.1. The theorem follows from the above lemmas, much the same as Theorems 3.1–3.3 follow from the lemmas in Section 3. We omit the details. □

In Theorem 4.1 we omitted the cases $\beta = -1$ and $\beta = 1$. If $\beta = -1$, then $p(t, x) = 0$ whenever $t > -x/b$. In particular, $p(t, x) = 0$ for all $x \geq 0$. For these processes, $\int_0^\infty r(t)\,dt < \infty$ and $\exists\ t_0(B)$ such that

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + \varepsilon(t),$$

where $\varepsilon(t) = 0$ for $t > t_0(B)$. For $B = \{a\}$, $E_{\{a\}}(t) = bt$ exactly.

If $\beta = 1$, it was shown in ([3], Theorem 5) that, uniformly in x on compacts,

$$p(t,x)\sqrt{t}\,e^{\gamma_1 t} \to F_1, \tag{4.12}$$

where

$$\gamma_1 = (1-\alpha)\alpha^{\alpha/(1-\alpha)}\cos\left(\frac{\pi\alpha}{2}\right)^{-1/(1-\alpha)} b^{-\alpha/(1-\alpha)}$$

and

$$F_1 = \alpha\frac{1}{2(1-\alpha)}(2\pi(1-\alpha))^{-1/2} b^{\alpha/2(1-\alpha)-1}.$$

Using (4.12), it follows that

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + O(t^{-1/2}e^{-\gamma_1 t}).$$

If we take $B = \{0\}$ and set $E_{\{0\}}(t) = E(t)$, then $E(t)$ is the expected value of the Lebesgue measure of the range of the process up till time t. The constants entering into the expansion of $E(t)$ can be explicitly determined. These are given in the appendix.

5. Expansions for stable processes with drift on $\Re$ for $1 < \alpha < 2$. Throughout this section, we will consider a stable process on $\Re$ with $b > 0$ and $\alpha > 1$. We will take $\lambda = 1$. Our results will depend on well known asymptotic expansions of $f(x)$, the density of the corresponding drift free process at $t = 1$, as $x \to \infty$. Let

$$A_1 = \Gamma(1+\alpha)\pi^{-1}(1+h^2)^{1/2}\sin\left(\frac{\pi\alpha}{2} + 2\tan^{-1}(h)\right)$$

and let

$$A_2 = -\Gamma(2\alpha+1)(2\pi)^{-1}(1+h^2)\sin(\pi\alpha + 2\tan^{-1}(h)).$$

THEOREM 5.1. *Assume $\alpha > 1$ and $\beta > -1$. Then the quantity*

$$H_6(z) = \int_0^\infty u[p(u,z) - p(u,0)]\,du$$

exists and $H_6(y-x)$ is $\hat{\mu}_B(dx)\mu_B(dy)$ integrable. Let $\hat{\mu}_B H_6 \mu_B$ be the integral. If $1 < \alpha < 3/2$,

$$\begin{aligned} E_B(t) = {} & tC(B) + C(B)^2\left[[(\alpha-1)(2-\alpha)]^{-1}b^{-(\alpha+1)}A_1\right]t^{2-\alpha} \\ & + C(B)^2\Big[[(3-2\alpha)(2\alpha-2)]^{-1}b^{-(2\alpha+1)}A_2 \\ & \qquad + C(B)\Gamma(2-\alpha)^2\Gamma(4-2\alpha)^{-1}b^{-2(\alpha+1)}A_1^2\Big]t^{3-2\alpha} + o(t^{3-2\alpha}). \end{aligned}$$

If $\alpha = 3/2$, let

$$A_3 = \int_0^\infty \left[up(u,0) - A_1 b^{-5/2}u^{-1/2} - A_2 b^{-4}u^{-1}\right]du.$$

Then $|A_3| < \infty$ *and*

$$E_B(t) = tC(B) + C(B)^2\left[4A_1 b^{-5/2}\sqrt{t} + A_3 + A_2 b^{-4}\ln t + b^{-7/2} + C(B)A_1^2 b^{-5}\pi\right] + \hat{\mu}_B H_6 \mu_B + o(1).$$

If $3/2 < \alpha < 2$, *let*

$$A_4 = \int_0^\infty \left[up(u,0) - A_1 b^{-(\alpha+1)}u^{-\alpha}\right] du.$$

Then $|A_4| < \infty$ *and*

$$E_B(t) = tC(B) + C(B)^2[(\alpha-1)(2-\alpha)]^{-1}b^{-(\alpha+1)}t^{2-\alpha} + \hat{\mu}_B H_6 \mu_B + C(B)^2\left[A_4 + A_2 b^{-(2\alpha+1)}[(3-2\alpha)(2\alpha-2)]^{-1} + C(B)A_1^2 b^{-2(\alpha+1)}\Gamma(2-\alpha)^2\Gamma(4-2\alpha)^{-1}\right]t^{3-2\alpha} + o(t^{3-2\alpha}).$$

LEMMA 5.1. *As* $x \to \infty$,

$$f(x) = A_1 x^{-(\alpha+1)} + A_2 x^{-(2\alpha+1)} + O(x^{-(3\alpha+1)}).$$

Also,

$$\sup_x |f'(x)|\,|x|^{(\alpha+2)} = A_5 < \infty.$$

PROOF. These well known facts can be found in [6]. □

LEMMA 5.2. $H_6(z)$ *and* $\hat{\mu}_B H_6 \mu_B$ *exist.*

PROOF. Choose M such that $\sup_{z\in\bar{B}}|z|/M < b/2$. Using Lemma 5.1 and the mean value theorem, we find that for some constant A_6,

$$\int_M^\infty u|p(u,z) - p(u,0)|\,du \le A_6\int_M^\infty u^{-\alpha}\,du \le (b/2)^{-(\alpha+2)}.$$

