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LIMITING DISTRIBUTIONS OF NONLINEAR VECTOR FUNCTIONS OF STATIONARY GAUSSIAN PROCESSES

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In memory of the late Professor C. T. Hsiao

Given a stationary Gaussian vector process (X_m, Y_m), $m \in Z$, and two real functions $H(x)$ and $K(x)$, we define $Z_H^n = A_n^{-1}\Sigma_{m=0}^{n-1} H(X_m)$ and $Z_K^n = B_n^{-1}\Sigma_{m=0}^{n-1} K(Y_m)$, where A_n and B_n are some appropriate constants. The joint limiting distribution of (Z_H^n, Z_K^n) is investigated. It is shown that Z_H^n and Z_K^n are asymptotically independent in various cases. The application of this to the limiting distribution for a certain class of nonlinear infinite-coordinated functions of a Gaussian process is also discussed.

1. Introduction. Let (X_m, Y_m), $m \in Z$, be a sequence of stationary Gaussian vectors. We assume that $EX_m = EY_m = 0$, $EX_m^2 = EY_m^2 = 1$ and

$$r_1(m) = EX_0 X_m \sim |m|^{-\beta_1},$$

$$r_2(m) = EY_0 Y_m \sim |m|^{-\beta_2}$$

as $|m| \to \infty$, and

$$r_3(m) = EX_0 Y_m \sim m^{-\beta_3},$$

$$r_3(-m) = EY_0 X_m \sim m^{-\beta_4}$$

as $m \to \infty$, where $\beta_1, \beta_2, \beta_3, \beta_4 > 0$. With their correlation functions assumed above, $\{X_m\}$ and $\{Y_m\}$ are usually called processes of long-range dependence if $\beta_1, \beta_2 < 1$. Let $G_1(x)$ and $G_2(x)$ be the spectral distribution of $\{X_m\}$ and $\{Y_m\}$, and let Z_{G_1} and Z_{G_2} be their corresponding random measures. Since $\{(X_m, Y_m)\}$ is stationary, there always exists a complex-valued function $G_3(x)$ such that

$$r_3(m) = \int e^{-imx}\, dG_3(x), \qquad \forall\, m \in Z.$$

Since the matrix

$$\begin{bmatrix} G_1(dx) & G_3(dx) \\ \overline{G}_3(dx) & G_2(dx) \end{bmatrix}$$

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is positive definite, it follows that

$$|G_3(dx)|^2 \le G_1(dx)G_2(dx).$$

Given two functions $H(x)$ and $K(x)$, satisfying $EH(X_m) = EK(Y_m) = 0$, $EH^2(X_m) < \infty$ and $EK^2(Y_m) < \infty$, and having their Hermite expansions as

$$H(x) = \sum_{j=\nu_1}^{\infty} c_j H_j(x) \quad \text{and} \quad K(x) = \sum_{j=\nu_2}^{\infty} d_j H_j(x),$$

where $c_{\nu_1} \neq 0$, $d_{\nu_2} \neq 0$, we define

$$Z_H^n = A_n^{-1} \sum_{m=0}^{n-1} H(X_m) \quad \text{and} \quad Z_K^n = B_n^{-1} \sum_{m=0}^{n-1} K(Y_m).$$

It has been proved that, with proper choice of the norming factors A_n and B_n, Z_H^n and Z_K^n have marginal limiting distributions as $n \to \infty$. When $\nu_1\beta_1 < 1$ (or $\nu_2\beta_2 < 1$), the limiting distribution of Z_H^n (or Z_K^n) has a multiple Wiener integral representation and is non-Gaussian when $\nu_1 > 1$ (or $\nu_2 > 1$) [3], [6], [9]. On the other hand, when $\nu_1\beta_1 \ge 1$ (or $\nu_2\beta_2 \ge 1$), the limiting distributions of Z_H^n (or Z_K^n) is, in fact, Gaussian [1], [2]. The purpose of this paper is to study the joint limiting distribution of (Z_H^n, Z_K^n). Our main results can be summarized in the following three theorems.

THEOREM 1. *Assume $\nu_1\beta_1 < 1$ and $\nu_2\beta_2 > 1$. When $\nu_2 = 1$ we also assume*

$$\beta \equiv \beta_3 \wedge \beta_4 > \frac{1+\beta_2}{2}.$$

Then, with $A_n = n^{1-\nu_1\beta_1/2}$ and $B_n = n^{1/2}$,

$$(Z_H^n, Z_K^n) \to_d (Z_H^*, Z_K^*) \quad \text{as } n \to \infty,$$

where:

(i) *$Z_H^n \to_d Z_H^*$ as $n \to \infty$, where Z_H^* has a multiple Wiener integral representation and is non-Gaussian when $\nu_1 > 1$.*

(ii) *$Z_K^n \to_d Z_K^*$ as $n \to \infty$, where Z_K^* has a Gaussian distribution.*

(iii) *Z_H^* and Z_K^* are independent.*

THEOREM 2. *Assume $\nu_1\beta_1 < 1$ and $\nu_2\beta_2 < 1$. Then, with $A_n = n^{1-\nu_1\beta_1/2}$ and $B_n = n^{1-\nu_2\beta_2/2}$,*

$$(Z_H^n, Z_K^n) \to_d (Z_H^*, Z_K^*) \quad \text{as } n \to \infty,$$

where:

