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ASYMPTOTIC STATIONARITY OF QUEUES IN SERIES AND THE HEAVY TRAFFIC APPROXIMATION

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A tandem queue with m single server stations and unlimited interstage storage is considered. Such a tandem queue is described by a generic sequence of nonnegative random vectors in R^{m+1}. The first m coordinates of the kth element of the generic sequence represent the service times of the kth unit in m single server queues, respectively, and the $(m+1)$th coordinate represents the interarrival time between the kth and $(k+1)$th units to the tandem queue. The sequences of vectors $\tilde{w}_k = (w_k(1), w_k(2), \dots, w_k(m))$ and $\tilde{W}_k = (W_k(1), W_k(2), \dots, W_k(m))$, where $w_k(i)$ represents the waiting time of the kth unit in the ith queue and $W_k(i)$ represents the sojourn time of the kth unit in the first i queues, are studied. It is shown that if the generic sequence is asymptotically stationary in some sense and it satisfies some natural conditions then $\mathbf{w} = \{\tilde{w}_k, k \geq 1\}$ and $\mathbf{W} = \{\tilde{W}_k, k \geq 1\}$ are asymptotically stationary in the same sense. Moreover, their stationary representations are given and the heavy traffic approximation of that stationary representation is given.

1. Introduction. This paper deals with a system of queues in series: that is to say, a number of queues through which a unit passes in turn, spending a waiting time including service in any particular one, and proceeding to the next succeeding queue immediately the service time is completed. Such systems were considered by Loynes [6, 7], Harrison [3] and others. Our first main result (Theorem 1) deals with the asymptotic behaviour of the vector of waiting times of a unit in each queue and of the time which a unit spends in the tandem queue (Theorem 2). Our second main result (Theorem 3 and Corollary 3) deals with the heavy traffic approximation of the mentioned characteristics.

The system we consider can be described by the sequence

$$\{(v_k(1), v_k(2), \dots, v_k(m), u_k(1)), k \geq 1\}$$

of nonnegative random vectors in R^{m+1}, where $v_k(i)$, $1 \leq i \leq m$, $k \geq 1$, represents the service time of the kth unit in the ith queue and $u_k(1)$, $k \geq 1$, represents the interarrival time between the kth and $(k+1)$th units to the tandem queue. Henceforth this sequence is denoted by $(\mathbf{v}, \mathbf{u}(1))$ and it is called the generic sequence and the vector $(v_k(1), v_k(2), \dots, v_k(m))$ is denoted by v_k. For each i, $1 \leq i \leq m$, let us denote

$$(\mathbf{v}, \mathbf{u}(1), \mathbf{w}(1), \mathbf{u}(2), \dots, \mathbf{w}(i), \mathbf{u}(i+1)) =_{\text{df}} \{(v_k, u_k(1), w_k(1), u_k(2), \dots, w_k(i), u_k(i+1)), k \geq 1\},$$

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where $w_k(i)$, $k \geq 1$, $1 \leq i \leq m$, represents the waiting time of the kth unit in the ith queue and $u_k(i)$, $k \geq 1$, $1 \leq i \leq m$, represents the interarrival time between the kth and $(k+1)$th units to the ith queue [it is also the interdeparture time between the kth and $(k+1)$th units from the $(i-1)$th queueing system]. This sequence for $i = m$ is called the exit process of the tandem queue.

Besides that let us denote by $W_k(i)$, $1 \leq i \leq m$, $k \geq 1$, the total time which the kth unit spends in the first i queues (let the *sojourn time* of the kth unit in the first i queues be the time which elapses between the arrival of the kth unit at the tandem queue and its departure from the ith queue).

One of the main results of the paper (Theorem 1) gives conditions on the generic sequence under which the exit process of the tandem queue is asymptotically stationary in some sense. Moreover, the form of the stationary representation of the exit process is given. As a consequence of Theorem 1 we obtain the existence of a limit distribution of the sojourn time of the kth unit, its form and the type of that convergence (Theorem 2). Theorems 1 and 2 assume that the generic sequence is ergodic and it is either strongly asymptotically stationary or strongly asymptotically stationary in mean or asymptotically stationary in variation or asymptotically stationary in variation in mean. These conditions are weaker than the conditions assumed by Harrison in [3]. In [3] it is assumed that the sequences $\mathbf{v}(1), \mathbf{v}(2), \dots, \mathbf{v}(m), \mathbf{u}(1)$ are independent and each of them is a sequence of independent and identically distributed random variables. Thus the model of [3] does not include the case $\mathbf{v}(1) = \mathbf{v}(2) = \cdots = \mathbf{v}(m)$ or the case when these sequences are jointly dependent with $\mathbf{u}(1)$ in some way. For a discussion of application areas where such dependencies arise naturally, the reader is referred to Boxma [2] and Kelly [4].

The second result (Theorem 3 and Corollary 3) gives the heavy traffic approximation of the steady-state distribution of the vector of sojourn times and the vector of waiting times of a unit in the m stations. Similarly as in [3] this approximation is based on a Brownian approximation of some vector-valued process which is a function of a stationary representation of the generic sequence. Of course our multidimensional Brownian motion may be such that its coordinates are dependent. This result contains Harrison's result as a special case. Some fragments of the proof of Theorem 3 are based on Lemmas 1–3 from [11].

2. Preliminaries. The main notion of the paper is taken from [1], [5] and [9]. Here we recall some of that notation and we reformulate some results from [8] and [9] in a form useful for our further considerations.

Let $D[0,\infty)$ denote the set of real-valued right-continuous functions on $[0,\infty)$ which have left limits everywhere, endowed with Lindvall's metric d defined in [5]. This space is a Polish metric space and the topology generated by d is equivalent to the Stone topology on $D[0,\infty)$. Let $D^m[0,\infty)$ and d^m, $m \geq 1$, denote the m-fold product of $D[0,\infty)$ and of d, respectively.

Let $\mathbf{X} = \{X_k,\ k \geq 1\}$ be a sequence of random elements of a Polish metric space S and $\mathbf{X}_n =_{\mathrm{df}} \{X_{n+k},\ k \geq 1\}$. If the sequence of distributions $\mathscr{L}(\mathbf{X}_n)$,

$n \geq 1$, either weakly converges or strongly converges or converges in variation to a probability measure μ on S, then $\mathbf{X}$ is said to be either weakly asymptotically stationary or strongly asymptotically stationary or asymptotically stationary in variation, respectively. If the sequence of distributions $n^{-1}\sum_{k=1}^{n}\mathscr{L}(\mathbf{X}_k)$, $n \geq 1$, either weakly converges or strongly converges or converges in variation then $\mathbf{X}$ is said to be either weakly asymptotically stationary in mean or strongly asymptotically stationary in mean or asymptotically stationary in variation in mean, respectively. These are the six types of asymptotic stationarity studied in [9]. A sequence $\mathbf{X}^0 = \{X_k^0,\ k \geq 1\}$ of random elements of S having distribution μ is called a stationary representation of $\mathbf{X}$. By $\mathbf{X}^* = \{X_k^*,\ -\infty < k < \infty\}$ is denoted a two-sided stationary extension of $\mathbf{X}^0$.