But

$$\int_0^M u[p(u,z) - p(u,0)]\,du \le \int_0^M u^{1-1/\alpha}\,du \sup_x f(x).$$

It follows that

$$\sup_{z\in\bar{B}}\left|\int_0^\infty u[p(u,z) - p(u,0)]\,du\right| < \infty.$$

Consequently, $\hat{\mu}_B H_6 \mu_B$ exists. □

LEMMA 5.3.

$$t\int_t^\infty p(u,0)\,du = \frac{A_1}{\alpha-1}b^{-(\alpha+1)}t^{2-\alpha} + A_2(2\alpha-2)^{-1}b^{-(\alpha+2)}t^{3-2\alpha} + O(t^{4-3\alpha}).$$

PROOF. This follows at once from Lemma 5.1 via the scaling property (4.2) of $p(t,0)$. □

LEMMA 5.4. *Let*

$$I = \int_0^t up(u,0)\,du - A_1 b^{-(\alpha+1)}(2-\alpha)^{-1}t^{2-\alpha}.$$

If $\alpha < 4/3$,

$$I = A_2(3-2\alpha)^{-1}b^{-(2\alpha+1)}t^{3-2\alpha} + O(t^{4-3\alpha}).$$

If $\alpha = 4/3$,

$$I = 3A_2 b^{-11/3}t^{1/3} + O(\ln t).$$

If $4/3 < \alpha < 3/2$,

$$I = \int_0^\infty \left[up(u,0) - A_1 u^{1-\alpha}b^{-(\alpha+1)} - A_2 u^{2(1-\alpha)}b^{-(2\alpha+1)}\right]du$$
$$+ A_2(3-2\alpha)^{-1}b^{-(2\alpha+1)}t^{3-2\alpha} + O(t^{4-3\alpha}).$$

If $\alpha = 3/2$,

$$I = \int_0^\infty \left[up(u,0) - A_1 u^{-1/2}b^{-5/2} - A_2 u^{-1}b^{-4}\right]du + A_2 b^{-4}\ln t + O(t^{-1/2}).$$

If $\alpha > 3/2$,

$$I = \int_0^\infty \left[up(u,0) - A_1 b^{-(\alpha+1)}u^{1-\alpha}\right]du - \frac{A_2 b^{-(2\alpha+1)}}{2\alpha-3}t^{3-2\alpha} + O(t^{4-3\alpha}).$$

PROOF. These follow by routine calculations from Lemma 5.1. We illustrate by proving the assertion for $\alpha > 3/2$. If $\alpha > 3/2$, it follows from Lemma 5.1 that

$$u^{1-1/\alpha}f(bu^{1-1/\alpha}) - A_1 b^{-(\alpha+1)}u^{1-\alpha}$$

is integrable on $(0,\infty)$. Thus,

$$\int_0^t up(u,0)\,du = \int_0^t A_1 b^{-(\alpha+1)}u^{1-\alpha}\,du$$
$$+ \int_0^\infty \left[u^{1-1/\alpha}f(bu^{1-1/\alpha}) - A_1 b^{-(\alpha+1)}u^{1-\alpha}\right]du$$
$$- \int_t^\infty \left[u^{1-1/\alpha}f(bu^{1-1/\alpha}) - A_1 b^{-(\alpha+1)}u^{1-\alpha}\right]du.$$

Using Lemma 5.1 once again, we find

$$\int_t^\infty \left[u^{1-1/\alpha}f(bu^{1-1/\alpha}) - A_1 b^{-(\alpha+1)}u^{1-\alpha}\right]du$$
$$= \int_t^\infty A_2 u^{2(1-\alpha)}b^{-(2\alpha+1)}\,du + O(t^{4-3\alpha}).$$ □

LEMMA 5.5. *For any compact set K,*

$$\lim_{t\to\infty} \sup_{z\in K} t^{\alpha-1}\int_t^{\infty}(u-t)|p(u,z)-p(u,0)|\,du<\infty.$$

PROOF. This follows from Lemma 5.1 and the mean value theorem. □

LEMMA 5.6. *Let*

$$J=\int_0^{\infty}p(u,0)(u\wedge t)\,du-A_1b^{-(\alpha+1)}[(\alpha-1)(2-\alpha)]^{-1}t^{2-\alpha}.$$

For $\alpha<3/2$,

$$J=A_2b^{-(2\alpha+1)}[(3-2\alpha)(2\alpha-2)]^{-1}t^{3-2\alpha}+o(t^{3-2\alpha}).$$

For $\alpha=3/2$, let

$$A_3=\int_0^{\infty}\left[up(u,0)-A_1b^{-5/2}u^{-1/2}-A_2b^{-4}u^{-1}\right]du.$$

Then

$$J=A_3+A_2[b^{-4}\ln t+b^{-7/2}]+o(1).$$

For $2>\alpha>3/2$, let

$$A_4=\int_0^{\infty}\left[up(u,0)-A_1b^{-(\alpha+1)}u^{-\alpha}\right]du.$$

Then

$$J=A_4+A_2b^{-(2\alpha+1)}[(2\alpha-3)(2\alpha-2)]^{-1}t^{3-2\alpha}+o(t^{3-2\alpha}).$$

PROOF. The lemma follows at once from Lemmas 5.3 and 5.4. □

LEMMA 5.7.

$$G_B(t)\sim\frac{\Gamma(2-\alpha)^2}{\Gamma(4-2\alpha)}C(B)^3A_1^2b^{-2(\alpha+1)}t^{3-2\alpha}.$$

