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THE FIRST EXIT TIME OF A TWO-DIMENSIONAL SYMMETRIC STABLE PROCESS FROM A WEDGE[1]

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Let T_θ be the first exit time of a symmetric stable process [with parameter $\alpha \in (0,2)$] from a wedge of angle 2θ, $0 < \theta < \pi$. Then there are constants $p_{\theta,\alpha} > 0$ such that for starting points x in the wedge, $E_x T_\theta^p < \infty$ if $0 < p < p_{\theta,\alpha}$ and $E_x T_\theta^p = \infty$ if $p > p_{\theta,\alpha}$. We characterize $p_{\alpha,\theta}$ and obtain upper and lower bounds.

1. Introduction. Let $\{X_t: t \geq 0\}$ be the symmetric stable process in $\mathbb{R}^2$ of index $\alpha \in (0,2)$; namely that process with stationary independent increments, whose transition density $f(t, x-y)$ (with respect to Lebesgue measure) is determined by its characteristic function

$$\exp\{-t|\xi|^\alpha / 2\Gamma(\alpha/2)\} = \int_{\mathbb{R}^2} e^{ix\cdot\xi} f(t,x)\,dx.$$

For $x \in \mathbb{R}^2$ let $\varphi(x)$ be the magnitude of the angle between x and $(0,1)$. Given $\theta \in (0,\pi)$, define

$$W_\theta := \{x \in \mathbb{R}^2 : x \neq 0, \pi - \theta < \varphi(x) \leq \pi\}$$

and call it a *wedge of angle* 2θ. Define

$$T_\theta := \inf\{t > 0: X_t \notin W_\theta\},$$

the first exit time of X_t from W_θ. In this paper we study the distribution of T_θ.

In the case of a two-dimensional Brownian motion $B(t)$, it is known that for $p > 0$ and $x \in W_\theta$,

$$E_x T_\theta^p < \infty \Leftrightarrow p < \frac{\pi}{4\theta} \tag{1.1}$$

[Burkholder (1977), page 192, and Spitzer (1958)]. There are several ways to obtain this result. The heat equation can be solved explicitly for $P_x(T_\theta > t)$ by separation of variables. Spitzer (1958) solves the heat equation using an integral transform from which he deduces (1.1). Burkholder's approach involves use of his two-sided L^p inequalities for stopping times of Brownian motion—he is able to reduce consideration to solution of a simple Dirichlet problem for the Laplacian. It is not possible to mimic these techniques for the symmetric stable process X_t because its generator is an integral operator. The trouble is all the techniques described involve separation of variables.

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However, it is known [Blumenthal and Getoor (1960), pages 264 and 265] X_t can be represented as a two-dimensional Brownian motion run with an independent clock (namely, a stable subordinator). The important observation, due to Molchanov and Ostrovskii (1969), is that this representation can be interpreted as the trace of a degenerate diffusion (described in detail below). This is our starting point. We will prove the following theorem, making essential use of the results of Bass and Cranston (1983). Here $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n: x_1 \geq 0\}$.

THEOREM 1.1. *Let $\theta \in (0, \pi)$. Then there exists a constant $p_{\theta,\alpha} > 0$ such that for $x \in W_\theta$,*

$$E_x T_\theta^p < \infty \quad \text{if } p < p_{\theta,\alpha},$$
$$E_x T_\theta^p = \infty \quad \text{if } p > p_{\theta,\alpha}.$$

Moreover,

(i) $\theta \in (0, \pi) \to p_{\theta,\alpha}$ *is continuous and decreasing*;
(ii) *for* $\theta \in (0, \pi/2)$, $p_{\theta,\alpha} > \frac{1}{2} = p_{\pi/2,\alpha}$;
(iii) *for* $\theta \in (\pi/2, \pi)$,

$$p_{\theta,\alpha} \geq \left\{\alpha - 2 + \left[(\alpha - 2)^2 + \left[(1 + \cos\theta)/\sin\theta\right]^2 \alpha(4 - \alpha)\right]^{1/2}\right\}/2\alpha;$$

(iv) *for* $\theta \in (0, \pi)$, $p_{\theta,\alpha} < 1$;
(v) *for* $\alpha = 1$, $E_x T_\theta^{p_{\theta,1}} = \infty$.

In fact, let $\delta = \delta(\theta) = \sin\theta/(1 + \cos\theta)$ and $H_\delta = \mathbb{R}^2_+ \setminus [\{0\} \times [\mathbb{R} \setminus (-\delta, \delta)]]$. Then $p_{\theta,\alpha} = \{-(2 - \alpha) + [(2 - \alpha)^2 + 4\lambda_\delta]^{1/2}\}/2\alpha$, where λ_δ is the principal eigenvalue of the differential operator

$$L = \frac{1}{4}\left(v_1 + v_2^2 + 1\right)^2 \left[4v_1 \frac{\partial^2}{\partial v_1^2} + \frac{\partial^2}{\partial v_2^2} + 2\left(2 - \alpha - \frac{2(1-\alpha)v_1}{v_1 + v_2^2 + 1}\right)\frac{\partial}{\partial v_1} - \frac{2(1-\alpha)v_2}{v_1 + v_2^2 + 1}\frac{\partial}{\partial v_2}\right]$$

on H_δ. More precisely,

$$\lambda_\delta = -\sup_{U \in \mathscr{K}(\delta)} \sup_{\mu(\overline{U}) = 1} \left[-I(\mu)\right],$$

where the μ's are probability measures, $\mathscr{K}(\delta) = \{U \subseteq \mathbb{R}^2_+$: U is bounded and open in $\mathbb{R}^2_+$ with C^∞ boundary in $\mathbb{R}^2$ and $U \subseteq \overline{U} \subseteq H_\delta$ in $\mathbb{R}^2_+\}$ and $I(\mu)$ is the Donsker–Varadhan I-function associated to L,

$$I(\mu) = -\inf\left\{\int \frac{Lf}{f}\, d\mu : f \in C^2(\mathbb{R}^2_+) \text{ and } f \equiv \text{constant outside a compact subset of } \mathbb{R}^2_+\right\}.$$

Now we describe the method of solution. Let $Y_t = (Y_t^{(1)}, Y_t^{(2)}, Y_t^{(3)}) \in \mathbb{R}^3$, where $Y_t^{(1)}$ is a Bessel process with parameter $2-\alpha$, $Y_t^{(2)}$ and $Y_t^{(3)}$ are one-dimensional Brownian motions and $Y^{(1)}, Y^{(2)}, Y^{(3)}$ are independent. The generator of Y_t is

$$(1.2)\qquad G = \frac{1}{2}\left\{\frac{\partial^2}{\partial y_1^2} + \frac{1-\alpha}{y_1}\frac{\partial}{\partial y_1} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}\right\}, \qquad y \in \mathbb{R}_+^3,$$

and Y_t has state space $\mathbb{R}_+^3$ if $Y_0^{(1)} \geq 0$.