Let us consider a single server queueing system which operates in a first-in–first-out manner. Such a system can be described by the generic sequence $(\mathbf{v}, \mathbf{u}) = \{(v_k, u_k),\ k \geq 1\}$, where v_k, $k \geq 1$, represents the service time of the kth unit and u_k, $k \geq 1$, represents the interarrival time between the kth and $(k+1)$th units. Let us denote by w_k, $k \geq 1$, the waiting time of the kth unit and $\mu_k(2)$, $k \geq 1$, the interdeparture time between the kth and $(k+1)$th units. Obviously,

$$w_{k+1} = \max(0, w_k + v_k - u_k), \qquad k \geq 1,$$

for all initial conditions $w_1 \geq 0$, and

$$u_k(2) = u_k + v_{k+1} + w_{k+1} - v_k - w_k, \qquad k \geq 1.$$

Let $\mathbf{Y} = \{Y_k,\ k \geq 1\}$ be a sequence of random vectors defined on the same probability space as the generic sequence. The sequence $(\mathbf{Y}, \mathbf{v}, \mathbf{u}) = \{(Y_k, v_k, u_k),\ k \geq 1\}$ is called the enter process of the single server queue and $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{u}(2)) = \{(Y_k, v_k, u_k, w_k, u_k(2)),\ k \geq 1\}$ is called the exit process of this system. This notation is introduced for the convenience of our considerations in Section 3.

Studying the proof method of Theorems 1–3 in [9], we notice that if $(\mathbf{Y}, \mathbf{v}, \mathbf{u})$ is asymptotically stationary in one of six senses defined in [9] and $(\mathbf{v}, \mathbf{u})$ and $(\mathbf{v}^0, \mathbf{u}^0)$ satisfy conditions which are needed for the appropriate type of asymptotic stationarity of $(\mathbf{v}, \mathbf{u}, \mathbf{w})$, then $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w})$ is asymptotically stationary in the same sense as $(\mathbf{Y}, \mathbf{v}, \mathbf{u})$.

Now let us notice that the process $\mathbf{u}(2)$ is a function, say f, of $(\mathbf{v}, \mathbf{u}, \mathbf{w})$, namely

$$\mathbf{u}(2) = f(\mathbf{v}, \mathbf{u}, \mathbf{w}) = \mathbf{u} + T\mathbf{v} + T\mathbf{w} - \mathbf{v} - \mathbf{w},$$

where T is the shift transformation in R. Moreover,

$$T\mathbf{u}(2) = Tf(\mathbf{v}, \mathbf{u}, \mathbf{w}) = f(T\mathbf{v}, T\mathbf{u}, T\mathbf{w}).$$

Since the operation of addition in R^∞ and the operation T are continuous in R^∞, by the above facts and by Propositions 1–3 in [9] we obtain the following.

REMARK 1. If $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w})$ is asymptotically stationary in one of the six senses defined in [9], then $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{u}(2))$ is asymptotically stationary in the same sense.

Compiling the above considerations with Theorems 2 and 3 in [9] and Corollary 2 in [8], we get the following fact.

LEMMA 1. *Let the enter process* $(\mathbf{Y}, \mathbf{v}, \mathbf{u})$ *be either strongly asymptotically stationary or strongly asymptotically stationary in mean or asymptotically stationary in variation or asymptotically stationary in variation in mean and let its stationary representation be ergodic and* $Ev_1^0 < Eu_1^0$. *Then the exit process* $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{u}(2))$ *is asymptotically stationary in the same sense as the enter process* $(\mathbf{Y}, \mathbf{v}, \mathbf{u})$. *Furthermore, the exit process and its stationary representation are ergodic and* $Eu_1^0(2) = Eu_1^0$. *Moreover, the stationary representation of the exit process* $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{u}(2))$ *is the same as the stationary representation of* $(\mathbf{Y}^0, \mathbf{v}^0, \mathbf{u}^0, \mathbf{w}, \mathbf{u}(2))$ *and the two-sided stationary extension of*

$$\left(\mathbf{Y}^0, \mathbf{v}^0, \mathbf{u}^0, \mathbf{w}^0, \mathbf{u}^0(2)\right)$$

has the following form:

$$w_{k+1}^* = \sup_{-\infty < n \le k} S_{n,k} \qquad \text{for } -\infty < k < \infty,$$

$$u_k^*(2) = u_k^* + v_{k+1}^* + w_{k+1}^* - v_k^* - w_k^* \quad \text{for } -\infty < k < \infty,$$

where

$$S_{n,k} = \sum_{j=n+1}^{k} (v_j^* - u_j^*) \quad \text{for } n < k,$$

$$S_{k,k} = 0 \qquad \text{for } -\infty < k < \infty,$$

and $(\mathbf{Y}^*, \mathbf{v}^*, \mathbf{u}^*) = \{(Y_k^*, v_k^*, u_k^*),\ -\infty < k < \infty\}$ *is a two-sided stationary extension of* $(\mathbf{Y}^0, \mathbf{v}^0, \mathbf{u}^0)$.

PROOF. The appropriate type of asymptotic stationarity of $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{u}(2))$ follows from Remark 1 and Theorems 2 and 3 in [9]. The ergodicity of the sequences $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{u}(2))$ and $(\mathbf{Y}^0, \mathbf{v}^0, \mathbf{u}^0, \mathbf{w}^0, \mathbf{u}^0(2))$ and the equality $Eu_1^0(2) = Eu_1^0$ follow from Corollary 2 in [8].

Now $(\mathbf{Y}^*, \mathbf{v}^*, \mathbf{u}^*, \mathbf{w}^*, \mathbf{u}^*(2))$ is stationary and $\{(Y_k^*, v_k^*, u_k^*, w_k^*, u_k^*(2)),\ k \ge 1\}$ is distributed as $(\mathbf{Y}^0, \mathbf{v}^0, \mathbf{u}^0, \mathbf{w}^0, \mathbf{u}^0(2))$, $(\mathbf{v}^0, \mathbf{u}^0, \mathbf{w}^0)$ is given in Theorem 1 in [9], and so we have the assertion. □

3. Asymptotic stationarity. It is assumed throughout the paper that the generic sequence is asymptotically stationary in some sense, and for some parameters of its stationary representation the following notation is used:

$$a = Eu_1^0(1), \qquad b_i = Ev_1^0(i) \quad \text{for } 1 \le i \le m,$$

$$\alpha = \min_{1 \le i \le m} (a - b_i) \quad \text{and} \quad c_i = (a - b_i)/\alpha \quad \text{for } 1 \le i \le m.$$

Furthermore, let us denote

$$\mathbf{Z}(i) =_{\mathrm{df}} (\mathbf{v}, \mathbf{u}(1), \mathbf{w}(1), \mathbf{u}(2), \dots, \mathbf{w}(i), \mathbf{u}(i+1)), \qquad 1 \le i \le m.$$

The sequence $\mathbf{Z}(i)$, for $1 \le i \le m$, is called the exit process of the tandem queue which consists of the first i queueing systems and it is called the enter process of the tandem queue which consists of the first $i + 1$ queueing systems.