PROOF. Using (4.2) and Lemma 5.1, we see that $r(t)\sim A_1b^{-(\alpha+1)}t^{1-\alpha}$. The lemma now follows from Lemma 2.1. □

PROOF OF THEOREM 5.1. The theorem follows easily from Lemmas 5.5–5.7 via (2.8). If $\beta=-1$ it was shown in Theorem 8 of [3] that

$$p(t,x)\sqrt{t}\,e^{\gamma_2 t}\to F_2,\qquad t\to\infty, \tag{5.1}$$

where

$$\gamma_2=(\alpha-1)\left|\frac{b}{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\right|^{\alpha/(\alpha-1)}$$

and

$$F_2 = [2\pi(\alpha - 1)]^{-1/2}\alpha^{-1/2(\alpha-1)}b^{(1-\alpha/2(\alpha-1))}.$$

Using (5.1), it easily follows that

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + O(t^{-1/2}e^{-\gamma_2 t}).$$

Let $E(t) = E_{\{0\}}(t)$. Then the constants entering into the expansion can be explicitly determined. These can be found in the Appendix. □

6. Expansions for asymmetric Cauchy processes on $\mathfrak{R}$. In this section we will consider asymmetric Cauchy processes on $\mathfrak{R}$. We will take $\lambda = \pi/2$. This will enable us to use the asymptotic expansion of $p(1, x) = f(x)$ given by Zolotarev [9].

THEOREM 6.1. *If* $-1 < \beta < 0$,

$$\begin{aligned} E_B(t) = tC(B) + C(B)^2\left[\frac{(1+\beta)\beta^{-2}t}{2\ln t} + (1+\beta)\beta^{-2}\frac{(\ln\ln t)t}{4\ln^2 t}\right] \\ + \Bigg[\beta^{-2}\Bigg[(1+\beta)\ln(-\beta)\frac{11}{8}\beta^{-2}(1+\beta) \\ +2C + (B)\left(\frac{1+\beta}{2\beta^2}\right)^2\Bigg]\frac{t}{\ln^2 t}\Bigg] + o\left(\frac{t}{\ln^2 t}\right). \end{aligned}$$

If $0 < \beta < 1$, *the expansion of* $E_B(t)$ *is obtained from the above by replacing* β *with* $-\beta$. *Henceforth we assume* $\beta < 0$.

The following expansion of f can be found in Theorem 2.5.4 of [9].

LEMMA 6.1. *If* $\beta > -1$, *then as* $x \to \infty$,

$$f(x) = \tfrac{1}{2}(1+\beta)x^{-2} + \tfrac{1}{2}\beta(1+\beta)(\ln x)x^{-3} - \beta(1+\beta)x^{-3} + O((\ln^2 x)x^{-4}).$$

Also, for some constant B,

$$|f'(x)| \le Bx^{-3}, \qquad x > 0.$$

Set

$$\begin{aligned} B_1 &= (1+\beta)/2, \\ B_2 &= \beta(1+\beta)/2, \\ B_3 &= -\beta(1+\beta). \end{aligned}$$

Lemma 6.1 and the scaling property

$$p(t,x) = t^{-1}p\left(1, \frac{x}{t} - \beta\ln t\right) \tag{6.1}$$

form the basis for the proof of Theorem 6.1. Applying Lemma 6.1, we find for $u > 1$,

$$p(u,0) = \frac{1}{u}\left[\frac{B_1}{\beta^2 \ln^2 u} + \frac{B_2}{\beta^3}\frac{\ln\ln u}{\ln^3 u}\right]$$
$$+[B_3 + B_2 \ln(-\beta)]\beta^{-3}u^{-1}\ln^{-3} u + O\left(\frac{\ln^2 \ln u}{\ln^4 u}\right).$$

Integrating from t to ∞, we find

$$(6.2)\quad \begin{aligned} t\int_t^\infty p(u,0)\,du &= \frac{B_1\beta^{-2}t}{\ln t} - B_2\beta^{-3}\frac{\ln\ln t}{2\ln^2 t} \\ &\quad -\left[\frac{B_2}{4} + B_3 + B_2\ln(-\beta)\right]\beta^{-3}\frac{t}{\ln^2 t} + o\left(\frac{t}{\ln^2 t}\right), \end{aligned}$$
$$t \to \infty.$$

Now

$$(6.3)\quad \int_0^t up(u,0)\,du = \int_0^t uf(-\beta\ln u)\,du = \frac{B_1 t}{\beta^2\ln^2 t} + o\left(\frac{t}{\ln^2 t}\right).$$

LEMMA 6.2. *As $t \to \infty$,*

$$\int_0^\infty (u \wedge t)p(u,t)\,du$$
$$= B_1\beta^{-2}\frac{t}{\ln t} + B_2\beta^{-3}\frac{\ln\ln t}{2\ln^2 t}$$
$$+\left[\left[B_3 + B_2\left(\frac{1}{4} + \ln(-\beta)\right)\right]\beta^{-3} + B_1\beta^{-2}\right]\frac{t}{\ln U2\, t} + o\left(\frac{t}{\ln^2 t}\right).$$

PROOF. This follows at once from (6.2) and (6.3). □

LEMMA 6.3. *For any z*

$$H_7(z) = \int_0^\infty u|p(u,z) - p(u,0)|\,du < \infty$$

and for a compact set K

$$\sup_{z\in K} H_7(z) < \infty.$$

PROOF. This follows from Lemma 6.1 and the mean value theorem. □

LEMMA 6.4.

$$G_B(t) \sim 2\left(\frac{B_1}{\beta^2}\right)^2 C(B)^3\frac{t}{\ln^2 t}.$$

PROOF. This follows from Lemma 6.3 and the fact that here $r(t) \sim (B_1/\beta^2)/\ln t$. □