(i) *$Z_H \to_d Z_H^*$ and $Z_K^n \to_d Z_K^*$ as $n \to \infty$.*

(ii) *Z_H^* and Z_K^* both have multiple Wiener integral representations and are non-Gaussian when $\nu_1 > 1$ and $\nu_2 > 1$, respectively.*

(iii) *Z_H^* and Z_K^* are independent unless*

$$\beta = (\beta_1 + \beta_2)/2.$$

THEOREM 3. *Assume $\nu_1\beta_1 > 1$ and $\nu_2\beta_2 > 1$. Then, with $A_n = B_n = n^{1/2}$,*

$$(Z_H^n, Z_K^n) \to_d (Z_H^*, Z_K^*) \quad \text{as } n \to \infty,$$

where:

(i) $Z_H^n \to_d Z_H^*$ *and* $Z_K^n \to_d Z_K^*$ *as* $n \to \infty$.

(ii) Z_H^* *and* Z_K^* *have a joint normal distribution.*

(iii) Z_H^* *and* Z_K^* *are independent unless the coefficients* c_j *and* d_j *in the expansions of* $H(x)$ *and* $K(x)$ *satisfy the condition* $\{j | j \geq \nu_1 \text{ and } c_j \neq 0\} \cap \{j | j \geq \nu_2 \text{ and } d_j \neq 0\} \neq \varnothing$.

In the above theorems, the notation $\to_d$ means convergence in distribution.

This problem was first studied in a paper of Hsiao [5]. However, his proof was found to be false and his paper was never published. Our proof in this paper is much shorter. What motivated us to study this problem is the following. Given a square-integrable function L of a stationary Gaussian process with the form

$$L = I_{k_1}(f_1) + I_{k_2}(f_2), \qquad 1 \leq k_1 < k_2,$$

where $I_j(f)$ is the j-fold multiple Wiener integral, with kernel f, with respect to the random spectral measure generated by the Gaussian process. Write

$$\begin{aligned} Z_L^n &= C_n^{-1} \sum_{m=0}^{n-1} I_{k_1}(U_m \circ f_1) + C_n^{-1} \sum_{m=0}^{n-1} I_{k_2}(U_m \circ f_2) \\ &\equiv Z_{L_1}^n + Z_{L_2}^n, \end{aligned}$$

where U_m is the m-steps shift operator, i.e., $(U_m \circ f)(x_1, \dots, x_k) = \exp(im(x_1 + \cdots + x_k)) f(x_1, \dots, x_k)$.

When f_1 has a zero value at the origin, it is possible that $Z_{L_1}^n$, $Z_{L_2}^n$ and Z_L^n all have nonzero limiting distributions with $C_n = n^{1/2}$ [7] and, in particular, $Z_{L_1}^n$ has a Gaussian limit and $Z_{L_2}^n$ has a non-Gaussian limit. It is interesting to determine the limiting distribution of Z_L^n. In some cases, we can apply Theorem 1 to conclude that $Z_{L_1}^n$ and $Z_{L_2}^n$ are asymptotically independent, and hence the limiting distribution of Z_L^n is the convolution of the marginal limiting distributions of $Z_{L_1}^n$ and $Z_{L_2}^n$, one is Gaussian and one is non-Gaussian. More detailed results in this direction will appear in a subsequent paper.

There are already many publications discussing the cases where both $Z_{L_1}^n$ and $Z_{L_2}^n$ have Gaussian limiting distributions (e.g., [1], [4]) or both $Z_{L_1}^n$ and $Z_{L_2}^n$ have non-Gaussian limiting distributions (e.g., [8]). So far, there are no existing publications studying the case where one is Gaussian and one is non-Gaussian.

2. Proofs of the theorems. We shall prove Theorem 1 in detail and shall only sketch the proofs of Theorems 2 and 3.

Part (i) of Theorem 1 was proved in [3] and [9] and part (ii) of Theorem 1 was proved in [1]. Therefore, to prove Theorem 1 we only have to show that

$$(Z_H^n, Z_K^n) \to_d (Z_H^*, Z_K^*)$$

and Z_H^* and Z_K^* are independently distributed. We shall prove this fact only for the special case when $H(x)$ and $K(x)$ have one-term expansions:

$$H(x) = H_{\nu_1}(x) \quad \text{and} \quad K(x) = H_{\nu_2}(x).$$

The reduction of $H(x)$ to its first term is justified because when $\nu_1\beta_1 < 1$ only the first term is relevant to the distribution of Z_H^* [9]. In [1] it is made clear that when $\nu_2\beta_2 > 1$, we need only to consider the $K(x)$ with finite expansion to prove the central limit theorem. Though we prove the theorem only for the $K(x)$ with one-term expansion, the arguments in the proof can be extended easily to the finite expansion case.

The major tool we use to prove the theorems is the so-called diagram formula [6] on how to compute the expectation of a product of Hermite polynomials of standard Gaussian random variables. Prior to giving the statement of the formula, we need some notations and definitions. Let a given set of $l_1 + \cdots + l_p$ vertices be arranged into p levels such that the ith level has l_i vertices. A graph G is called a *diagram of order* $l_1, \dots, l_p$, if (i) each vertex is of degree 1 and (ii) edges may pass only between different levels. By a *regular* diagram we mean a diagram whose levels can be paired in such a way that its edges do not pass between levels of different pairs. For each edge $w \in G$ connecting the ith and jth level, $i < j$, define $d_1(w) = i$ and $d_2(w) = j$. The diagram formula states:

Lemma 1 (Diagram formula). *Let $(W_1, \dots, W_p)$ be a Gaussian vector with $EW_i = 0$, $EW_i^2 = 1$ and $EW_iW_j = r(i,j)$. Then for the Hermite polynomials $H_{l_1}(x), \dots, H_{l_p}(x)$, we have*

$$E\prod_{i=1}^{p} H_{l_i}(W_i) = \sum_G \prod_{w\in G} r(d_1(w), d_2(w)),$$

where the sum runs through all the diagrams G of order $(l_1, \dots, l_p)$.