THEOREM 1. *Let the generic sequence* $(\mathbf{v}, \mathbf{u}(1))$ *be ergodic and let it be either strongly asymptotically stationary or strongly asymptotically stationary in mean or asymptotically stationary in variation or asymptotically stationary in variation in mean. Furthermore, let its stationary representation be such that* $\alpha > 0$. *Then the exit process of the tandem queue is asymptotically stationary in the same sense as the generic sequence. Furthermore, the exit process and its stationary representation are ergodic and* $Eu_1^0(i) = Eu_1^0(1)$ *for* $1 \le i \le m$. *Moreover, the two-sided stationary extension of the stationary representation of the exit process has the following form*:

$$(3.1) \quad w_{k+1}^*(i) = \sup_{-\infty < n \le k} S_{n,k}(i),$$

$$(3.2) \qquad u_k^*(i) = u_k^*(1) + \sum_{j=1}^{i-1} \left(v_{k+1}^*(j) + w_{k+1}^*(j) - v_k^*(j) - w_k^*(j)\right)$$

for all k, $-\infty < k < \infty$, *and* $1 \le i \le m$, *where*

$$S_{n,k}(i) = \sum_{j=n+1}^{k} \left(v_j^*(i) - u_j^*(1)\right)$$
$$+ \sum_{j=1}^{i-1} \left(v_{n+1}^*(j) + w_{n+1}^*(j) - v_{k+1}^*(j) - w_{k+1}^*(j)\right)$$

for all $n < k$ *and* $S_{k,k}(i) = 0$ *for all* k, $-\infty < k < \infty$, $1 \le i \le m$, *while* $(\mathbf{v}^*, \mathbf{u}^*(1)) = \{(v_k^*(1), v_k^*(2), \ldots, v_k^*(m), u_k^*(1)),\ -\infty < k < \infty\}$ *is a two-sided stationary extension of* $(\mathbf{v}^0, \mathbf{u}^0(1))$.

PROOF. The proof is obtained by applying Lemma 1 m times. At the first step we use Lemma 1, taking as the enter process the generic sequence $(\mathbf{v}, \mathbf{u}(1))$ and as the exit process the sequence $\mathbf{Z}(1)$. As a result we obtain that the exit process $\mathbf{Z}(1)$ is asymptotically stationary in the same sense as the enter process. Furthermore, the exit process and its stationary representation are ergodic and $Eu_1^0(2) = Eu_1^0(1)$. Moreover, the stationary representation of the exit process $\mathbf{Z}(1)$ is the same as the stationary representation of the process $(\mathbf{v}^0, \mathbf{u}^0(1), \mathbf{w}(1), \mathbf{u}(2))$. The two-sided stationary extension of $(\mathbf{v}^0, \mathbf{u}^0, \mathbf{w}^0(1), \mathbf{u}^0(2))$ has the following form:

$$w_{k+1}^*(1) = \sup_{-\infty < n \le k} S_{n,k}(1), \qquad -\infty < k < \infty,$$

$$u_k^*(2) = u_k^*(1) + v_{k+1}^*(1) + w_{k+1}^*(1) - v_k^*(1) - w_k^*(1), \qquad -\infty < k < \infty,$$

where

$$S_{n,k}(1) = \sum_{j=n+1}^{k} \left(v_j^*(1) - u_j^*(1)\right) \quad \text{for } n < k,$$

$$S_{k,k}(1) = 0 \qquad \text{for } -\infty < k < \infty,$$

while $(\mathbf{v}^*, \mathbf{u}^*(1))$ is a two-sided stationary extension of $(\mathbf{v}^0, \mathbf{u}^0(1))$. The form of $\mathbf{Z}^*(1)$ given here agrees with the form of $\mathbf{Z}^*(i)$ given in the theorem for $i = 1$.

At the second step we use Lemma 1, taking as the enter process the sequence $\mathbf{Z}(1)$ and as the exit process the sequence $\mathbf{Z}(2)$. Of course, by the first step, the enter process $\mathbf{Z}(1)$ satisfies the conditions of Lemma 1. Thus the exit process $\mathbf{Z}(2)$ is asymptotically stationary in the same sense as the enter process $\mathbf{Z}(1)$. Furthermore, the exit process $\mathbf{Z}(2)$ and its stationary representation $\mathbf{Z}^0(2)$ are ergodic and $Eu_1^0(3) = Eu_1^0(2)$. Moreover, the stationary representation of the exit process is the same as the stationary representation of the process $(\mathbf{v}^0, \mathbf{u}^0(1), \mathbf{w}^0(1), \mathbf{u}^0(2), \mathbf{w}(2), \mathbf{u}(3))$.

The two-sided stationary extension of $(\mathbf{v}^0, \mathbf{u}^0(1), \mathbf{w}^0(1), \mathbf{u}^0(2), \mathbf{w}^0(2), \mathbf{u}^0(3))$ has the following form:

$$w_{k+1}^*(2) = \sup_{-\infty < n \le k} S_{n,k}(2), \qquad -\infty < k < \infty,$$

$$u_k^*(3) = u_k^*(2) + v_{k+1}^*(2) + w_{k+1}^*(2) - v_k^*(2) - w_k^*(2), \qquad -\infty < k < \infty,$$

where

$$S_{n,k}(2) = \sum_{j=n+1}^{k} \left(v_j^*(2) - u_j^*(2)\right) \quad \text{for } n < k,$$

$$S_{k,k}(2) = 0 \qquad \text{for } -\infty < k < \infty.$$

Hence for $n < k$,

$$S_{n,k}(2) = \sum_{j=n+1}^{k} \left(v_j^*(2) - u_j^*(1)\right) + v_{n+1}^*(1) + w_{n+1}^*(1) - v_{k+1}^*(1) - w_{k+1}^*(1).$$

Thus the form of $\mathbf{Z}^*(2)$ given here agrees with the form of $\mathbf{Z}^*(i)$ given in the theorem for $i = 2$.

Continuing the above procedure $m - 1$ times, we arrive to the mth step in which as the enter process we take the sequence $\mathbf{Z}(m - 1)$ and as the exit process we take the sequence $\mathbf{Z}(m)$. Of course, by the $(m - 1)$th step, the enter process $\mathbf{Z}(m - 1)$ satisfies the conditions of Lemma 1. Thus, applying Lemma 1, we obtain that the exit process $\mathbf{Z}(m)$ is asymptotically stationary in the same sense as the enter process $\mathbf{Z}(m - 1)$. Furthermore, the exit process and its stationary representation are ergodic and $Eu_1^0(m) = Eu_1^0(1)$. Moreover, the stationary representation of the exit process $\mathbf{Z}(m)$ is the same as the stationary representation of the sequence $(\mathbf{v}^0, \mathbf{u}^0(1), \mathbf{w}^0(1), \mathbf{u}^0(2), \dots, \mathbf{u}^0(m), \mathbf{w}(m), \mathbf{u}(m + 1))$. But the two-sided stationary extension of $\mathbf{Z}^0(m)$ has the following form:

$$w^*_{k+1}(m) = \sup_{-\infty < n \le k} S_{n,k}(m), \qquad -\infty < k < \infty,$$

$$u^*_k(m+1) = u^*_k(m) + v^*_{k+1}(m) + w^*_{k+1}(m) - v^*_k(m) - w^*_k(m), \qquad -\infty < k < \infty,$$

where

$$S_{n,k}(m) = \sum_{j=n+1}^{k} \big(v^*_j(m) - u^*_j(m)\big) \quad \text{for } n < k,$$

$$S_{k,k}(m) = 0, \qquad -\infty < k < \infty,$$

while $(\mathbf{v}^*, \mathbf{u}^*(1), \mathbf{w}^*(1), \mathbf{u}^*(2), \dots, \mathbf{w}^*(m-1), \mathbf{u}^*(m))$ is a two-sided stationary extension of $\mathbf{Z}^0(m-1)$.