PROOF OF THEOREM 6.1. Note that

$$\hat{P}_{\phi B}(L_B \leq t) = \int\int \hat{\mu}_B(dx)\mu_B(dy)\int_0^\infty (u \wedge t)p(u, y - x)\,du$$

$$= C(B)^2\int_0^\infty (u \wedge t)p(u, 0)\,du + \varepsilon,$$

where

$$\varepsilon = \int\int \hat{\mu}_B(dx)\mu_B(dy)\int_0^\infty [p(u, y - x) - p(u, 0)](u \wedge t)\,du.$$

By Lemma 6.3

$$|\varepsilon| \leq C(B)^2 \sup_{z \in \overline{B}} H_7(z).$$

Using Lemmas 6.2 and 6.4, Equations (6.4) and (2.8) and the fact that $t/\ln^2 t \to \infty$, we obtain Theorem 6.1.

We will now consider the case when $\beta = -1$.

Theorem 2.5.2 of [9] shows that

$$f(x) \sim \frac{1}{\sqrt{2\pi e}} e^{x/2 - e^{x-1}}, \qquad x \to \infty.$$

Thus, uniformly in z on compacts,

$$\int_t^\infty p(u, z)(u - t)\,du \sim \frac{1}{\sqrt{2\pi e}}\int_t^\infty (u - t)u^{-1/2}e^{-u/e}\,du$$

$$= \left[\int_0^\infty s e^{-s/e}(t + s)^{-1/2}\,ds\right]\frac{e^{-t/e}}{\sqrt{2\pi e}} = O(t^{-1/2}e^{-t/e}).$$

Using (2.7), $r(t) = O(t^{-1/2}e^{-t/e})$ and the above, we find

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + O(t^{-1/2}e^{-t/e}). \qquad \square$$

7. Linear Brownian motion with drift. The stable process with drift b for $\alpha = 2$ is linear Brownian motion with mean $-bt$ and transition density $p(t, x) = (4\pi t)^{-1/2}e^{-(x+bt)^2/4t}$. The function $E_{\{a\}}(t)$ can be explicitly computed. Using Proposition A9, we find

$$(7.1) \quad E_a(t) = \int_0^t \frac{2s^{-1/2}}{\sqrt{\pi}} e^{-b^2 s/4}\,ds + \frac{b^2}{2\sqrt{\pi}}\int_0^t ds\int_0^s u^{-1/2}e^{-b^2 u/4}\,du.$$

A simple computation shows

$$\int_0^t \frac{2s^{-1/2}}{\sqrt{\pi}} e^{-b^2 s/4}\, ds = \frac{4}{b} - \int_t^\infty \frac{2s^{-1/2}e^{-b^2 s/4}}{\sqrt{\pi}}\, ds.$$

Another computation shows

$$\begin{aligned} &\frac{b^2}{2\sqrt{\pi}} \int_0^t ds \int_0^s u^{-1/2} e^{-b^2 u/4}\, du \\ &\quad = bt - \frac{b^2}{2\sqrt{\pi}} \int_0^t ds \int_0^s u^{-1/2} e^{-b^2 u/4}\, du \\ &\quad = bt - \frac{2}{b} + \frac{b^2}{2\sqrt{\pi}} \left[\int_t^\infty u^{-1/2} e^{-b^2 u/4}\, du - t \int_t^\infty u^{-1/2} e^{-b^2 u/4}\, du \right]. \end{aligned}$$

Hence

$$\begin{aligned} E_a(t) &= bt - \frac{2}{b} \\ &\quad - e^{-b^2 t/4} \int_0^\infty e^{-b^2 s/4} \left[\frac{2(s+t)^{-1/2}}{\sqrt{\pi}} \right. \\ &\qquad \left. + \frac{b^2}{2\sqrt{\pi}} \left[(s+t)^{1/2} - t(s+t)^{-1/2} \right] \right] ds. \end{aligned} \tag{7.2}$$

If we now expand the integrand in (7.2), we find

$$E_a(t) \sim bt - \frac{2}{b} + e^{-b^2 t/4} \sum_{j=0}^{\infty} a_j t^{-(j+1/2)},$$

where $a_0 = -2/\sqrt{\pi}$ and for $j > 0$,

$$a_j = -\frac{2}{\sqrt{\pi}} \binom{-1/2}{j} \left(\frac{b^2}{4} \right)^{-j} j! + \frac{b^2}{2\sqrt{\pi}} \left[\binom{1/2}{j+1} - \binom{-1/2}{j+1} \right] (j+1)!.$$

Let B be a compact set and let $p =$ g.l.b. of B and $q =$ l.u.b. of B. Then

$$E_B(t) = E_{\{0\}}(t) + (q-p) - \int_p^q P_x(T_B > t)\, dx.$$

If B is the closed interval $[p, q]$, then $P_x(T_B > t) = 0$ for all $x \in B$. For a general compact B

$$P_x(T_B > t) \le P_x(T_p > t).$$

Using Proposition A9, we see that

$$P_x(T_B > t) = O(t^{-3/2} e^{-b^2 t/4}).$$

Thus, in general,

$$E_B(t) = E_{\{0\}}(t) + (q-p) + O(t^{-3/2}e^{-b^2t/4}). \tag{7.3}$$

Using the expansion of $E_{\{0\}}(t)$ given in (7.2), we can write

$$E_B(t) = bt + \frac{2}{b} + (q-p)a_0 e^{-b^2t/4}t^{-1/2} + O(t^{-3/2}e^{-b^2t/4}).$$