THEOREM 1.2 [Molchanov and Ostrovskii (1969)]. *Let β_t be the inverse local time of the Bessel process $Y_t^{(1)}$ and 0. If $Y_0^{(1)} = 0$ then*

$$X_t = \left(0, Y^{(2)}(\beta_t), Y^{(3)}(\beta_t)\right).$$

is a symmetric stable process with index α.

Define

$$(1.3)\qquad \begin{aligned} Z_t &= \left(\left(Y_t^{(1)}\right)^2, Y_t^{(2)}, Y_t^{(3)}\right), \qquad t \geq 0,\\ \tau_\theta &:= \inf\{t > 0\colon Z_t \in \{0\} \times W_\theta^c\}. \end{aligned}$$

We consider Z_t rather than Y_t because it is easier to get a martingale characterization of Z_t. Now T_θ is the first time X_t hits W_θ^c and τ_θ is the first time Z_t (or Y_t) hits $\{0\} \times W_\theta^c$. Since $Z_t \in \{0\} \times W_\theta^c$ only if $Z_t^{(1)} = 0$, we see that $(0, X(T_\theta))$ and $Z(\tau_\theta)$ have the same distribution. If $\mathscr{F}_t = \sigma(Z_t)$ then $(Z_t^{(2)}, Z_t^{(3)})$ is a two-dimensional $\mathscr{F}_t$-Brownian motion and τ_θ is an $\mathscr{F}_t$-stopping time. Thus Burkholder's (1977) Theorem 3.1 and Remark 3.1 results hold, and in particular for $q > 0$,

$$(1.4)\qquad E_z|Z(\tau_\theta)|^{2q} < \infty \quad \text{iff } E_z\tau_\theta^q < \infty, z = (0, x).$$

Hence we get

$$(1.5)\qquad E_x|X(T_\theta)|^{2q} < \infty \quad \text{iff } E_{(0,x)}\tau_\theta^q < \infty.$$

Bass and Cranston (1983) have analogues to Burkholder's result (1.4) for X_t. Thus the problem is reduced to the study of τ_θ, a stopping time of a *diffusion*. We find and study $p_{\theta,\alpha} > 0$ such that for $z \in \{0\} \times W_\theta$, $E_z\tau_\theta^p$ is finite if $0 < p < p_{\theta,\alpha}$ and infinite if $p > p_{\theta,\alpha}$. Our method is not refined enough to settle the case $p = p_{\theta,\alpha}$ except when $\alpha = 1$.

Under the change of coordinates (on $\mathbb{R}_+^3$),

$$(1.6)\qquad z = \left(y_1^2, y_2, y_3\right), \qquad y \in \mathbb{R}_+^3,$$

the operator G in (1.2) becomes the generator $\overline{G}$ of Z_t, where

$$(1.7)\qquad \overline{G} = \frac{1}{2}\left\{4z_1\frac{\partial^2}{\partial z_1^2} + 2(2-\alpha)\frac{\partial}{\partial z_1} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}\right\}, \qquad z \in \mathbb{R}_+^3.$$

Introduce the new coordinates $(\rho, v) = \Phi(z)$, $z \in \mathbb{R}^3_+ \backslash \{(0,0,a): a \ge 0\}$, where

$$\rho = \left(z_1 + z_2^2 + z_3^2\right)^{1/2}, \quad v = \left(z_1/(\rho - z_3)^2, z_2/(\rho - z_3)\right). \tag{1.8}$$

Then in these coordinates, the operator $\overline{G}$ in (1.7) becomes

$$\begin{aligned}\tilde{G} = \frac{1}{2}\Bigg\{ \frac{\partial^2}{\partial\rho^2} + \frac{3-\alpha}{\rho}\frac{\partial}{\partial\rho} + \frac{\left(v_1 + v_2^2 + 1\right)^2}{4\rho^2}\Bigg[4v_1 \frac{\partial^2 f}{\partial v_1^2} + \frac{\partial^2 f}{\partial v_2^2} \\ + 2\left[2 - \alpha - \frac{2(1-\alpha)v_1}{v_1 + v_2^2 + 1}\right]\frac{\partial}{\partial v_1} - \frac{2(1-\alpha)v_2}{v_1 + v_2^2 + 1}\frac{\partial}{\partial v_2}\Bigg]\Bigg\},\end{aligned} \tag{1.9}$$

$\mathbb{R}^3_+ \backslash [\{0\} \times W_\theta^c]$ is taken to $(0,\infty) \times H_\delta$ and $[\{0\} \times W_\theta^c] \setminus \{(0,0,a): a \ge 0\}$ is taken to $(0,\infty) \times [\mathbb{R}^2_+ \setminus H_\delta]$ (details in Section 4), where H_δ is as in Theorem 1.1. Now $u(t,z) = P_z(\tau_\theta > t)$ solves $(\partial/\partial t - \overline{G})u(t,z) = 0$ for $(t,z) \in (0,\infty) \times [\mathbb{R}^3_+ \backslash (\{0\} \times W_\theta^c)]$ with initial data $u(0,z) = 1$, $z \notin \{0\} \times W_\theta^c$ and boundary data $u(t,z) = 0$ for all $(t,z) \in (0,\infty) \times (\{0\} \times W_\theta^c)$. Expressed in the coordinates (1.8), separation of variables is the obvious means of solution, but the v-eigenvalue problem involves a degenerate non-self-adjoint differential operator. Thus the classical means of resolving the eigenvalue problem are not available. However, Donsker and Varadhan have a way to characterize the principal eigenvalue for such operators and it is their machinery that we employ.

We see from (1.9) that Z_t has a skew product representation $(R(t), V(A(t)))$, where R and V are independent and $A(t)$ is continuous, strictly increasing, and depends on R not V. Below we show $V(\cdot)$ can explode in some cases, but ignoring this for the moment, for $\eta_\delta = \inf\{t > 0: V(t) \in \mathbb{R}^2_+ \setminus H_\delta\}$ [where $\delta = \sin\theta/(1 + \cos\theta)$]

$$\begin{aligned} P_z(\tau_\theta > t) &= P_z\left(V(A(s)) \notin \mathbb{R}^2_+ \setminus H_\delta \text{ for } s \in [0,t]\right) \\ &= P_z(\eta_\delta > A(t)) \\ &= \int_0^\infty P_z(\eta_\delta > a)\, d_a P_z(A(t) \le a). \end{aligned}$$

$A(t)$ is easy to analyze and the Donsker–Varadhan theory gives us information about $P_z(\eta_\delta > u)$ as $u \to \infty$. Thus we can decide when $E_z\tau_\theta^q < \infty$.