Independent Gaussian random variables have the moments given in Lemma 2, and these moments determine their distribution.

Lemma 2. *Given two r.v.'s Z and W with $EZ = EW = 0$ and $EZ^2 = \sigma_1^2$ and $EW^2 = \sigma_2^2$, then Z and W are independent Gaussian r.v.'s if and only if*

$$EZ^lW^m = \begin{cases} \dfrac{l!\,m!}{2^{l/2+m/2}(l/2)!(m/2)!}\sigma_1^l\sigma_2^m, & \text{if } l \text{ and } m \text{ are even},\\ 0, & \text{otherwise}. \end{cases}$$

In the following, Δ always denotes bounded intervals in $\mathscr{R}$. Then because $r_3(n) = \int e^{inx}\, dG_3(x)$, we have

$$E\int f(x) Z_{G_1}(dx) \overline{\int g(x) Z_{G_2}(dx)} = \int f(x) \bar{g}(x)\, dG_3(x) \tag{1}$$

for $f \in L^2(G_1)$ and $g \in L^2(G_2)$. By Proposition 1 of [3] or similar arguments, it can be proved that there exist $G_1^*(x)$ and $G_2^*(x)$ such that

$$n^{\beta_1} G_1\left(\frac{dx}{n}\right) \to G_1^*(dx) \quad \text{weakly} \tag{2}$$

and

$$n^{\beta_2^*} (\log n)^{-\delta(\beta_2)} G_2\left(\frac{dx}{n}\right) \to G_2^*(dx) \quad \text{weakly} \tag{3}$$

as $n \to \infty$, where

$$\beta_2^* = \beta_2 \wedge 1 \quad \text{and} \quad \delta(x) = 1 \quad \text{if } x = 1 \text{ and } 0 \quad \text{if } x \neq 1.$$

We shall need Lemma 3 to prove the theorems. Recall that $\beta = \beta_3 \wedge \beta_4$.

LEMMA 3. *Assume $\beta \leq 1$. There exists a function $G_3^*(x)$ of locally bounded variation such that for each bounded interval Δ,*

$$\lim_{m \to \infty} m^{\beta} (\log m)^{-\delta(\beta)} G_3\left(\frac{\Delta}{m}\right) = G_3^*(\Delta). \tag{4}$$

Moreover G_3^* satisfies

$$G_3^*([0, y]) = \overline{G_3^*([-y, 0])} = y^{\beta} D,$$

where D is some complex constant.

PROOF. It is sufficient to show that (4) holds for $\Delta = [0, y]$ or $[-y, 0]$. Define

$$F_n(x) = \frac{1}{2\pi} \sum_{|s| \leq n} r_3(s) \int_{-\pi}^{x} e^{isy}\, dy$$

for $x \in [-\pi, \pi]$. Since each term in the above sum is bounded by $C|s|^{-\beta-1}$ for some constant C, $F_n(x)$ converges to $G_3(x)$ for all x, i.e.,

$$G_3(x) = \lim_{n \to \infty} F_n(x) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} r_3(s) \frac{e^{isx} - e^{-is\pi}}{is}.$$

Let $\Delta = [0, y]$. Define

$$S_{m,y} = m^{\beta} (\log m)^{-\delta(\beta)} \left[G_3\left(\frac{\Delta}{m}\right) + \overline{G_3\left(\frac{\Delta}{m}\right)} \right].$$

By $\overline{G_3(\Delta)} = G_3(-\Delta)$,

$$S_{m,y} = \frac{1}{\pi}(\log m)^{-\delta(\beta)} \sum_{s=-\infty}^{\infty} \left(r_3(s) m^{\beta}\right) \frac{\sin(s/m)y}{s},$$

which as $m \to \infty$ tends to

$$\lim_{m \to \infty} S_{m,y} = \begin{cases} y^{\beta} R_{\beta}, & \text{if } \beta_3 \neq \beta_4, \\ y^{\beta} 2 R_{\beta}, & \text{if } \beta_3 = \beta_4, \end{cases} \tag{5}$$

where $R_{\beta} = (1/\pi)\lim_{\varepsilon \to 0}(-\log \varepsilon)^{-\delta(\beta)} \int_{\varepsilon}^{\infty} x^{-\beta-1} \sin x \, dx$. Similarly, if we define

$$C_{m,y} = m^{\beta}(\log m)^{-\delta(\beta)} \left[G_3\left(\frac{\Delta}{m}\right) - \overline{G_3\left(\frac{\Delta}{m}\right)} \right]$$
$$= i\frac{1}{\pi}(\log m)^{-\delta(\beta)} \sum_{s=-\infty}^{\infty} \left(r_3(s) m^{\beta}\right) \frac{(1 - \cos(s/m)y)}{s},$$