8. Expansions for processes with drift on $\mathfrak{R}^d$, $d \geqslant 2$. In this section we will consider a stable process with drift on $\mathfrak{R}^d$, $d \geq 2$. As usual, $p(t,x)$ will be the transition density and $f(x)$ will be the density of the corresponding drift free process at time 1. Then

$$p(t,x) = t^{-d/\alpha} f\big(t^{1-1/\alpha}(b + x/t)\big). \tag{8.1}$$

THEOREM 8.1. *Suppose $\alpha < 1$. Let $b = (b_1, \dots, b_d)$ and let*

$$c_j = \sum_{i_1=1}^{d} \cdots \sum_{i_j=1}^{d} \frac{\partial^i}{\partial x_{i_1} \cdots \partial x_{i_j}} f(0) b_{i_1} \cdots b_{i_j}.$$

Let n be such that $(n-1)/n \leq \alpha < n/(n+1)$. Then

$$\begin{aligned} E_B(t) &= tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx \\ &\quad + \sum_{j=0}^{n-1} D_j C(B)^j t^{2+j-(d+j)/\alpha} + O(t^{1-d/\alpha}), \end{aligned} \tag{8.2}$$

where

$$D_j = \left[\left(\frac{d+j}{\alpha} - j - 1\right)\left(\frac{d+j}{\alpha} - j - 2\right)\right]^{-1} c_j.$$

PROOF. Using 8.1, we find

$$p(u,x) = \sum_{j=0}^{n-1} c_j u^{j-(d+j)/\alpha} + O(u^{n-(d+n)/\alpha}).$$

Hence,

$$\int_t^\infty (u-t)p(u,x)\,du = \sum_{j=0}^{n-1} D_j t^{j+2-(d+j)/\alpha} + O(t^{1-d/\alpha}). \tag{8.3}$$

Equation (8.2) now follows from (8.3) and (2.7). □

THEOREM 8.2. *Let $\alpha > 1$ and assume the corresponding drift free process is isotropic. Let*

$$\begin{aligned} B_n = \Big[&(-1)^{n-1}\Gamma(n\alpha/2 + 1)4^{(\alpha+d)/2}\big(1 + \tan^2(\pi\alpha/4)\big)^{n/2} \\ &\times \sin\big(n(\pi\alpha/4 + \tan^{-1}(\pi\alpha/4))\big)\Big]\Big/\Big[\pi n!(4\pi)^{d/2}\Big]. \end{aligned}$$

Let n be such that $(n+1)/n \le \alpha$ *but* $n/(n-1) > \alpha$. *Then*

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx$$
$$+ \sum_{j=1}^{n} \frac{B_j b^{-(d+\alpha j)} C(B)^j t^{j+2-d-\alpha j}}{(d+\alpha j - j - 1)(d + \alpha j - j - 2)} + O(t^{2-d-\alpha}).$$

PROOF. This follows from the asymptotic expansion of $f(x)$, given in Proposition A.13, and (2.7). □

REMARK 3. What can be said of the expansion of $E_B(t)$ when the associated drift free process is nonisotropic and $\alpha > 1$? The difficulty here is lack of knowledge about the asymptotic behavior of $f(x)$ as x tends to ∞ along the direction b. Examples show that it can be of the order $|x|^{-(1+\alpha)}$ instead of $|x|^{-(d+\alpha)}$. Pruitt and Taylor [5] investigated the behavior of $f(x)$ as x tends to ∞. In general, no asymptotics seem possible except in special cases, but they do show that $f(x) = O(|x|^{-(1+\alpha)})$. This suffices to show that all these processes are such that $\int_0^\infty r(t)\,dt < \infty$ and to yield the expansion

$$E_B(t) = tC(B) + \int \hat{\phi}_B(x)\phi_B(x)\,dx + O(t^{-[(\alpha-1)^2+(d-2)]/\alpha}).$$

APPENDIX

The stable subordinator with exponent $\alpha/2$ is the stable process with exponent $\alpha/2$ and $\beta = 1$. Let its transition density be $h_{\alpha/2}(t,u)$. Then

$$\text{(A.1)} \qquad \int_0^\infty e^{-su} h_{\alpha/2}(t,u)\,du = e^{-ts^{\alpha/2}}.$$

Let $p(t,x)$ be the density of the isotropic stable process with exponent α and $\lambda = 1$. Then

$$\text{(A.2)} \qquad p(t,x) = \int_0^\infty h_{\alpha/2}(t,u) e^{-|x|^2/4u} (4\pi u)^{-d/2}\,du.$$

PROPOSITION A.1. *For any* $N > 0$

$$\text{(A.3)} \qquad \int_0^\infty h_{\alpha/2}(t,u) u^{-N}\,du = \left(\frac{2}{\alpha}\right)\Gamma\left(\frac{2N}{\alpha}\right)\Gamma(N)^{-1} t^{-2N/\alpha}.$$

PROOF. Write

$$u^{-N} = \frac{1}{\Gamma(N)} \int_0^\infty e^{-su} s^{N-1}\,ds.$$

Then the left-hand side of (A.3) is

$$\frac{1}{\Gamma(N)}\int_0^\infty s^{N-1}e^{-ts^{\alpha/2}}\,ds.$$

The change of variable $v = s^{\alpha/2}$ in the above integral now yields the right-hand side of (A.3). □

REMARK 4. In order not to have to single out Brownian motion from the other isotropic stable processes, we will interpret the stable subordinator with exponent 1 to be uniform motion to the right with unit speed. So done, all of the formulas derived for the isotropic stable processes via (A.2) are then valid for $\alpha = 2$.