The paper is organized as follows. In Section 2 we impose a convenient framework: the martingale problem formulation. A skew product representation of Z_t comprises the content of Sections 3 and 4. In Section 5 we establish the groundwork for the application of the Donsker–Varadhan theory to characterize the principal eigenvalue of the v-part of $\tilde{G}$ on H_δ. The Donsker–Varadhan results and Pinsky's theorem are used in Section 6 to obtain lower bounds on $P_z(\eta_\delta > u)$ as $u \to \infty$. In Section 7 we obtain upper bounds. Using techniques of Donsker and Varadhan, we prove equality of the upper and lower bounds in Section 8. In Section 9 we study the ρ-part of Z_t.

Section 10 is concerned with properties of the principal eigenvalue of the v-part of $\tilde{G}$ on H_δ as a function of δ. Theorem 1.1 is proved in Section 11 and there we also discuss an application.

2. A convenient setup. Let S be a complete separable metric space and define $C_b(S) = \{f\colon S \to \mathbb{R} | f \text{ is bounded and continuous}\}$. Give it the sup norm topology. Denote by Ω_S the set $C([0,\infty), S)$ of continuous functions from $[0,\infty)$ into S endowed with the topology of uniform convergence on compacta. Let $x_t(\omega) = \omega(t)$, $\omega \in \Omega_S$, be the t-coordinate map and define $\mathscr{M}_t = \sigma(x_s\colon s \le t)$, $\mathscr{M} = \sigma(x_s\colon s \ge 0)$. Suppose $\tilde{D} \subseteq C_b(S)$ and $L\colon \tilde{D} \to C_b(S)$ is an operator. A probability measure P on $(\Omega_S, \mathscr{M})$ solves the *$(L, \tilde{D})$-martingale problem*, starting at x, iff

(i) $P(x(0) = x) = 1$;
(ii) $f(x(t)) - \int_0^t Lf(x(s))\,ds$ is an $\mathscr{M}_t$-martingale for any $f \in \tilde{D}$.

The $(L, \tilde{D})$ martingale problem is *well posed* iff there is a unique solution for each $x \in S$. In the case when $S = \mathbb{R}^n$ we will write Ω_n for Ω_S.

3. Representation of the v-process. Let L be the v-part of $\tilde{G}$ [see (1.9)],

$$
\begin{aligned}
L = \frac{1}{4}\big((v_1 \vee 0) + v_2^2 + 1\big)^2 \Bigg\{ & 4(v_1 \vee 0)\frac{\partial^2}{\partial v_1^2} + \frac{\partial^2}{\partial v_2^2} \\
& + 2\left[2 - \alpha - \frac{2(1-\alpha)(v_1 \vee 0)}{(v_1 \vee 0) + v_2^2 + 1}\right]\frac{\partial}{\partial v_1} - \frac{2(1-\alpha)v_2}{(v_1 \vee 0) + v_2^2 + 1}\frac{\partial}{\partial v_2}\Bigg\}.
\end{aligned}
\tag{3.1}
$$

Notice it only behaves "badly" near $v_1 = 0$, and there the $[4(1-\alpha)(v_1 \vee 0)/(v_1 \vee 0) + v_2^2 + 1]\partial/\partial v_1$ part is negligible. So we eliminate that part and also the $\partial/\partial v_2$ part via a transformation of drift. The $\frac{1}{4}((v_1 \vee 0) + v_2^2 + 1)^2$ can be eliminated via time change and we are left with a nice operator. We start there and reverse this procedure to get a representation of the process governed by L. Set

$$
L_\varepsilon = \frac{2\varepsilon}{\big((v_1 \vee 0) + v_2^2 + 1\big)^2} L, \tag{3.2}
$$

$$
\bar{L}_\varepsilon = \frac{\varepsilon}{2}\left\{4(v_1 \vee 0)\frac{\partial^2}{\partial v_1^2} + \frac{\partial^2}{\partial v_2^2} + 2(2-\alpha)\frac{\partial}{\partial v_1}\right\}. \tag{3.3}
$$

Since $L = [((v_1 \vee 0) + v_2^2 + 1)^2/2\varepsilon]L_\varepsilon$, we see the ε should be irrelevant in the representation of the L-process. Of course it is in the sense that the representation will be independent of ε. However, in Section 10, ε plays a crucial role in obtaining bounds on the principal eigenvalue of L on H_δ.

By the results of Ikeda and Watanabe (1981), Example 8.3, pages 223–225, for each $v \in \mathbb{R}_+^2$ there is a (pathwise) unique strong solution $\overline{V}_\varepsilon(t) = (\overline{V}_\varepsilon^{(1)}(t), \overline{V}_\varepsilon^{(2)}(t))$ to

$$(3.4)\qquad \begin{aligned} dV(t) &= \bar{\sigma}_\varepsilon(V(t))\,dB(t) + \bar{b}_\varepsilon(V(t))\,dt, \\ V(0) &= v, \end{aligned}$$

where $B(\cdot)$ is a Brownian motion on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$,

$$(3.5)\qquad \begin{aligned} \bar{\sigma}_\varepsilon(y) &= \begin{pmatrix} [4\varepsilon(y_1 \vee 0)]^{1/2} & 0 \\ 0 & \varepsilon^{1/2} \end{pmatrix}, \\ \bar{b}_\varepsilon(y) &= \begin{pmatrix} \varepsilon(2-\alpha) \\ 0 \end{pmatrix}, \end{aligned}$$

for $y \in \mathbb{R}^2$. Moreover, $\overline{V}_\varepsilon^{(1)}(t) \ge 0$ a.s. and so the law induced by $\overline{V}_\varepsilon(\cdot)$ on $(\Omega_2, \mathscr{M})$ furnishes the unique solution to the $(\overline{L}_\varepsilon, C_0^2(\mathbb{R}^2))$-martingale problem starting at $v \in \mathbb{R}_+^2$, and the law is supported on $C([0,\infty), \mathbb{R}_+^2)$. Here $C_0^2(\mathbb{R}^2)$ is the set of C^2 functions of compact support on $\mathbb{R}^2$.