then we obtain

$$\lim_{m \to \infty} C_{m,y} = \begin{cases} -iy^{\beta} I_{\beta}, & \text{if } \beta_3 < \beta_4, \\ iy^{\beta} I_{\beta}, & \text{if } \beta_4 < \beta_3, \\ 0, & \text{if } \beta_3 = \beta_4, \end{cases} \tag{6}$$

where $I_{\beta} = (1/\pi)\lim_{\varepsilon \to 0}(-\log \varepsilon)^{-\delta(\beta)} \int_{\varepsilon}^{\infty} x^{-\beta-1}(1 - \cos x)\, dx$. (5) and (6) imply

$$\lim_{m \to \infty} m^{\beta}(\log m)^{-\delta(\beta)} G_3\left(\frac{[0,y]}{m}\right) = y^{\beta} D \equiv G_3^*([0,y]),$$

where

$$D \equiv \begin{cases} R_{\beta}, & \text{if } \beta_3 = \beta_4, \\ (R_{\beta} - iI_{\beta})/2, & \text{if } \beta_3 < \beta_4, \\ (R_{\beta} + iI_{\beta})/2, & \text{if } \beta_4 < \beta_3. \end{cases}$$

Since the property $\overline{G_3(\Delta)} = G_3(-\Delta)$ is preserved by passing to the limit G_3^*, we have

$$G_3^*([-y,0]) = y^{\beta} \overline{D}.$$

The proof is complete. □

When $\beta \leq 1$, observe that

$$m^{\beta}(\log m)^{-\delta(\beta)} \left| G_3\left(\frac{\Delta}{m}\right) \right| \leq m^{\beta-(\beta_1+\beta_2^*)/2} (\log m)^{\delta(\beta_2)-\delta(\beta)}$$
$$\times \left[m^{\beta_1} G_1\left(\frac{\Delta}{m}\right) m^{\beta_2^*} (\log m)^{-\delta(\beta_2)} G_2\left(\frac{\Delta}{m}\right) \right]^{1/2}.$$

Thus we have an immediate corollary from (2), (3) and (4):

$$\beta \geq (\beta_1 + \beta_2^*)/2. \tag{7}$$

When $\beta > 1$, $G_3(dx)$ is absolutely continuous and its density is continuous. Let $G_3(dx) = f(x)\,dx$. Then

$$\lim_{m \to \infty} mG_3\left(\frac{\Delta}{m}\right) = \lambda(\Delta)\, f(0), \tag{8}$$

where λ is the Lebesgue measure. Also (7) is clearly satisfied for $\beta > 1$.

By (2),

$$n^{\beta_1/2} Z_{G_1}\left(\frac{\Delta}{n}\right) \to_d Z_{G_1^*}(\Delta) \tag{9}$$

as $n \to \infty$ [3], where $Z_{G_1^*}$ is the random measure induced by $G_1^*(dx)$. Since the distribution of Z_H^* can be represented by the ν_1-fold Wiener integral, to prove Theorem 1 we need to show that, for disjoint Δ_i's, $i = 1, 2, \dots, \nu_1$,

$$\left(Z_{G_1}^*(\Delta_1), \dots, Z_{G_1}^*(\Delta_{\nu_1})\right) \perp Z_K^*,$$

where $\perp$ denotes independence. This is equivalent to the statement that for any linear combination $\sum_{i=1}^n a_i Z_{G_1}^*(\Delta_i)$ of $Z_{G_1}^*(\Delta_i)$'s,

$$E\left(\sum_{i=1}^n a_i Z_{G_1}^*(\Delta_i)\right)^l (Z_K^*)^m = E\left(\sum_{i=1}^n a_i Z_{G_1}^*(\Delta_i)\right)^l E(Z_K^*)^m$$

for all integers $l \geq 0$ and $m \geq 0$.

Clearly it is true if and only if for any Δ,

$$E\left(Z_{G_1}^*(\Delta)\right)^l (Z_K^*)^m = E\left(Z_{G_1}^*(\Delta)\right)^l E(Z_K^*)^m, \quad \text{for all integers } l \geq 0 \text{ and } m \geq 0$$

or equivalently

$$Z_{G_1}^*(\Delta) \perp Z_K^*. \tag{10}$$

We know already that the vector $(Z_{G_1}^*(\Delta_1), \dots, Z_{G_1}^*(\Delta_{\nu_1}))$ and the random variable Z_k^* are Gaussian, but we have to prove that they are also jointly Gaussian. We do this with the help of Lemma 2. It is not difficult to see that (10) is also equivalent to

$$W(\Delta) = \int_\Delta \frac{e^{ix} - 1}{ix} Z_{G_1}^*(dx) \perp Z_K^*. \tag{11}$$

It is a mere technical convenience for us to replace $Z_{G_1^*}(\Delta)$ by $W(\Delta)$ [5]. Define

$$K_n(x) \equiv \frac{e^{ix} - 1}{(e^{ix/n} - 1)n} = \frac{1}{n} \sum_{j=0}^{n-1} e^{ijx/n} \tag{12}$$

and

$$W_n(\Delta) \equiv n^{-(1-\beta_1/2)} \sum_{j=0}^{n-1} \int_{\Delta/n} e^{ijx} Z_{G_1}(dx)$$

$$= \int_{\Delta} K_n(x) n^{(\beta_1/2)} Z_{G_1}\left(\frac{dx}{n}\right).$$

(9), (2) and the fact that $K_n(x)$ converges to $(e^{ix}-1)/ix$ uniformly on every bounded set imply

$$W_n(\Delta) \to_d W(\Delta) \tag{13}$$

as $n \to \infty$ [3]. If we can show

$$\lim_{n\to\infty} E(W_n(\Delta))^l (Z_K^n)^m = \begin{cases} \dfrac{l!m!}{2^{l/2+m/2}(l/2)!(m/2)!}\sigma_1^l \sigma_2^m, & \text{if } l \text{ and } m \text{ are even,} \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

where $EW^2(\Delta) = \sigma_1^2$ and $E(Z_K^*)^2 = \sigma_2^2$, then by (13) and Lemma 2, (11) follows.