Now solving the system of recurrent equations

$$u^*_k(i+1) = u^*_k(i) + v^*_{k+1}(i) + w^*_{k+1}(i) - v^*_k(i) - w^*_k(i), \qquad 1 \le i \le m,$$

we obtain

$$u^*_k(i+1) = u^*_k(1) + \sum_{j=1}^{i} \big(v^*_{k+1}(j) + w^*_{k+1}(j) - v^*_k(j) - w^*_k(j)\big).$$

Hence for $n < k$,

$$S_{n,k}(i) = \sum_{j=n+1}^{k} \big(v^*_j(j) - u^*_j(1)\big) + \sum_{j=1}^{i-1} \big(v^*_{n+1}(j) + w^*_{n+1}(j) - v^*_{k+1}(j) - w^*_{k+1}(j)\big)$$

and $S_{k,k}(i) = 0$ for all k, $-\infty < k < \infty$, $1 \le i \le m$. Furthermore, we see that the form of $\mathbf{Z}^*(m)$ given here agrees with the form of $\mathbf{Z}^*(i)$ given in the theorem for $i = m$. This finishes the proof. □

COROLLARY 1. *Under the conditions of Theorem* 1 *the following relations hold*:

$$w^*_{k+1}(i) = \max\big(0, w^*_k(i) + v^*_k(i) - u^*_k(i)\big), \qquad -\infty < k < \infty,\ 1 \le i \le m, \tag{3.3}$$

$$E\big(v^*_k(i) - u^*_k(i)\big)^2 \le C(i)\left\{\sum_{j=1}^{i} E\big(v^*_k(j) - u^*_k(1)\big)^2 + 4\sum_{j=1}^{i-1} E\big(v^*_k(j) - b_j\big)^2\right\}, \tag{3.4}$$

where $C(i)$ *depends only on* i.

PROOF. The relation (3.3) follows immediately from (3.1). To prove (3.4), let us notice that by (3.3) we have

$$(w_{k+1}^*(i) - w_k^*(i))^2 = (\max(-w_k^*(i), v_k^*(i) - u_k^*(i)))^2 \le (v_k^*(i) - u_k^*(i))^2.$$

Hence and from the relation (3.2) we get

$$E(v_k^*(i) - u_k^*(i))^2 \le (2i-1)^2 \Bigg\{ E(v_k^*(i) - u_k^*(1))^2 + 4 \sum_{j=1}^{i-1} E\big(v_k^*(j) - b_j\big)^2 + \sum_{j=1}^{i-1} E\big(v_k^*(j) - u_k^*(j)\big)^2 \Bigg\}$$

for $1 \le i \le m$. Now solving this scheme of recurrent relations we get (3.4), where $C(i)$ depends only on i. □

THEOREM 2. *Under the conditions of Theorem* 1 *the sequence*

$$\theta =_{\mathrm{df}} (\mathbf{v}, \mathbf{u}(1), \mathbf{W}(1), \mathbf{u}(2), \dots, \mathbf{W}(m), \mathbf{u}(m+1))$$

is asymptotically stationary in the same sense as the generic sequence $(\mathbf{v}, \mathbf{u}(1))$. *Moreover, the two-sided stationary extension of the stationary representation* θ^0 *has the following form*:

$$W_{k+1}^*(i) = \sup_{-\infty < n_1 \le n_2 \le \cdots \le n_i \le n_{i+1} = k} \sum_{j=1}^{i} \Bigg\{ \sum_{s=n_j+1}^{n_{j+1}} (v_s^*(j) - u_s^*(1)) + v_{n_{j+1}+1}^*(j) \Bigg\}, \tag{3.5}$$

$$u_k^*(i) = u_k^*(1) + W_{k+1}^*(i-1) - W_k^*(i-1),$$

for $-\infty < k < \infty$, $1 \le i \le m$, *where* $(\mathbf{v}^*, \mathbf{u}^*(1))$ *is a two-sided stationary extension of* $(\mathbf{v}^0, \mathbf{u}^0(1))$.

PROOF. Let us notice that

$$W_k(i) = \sum_{j=1}^{i} (w_k(j) + v_k(j)) \quad \text{for } k \ge 1, 1 \le i \le m.$$

Hence θ is a function, say f, of the exit process $\mathbf{Z}(m)$. Moreover,

$$T_{3m}\theta = f(T_{3m} Z(m)), \tag{3.6}$$

where T_{3m} is the shift transformation in $(R^{3m})^\infty$. Hence the first assertion, i.e., the asymptotic stationarity of θ, follows from Propositions 2 and 3 in [9] and from Theorem 1.

To obtain the second assertion, i.e., the form of θ^0, let us notice that by (3.6) and Propositions 2 and 3 in [9] we obtain

$$\theta^0 = f(\mathbf{Z}^0(m)).$$

Now let us notice that $w_{k+1}^*(i)$ given in Theorem 1 can be rewritten as

$$w_{k+1}^*(i) = \sup_{-\infty<n\le k}\left\{\sum_{j=n+1}^{k}\left(v_j^*(i) - u_j^*(1)\right) + W_{n+1}^*(i-1) - W_{k+1}^*(i-1)\right\}. \tag{3.7}$$

Since

$$W_k^*(i) = W_k^*(i-1) + w_k^*(i) + v_k^*(i), \qquad -\infty < k < \infty, 1 \le i \le m,$$

thus by (3.7) we have

$$W_{k+1}^*(i) = \sup_{-\infty<n\le k}\left\{\sum_{j=n+1}^{k}\left(v_j^*(i) - u_j^*(1)\right) + v_{k+1}^*(i) + W_{n+1}^*(i-1)\right\}$$

for $-\infty < k < \infty$ and $1 \le i \le m$. This scheme of recurrent equations with respect to i, jointly with the initial condition $W_k^*(1) = w_k^*(1) + v_k^*(1)$, imply the form of $W_{k+1}^*(i)$ given by (3.5). This finishes the proof. □

Corollary 2. *If* $(\mathbf{v}, \mathbf{u}(1))$ *satisfies conditions of Theorem* 1 *and* $\mathbf{v}(1) = \mathbf{v}(2) = \cdots = \mathbf{v}(m)$ *or* $\mathbf{v}^*(1) = \mathbf{v}^*(2) = \cdots = \mathbf{v}^*(m)$, *then*

$$W_{k+1}^*(i) = \sup_{-\infty<n_1\le n_2\le\cdots\le n_i\le n_{i+1}=k}\left\{\sum_{j=n_1+1}^{k}\left(v_j^*(1) - u_j^*(1)\right) + v_{n_2+1}^*(1) + v_{n_3+1}^*(1) + \cdots + v_{n_i+1}^*(1) + v_{n_{i+1}+1}^*(1)\right\}.$$

Remark 2. The random variables $W_1^*(i)$, $1 \le i \le m$, can be rewritten as

$$W_1^*(i) = \sup_{t\ge 0}\ \sup_{0=t_{i+1}\le t_i\le\cdots\le t_2\le t_1=t}\ \sum_{j=1}^{i}\left\{\sum_{k=-[t_j]+1}^{-[t_{j+1}]}\left(v_k^*(j) - u_k^*(1)\right) + v_{-[t_{j+1}]+1}^*(j)\right\}.$$

Notice that the vector of waiting times can be obtained from the vector of sojourn times by a continuous mapping. Namely, let G be the mapping of R^m into R^m defined at $x = (x^1, x^2, \dots, x^m) \in R^m$ as $G(x) = (x^1, x^2 - x^1, \dots, x^m - x^{m-1})$.