Define

$$(3.6)\qquad \begin{aligned} \bar{a}_\varepsilon(y) &= \bar{\sigma}_\varepsilon(y)\bar{\sigma}_\varepsilon^*(y), \\ \tilde{b}_\varepsilon(y) &= \varepsilon\left(2 - \alpha - \frac{2(1-\alpha)y_1}{y_1 + y_2^2 + 1}, -\frac{(1-\alpha)y_2}{y_1 + y_2^2 + 1}\right)^*, \\ \bar{c}_\varepsilon(y) &= \begin{cases} \bar{a}_\varepsilon^{-1}(y)\big(\tilde{b}_\varepsilon(y) - \bar{b}_\varepsilon(y)\big), & y_1 > 0, \\ 0, & y_1 \le 0, \end{cases} \end{aligned}$$

for $y \in \mathbb{R}^2$. Then writing $\langle\ ,\ \rangle$ for the usual Euclidean inner product, we have

$$(3.7)\qquad \sup_{y_1 > 0} \langle \bar{a}_\varepsilon \bar{c}_\varepsilon(y), \bar{c}_\varepsilon(y) \rangle \le (1-\alpha)^2 \varepsilon.$$

Let

$$(3.8)\qquad \begin{aligned} \overline{R}_\varepsilon^p(t) = \exp\bigg\{ &\int_0^t \langle p\bar{c}_\varepsilon(\overline{V}_\varepsilon(u)), d\overline{V}_\varepsilon(u) - \bar{b}_\varepsilon(\overline{V}_\varepsilon(u))\,du \rangle \\ &- \frac{1}{2}\int_0^t \langle p\bar{c}_\varepsilon, \bar{a}_\varepsilon p\bar{c}_\varepsilon \rangle(\overline{V}_\varepsilon(u))\,du \bigg\}. \end{aligned}$$

Then by the Cameron–Martin–Girsanov transformation, $\overline{R}_\varepsilon^p$ is an $\mathscr{F}_t$-martingale and for $p = 1$ gives rise to a new process, call it $V_\varepsilon(t)$, such that

(3.9) the law of $V_\varepsilon(t)$ on $(\Omega_2, \mathscr{M})$ is the unique solution

to the $(L_\varepsilon, C_0^2(\mathbb{R}^2))$-martingale problem starting at $v \in \mathbb{R}_+^2$, and it is supported on $(C[0,\infty),\mathbb{R}_+^2)$;

$$R_\varepsilon^p = \exp\left\{\int_0^t \langle -p\bar{c}_\varepsilon(V_\varepsilon(u)), dV_\varepsilon(u) - \bar{b}_\varepsilon(V_\varepsilon(u))\,du\rangle + \frac{1}{2}\int_0^t \langle p\bar{c}_\varepsilon, \bar{a}_\varepsilon p\bar{c}_\varepsilon\rangle(V_\varepsilon(u))\,du\right\} \tag{3.10}$$

is a martingale;

$$\text{for any } A \in \mathscr{F}_t,\quad P_v(V_\varepsilon \in A) = E_v\bar{R}_\varepsilon^1(t)I(\bar{V}_\varepsilon \in A),\quad P_v(\bar{V}_\varepsilon \in A) = E_v R_\varepsilon^1(t) I(V_\varepsilon \in A); \tag{3.11}$$

$$\text{for any } p > 0,\quad E_v[\bar{R}_\varepsilon^p(t)] = E_v[R_\varepsilon^p(t)] = 1. \tag{3.12}$$

LEMMA 3.1. *For* $p \geq 1$,

$$1 \leq E_v[\bar{R}_\varepsilon^1(t)]^p \leq \exp\{\tfrac{1}{2}p(p-1)(1-\alpha)^2\varepsilon t\},$$

$$E_v[R_\varepsilon^1(t)]^p \leq 1.$$

PROOF. We have

$$E_v[\bar{R}_\varepsilon^1(t)]^p = E_v\left[\bar{R}_\varepsilon^p(t)\exp\left\{\frac{p(p-1)}{2}\int_0^t \langle\bar{c}_\varepsilon, \bar{a}_\varepsilon\bar{c}_\varepsilon\rangle(\bar{V}_\varepsilon(u))\,du\right\}\right],$$

$$E_v[R_\varepsilon^1(t)]^p = E_v\left[R_\varepsilon^p(t)\exp\left\{\frac{p(1-p)}{2}\int_0^t \langle\bar{c}_\varepsilon, \bar{a}_\varepsilon\bar{c}_\varepsilon\rangle(V_\varepsilon(u))\,du\right\}\right].$$

The lemma follows from these using (3.7) and (3.12). □

Now we make the time change. Define $\tau_\varepsilon(t,\omega)$ by

$$t = \int_0^{\tau_\varepsilon(t)} 2\varepsilon\left[V_\varepsilon^{(1)}(s) + V_\varepsilon^{(2)}(s)^2 + 1\right]^{-2} ds. \tag{3.13}$$

Unfortunately, it is possible for $\int_0^\infty 2\varepsilon[V_\varepsilon^{(1)}(s) + V_\varepsilon^{(2)}(s)^2 + 1]^{-2}\,ds$ to be finite, in which case the process

$$V(t) := V_\varepsilon(\tau_\varepsilon(t)),\qquad t \geq 0, \tag{3.14}$$

explodes to ∞ in finite time. However, it is easy to see we have the following theorem.

THEOREM 3.2. *Up to a possibly finite explosion time* $\hat{e}$, *the law of* $V(\cdot)$ *on* $(\Omega_2, \mathscr{M})$ *furnishes the unique solution to the* $(L, C_0^2(\mathbb{R}_+^2))$*-martingale problem starting at* $v \in \mathbb{R}_+^2$. *Thus* $V(\cdot)$ *is independent of* $\varepsilon > 0$.

4. Skew product representation of Z_t. By the results of Ikeda and Watanabe (1981), pages 223–225, Z_t has a representation as the (pathwise) unique strong solution to the stochastic differential equation (* = transpose)

$$dZ_t^* = \begin{pmatrix} \left[4\left(Z_t^{(1)} \vee 0\right)\right]^{1/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dB_t^* + \begin{pmatrix} 2-\alpha \\ 0 \\ 0 \end{pmatrix} dt,$$

$$Z_0 = z \in \mathbb{R}^3_+,$$

with state space $\mathbb{R}^3_+$. The important point is that as a consequence of this, the law of Z_t induced on $(\Omega_2, \mathscr{M})$ is the *unique* solution to the $(\overline{G}, C_0^2(\mathbb{R}^3_+))$-martingale problem starting at $z \in \mathbb{R}^3_+$. Let us emphasize that $C_0^2(\mathbb{R}^3_+)$ is the set of all C^2 functions on a neighborhood of R^3_+ whose support is a compact subset of $\mathbb{R}^3_+$.

In the Appendix we show that Z_t never hits 0, but it can hit the half-line $\{(0,0,a): a > 0\}$ with positive probability when $1 < \alpha < 2$. This causes some minor technical difficulties. As we shall see, Z_t hitting $\{(0,0,a): a > 0\}$ corresponds to $\Phi(Z_t)$ exploding to ∞ in the v-coordinate.