PROPOSITION A.2.

$$\int_0^\infty t h_{\alpha/2}(t,u)\,dt = \frac{1}{\Gamma(\alpha)}u^{\alpha-1}. \tag{A.4}$$

PROOF. The function $\int_0^\infty t h_{\alpha/2}\,dt$ is continuous. Using (A.1), we find its Laplace transform to be $s^{-\alpha}$. It follows that (A.4) holds. □

PROPOSITION A.3. *If $\alpha > d/2$ and the process is a strictly stable isotropic process,*

$$\begin{aligned}&\int_0^\infty t[p(t,0) - p(t,x)]\,dt\\ &\quad= \left[\pi^{-d/2}4^{-d}\Gamma(\alpha)^{-1}\left(\alpha - \frac{d}{2}\right)^{-1}\Gamma\left(\frac{d}{2} + 1 - \alpha\right)\right]|x|^{2\alpha-d}.\end{aligned} \tag{A.5}$$

PROOF. Using (A.2) and (A.4), we find

$$\int_0^\infty t[p(t,0) - p(t,x)]\,dt = \int_0^\infty \frac{u^{\alpha-1}}{\Gamma(\alpha)}\left[1 - e^{-|x|^2/4u}\right](4\pi u)^{-d/2}\,du.$$

Making the change of variable $|x|^2/4u = s$ and using the fact that

$$\int_0^\infty (1 - e^{-s})s^{(d/2)-\alpha-1}\,ds = \left(\alpha - \frac{d}{2}\right)^{-1}\Gamma\left(\frac{d}{2} - \alpha + 1\right),$$

we find that (A.5) holds. □

PROPOSITION A.4. *For any strictly stable process with $\alpha = d/2$ and $z \neq 0$,*

$$\begin{aligned}\left(\frac{2}{d}\right)H_2(z) &= \int_1^\infty p\left(1, \frac{z}{|z|}u\right)u^{-1}\,du\\ &\quad+ \int_0^1\left[p\left(1, u\frac{z}{|z|}\right) - p(1,0)\right]u^{-1}\,du - p(1,0)\ln|z|.\end{aligned} \tag{A.6}$$

PROOF. Make the change of variable $s|z| = u$ in the integrals defining $H_2(z)$. □

For an isotropic stable process with $\alpha = d/2$ (d must be 1, 3 or 4), the integral expression in (A.6) is a constant I_α that can be determined explicitly.

PROPOSITION A.5. *Let γ be Euler's constant and let $\psi(x) = \Gamma'(x)/\Gamma(x)$. Then*

$$I_d = (2\pi)^{-d/2}(2/d)\Gamma(d/2)^{-1}[\ln 4 + ((4/d) - 1)\gamma - 4/d + \psi(d/2)],$$

$$I_1 = \left(\frac{1}{\pi}\right)[\ln 4 - 4 - 2\ln 2 - 4\gamma],$$

(A.7) $$I_3 = \left(\frac{1}{6\pi^2}\right)\left[\ln 4 - 2\ln 2 - \frac{2}{3} - \frac{4}{3}\gamma\right],$$

$$I_4 = \frac{(4\pi)^{-2}}{2}[\ln 4 - \gamma].$$

PROOF. Using (A.2), we find after a small calculation that

$$I_d = (1/2)(4\pi)^{-d/2}\left[A\int_0^\infty h(u)u^{-d/2}\,du + \int_0^\infty \ln(4u)u^{-d/2}h(u)\,du\right],$$

where

$$A = \int_0^1 (e^{-s} - 1)s^{-1}\,ds + \int_1^\infty e^{-s}s^{-1}\,ds.$$

Integration by parts shows

$$A = \int_0^\infty e^{-s}\ln s\,ds = -\gamma.$$

Using (A.3), we see that

$$I_d = (4\pi)^{-d/2}(2/d)\Gamma(d/2)^{-1}[\ln 4 - \gamma]$$
$$+(1/2)(4\pi)^{-d/2}\int_0^\infty (\ln u)u^{-d/2}h(u)\,du.$$

To evaluate

$$J = \int_0^\infty h(u)u^{-d/2}\ln u\,du,$$

observe that

$$\int_0^\infty e^{-su}s^{(d/2)-1}\ln(1/s)\,ds$$
$$= \left[\int_0^\infty e^{-t}t^{(d/2)-1}\,dt\right]u^{-d/2}\ln u - \left[\int_0^\infty e^{-t}t^{-(d/2)-1}\ln t\,dt\right]u^{-d/2}$$
$$= \Gamma(d/2)u^{-d/2}\ln u - \Gamma'(d/2)u^{-d/2}.$$

Hence,

$$J = \frac{1}{\Gamma(d/2)} \int_0^\infty \int_0^\infty h(u) e^{-su} s^{(d/2)-1} \ln(s)\, ds + \frac{\Gamma'(d/2)}{\Gamma(d/2)} \int_0^\infty u^{-d/2} h(u)\, du.$$

Using (A.3) and (A.1), we find

$$J = -\frac{1}{\Gamma(d/2)} \int_0^\infty e^{-s^{d/4}} s^{(d/2)-1} \ln s\, ds + \psi\left(\frac{d}{2}\right)\left(\frac{4}{d}\right)\Gamma\left(\frac{d}{2}\right)^{-1}.$$

The change of variable $t = s^{d/4}$ now shows

$$\begin{aligned} J &= -\frac{1}{\Gamma(d/2)}\left(\frac{4}{d}\right)^2 \int_0^\infty e^{-t} t \ln t\, dt + \left(\frac{4}{d}\right)\Gamma\left(\frac{d}{2}\right)^{-1} \psi\left(\frac{d}{2}\right) \\ &= \left(\frac{4}{d}\right)\Gamma\left(\frac{d}{2}\right)^{-1}\left[\left(\frac{4}{d}\right)(\gamma - 1) + \psi\left(\frac{d}{2}\right)\right]. \end{aligned}$$