Remark 3. The following relation holds:

$$\left(w_1^*(1), w_1^*(2), \dots, w_1^*(m)\right) = G\left(W_1^*(1), W_1^*(2), \dots, W_1^*(m)\right) - \left(v_1^*(1), v_1^*(2), \dots, v_1^*(m)\right).$$

4. The heavy traffic result. We consider now a sequence of systems indexed by $r \geq 1$. It is assumed that each system is composed of m stations and is generated by a generic sequence satisfying the conditions of Theorem 1. We shall maintain all of the notation established earlier for the sundry random variables of interest, except that a functional dependence on r will be added to indicate a random variable associated with the rth system. Thus, for example,

$$(\mathbf{v}(r), \mathbf{u}(1,r)) = \{(v_k(1,r), v_k(2,r), \ldots, v_k(m,r), u_k(1,r)),\ k \geq 1\},$$
$$(\mathbf{v}^0(r), \mathbf{u}^0(1,r)) = \{(v_k^0(1,r), v_k^0(2,r), \ldots, v_k^0(m,r), u_k^0(1,r)),\ k \geq 1\},$$
$$(\mathbf{v}^*(r), \mathbf{u}^*(1,r)) = \{(v_k^*(1,r), v_k^*(2,r), \ldots, v_k^*(m,r), u_k^*(1,r)),\ -\infty < k < \infty\}$$

denote the generic sequence, the stationary representation of $(\mathbf{v}(r), \mathbf{u}(1,r))$ and the two-sided stationary extension of the stationary representation $(\mathbf{v}^0(r), \mathbf{u}^0(1,r))$, respectively, for the rth system. Furthermore,

$$\tilde{w}_k(r) =_{\mathrm{df}} (w_k(1,r), w_k(2,r), \ldots, w_k(m,r))$$

and

$$\tilde{W}_k(r) =_{\mathrm{df}} (W_k(1,r), W_k(2,r), \ldots, W_k(m,r))$$

denote the vectors of waiting times and of sojourn times of the kth unit. Under the assumptions made in Theorem 1 the sequences of distributions $\mathscr{L}(\tilde{w}_k(r))$ and $\mathscr{L}(\tilde{W}_k(r))$, $k \geq 1$, for each fixed $r \geq 1$, are convergent as $k \to \infty$, in the senses described in Theorem 1. These limiting distributions are the same as the distributions of the random vectors $\tilde{w}(r) =_{\mathrm{df}} (w_1^*(1, r), w_1^*(2, r), \ldots, w_1^*(m, r))$ and $\tilde{W}(r) =_{\mathrm{df}} (W_1^*(1, r), W_1^*(2, r), \ldots, W_1^*(m, r))$, respectively, defined in Theorems 1 and 2. For example, the form of $\tilde{W}(r)$ is the following:

$$W_1^*(i,r) = \sup_{t \geq 0} \sup_{0 = t_{i+1} \leq t_i \leq \cdots \leq t_2 \leq t_1 = t} \sum_{j=1}^{i} \left\{ \sum_{k=-[t_j]+1}^{-[t_{j+1}]} (v_k^*(j,r) - u_k^*(1,r)) + v^*_{-[t_{j+1}]+1}(j,r) \right\}. \tag{4.1}$$

(Throughout the paper square brackets are used exclusively to denote the integer part and sums of the type $\sum_{j=i}^{k}$ where $k < i$ are taken as 0.)

The same convention as above is used in denoting the important constants associated with the rth system. It is assumed throughout that

A_1 $\quad \alpha(r) \downarrow 0$ as $r \to \infty$;

A_2 $\quad (a(r) - b_k(r))/\alpha(r) \to c_k$ as $r \to \infty$, $1 \leq k \leq m$, where $c_1, c_2, \ldots, c_m$ are all positive and finite;

A_3 $\quad \max_{1 \leq i \leq m} \sup_r \{E(v_1^*(i,r) - u_1^*(1,r))^2 + \operatorname{var} v_1^*(i,r)\} < \infty.$

Let us denote by $\xi = (\xi^1, \xi^2, \ldots, \xi^m, \xi^{m+1})$, the $(m + 1)$-dimensional Wiener process, i.e., the $(m + 1)$-dimensional Gaussian process being a random element of $D^{m+1}[0,\infty)$ and such that ξ^i, $1 \le i \le m+1$, are Wiener processes with $E\xi^i(t)\xi^j(s) = E\xi^j(t)\xi^i(s) = \sigma_{i,j}\min(s,t)$, $1 \le i, j \le m$ ($\sigma_{i,i} = 1$). Thus almost all sample paths of ξ are continuous. For the process ξ let us define the process $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^m, \zeta^{m+1})$ as

$$\zeta^i(t) = \sigma_i \xi^i(t) - \sigma_{m+1}\xi^{m+1}(t) - c_i t, \qquad t \ge 0, 1 \le i \le m,$$

where σ_i, $1 \le i \le m$, are some positive and finite constants and c_i, $1 \le i \le m$, are defined in assumption A_2.

As in [3] let us define the random vector $Y = (Y^1, Y^2, \dots, Y^m)$ by

$$Y^i = \sup_{0 = t_{i+1} \le t_i \le \cdots \le t_2 \le t_1 < \infty} \sum_{j=1}^{i} \left(\zeta^j(t_j) - \zeta^j(t_{j+1})\right), \qquad 1 \le i \le m,$$

and the process $\chi = (\chi^1, \chi^2, \dots, \chi^m)$ by

$$\chi^i(t) = \sup_{0 = t_{i+1} \le t_i \le \cdots \le t_1 = t} \sum_{j=1}^{i} \left(\zeta^j(t_j) - \zeta^j(t_{j+1})\right), \qquad t \ge 0, 1 \le i \le m.$$

Analogously as in [3] (see Lemma 3 in [3]), we can prove the following convergences:

$$\begin{aligned} &t^{-1}\zeta^i(t) \to -c_i && \text{a.e., as } t \to \infty, 1 \le i \le m, \\ &t^{-1}\chi^i(t) \to \max_{1 \le j \le m}\{-c_j\} && \text{a.e., as } t \to \infty, 1 \le i \le m. \end{aligned} \tag{4.2}$$

Hence it follows that Y is almost surely finite.