PROOF OF THEOREM 1. To prove (14), first we have to use Lemma 1 (diagram formula) to compute $E(W_n(\Delta))^l (Z_K^n)^m$ for any given l and m. For this we need the following notation.

Consider a set V of $l + m\nu_2$ vertices having $l + m$ levels with the configuration

$$l\begin{cases} 0 \\ 0 \\ \vdots \\ 0 \end{cases}$$

$$m\begin{cases} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{cases}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\nu_2}$$

Let us use the notation $V \equiv (1,\dots,1,\nu_2,\dots,\nu_2)$. Define Γ to be the set of all regular diagrams of order V and Γ^c the complement of Γ, i.e., the set of all nonregular diagrams of order V. Any subgraph of a diagram is called a *subdiagram* if it is itself a diagram, its vertices consist of all the vertices in some levels of the original diagram and no edge exists between this subgraph and its complement in the original diagram.

Any diagram $G \in \Gamma^c$ can be partitioned into three disjoint subdiagrams $V_{G,1}$, $V_{G,2}$ and $V_{G,3}$, which are defined as

$V_{G,1}$ = the maximal subdiagram of G which is regular within itself, and all its edges satisfy $1 \le d_1(w) < d_2(w) \le l$ or $l+1 \le d_1(w) < d_2(w) \le l+m$,

$V_{G,2}$ = the maximal subdiagram of $G - V_{G,1}$ whose edges satisfy $l+1 \le d_1(w) < d_2(w) \le l+m$,

$V_{G,3} = G - (V_{G,1} \cup V_{G,2})$.

For each subdiagram $V_{G,i}$ of G, $i = 1,2,3$, define

$$V_{G,i}^* = \{j \mid \text{the } j\text{th level of } V \text{ is in } V_{G,i}\},$$

$$V_{G,i}^*(1) = \{j \in V_{G,i}^* \mid 1 \le j \le l\},$$

$$V_{G,i}^*(2) = \{j \in V_{G,i}^* \mid l+1 \le j \le l+m\}.$$

Also in the following $E(G)$ will denote the set of all edges contained in the diagram G.

Note that, in the diagram formula, each summand corresponding to a diagram G is a product of numbers. Now apply the formula to $E(W_n(\Delta))^l(Z_K^n)^m$. If G is regular, the product is equal to $[E(W_n(\Delta))^2]^{l/2}[E(Z_K^n)^2]^{m/2}$. If G is nonregular, then the product can be partitioned into three subproducts A_1^n, A_2^n and A_3^n corresponding to the three subdiagrams $V_{G,1}$, $V_{G,2}$ and $V_{G,3}$ of the diagram G. To be more precise, we have

$$E(W_n(\Delta))^l(Z_K^n)^m = \sum_{G\in\Gamma} + \sum_{G\in\Gamma^c} A_1^n \times A_2^n \times A_3^n, \tag{15}$$

where

$$\sum_{G\in\Gamma} = \sum_{G\in\Gamma} \left[E(W_n(\Delta))^2\right]^{l/2}\left[E(Z_K^n)^2\right]^{m/2},$$

$$A_1^n = \left[E(W_n(\Delta))^2\right]^{|V_{G,1}^*(1)|/2}\left[E(Z_K^n)^2\right]^{|V_{G,1}^*(2)|/2},$$

$$A_2^n = n^{-|V_{G,2}^*(2)|/2} \sum_{\substack{0\le p_i\le n-1\\ i\in V_{G,2}^*}} \prod_{w\in E(V_{G,2})} r_2(p_{d_1(w)} - p_{d_2(w)}),$$

$$A_3^n = n^{-|V_{G,3}^*(2)|/2} \sum_{\substack{0\le p_i\le n-1\\ i\in V_{G,3}^*}} \prod_{\substack{w\in E(V_{G,3})\\ d_1(w)\in V_{G,3}^*(2)}} r_2(p_{d_1(w)} - p_{d_2(w)})$$

$$\times \prod_{\substack{e\in E(V_{G,3})\\ d_1(e)\in V_{G,3}^*(1)}} n^{-(1-\beta_1/2)}$$

$$\times E\left[\int_{\Delta/n} \exp(ip_{d_1(e)}x) Z_{G_1}(dx) \overline{\int \exp(ip_{d_2(e)}x) Z_{G_2}(dx)}\right].$$