Thus,

$$I_d = (4\pi)^{-d/2}(2/d)\Gamma(d/2)^{-1}[\ln 4 + (4/d - 1)\gamma - 4/d + \psi(d/2)].$$

Using the fact that

$$\begin{aligned} \psi(1/2) &= -(\gamma + 2\ln 2), \\ \psi(3/2) &= 2 - \gamma - 2\ln 2, \\ \psi(2) &= 1 - \gamma, \end{aligned}$$

we find (A.7) holds. □

PROPOSITION A.6. *For an isotropic stable process with exponent* α

$$p(1,0) = (4\pi)^{-d/2}(2/\alpha)\Gamma(d/\alpha)\Gamma(d/2)^{-1}.$$

PROOF. This known fact follows from (A.2) and (A.3). □

PROPOSITION A.7. *For an isotropic stable process with exponent* $\alpha < d$,

$$g(x) = k_1 |x|^{\alpha - d},$$

where

$$k_1 = \Gamma\left(\frac{d-\alpha}{2}\right)\left[4^{\alpha/2}\pi^{d/2}\Gamma(\alpha/2)\right]^{-1}.$$

PROOF. This is a well known fact. It follows easily from the fact that $\int_0^\infty h_{\alpha/2}(t,u)\, dt = [1/(\Gamma(\alpha/2))] u^{(\alpha/2)-1}$ and (A.2). □

PROPOSITION A.8. *For an isotropic stable process with* $\alpha < d/2$,

$$\int \phi_B(x)^2\, dx = k_2 \int\int |a - b|^{2\alpha - d} \mu_B(da)\mu_B(db),$$

where

$$k_2 = k_1^2\pi^{d/2}\Gamma\left(\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}-\alpha\right)\left[\Gamma\left(\frac{d-\alpha}{2}\right)^2\Gamma(\alpha)\right]^{-1}.$$

PROOF. This follows from the fact that

$$\phi_B(x) = \int k_1|x-a|^{\alpha-d}\mu_B(da)$$

and the Riesz composition formula. □

PROPOSITION A.9. *Let X_t be linear Brownian motion with mean $-bt$, $b>0$, and transition density $p(t,x) = (4\pi t)^{-1/2}e^{-(x+bt)^2/4t}$. Let $E(t) = E_{\{a\}}(t)$. Then $E(dt)$ has density*

$$e(t) = \frac{2}{\sqrt{\pi}}t^{-1/2}e^{-b^2t/4} + \frac{b^2}{2\sqrt{\pi}}\int_0^t s^{-1/2}e^{-b^2s/4}\,ds \tag{A.8}$$

and $P_x(T_{\{y\}}\in dt)$ has density $f_x(t,y)$ given by

$$f_x(t,y) = e^{-b(y-x)/2}e^{-b^2t/4}|y-x|(4\pi t)^{-3/2}e^{-(y-x)^2/4t}. \tag{A.9}$$

PROOF. Using the fact that

$$\int_0^\infty e^{-\lambda t}e^{-x^2/4t}(4\pi t)^{-1/2}\,dt = e^{-\sqrt{\lambda}|x|}(2\sqrt{\lambda})^{-1},$$

it follows that

$$g^\lambda(x) = \int_0^\infty e^{-\lambda t}p(t,x)\,dt = e^{-bx/2}e^{-|x|\sqrt{b^2/4+\lambda}}\left(2\sqrt{\frac{b^2}{4}+\lambda}\right).$$

Now

$$E_xe^{-\lambda T_{\{y\}}} = \frac{g^\lambda(y-x)}{g^\lambda(0)}. \tag{A.10}$$

Integrating over x, we find

(A.11)

$$\int_0^\infty E(dt)e^{-\lambda t} = \frac{1}{\lambda g^\lambda(0)} = \frac{2}{\lambda}\sqrt{\frac{b^2}{4}+\lambda} = \frac{2}{\sqrt{\frac{b^2}{4}+\lambda}}\left(1+\frac{b^2}{4\lambda}\right).$$

The right-hand side of (A.11) is the Laplace transform of the right-hand side of (A.8). Thus, (A.8) holds. Using (A.10), we see that

$$E_xe^{-\lambda T_{\{y\}}} = e^{-b(y-x)/2}e^{-\sqrt{b^2/4+\lambda}|y-x|}. \tag{A.12}$$

Now

$$e^{-\sqrt{\lambda}|y-x|}$$

is the Laplace transform of $|y-x|(4\pi t)^{-3/2}e^{-(y-x)^2/4t}$. Thus, the right-hand side of (A.12) is the Laplace transform of the right-hand side of (A.9). Thus, (A.9) holds. □

PROPOSITION A.10. *For any strictly stable process with $\alpha < d$, for $z \neq 0$,*

$$F(z) = \int_0^\infty p(u,z)u\,du = \alpha|z|^{2\alpha-d}U\left(\frac{z}{|z|}\right), \tag{A.13}$$

where

$$U\left(\frac{z}{|z|}\right) = \int_0^\infty s^{d-2\alpha-1}p\left(1, \frac{z}{|z|}s\right)ds. \tag{A.14}$$

In particular, for $d = 1$,

$$\alpha U\left(\frac{z}{|z|}\right) = \frac{\Gamma(1-2\alpha)}{\pi(1+h^2)}\cos\left[2\tan^{-1}(h) - \operatorname{sgn}(z)\frac{\pi(1-2\alpha)}{2}\right]. \tag{A.15}$$

Additionally,

$$\int \hat{\phi}_B(x)\phi_B(x)\,dx = \int\int \hat{\mu}_B(dx)\mu_B(dy)F(y-x). \tag{A.16}$$