THEOREM 4.1. *Up to the first time Z_t hits the half-line $\{(0,0,a): a \geq 0\}$, the law of $\Phi(Z_t)$ on $(\Omega_3, \mathscr{M})$ furnishes the unique solution to the $(\tilde{G}, C_0^2((0,\infty) \times \mathbb{R}^2_+))$-martingale problem up to its explosion time, starting at $\Phi(z)$ [where $Z_0 = z \in \mathbb{R}^3_+ \setminus \{(0,0,a): a \geq 0\}$].*

PROOF. We need only verify that the mapping

$$(4.1) \qquad z \in \mathbb{R}^3_+ \setminus \{(0,0,a): a \geq 0\} \to \Phi(z) = (\rho, v) \in (0,\infty) \times \mathbb{R}^2_+,$$

from (1.8) is a diffeomorphism onto, that the image under v of a neighborhood of $\{(0,0,a): a \geq 0\}$ corresponds to a neighborhood of the point at ∞ for $\mathbb{R}^2_+$, and that $\tilde{G}$ is just $\overline{G}$ expressed in the coordinates (ρ, v).

For this, let Ψ: $S^2 \setminus (0,0,1)$ be the stereographic projection of the unit sphere S^2 in $\mathbb{R}^3$ centered at the origin,

$$\Psi(y) = (y_1, y_2)/(1 - y_3), \qquad y \in S^2 \setminus (0,0,1),$$

and define the *stereographic coordinates*

$$(4.2) \qquad (r, u) = (|y|, \Psi(y/|y|)), \qquad y \in \mathbb{R}^3 \setminus \{(0,0,a): a \geq 0\}.$$

Then $y \in \mathbb{R}^3_+ \setminus \{(0,0,a): a \geq 0\} \to (r, u) \in (0,\infty) \times \mathbb{R}^2_+$ is one to one and onto with inverse

$$y = \frac{\left(2ru_1, 2ru_2, r\left(|u|^2 - 1\right)\right)}{|u|^2 + 1}, \qquad (r, u) \in (0,\infty) \times \mathbb{R}^2_+.$$

In these coordinates the operator G [from (1.2)] becomes

$$\frac{1}{2}\left\{\frac{\partial^2}{\partial r^2}+\frac{3-\alpha}{r}\frac{\partial}{\partial r}+\frac{\left(|u|^2+1\right)^2}{4r^2}\left[\frac{\partial^2}{\partial u_1^2}+\frac{\partial^2}{\partial u_2^2}+(1-\alpha)\frac{u_2^2-u_1^2+1}{u_1\left(|u|^2+1\right)}\frac{\partial}{\partial u_1}\right.\right.$$
$$\left.\left.-(1-\alpha)\frac{2u_2}{|u|^2+1}\frac{\partial}{\partial u_2}\right]\right\} \tag{4.3}$$

[cf. DeBlassie (1987a), Section 3]. Hence by (1.8), (1.6) and (4.2) we see that

$$(\rho, v)=\Phi(z)=\left(|y|, \frac{y_1^2}{\left(|y|-y_3\right)^2}, \frac{y_2}{|y|-y_3}\right) \tag{4.4}$$
$$=\left(r, u_1^2, u_2\right).$$

From this, (1.7) and (4.3) we see that $\tilde{G}$ is just $\overline{G}$ expressed in the coordinates (ρ, v). Also, it is clear that the map in (4.1) is one to one and onto with inverse

$$\Phi^{-1}(\rho, v)=z=\left(y_1^2, y_2, y_3\right)$$
$$=\left(\frac{4r^2u_1^2}{\left(|u|^2+1\right)^2}, \frac{2ru_2}{|u|^2+1}, \frac{r\left(|u|^2-1\right)}{|u|^2+1}\right)$$
$$=\left(\frac{4\rho^2 v_1}{\left(v_1+v_2^2+1\right)^2}, \frac{2\rho v_2}{v_1+v_2^2+1}, \frac{\rho\left(v_1+v_2^2-1\right)}{v_1+v_2^2+1}\right).$$

That the map in (4.1) is a diffeomorphism is immediate. □

REMARK 4.2. Let N be a neighborhood of $\{(0,0,a): a>0\}$ in $\mathbb{R}_+^3$. Then by (4.4) the projection onto the v-plane of $\Phi(N)=\rho(N)\times v(N)$ corresponds to a neighborhood of ∞ in $\mathbb{R}_+^2$. Thus Z_t hitting $\{(0,0,a): a>0\}$ corresponds to $v(Z_t)$ exploding to ∞.

In Section 8 we will need the following result which follows from the preceding proof.

COROLLARY 4.3. *Under the change of coordinates $u \in \mathbb{R}_+^2 \to v \in \mathbb{R}_+^2$, where $v=(u_1^2, u_2)$, the v-part L of $\tilde{G}$ [from (3.1)] expressed in the u-coordinates becomes*

$$\tilde{L}=\frac{\left(|u|^2+1\right)^2}{4}\left[\frac{\partial^2}{\partial u_1^2}+\frac{\partial^2}{\partial u_2^2}+(1-\alpha)\frac{u_2^2-u_1^2+1}{u_1\left(|u|^2+1\right)}\frac{\partial}{\partial u_1}\right.$$
$$\left.-(1-\alpha)\frac{2u_2}{|u|^2+1}\frac{\partial}{\partial u_2}\right].$$

Now we can give the skew product representation of Z_t. Let $R(t)$ be a Bessel process with parameter $4-\alpha$. We take $R(\cdot)$ to be independent of the process $V(\cdot)$. Thus $R(t)$ is that unique diffusion governed by

$$G_R = \frac{1}{2}\left\{\frac{d^2}{d\rho^2} + \frac{3-\alpha}{\rho}\frac{d}{d\rho}\right\}, \qquad \rho > 0. \tag{4.5}$$

Since $4 - \alpha > 2$, $R(t) > 0$ a.s. if $R(0)$. Thus for each $r > 0$, the law of $R(\cdot)$ on $(\Omega_1, \mathscr{M})$ uniquely solves the $(G_R, C_0^2(0,\infty))$-martingale problem starting at r.

Define

$$A(t) = \frac{1}{2}\int_0^t R(s)^{-2}\,ds, \qquad t \ge 0.$$

Then $t \to A(t)$ is continuous, strictly increasing and independent of $V(\cdot)$. Recall $\hat{e}$ is the explosion time of $V(\cdot)$.

THEOREM 4.4. *For any* $z \in \mathbb{R}_+^3 \backslash \{(0,0,a)\colon a \ge 0\}$, *if* $(R(0), V(0)) = \Phi(z)$ *and* $Z_0 = z$, *the law of* $(R(t), V(A(t)))$ *up to time* $A^{-1}(\hat{e})$ *on* $(\Omega_3, \mathscr{M})$ *is the same as that of* $\Phi(Z_t)$ *up to the first time* Z_t *hits* $\{(0,0,a)\colon a > 0\}$.