Let us define the processes

$$\begin{aligned} \tilde{S}_r &= \left(\tilde{S}_r^1, \tilde{S}_r^2, \dots, \tilde{S}_r^{m+1}\right), \\ \beta_r &= \left(\beta_r^1, \beta_r^2, \dots, \beta_r^{m+1}\right), \\ \hat{S}_r &= \left(\hat{S}_r^1, \hat{S}_r^2, \dots, \hat{S}_r^m\right), \\ \gamma_r &= \left(\gamma_r^1, \gamma_r^2, \dots, \gamma_r^m\right) \end{aligned}$$

as follows:

$$\tilde{S}_r^i(t) = \alpha(r) \sum_{j=-[t/\alpha^2(r)]+1}^{0} \left(v_j^*(i,r) - b_i(r)\right), \qquad t \ge 0, 1 \le i \le m,$$

$$\tilde{S}_r^{m+1}(t) = \alpha(r) \sum_{j=-[t/\alpha^2(r)]+1}^{0} \left(u_j^*(1,r) - a(r)\right), \qquad t \ge 0,$$

$$\beta_r^i(t) = \alpha(r)\left[t/\alpha^2(r)\right] b_i(r), \qquad t \ge 0, 1 \le i \le m,$$

$$\beta_r^{m+1}(t) = \alpha(r)\left[t/\alpha^2(r)\right] a(r), \qquad t \ge 0,$$

$$\hat{S}_r^i = \tilde{S}_r^i - \tilde{S}_r^{m+1} + \beta_r^i - \beta_r^{m+1}, \qquad 1 \le i \le m,$$

$$\gamma_r^i(t) = \sup_{0 = t_{i+1} \le t_i \le \cdots \le t_1 = t} \sum_{j=1}^{i} \left\{ \frac{1}{\alpha(r)}\left(\hat{S}_r^j(t_j) - \hat{S}_r^j(t_{j+1})\right) + v^*_{-[t_{j+1}/\alpha^2(r)]+1}(j,r) \right\},$$
$$t \ge 0, 1 \le i \le m.$$

In view of the above and (4.1) we obtain

$$W_1^*(i,r) = \sup_{t\ge 0}\gamma_r^i(t), \qquad 1\le i\le m. \tag{4.3}$$

Now let us define the mapping g: $D^m[0,\infty)\to D^m[0,\infty)$ as $g = (g^1, g^2, \dots, g^m)$, where g^i on $x = (x^1, x^2, \dots, x^m)\in D^m[0,\infty)$ ($x^i\in D[0,\infty)$) is defined as

$$g^i(x)(t) = \sup_{0=t_{i+1}\le t_i\le \cdots \le t_1 = t}\sum_{j=1}^{i}\left(x^j(t_j) - x^j(t_{j+1})\right), \qquad t\ge 0.$$

Each of g^i is a measurable mapping of $D^m[0,\infty)$ into $D[0,\infty)$. Moreover, each g^i is continuous at $x = (x^1, x^2, \dots, x^m)$ if x^j, $1\le j\le m$, are continuous functions on $[0,\infty)$. Denote by C^i, $1\le i\le m+1$, the set of all those $x = (x^1, x^2, \dots, x^i)\in D^i[0,\infty)$ for which all x^i are continuous functions on $[0,\infty)$. The mapping g was considered and used by Harrison in [3]. Here we prove that each of the mappings g^i has the following property:

for $0\le s\le t<\infty$,

$$g^i(x)(t)\le g^i(x)(s) + \sup_{s=t_{i+1}\le t_i\le \cdots\le t_1=t}\sum_{j=1}^{i}\left(x^j(t_j) - x^j(t_{j+1})\right). \tag{4.4}$$

To prove (4.4), let us notice that by the definition of the supremum it follows that there exists a sequence $\mathbf{t}_n = (0 = t_{i+1,n}\le t_{i,n}\le\cdots\le t_{2,n}\le t_{1,n} = t)$, $n\ge 1$, such that

$$\sup_{0=t_{i+1}\le t_i\le\cdots\le t_1=t}\sum_{j=1}^{i}\left(x^j(t_j) - x^j(t_{j+1})\right) = \lim_{n\to\infty}\sum_{j=1}^{i}\left(x^j(t_{j,n}) - x^j(t_{j+1,n})\right).$$

But

$$\begin{aligned}
&\sum_{j=1}^{i}\left(x^j(t_{j,n}) - x^j(t_{j+1,n})\right)\\
&= \left\{\sum_{j=1}^{i_n-1}\left(x^j(t_{j,n}) - x^j(t_{j+1,n})\right) + \left(x^{i_n}(t_{i_n}, n) - x^{i_n}(s)\right)\right\}\\
&\quad + \left\{\left(x^{i_n}(s) - x^{i_n}(t_{i_n+1,n})\right) + \sum_{j=i_n+1}^{i}\left(x^j(t_{j,n}) - x^j(t_{j+1,n})\right)\right\},
\end{aligned} \tag{4.5}$$

where $t_{i_n+1,n}\le s< t_{i_n,n}$. Now let us notice that the first expression in the right-hand side of (4.5) does not exceed

$$\sup_{s=t_{i+1}\le t_i\le\cdots\le t_1=t}\sum_{j=1}^{i}\left(x^j(t_j) - x^j(t_{j+1})\right)$$

and the second expression in the right-hand side of (4.5) does not exceed

$$\sup_{0=t_{i+1}\le t_i\le\cdots\le t_1=s}\sum_{j=1}^{i}\left(x^j(t_j) - x^j(t_{j+1})\right).$$

Hence and from (4.5) we obtain (4.4).

Analogously to property (4.4) we can prove the following property of the processes γ_r^i:

for $0 \le s \le t < \infty$,

$$\gamma_r^i(t) \le \gamma_r^i(s) + \sup_{s=t_{i+1}\le t_i \le \cdots \le t_1 = t} \sum_{j=1}^{i} \Big\{\alpha^{-1}(r)\big(\hat{S}_r^j(t_j) - \hat{S}_r^j(t_{j+1})\big) + v^*_{-[t_{j+1}/\alpha^2(r)]+1}(j,r)\Big\}. \tag{4.6}$$

Furthermore, let us notice that

$$\begin{aligned} |\gamma_r^i(t) - g^i(\hat{S}_r)(t)\alpha^{-1}(r)| &\le \sum_{j=1}^{i} \max_{-[t/\alpha^2(r)]\le k \le 0} v^*_{k+1}(j,r) \\ &\le \sum_{j=1}^{i} \Big\{\sup_{0\le s\le t} \big(\tilde{S}_r^j(s) - \tilde{S}_r^j(s-)\big)\alpha^{-1}(r) + b_j(r)\Big\}. \end{aligned} \tag{4.7}$$

Now let us define the mapping $\tilde{h}$: $D^m[0,\infty) \to R^m$ as $\tilde{h}(x) = (h(x^1), h(x^2),\dots,h(x^m))$, where $x^i \in D[0,\infty)$, $x = (x^1, x^2,\dots,x^m)$ and $h(x^i) = \sup_{t\ge 0} x^i(t)$. It is obvious that $\tilde{h}$ is measurable and in view of (4.3) the following relations hold:

$$\begin{aligned} W_1^*(i,r) &= h(\gamma_r^i), \qquad 1 \le i \le m, \\ \tilde{W}(r) &= \tilde{h}(\gamma_r). \end{aligned} \tag{4.8}$$