Since $\sum_{G\in\Gamma}$ converges to the right-hand side of (14), it is sufficient to show that, for fixed $G \in \Gamma^c$, the triple product in the second term of (15) vanishes, i.e.,

$$\lim_{n\to\infty} A_1^n \times A_2^n \times A_3^n = 0. \tag{16}$$

Recall the definition of $W_n(\Delta)$. We have

$$EW_n^2(\Delta) \to \int_\Delta \left|\frac{e^{ix}-1}{ix}\right|^2 dG_1^*(x) = EW^2(\Delta) \tag{17}$$

as $n \to \infty$. (17) and the central limit theorem for Z_k^n imply

$$\lim_{n\to\infty} A_1^n = \left(EW^2(\Delta)\right)^{|V_{G,1}^*(1)|/2}\left(\sigma_2^2\right)^{V_{G,1}^*(2)|/2} < \infty.$$

Using (1), we can rewrite

$$\begin{aligned} A_3^n = n^{-|V_{G,3}^*(2)|/2} &\sum_{\substack{0\le p_i\le n-1\\ i\in V_{G,3}^*(2)}} \prod_{\substack{w\in E(V_{G,3})\\ d_1(w)\in V_{G,3}^*(2)}} r_2(p_{d_1(w)} - p_{p_2(w)}) \\ &\times \prod_{\substack{e\in E(V_{G,3})\\ d_1(e)\in V_{G,3}^*(1)}} \left[\sum_{0\le p_{d_1(e)}\le n-1} n^{-(1-\beta_1/2)}\int_\Delta \exp i\left(\frac{p_{d_1}(e) - p_{d_2}(e)}{n}\right)x\, dG_3\left(\frac{x}{n}\right)\right]. \end{aligned} \tag{18}$$

For each $e \in E(V_{G,3})$ with $d_1(e) \le l$, we can obtain, as a result of (4) in Lemma 3 [or (8) if $\beta > 1$], an asymptotic bound (denoted by Σ_n^*) for the second summation in (18). Using the notation

$$\alpha = \beta \wedge 1,$$

we have

$$\begin{aligned} \sum_n{}^* &= n^{(\beta_1-2\alpha)/2}(\log n)^{\delta(\beta)}\int_\Delta\left[\sum_{0\le p_{d_1(e)}\le n-1}\exp\left(\frac{ip_{d_1(e)}x}{n}\right)\frac{1}{n}\right] \\ &\qquad\times\exp\left(\frac{-ip_{d_2(e)}x}{n}\right)n^\alpha(\log n)^{-\delta(\beta)}\, dG_3\left(\frac{x}{n}\right) \\ &= O\left(n^{(\beta_1-2\alpha)/2}(\log n)^{\delta(\beta)}\int_\Delta\left|\frac{e^{ix}-1}{ix}\right||dG_3^*(x)|\right). \end{aligned}$$

If $\beta \le 1$, then, by (7),

$$\beta_1 - 2\alpha = \beta_1 - 2\beta \le -\beta_2^* < 0. \tag{19}$$

If $\beta > 1$, clearly (19) still holds. By (19) it follows that

$$\sum_n{}^* = O(n^{(\beta_1-2\alpha)/2}) = o(1). \tag{20}$$

Define

$k(i)$ = the number of edges w satisfying $d_1(w) = i$

and

$g(i)$ = the number of vertices in the ith level not connected by edges to any of the first l levels.

First we assume that $\nu_2 > 1$. When $V_{G,2}$ is nonempty, by a similar argument employed to prove the proposition in [1], (page 433), we can show that

$$\lim_{n\to\infty} A_2^n = 0. \tag{21}$$

When $V_{G,3}$ is nonempty, we are going to show that $A_3^n \to 0$ as $n \to \infty$. We shall decompose the set $V_{G,3}^*(2)$ of levels into three parts by defining

$$C_G = \{i \in V_{G,3}^*(2) | g(i)\beta_2 > 1\},$$
$$A_G = \{i \in V_{G,3}^*(2) | g(i) = 0\},$$
$$B_G = V_{G,3}^*(2) - (A_G \cup C_G).$$

We shall rearrange the levels in $V_{G,3}^*(2)$ in such a way that the levels of B_G are preceded by the levels of A_G and followed by the levels of C_G. Within B_G and C_G, their levels are also rearranged so that the levels with smaller $g(i)$ come first. We also need the following notation later:

$\tilde{V}_{G,3} =$ the subgraph of $V_{G,3}$ which consists of all the edges that connect only between the levels in $V_{G,3}^*(2)$.

If $C_G = V_{G,3}^*(2)$, i.e., $A_G = B_G = \varnothing$, we can obtain an asymptotic bound for the first summation in (18). We have

$$\begin{aligned}\alpha_n &\equiv \sum_{\substack{0\le p_i \le n-1\\ i\in V_{G,3}^*(2)}} \prod_{\substack{w\in E(V_{G,2})\\ d_1(w)\in V_{G,3}^*(2)}} |r_2(p_{d_1(w)} - p_{d_2(w)})| \\ &= O\big(n^{|V_{G,3}^*(2)| - \sum_{i\in V_{G,3}^*(2)} k(i)/g(i)}\big).\end{aligned} \tag{22}$$

Note that the α_n given above is well defined because it is assumed that $\nu_2 > 1$. As shown in (2.20) in [1], we have the inequality

$$\sum_{i\in V_{G,3}^*(2)} \frac{k(i)}{g(i)} \ge \tfrac{1}{2}|V_{G,3}^*(2)|. \tag{23}$$