PROOF. By Theorem 4.1 and Remark 4.2 it suffices to show $(R(t), V(A(t)))$ solves the $(\tilde{G}, C_0^2((0,\infty) \times \mathbb{R}_+^2))$-martingale problem up to time $A^{-1}(\hat{e})$. It is no loss to restrict attention to $f \in C_0^2((0,\infty) \times \mathbb{R}_+^2)$ of the form $f(\rho, v) = f_1(\rho)\tilde{f}(v)$, where $f_1 \in C_0^2(0,\infty)$ and $\tilde{f} \in C_0^2(\mathbb{R}_+^2)$. By Theorem 3.2 and the independence of $R(\cdot)$ and $V(\cdot)$, the rest of the proof is straightforward. □

In the sequel, we will abuse the P_z, P_x, etc., notation, letting the subscript denote the starting points of the processes inside the P.

Next we determine the image of $J := (\{0\} \times W_\theta^c) \setminus \{(0,0,0)\colon a \ge 0\}$ under the mapping $z \to (\rho, v)$ in (1.8). Observe the image of $\{y \in J\}$ in the stereographic coordinates (r, u) in (4.2) is $(0,\infty) \times \{0\} \times (\mathbb{R} \setminus (-\delta, \delta))$, where

$$\delta = \delta(\theta) = \frac{\sin\theta}{1 + \cos\theta}. \tag{4.6}$$

Thus by (1.6), (4.2) and (4.4) we have $z \in J \Leftrightarrow y \in J \Leftrightarrow (r, u) \in (0,\infty) \times \{0\} \times (\mathbb{R} \setminus (-\delta, \delta)) \Leftrightarrow (\rho, v) \in (0,\infty) \times \{0\} \times (\mathbb{R} \setminus (-\delta, \delta))$. Consequently, the first time τ_θ that Z_t hits $\{0\} \times W_\theta^c$ is the same as the first time $v(Z_t)$ explodes or *exits* the set

$$H(\delta) = H_\delta = \mathbb{R}_+^2 \backslash [\{0\} \times (\mathbb{R} \setminus (-\delta, \delta))].$$

Thus if we define

$$\eta_\delta = \inf\{t > 0\colon V(t) \notin H_\delta\}$$

then for $z \in W_\theta$ and $(r, v) = \Phi(z)$ we have by Theorem 4.4 and independence

$$\begin{aligned} P_z(\tau_\theta > t) &= P_z(v(Z_s) \in H_\delta \text{ for all } s \le t) \\ &= P_{(r,v)}(V(A(s)) \in H_\delta \text{ for all } s \le t) \\ &= P_{(r,v)}(\eta_\delta \wedge \hat{e} > A(t)) \\ &= \int_0^\infty P_v(\eta_\delta \wedge \hat{e} > s)\, d_s P_r(A(t) \le s). \end{aligned} \tag{4.7}$$

REMARK 4.5. When $\alpha = 1$, Y_t is a three-dimensional Brownian motion [see (1.2)] and hence Y_t has a skew product representation $(R(t), \Theta(A(t)))$, where $R(t)$ is a Bessel process with parameter $4 - \alpha = 3$, $\Theta(t)$ is an independent Brownian motion on S^2 governed by the Laplace–Beltrami operator L_{S^2} on S^2 and $A(t) = \frac{1}{2}\int_0^t R(s)^{-2}\,ds$. Thus Θ and V represent the same process in different coordinates and $\hat{e} = \infty$ a.s. Hence for $y = (\sqrt{z_1}, z_2, z_3)$ and $(r, u) = (|y|, \Psi(y/|y|))$, where Ψ is the stereographic projection, we have

$$\begin{aligned} &P_u\big(\Theta(s) \in \Psi((\{0\} \times W_\theta^c) \setminus \{(0,0,a)\colon a \geq 0\}) \text{ for all } s \in [0,t]\big) \\ &\quad = P_v(\eta_\delta > t). \end{aligned}$$

Since L_{S^2} is self-adjoint with respect to Haar measure on S^2, once we know $\lim_{t\to\infty} t^{-1} \log P_v(\eta_\delta > t) = -\lambda_\delta < 0$ then (obviously)

$$\begin{aligned} &\lim_{t\to\infty} t^{-1} \log P_u\big(\Theta(s) \in \Psi((\{0\} \times W_\theta^c) \setminus \{(0,0,a)\colon a \geq 0\}) \text{ for all } s \in [0,t]\big) \\ &\quad = -\lambda_\delta < 0 \end{aligned}$$

and it will follow that

$$\begin{aligned} &P_u\big(\Theta(s) \in \Psi((\{0\} \times W_\theta^c) \setminus \{(0,0,a)\colon a \geq 0\}) \text{ for all } s \in [0,t]\big) \\ &\quad \sim C(u)\exp(-\lambda_\delta t) \quad \text{as } t \to \infty \end{aligned}$$

[cf. Port and Stone (1978), pages 121–127]. In particular, we have

$$P_v(\eta_\delta > t) \sim C(v)\exp(-\lambda_\delta t) \quad \text{as } t \to \infty,\ \alpha = 1. \tag{4.8}$$

5. Preliminaries to the study of $\eta_\delta \wedge \hat{e}$. For any locally compact Hausdorff space D with countable base define

$$\begin{aligned} C(D) &= \{f\colon D \to \mathbb{R} \mid f \text{ is continuous}\}, \\ B(D) &= \{f\colon D \to \mathbb{R} \mid f \text{ is bounded and measurable with respect} \\ &\qquad \text{to the topological Borel } \sigma\text{-algebra}\}, \\ C_b(D) &= C(D) \cap B(D), \\ C_0(D) &= C(D) \cap \{f\colon \operatorname{supp} f \text{ is a compact subset of } D\}. \end{aligned}$$

For any $D \subseteq \mathbb{R}_+^2$ let $\eta_D = \inf\{t > 0\colon V(t) \notin D\}$. The process $V(\cdot)$ induces the following semigroups:

$$T_t f(v) = E_v[f(V(t)) I(\hat{e} > t)], \qquad f \in B(\mathbb{R}_+^2), \tag{5.1}$$

$$T_t^D f(v) = E_v[f(V(t)) I(\eta_D \wedge \hat{e} > t)], \qquad f \in B(\mathbb{R}_+^2). \tag{5.2}$$

We describe various properties of these semigroups via the following sequence of lemmas. In the sequel we will say "D is an open subset of $\mathbb{R}_+^2$" and we will mean that $\mathbb{R}_+^2$ is endowed with the relative topology of $\mathbb{R}^2$ and D is an open set in the relative topology. We will write $\partial_+ D$ for the boundary of D in $\mathbb{R}_+^2$ and ∂D for the boundary of D in $\mathbb{R}^2$.