THEOREM 3. *Let the assumptions* A_1, A_2 *and* A_3 *hold and let there exist positive and finite constants* σ_i, $1 \le i \le m+1$, *such that*

B $$\tilde{S}_r \to_D \big(\sigma_1\xi^1, \sigma_2\xi^2,\dots,\sigma_{m+1}\xi^{m+1}\big) \quad \text{as } r \to \infty,$$

where $\xi = (\xi^1, \xi^2,\dots,\xi^{m+1})$ *is an* $(m+1)$*-dimensional Wiener process. Then*

$$\alpha(r)\tilde{W}(r) \to_D \tilde{h}g(\zeta) \quad \text{as } r \to \infty,$$

where $\zeta = (\zeta^1, \zeta^2,\dots,\zeta^m)$ *and*

$$\zeta^i(t) = \sigma_i\xi^i(t) - \sigma_{m+1}\xi^{m+1}(t) - c_i t, \qquad t \ge 0,\ 1 \le i \le m.$$

Theorem 3 and Remark 3 give the following corollary.

COROLLARY 3. *If the conditions of Theorem* 3 *are satisfied then*

$$\alpha(r)\tilde{w}(r) \to_D (Y^1, Y^2 - Y^1, Y^3 - Y^2,\dots,Y^m - Y^{m-1}) \quad \text{as } r \to \infty,$$

where $Y = (Y^1, Y^2,\dots,Y^m) = \tilde{h}g(\zeta)$.

PROOF OF THEOREM 3. Let π be the mapping of $D^{m+1}[0,\infty)$ into $D^m[0,\infty)$ defined as $\pi(x^1, x^2,\dots,x^{m+1}) = (x^1 - x^{m+1},\dots,x^m - x^{m+1})$. The mapping π is measurable and continuous on C^{m+1}. Furthermore, $\hat{S}_r = \pi\tilde{S}_r + \pi\beta_r$. Hence using the assumptions A_1, A_2 and B and next Theorem 5.1 from [1] we obtain

$$\hat{S}_r \to_D \pi\xi - c \cdot e = \zeta \quad \text{as } r \to \infty, \tag{4.9}$$

where $c = (c_1, c_2, \ldots, c_m)$, $e(t) = t$, $t \geq 0$. Since g is continuous on C^m thus (4.9) and Theorem 5.1 from [1] give

$$g(\hat{S}_r) \to_D g(\zeta) \quad \text{as } r \to \infty. \tag{4.10}$$

Now using condition B and Theorem 5.1 from [1] leads to

$$\sup_{0 \leq s \leq t} |\tilde{S}_r^i(s) - \tilde{S}_r^i(s-) + \alpha(r) b_i(r)| \to_p 0 \quad \text{as } r \to \infty, t \geq 0, 1 \leq i \leq m.$$

Hence and from the inequality (4.7) we have

$$\sup_{0 \leq s \leq t} |\alpha(r)\gamma_r^i(s) - g^i(\hat{S}_r)(s)| \to_p 0 \quad \text{as } r \to \infty, t \geq 0, 1 \leq i \leq m,$$

which in turn gives

$$d\big(g^i(\hat{S}_r), \alpha(r)\gamma_r^i\big) \to_p 0 \quad \text{as } r \to \infty, 1 \leq i \leq m,$$

where we recall that d is the metric in $D[0,\infty)$ defined by Lindvall in [5]. The last convergence in turn gives

$$d^m\big(g(\hat{S}_r), \alpha(r)\gamma_r\big) \to_p 0 \quad \text{as } r \to \infty,$$

where we recall that d^m is the m-fold product metric of d. Hence

$$\alpha(r)\gamma_r \to_D g(\zeta) \quad \text{as } r \to \infty. \tag{4.11}$$

Let

$$\theta_i(s,t) =_{\text{df}} \sup_{s = t_{i+1} \leq t_i \leq \cdots \leq t_1 = t} \sum_{j=1}^{i} \Big\{\alpha^{-1}(r)\big(\hat{S}_r^j(t_j) - \hat{S}_r^j(t_{j+1})\big) + v^*_{-[t_{j+1}/\alpha^2(r)]+1}(j,r)\Big\}.$$

Then $\theta_i(s,t)$ can be rewritten as

$$\theta_i(s,t) = \sup_{s = t_{i+1} \leq t_i \leq \cdots \leq t_1 = t} \sum_{j=1}^{i} \left\{ \sum_{k=-[t_j/\alpha^2(r)]+1}^{[t_{j+1}/\alpha^2(r)]} \big(v_k^*(j,r) - u_k^*(1,r)\big) + v^*_{-[t_{j+1}/\alpha^2(r)]+1}(j,r) \right\}. \tag{4.12}$$

Since $(\mathbf{v}^*(r), \mathbf{u}^*(1,r))$ is stationary for each r thus by (4.12) we have

$$\{\theta_i(s,t), t \geq s\} =_D \{\theta_i(0, t-s), t \geq s\}, \qquad 1 \leq i \leq m.$$

Hence and from (4.6) we get

$$P\Big\{\sup_{t>0}\big(\gamma_r^i(t+s)-\gamma_r^i(s)\big)>x\Big\}$$

$$(4.13)\qquad \le P\Big\{\sup_{t>0}\theta_i(s,t)>x\Big\}=P\Big\{\sup_{t>0}\theta_i(0,t)>x\Big\}$$

$$=P\Big\{\sup_{t>0}\gamma_r^i(t)>x\Big\}\le P\{W_1^*(i,r)>x\}$$

for $1\le i\le m$. Now let us notice that assumption A_3, Corollary 1 and next Lemma 1 from [10] give the tightness of the sequences

$$\{\alpha(r)w_1^*(i,r),\,r\ge 1\},\qquad 1\le i\le m.$$

This and the relations

$$W_1^*(i,r)=\sum_{j=1}^{i}\big(w_1^*(j,r)+v_1^*(j,r)\big),\qquad 1\le i\le m,$$

and next the assumption A_3 give the tightness of the sequences

$$\{\alpha(r)W_1^*(i,r),\,r\ge 1\},\qquad 1\le i\le m.$$

Hence and from (4.13) we get the tightness of the sequences

$$\{\alpha(r)W_1^*(i,r),\,r\ge 1\},\qquad 1\le i\le m,$$

$$\Big\{\alpha(r)\sup_{t\ge 0}\big(\gamma_r^i(t+k)-\gamma_r^i(k)\big),\,r\ge 1,\,k\ge 1\Big\},\qquad 1\le i\le m.$$

These facts together with (4.11) and (4.2) give by Lemmas 2a and 3a in [10] the convergence $\tilde{h}(\alpha(r)\gamma_r)\to_D \tilde{h}g(\zeta)$ as $r\to\infty$, which in view of (4.8) gives the assertion. □

Immediately from Theorem 3 and Corollary 2 we get the following corollary.