(22) and (23) imply that

$$\alpha_n = O(n^{|V_{G,3}^*(2)|/2}). \tag{24}$$

Then (18), (20) and (24) imply

$$A_3^n = (o(1))^{|V_{G,3}^*(1)|}.$$

Hence if $V_{G,3} \ne \varnothing$, then, because $|V_{G,3}^*(1)| > 0$, we have

$$\lim_{n\to\infty} A_3^n = 0. \tag{25}$$

If $C_G \subsetneqq V_{G,3}^*(2)$, we have

$$\begin{aligned}A_3^n = O\Bigg(& n^{-|V_{G,3}^*(2)|/2} \prod_{i\in A_G} n^{(\beta_1-2\alpha)\nu_2/2} \prod_{\substack{i\in B_G\\ g(i)\beta_2\le 1}} n^{(\beta_1-2\alpha)(\nu_2-g(i))/2} n^{1-k(i)\beta_2} n^{\varepsilon\delta(k(i)\beta_2)} \\ & \times \prod_{\substack{i\in B_G\\ g(i)\beta_2>1}} n^{1-k(i)/g(i)} \prod_{i\in C_G} n^{1-k(i)/g(i)}\Bigg),\end{aligned} \tag{26}$$

where ε is some positive number chosen to handle the case when $k(i)\beta_2 = 1$. By (19) and the assumption that $\beta_2\nu_2 > 1$,

$$\prod_{i \in A_G} n^{(\beta_1 - 2\alpha)\nu_2/2} = o(n^{-|A_G|/2}),$$

where the order o should be replaced by the order O if $A_G = \varnothing$. By the same argument for deriving (24), we have

$$\prod_{i \in C_G} n^{1-k(i)/g(i)} = O(n^{|C_G|/2}).$$

It is clear that (25) will still hold if we can show that the asymptotic bound for the second product and the third product in (26) together is of $o(n^{|B_G|/2})$. Define

$$T = \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} [(\beta_1 - 2\alpha)(\nu_2 - g(i))/2 + (1 - k(i)\beta_2) + \varepsilon\delta(k(i)\beta_2)] + \sum_{\substack{i \in B_G \\ g(i)\beta_2 > 1}} [1 - k(i)/g(i)]$$

and

$b(i)$ = the number of vertices in the ith level whose edges w satisfy that $g(d_1(w))\beta_2 \le 1$ and $g(d_2(w))\beta_2 > 1$.

We make the following observation:

$$\begin{aligned}
\beta_2 \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} k(i) + \sum_{\substack{i \in B_G \\ g(i)\beta_2 > 1}} \frac{k(i)}{g(i)} &= \left[\sum_{\substack{w \in \vec{V}_{G,3} \\ g(d_1(w))\beta_2 \le 1 \\ g(d_2(w))\beta_2 \le 1}} \beta_2 + \sum_{\substack{w \in \vec{V}_{G,3} \\ g(d_1(w))\beta_2 \le 1 \\ g(d_2(w))\beta_2 > 1}} \frac{\beta_2}{2} \right] \\
&\quad + \sum_{\substack{i \in B_G \\ g(i)\beta_2 > 1}} \left[\frac{k(i)}{g(i)} + b(i)\frac{\beta_2}{2} \right] \\
&\ge \frac{\beta_2}{2} \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} g(i) + \sum_{\substack{i \in B_G \\ g(i)\beta_2 > 1}} \frac{k(i) + b(i)/2}{g(i)} \\
&\ge \frac{\beta_2}{2} \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} g(i) + \tfrac{1}{2}|\{i \in B_G | g(i)\beta_2 > 1\}|.
\end{aligned} \tag{27}$$

The second term in the second inequality of (27) is obtained by the same argument for showing (2.20) in [1]. Using (27), we have

$$-|B_G|/2 + T \le \tfrac{1}{2}\Bigg\{|B_G| + \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} \big[(\beta_1 - 2\alpha)(\nu_2 - g(i)) + \varepsilon\delta(k(i)\beta_2) - \beta_2 g(i)\big] - |\{i \in B_G | g(i)\beta_2 > 1\}|\Bigg\}$$

$$= \tfrac{1}{2} \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} \big[1 + (\beta_1 - 2\alpha)(\nu_2 - g(i)) + \varepsilon\delta(k(i)\beta_2) - \beta_2 g(i)\big]$$

$$\le \tfrac{1}{2} \sum_{\substack{i \in B_G \\ g(i)\beta_2 \le 1}} \big[(1 - \beta_2^* \nu_2) + (\beta_1 + \beta_2^* - 2\alpha)(\nu_2 - g(i)) + \varepsilon\delta(k(i)\beta_2)\big]$$

$$< 0 \quad (\text{by choosing an } \varepsilon > 0 \text{ small enough}).$$

Hence, (25) is true under the assumption that $\nu_2 > 1$.

For any nonregular diagram $G \in \Gamma^c$, if $\nu_2 > 1$, then its subdiagrams $V_{G,2}$ and $V_{G,3}$ cannot be empty at the same time; that is, either (21) or (25) must hold. Hence (16) is true.