PROOF. Note that

$$\int \hat{\phi}_B(x)\phi_B(x)\,dx = \int\int \hat{\mu}_B(dx)\mu_B(dy)\int g(y-z)g(z-x)\,dz.$$

Now

$$\int g(y-z)g(z-x)\,dz = \int_0^\infty \left[\int_0^\infty p(s,y-z)\,ds \int_0^\infty p(t,z-x)\,dt\right]dz$$
$$= \int_0^\infty up(u,y-x)\,du.$$

Hence, (A.16) holds. Using the scaling property and the change of variable $u^{-1/\alpha}|z| = s$ shows (A.13) holds. For $d = 1$ we use the fact that

$$p(1, \pm s) = \frac{1}{\pi}\int_0^\infty e^{-\theta^\alpha}\cos(\theta^\alpha h \mp \theta s)\,d\theta.$$

Multiplying both sides by $s^{-2\alpha}$, interchanging the order of integration and evaluating the integrals show that (A.15) holds. □

PROPOSITION A.11. *Let f be the density of a stable distribution on $\mathfrak{R}$ of exponent $\alpha \neq 1$. Then*

$$f(0) = \frac{1}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right)(1+h^2)^{-1/\alpha}\cos(\alpha^{-1}\tan^{-1}(h)),$$

$$f'(0) = -\frac{1}{\pi\alpha}\Gamma\left(\frac{2}{\alpha}\right)(1+h^2)^{-1/\alpha}\sin(\alpha^{-1}\tan^{-1}(h)).$$

PROOF. Note that

$$f(0) = \frac{1}{\pi}\int_0^\infty e^{-\theta^\alpha}\cos(\theta^\alpha h)\theta\,d\theta,$$

$$f'(0) = -\frac{1}{\pi}\int_0^\infty e^{-\theta\alpha}\sin(\theta^\alpha h)\theta\,d\theta.$$

The proposition now follows by evaluating the integrals. □

PROPOSITION A.12. *Let X_t be a stable process on $\mathfrak{R}$ with drift $b > 0$. Then for $\alpha < 1/2$*

$$\begin{aligned}\hat{\phi}_{\{0\}}(x)\phi_{\{0\}}(x)\,dx &= C^2\alpha b^{-1/1-\alpha}\Gamma\left(\frac{1-\alpha}{1-\alpha}\right)\Gamma\left(\frac{1+\alpha}{1-\alpha}\right)(1+h^2)^{-1/2(1-\alpha)}\\ &\quad\times\cos\left[\frac{1}{1-\alpha}\tan^{-1}(h)-\frac{\pi}{2}\left(1-\frac{1}{1-\alpha}\right)\right],\end{aligned} \tag{A.17}$$

where for $p = 1/2 + (1/\pi\alpha)\tan^{-1}[\beta h]$,

$$C = (1-\alpha)b[1-\alpha(1-p)]^{-1}. \tag{A.18}$$

PROOF. Observe that

$$\hat{\phi}_{\{0\}}(x)\phi_{\{0\}}(x)\,dx = C^2\int u^{1-1/\alpha}f(u^{-1/\alpha}b)\,du,$$

where C is the capacity of a point. It was shown in [3] that (A.18) holds. Evaluation of the integral above can be carried out as in Proposition A.10. This evaluation yields (A.17). □

PROPOSITION A.13. *Let $f(x)$ be the isotropic stable density on $\mathfrak{R}^d$ with $\alpha < 2$. Then*

$$f(x) = \sum_{n=1}^{\infty}(4\pi)^{-d/2}\Gamma\left(\frac{n\alpha+d}{2}\right)4^{(\alpha+d)/2}A_n|x|^{-(d+\alpha n)}, \qquad x\to\infty,$$

where

$$A_n = \frac{(-1)^{n-1}\Gamma\left(\frac{n\alpha}{2}+1\right)}{\pi n!}\left(1+\tan^2\frac{\pi\alpha}{4}\right)^{n/2}\sin\left[n\frac{\pi\alpha}{4}+n\tan^{-1}\left(\frac{\pi\alpha}{4}\right)\right].$$

PROOF. This follows from A2 for $t = 1$ and the asymptotic expansion of $h_{\alpha/2}(u)$ given in [6]. □

PROPOSITION A.14.

$$\int_0^\infty r(t)\,dt < \infty \Leftrightarrow \int\hat{\phi}_B(x)\phi_B(x)\,dx < \infty \tag{A.19}$$

for all bounded Borel sets B.

PROOF. Note that

$$r(t) \sim \int_t^\infty p(s,0)\,ds, \qquad t \to \infty,$$

so

$$\int_0^\infty r(t)\,dt < \infty \Leftrightarrow \int_0^\infty dt \int_t^\infty p(s,0)\,ds < \infty.$$

Now

$$\int_t^\infty p(s,y-x)\,ds \sim \int_t^\infty p(s,0), \qquad t \to \infty,$$

uniformly in x, y in compact sets. Hence

$$\int\int\left[\int_t^\infty p(s,y-x)\,ds\right]\hat{\mu}_B(dx)\mu_B(dy) \sim C(B)^2\int_t^\infty p(s,0)\,ds,$$

so

$$\int\int\left[\int_0^\infty dt\int_s^\infty p(s,y-x)\,ds\right]\hat{\mu}_B(dx)\mu_B(dy) < \infty \Leftrightarrow \int_0^\infty dt\int_t^\infty p(s,0)\,ds < \infty.$$

But

$$\int_0^\infty dt\int_s^\infty p(s,y-x)\,ds = \int g(z-x)g(y-z)\,dz,$$

so (A.19) holds. □

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