COROLLARY 4. *Let for each* $r\ge 1$, $\mathbf{v}^*(1,r)=\mathbf{v}^*(2,r)=\cdots=\mathbf{v}^*(m,r)$, $\alpha(r)\downarrow 0$, $(\alpha(r)-b_1(r))/\alpha(r)\to c$, $0<c<\infty$, *as* $r\to\infty$, *and*

$$\sup_r\Big(E\big(v_1^*(1,r)-u_1^*(1,r)\big)^2+\operatorname{var}v_1^*(1,r)\Big)<\infty.$$

Then $\alpha(r)\tilde{W}(r)\to_D Y$ *and* $\alpha(r)\tilde{w}(r)\to_D (Y^1,0,\dots,0)$, *where*

$$Y=(Y^1,Y^1,\dots,Y^1)\quad\textit{and}\quad Y^1=\sup_{t\ge 0}\big(\sigma_1\xi^1(t)-\sigma_2\xi^2(t)-c\cdot t\big).$$

REMARK 4. Let the conditions of Corollary 4 be satisfied and let $\sigma^2=_{\mathrm{df}}\sigma_1^2-2\sigma_{1,2}\sigma_1\sigma_2+\sigma_2^2>0$, where $\sigma_{1,2}\min(s,t)=E\xi^1(t)\xi^2(s)=E\xi^2(t)\xi^1(s)$. Then $(1/\sigma)(\sigma_1\xi^1-\sigma_2\xi^2)$ is the Wiener process and $P\{Y^1>x\}=\exp(-2xc/\sigma^2)$, $x\ge 0$.

PROOF. Since (ξ^1, ξ^2) is a two-dimensional Wiener process thus $Z = \sigma_1\xi^1 - \sigma_2\xi^2$ is a one-dimensional Gaussian process. Moreover, $EZ(t) = 0$, $EZ(t)Z(s) = (\sigma_1^2 - 2\sigma_{1,2}\sigma_1\sigma_2 + \sigma_2^2)\min(s,t)$. Hence $(1/\sigma)Z$ is the Wiener process. Now let us notice that for $x \geq 0$ we have

$$P\{Y^1 > x\} = P\Big\{\sup_{t\geq 0}\big(\sigma_1\xi^1(t) - \sigma_2\xi^2(t) - ct\big) > x\Big\}$$
$$= P\Big\{\sup_{t\geq 0}\big(\xi^1(t) - ct/\sigma\big) > x/\sigma\Big\} = \exp(-2xc/\sigma^2),$$

which finishes the proof. □

Now we give an example of a family of sequences $\{(v_k^*(1,r), v_k^*(2,r), \dots, v_k^*(m,r), u_k^*(1,r)),\ k \geq 0\}$, $\alpha(r)\downarrow 0$ as $r \to \infty$, for which condition B of Theorem 3 holds. For clarity let us denote

$$X_{k,i}(r) = v_{-k}^*(i,r), \qquad k \geq 0, r \geq 1, 1 \leq i \leq m,$$
$$X_{k,m+1}(r) = u_{-k}^*(1,r), \qquad k \geq 0, r \geq 1,$$

and

$$X(r) = \{X_k(r) =_{\text{df}} (X_{k,1}(r), X_{k,2}(r), \dots, X_{k,m+1}(r)),\ k \geq 1\}.$$

EXAMPLE. Denote by $\mathscr{F}_k(r)$ and $\mathscr{F}^k(r)$, $k \geq 1$, the σ-fields generated by $\{X_i(r),\ 0 \leq i \leq k-1\}$ and $\{X_i(r),\ i \geq k-1\}$, respectively. Assume that the family of sequences $\{(v_k^*(1,r), v_k^*(2,r), \dots, v_k^*(m,r), u_k^*(1,r)),\ k \geq 0\}$, where $\alpha(r)\downarrow 0$ as $r \to \infty$, is such that

(a_1) for each $r \geq 1$ the sequence $X(r)$ is ϕ-mixing with the same function $\phi = \{\phi_k\}$, $\phi_k \to 0$, as $k \to \infty$, i.e., $|P(E_2|E_1) - P(E_2)| < \phi_k$, for any $E_1 \in \mathscr{F}_i(r)$, $P(E_1) > 0$ and $E_2 \in \mathscr{F}^{k+i}(r)$, $i \geq 1$;
(a_2) $\sum_{k=1}^{\infty}\phi_k^{1/2} < \infty$;
(a_3) for some $\delta > 0$ the following hold:

$$\max_{1\leq i\leq m}\sup_r E|v_0^*(i,r) - b_i(r)|^{2+\delta} < M < \infty,$$
$$\sup_r E|u_0^*(1,r) - \alpha(r)|^{2+\delta} < M < \infty.$$

By [1] (see Lemmas 1 and 3, pages 170 and 172) the following constants are finite:

$$\sigma_{i,j}(r) =_{\text{df}} E\big(X_{0,i}(r) - EX_{0,i}(r)\big)\big(X_{0,j}(r) - EX_{0,j}(r)\big)$$
$$+ \sum_{k=1}^{\infty} E\big(X_{0,i}(r) - EX_{0,i}(r)\big)\big(X_{k,j}(r) - EX_{0,j}(r)\big)$$
$$+ \sum_{k=1}^{\infty} E\big(X_{0,j}(r) - EX_{0,j}(r)\big)\big(X_{k,i}(r) - EX_{0,i}(r)\big).$$

Proceeding analogously as in the proof of Lemma 5.2 in [10], page 48 (the condition assumed there, iv_2, can be dropped), we can prove the following fact (see also [1], page 177).

REMARK 5. The assumptions (a_1)–(a_3) imply condition B whenever $\sigma_{i,i}(r) \to \sigma_i^2 > 0$, as $r \to \infty$, $1 \le i, j \le m + 1$.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] BOXMA, O. J. (1979). On a tandem queueing model with identical service times at both counters. *Adv. in Appl. Probab.* **11** 616–659.

[3] HARRISON, J. M. (1973). The heavy traffic approximation for single server queues in series. *J. Appl. Probab.* **10** 613–629.

[4] KELLY, F. P. (1982). The throughput of a series of buffers. *Adv. in Appl. Probab.* **14** 633–653.

[5] LINDVALL, T. (1973). Weak convergence of probability measures and random functions in function space $D[0,\infty)$. *J. Appl. Probab.* **10** 109–121.

[6] LOYNES, R. M. (1962). The stability of queue with non-independent interarrival and service times. *Proc. Cambridge Philos. Soc.* **58** 497–520.

[7] LOYNES, R. M. (1965). On the waiting time distribution for queues in series. *J. Roy. Statist. Soc. Ser. B* **27** 491–496.

[8] ROLSKI, T. and SZEKLI, R. (1982). Networks of work-conserving normal queues. In *Applied Probability and Computer Science: The Interface* (R. L. Disney and T. J. Ott, eds.) **2** 477–497. Birkhaüser, Boston.

[9] SZCZOTKA, W. (1986). Stationary representation of queues. I. *Adv. in Appl. Probab.* **18** 815–848.

[10] SZCZOTKA, W. (1986). Joint distribution of waiting time and queue size for single server queue. *Dissertationes Math.* **248** 395–403.

[11] SZCZOTKA, W. (1990). Exponential approximation of waiting time and queue size for queues in heavy traffic. *Adv. in Appl. Probab.* **22** 230–240.

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