When $\nu_2 = 1$, then $V_{G,2}$ is empty, i.e., A_2^n is absent. Thus in order to assure (16) we have to show (25). Note that when $\nu_2 = 1$, the first product in (18) no longer exists. Also note in this case $|V_{G,3}^*(2)| \ge 1$. Hence, by Lemma 3, we have

$$A_3^n = n^{-|V_{G,3}^*(2)|/2} \prod_{w \in E(V_{G,3})} n^{(1+\beta_1/2)-\alpha} (\log n)^{\delta(\beta)} \int_\Delta \left[\sum_{0 \le p_{d_1(w)} \le n-1} \exp\left(\frac{i p_{d_1(w)} x}{n}\right)\frac{1}{n}\right]$$
$$\times \left[\sum_{0 \le p_{d_2(w)} \le n-1} \exp\left(\frac{-i p_{d_2(w)} x}{n}\right)\frac{1}{n}\right] n^{\alpha} (\log n)^{-\delta(\beta)}\, dG_3\left(\frac{x}{n}\right)$$
$$= O\left(n^{(\beta_1+1)/2-\alpha} (\log n)^{\delta(\beta)} \int_\Delta \left|\frac{e^{ix} - 1}{ix}\right|^2 |dG_3^*(x)|\right),$$

because $|V_{G,3}^*(2)| \ge 1$. By the assumption of the theorem, when $\nu_2 = 1$,

$$\frac{\beta_1 + 1}{2} < \beta \Rightarrow \frac{\beta_1 + 1}{2} - \alpha < 0$$
$$\Rightarrow \lim_{n \to \infty} A_3^n = 0.$$

The proof is complete. □

PROOF OF THEOREM 2. Part (i) and part (ii) are proved in [3] and [9].

For any bounded interval $\Delta \subset R$, define

$$Z_{G_{in}}(\Delta) = n^{\beta_i/2} Z_{G_i}\left(\frac{\Delta}{n}\right), \qquad i = 1, 2.$$

Then (2) and (3) imply

$$Z_{G_{in}}(\Delta) \to_d Z_{G_i^*}(\Delta), \qquad i = 1, 2.$$

Let A_i, $i = 1, 2, \dots, \nu_1$, and B_i, $i = 1, 2, \dots, \nu_2$, be bounded symmetric intervals such that $A_i \cap A_j = \varnothing$ and $B_i \cap B_j = \varnothing$ for $i \neq j$. To prove the theorem, we only have to show

$$\begin{aligned}&\left(Z_{G_{1n}}(A_1), \dots, Z_{G_{1n}}(A_{\nu_1}), Z_{G_{2n}}(B_1), \dots, Z_{G_{2n}}(B_{\nu_2})\right)\\ &\qquad \to_d \left(Z_{G_1^*}(A_1), \dots, Z_{G_1^*}(A_{\nu_1}), Z_{G_2^*}(B_1), \dots, Z_{G_2^*}(B_{\nu_2})\right),\end{aligned}$$

which is, in fact, equivalent to

$$(28) \qquad \left(Z_{G_{1n}}(\Delta_1), Z_{G_{2n}}(\Delta_2)\right) \to_d \left(Z_{G_1^*}(\Delta_1), Z_{G_2^*}(\Delta_2)\right)$$

for any bounded symmetric intervals Δ_1 and Δ_2. (28) can be proved by using the diagram formula and a procedure similar to the one used in the proof of Theorem 1 and we shall omit the details here. Suppose $\beta \neq (\beta_1 + \beta_2)/2$. Then by (7),

$$\beta > (\beta_1 + \beta_2)/2.$$

The correlation ρ of $Z_{G_1^*}(\Delta_1)$ and $Z_{G_2^*}(\Delta_2)$ is

$$\begin{aligned}\rho &= EZ_{G_1^*}(\Delta_1) Z_{G_2^*}(\Delta_2)\\ &= \lim_{n \to \infty} EZ_{G_{1n}}(\Delta_1) Z_{G_{2n}}(\Delta_2)\\ &= \lim_{n \to \infty} n^{(\beta_1 + \beta_2)/2} G_3\left(\frac{\Delta_1 \cap \Delta_2}{n}\right) \quad \text{by } (1)\\ &= \lim_{n \to \infty} n^{(\beta_1 + \beta_2)/2 - \beta} n^{\beta} G_3\left(\frac{\Delta_1 \cap \Delta_2}{n}\right)\\ &\to 0\end{aligned}$$

by Lemma 3. Thus $Z_{G_1^*}(\Delta_1)$ and $Z_{G_2^*}(\Delta_2)$ are independent for any Δ_1 and Δ_2, and hence Z_H^* and Z_K^* are independent. □

PROOF OF THEOREM 3. It is sufficient for us to consider only the special case

$$H(x) = H_{\nu_1}(x) \quad \text{and} \quad K(x) = H_{\nu_2}(x).$$

A procedure similar to the one used in the proof of Theorem 1 can be used to show

$$(Z_H^n, Z_K^n) \to_d (Z_H^*, Z_K^*).$$

Let

$$\begin{aligned}\rho' &= EZ_H^* Z_K^*\\ &= \lim_{n \to \infty} EZ_H^n Z_K^n.\end{aligned}$$

Clearly, if $\nu_1 \neq \nu_2$, then $\rho = 0$ due to the orthogonality property of the H_ν's.

Conversely if $\nu_1 = \nu_2 = \nu$, then by (7),

$$\nu\beta_3 > 1 \quad \text{and} \quad \nu\beta_4 > 1.$$

Hence

$$\sum_{m=-\infty}^{\infty} |r_3(m)|^{\nu} < \infty.$$

It follows that

$$\rho = \nu_1 \sum_{m=-\infty}^{\infty} r_3(m)^{\nu}. \qquad \square$$

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