It is desirable to have the semigroup T_t^D strong Feller for open subsets D of $\mathbb{R}_+^2$, but the proof seems difficult. However, we have a property very close to this and just as useful, at least for our purposes.

LEMMA 5.1. *For any open subset D of $\mathbb{R}^2_+$, $\sigma > 0$ and $f \in B(\mathbb{R}^2_+)$ the function*

$$v \in D \to \int_0^\infty e^{-\sigma t} T_t^D f(v)\, dt$$

is continuous.

PROOF. We have for $\eta_D^\varepsilon = \inf\{t > 0: V_\varepsilon(t) \notin D\}$,

$$\begin{aligned}\int_0^\infty e^{-\sigma t} T_t^D f(v)\, dt &= \int_0^\infty e^{-\sigma t} E_v f(V(t)) I(\eta_D \wedge \hat{e} > t)\, dt \\ &= \int_0^\infty e^{-\sigma t} E_v\big[f(V_1(\tau_1(t))) I(\eta_D^1 > \tau_1(t))\big]\, dt \quad \text{[by (3.13)]} \\ &= \int_0^\infty E_v \exp(-\sigma\tau_1^{-1}(s)) f(V_1(s)) I(\eta_D^1 > s) \\ &\quad \times \frac{2\, ds}{\big[V_1^{(1)}(s) + [V_1^{(2)}(s)]^2 + 1\big]^2}.\end{aligned} \tag{5.3}$$

Now the components of the $\overline{V}_1$ process are the square of a Bessel process with parameter $2 - \alpha$ and an independent one-dimensional Brownian motion. Hence it has an explicit density and enjoys various nice properties. Since V_1 is obtained from $\overline{V}_1$ from a transformation of drift, it enjoys many of the same properties. Thus it is not too hard to prove the ds-integrand in (5.3) is continuous as a function of $v \in D$. A little more argument shows that the integral itself is continuous as a function of $v \in D$. For the sake of brevity we leave the details to the reader. □

LEMMA 5.2. *For any $\delta > 0$,*

$$P_v\Big(\sup_{s \le t} |V(s) - v| \ge \delta\Big) \to 0 \quad \text{as } t \to 0$$

uniformly on compact subsets of $\mathbb{R}^2_+$.

PROOF. We have for $\gamma > 0$,

$$\begin{aligned}P_v\Big(\sup_{s\le t}|V(s) - v| \ge \delta\Big) &= P_v\Big(\sup_{s \le t}|V_1(\tau_1(s)) - v| \ge \delta\Big) \\ &\le P_v\Big(\sup_{s\le\gamma}|V_1(s) - v| \ge \delta\Big) + P_v(\tau_1(t) \ge \gamma)).\end{aligned}$$

The first term on the left goes to 0 uniformly on compacta as $\gamma \to 0$ by the Cameron–Martin–Girsanov formula (3.11) and that the corresponding result holds for the nice process $\overline{V}_1(\cdot)$. Once we know the second term is upper semicontinuous in v for given t and γ, then for fixed γ small and sufficiently small t, by compactness $P_v(\tau_1(t) \geq \gamma)$ is uniformly small for v in a given compact set. But upper semicontinuity is easy: For t and $\gamma > 0$, let $f_n \in C_b(\mathbb{R})$ satisfy $f_n(x) \downarrow I_{[0,t]}(x)$ as $n \to \infty$ with $f_n \leq 1$. The function

$$\omega \in \Omega_2 \to \int_0^\gamma \frac{2\, ds}{\left[x_1(s) + x_2(s)^2 + 1\right]^2}$$

is continuous and bounded, hence

$$\omega \in \Omega_2 \to f_n\left(\int_0^\gamma \frac{2\, ds}{\left[x_1(s) + x_2(s)^2 + 1\right]^2}\right)$$

is also continuous and bounded. Since the law of $V_1(\cdot)$ on $(\Omega_2, \mathscr{M})$ is the unique solution to the $(L_1, C_0^2(\mathbb{R}^2))$-martingale problem starting at $v \in \mathbb{R}_+^2$ [see (3.9)], $v \in \mathbb{R}_+^2 \to E_v f_n(\tau_1^{-1}(\gamma))$ is continuous and consequently $P_v(\tau_1(t) \geq \gamma) = P_v(t \geq \tau_1^{-1}(t))$ is a decreasing limit of continuous functions. □

LEMMA 5.3. *Let D be an open subset of $\mathbb{R}_+^2$. Then*

$$C_0^2(\mathbb{R}^2) \cap \{\text{supp}\, f \cap \mathbb{R}_+^2 \subseteq D\}$$
$$\subseteq \mathscr{D}(D) := \left\{f\colon \lim_{t\to 0} t^{-1}\left[T_t^D f - f\right] = Lf \text{ uniformly on } D\right\}.$$

PROOF. Let $f \in C_0^2(\mathbb{R}^2) \cap \{\text{supp}\, f \cap \mathbb{R}_+^2 \subseteq D\}$ and set $K = \mathbb{R}_+^2 \cap \text{supp}\, f$. Choose a compact subset $\tilde{K} \subseteq D$ such that $K \subseteq \tilde{K}$, $\tilde{K} \cap \partial_+ D = \varnothing$ and $d(\mathbb{R}_+^2 \setminus \tilde{K}, K) > 0$. Set

$$\xi = d\left(\tilde{K}, \partial_+ D\right) \wedge d\left(\mathbb{R}_+^2 \setminus \tilde{K}, K\right) \qquad (>0). \tag{5.4}$$

For $\tilde{K}^c = \mathbb{R}_+^2 \setminus K$ and $K^c = \mathbb{R}_+^2 \setminus K$, by the strong Markov property applied at time $\eta_{\tilde{K}^c}$,

$$\begin{aligned} \sup_{v \in \tilde{K}^c} P_v(V(s) \in K) &\leq \sup_{v \in \tilde{K}^c} P_v(\eta_{K^c} \leq s) \\ &\leq \sup_{y \in \partial_+ \tilde{K}} P_y(\eta_{K^c} \leq s) \\ &\leq \sup_{y \in \partial_+ \tilde{K}} P_y\left(\sup_{u \leq s} |V(u) - y| \geq \xi\right). \end{aligned} \tag{5.5}$$

Since $f = 0$ on $\partial_+ D \cup \{\infty\}$ we get

$$\begin{aligned} T_t^D f(v) &= E_v f(V(t)) I(\eta_D \wedge \hat{e} \geq t) \quad \text{[by (5.2)]} \\ &= E_v f(V(t \wedge \eta_D \wedge \hat{e})) \end{aligned}$$

and because the law of $V(\cdot)$ on $(\Omega_2, \mathscr{M})$ solves the $(L, C_0^2(\mathbb{R}_+^2))$-martingale problem up to time $\hat{e}$ (see Theorem 3.2) we have

$$\begin{aligned}\sup_{v\in D}\left|t^{-1}\left[T_t^D f(v)-f(v)\right]-Lf(v)\right| &= \sup_{v\in D}\left|t^{-1}E_v\int_0^{t\wedge\eta_D\wedge\hat{e}}Lf(V(s))\,ds-Lf(v)\right| \\ &\le \sup_{v\in\tilde{K}}('')\vee\sup_{v\in D\setminus\tilde{K}}('') \\ &= (1)\vee(2),\quad\text{say.}\end{aligned}$$

Now for $\gamma>0$,

$$\begin{aligned}(1) &\le t^{-1}\int_0^t\sup_{v\in\tilde{K}}E_v I(\eta_D\wedge\hat{e}>s)|Lf(V(s))-Lf(v)|\,ds \\ &\quad+t^{-1}\int_0^t\sup_{v\in\tilde{K}}P_v(s\ge\eta_D\wedge\hat{e})|Lf(v)|\,ds \\ &\le t^{-1}\int_0^t\sup_{v\in\tilde{K}}E_v I(\eta_D\wedge\hat{e}>s)|Lf(V(s))-Lf(v)|I(|V(s)-v|<\gamma)\,ds \\ &\quad+t^{-1}\int_0^t 2[\sup|Lf|]\sup_{v\in\tilde{K}}P_v(|V(s)-v|\ge\gamma)\,ds \\ &\quad+t^{-1}\int_0^t\sup_{v\in\tilde{K}}P_v(s\ge\eta_D\wedge\hat{e})\sup|Lf|\,ds.\end{aligned}$$

The first term on the right can be made arbitrarily small for γ sufficiently small independent of t (by uniform continuity of Lf). In the third term, by (5.4)

$$P_v(s\ge\eta_D\wedge\hat{e})\le P_v\Big(\sup_{u\le s}|V(u)-v|\ge\xi\Big).$$

Hence by Lemma 5.2, for $\gamma>0$ given (small), we can make the second and third terms arbitrarily small too. Thus we can choose $\gamma>0$ small and then $t>0$ small so that (1) is arbitrarily small.

As for (2), since $Lf=0$ off supp $f=K\subseteq\tilde{K}$,

$$\begin{aligned}(2) &\le t^{-1}\int_0^t\sup_{v\in D\setminus\tilde{K}}E_v\left[I(\eta_D\wedge\hat{e}>s)|Lf(V(s))|\right]ds \\ &\le[\sup|Lf|]t^{-1}\int_0^t\sup_{v\in D\setminus\tilde{K}}P_v(\eta_D\wedge\hat{e}>s,V(s)\in K)\,ds.\end{aligned}$$

By (5.5) and Lemma 5.2 the ds-integrand can be made arbitrarily small for t sufficiently small. Thus $f\in\mathscr{D}(D)$ as desired. □

LEMMA 5.4.

$$\begin{aligned}&C_b^2(\mathbb{R}_+^2)\cap\{f\equiv\textit{constant outside some compact subset of }\mathbb{R}_+^2\} \\ &\quad\subseteq\Big\{f\in C_b(\mathbb{R}_+^2):\lim_{t\to0}\sup_v\left[T_t|f-f(v)|\right](v)=0\Big\}.\end{aligned}$$

PROOF. Let

$$f \in C_b^2(\mathbb{R}_+^2) \cap \{f \equiv \text{constant outside compact subset of } \mathbb{R}_+^2\}.$$

By replacing f by $f - f(\infty)$ it is no loss to assume $f \in C_0^2(\mathbb{R}_+^2)$. Let $K = \operatorname{supp} f$, a compact subset of $\mathbb{R}_+^2$. Then

$$\begin{aligned} \sup_v \left[T_t|f - f(v)|\right](v) &\le \sup_{v \in K^c} (\text{''}) \vee \sup_{v \in K} (\text{''}) \\ &= (1) \vee (2), \quad \text{say}, \end{aligned} \tag{5.6}$$

and since $f = 0$ on K^c,

$$(1) = \sup_{v \in K^c} |T_t|f|(v) - |f|(v)|.$$

By Lemma 5.3 (with $D = \mathbb{R}_+^2$) the latter goes to 0 as $t \to 0$. Moreover, for $B_\gamma(v) = \{w: |w - v| < \gamma\}$,

$$\begin{aligned} (2) &= \sup_K \left\{T_t\left[|f(\cdot) - f(v)| I_{B_\gamma(v)}(0)\right](v) + T_t\left[|f(\cdot) - f(v)| I_{B_\gamma(v)^c}(\cdot)\right](v)\right\} \\ &\le (\text{''}) + (2 \sup|f|) \sup_K P_v\left(V(t) \in B_\gamma(v)^c\right) \\ &\le (\text{''}) + (2 \sup|f|) \sup_K P_v\left(\sup_{s \le t} |V(s) - v| \ge \gamma\right). \end{aligned}$$

The first term on the right can be made arbitrarily small if $\gamma > 0$ (independent of t) is small enough and then given γ, the second term goes to 0 as $t \to 0$ by Lemma 5.2. □

6. Lower bounds. To get lower bounds on $P_v(\eta_\delta \wedge \hat{e} > t)$ as $t \to \infty$, we use the results of Donsker and Varadhan (1976). Let $\mathscr{M}$ be the set of probability measures on $\mathbb{R}_+^2$ and endow it with the topology of weak convergence.

For any $\mu \in \mathscr{M}$ with $\operatorname{supp} \mu$ being a compact subset of $\mathbb{R}_+^2$, define the Donsker–Varadhan I-function

$$I(\mu) = -\inf_{f \in \mathscr{C}^+} \int \frac{Lf}{f} \, d\mu, \tag{6.1}$$

where

$$\mathscr{C} = \left\{f \in C^2(\mathbb{R}_+^2): f \equiv \text{constant outside a compact subset of } \mathbb{R}_+^2\right\}$$

and

$$\mathscr{C}^+ = \{f \in \mathscr{C}: \inf f > 0\}.$$

Our lower bound is given in the following theorem.

THEOREM 6.1. *Let W be a compact subset of $\mathbb{R}_+^2$. Then*

$$\liminf_{t \to \infty} t^{-1} \log \inf_{v \in W} P_v(\eta_\delta \wedge \hat{e} > t) \ge \sup_{U \in \mathscr{K}(\delta)} \sup_{\mu(\overline{U}) = 1} [-I(\mu)],$$

where

$$\mathscr{K}(\delta) = \{U \subseteq \mathbb{R}_+^2\colon U \text{ is bounded and open in } \mathbb{R}_+^2 \text{ with } C^\infty \text{ boundary in } \mathbb{R}^2 \text{ and } U \subseteq \overline{U} \subseteq H_\delta\}$$

PROOF. First observe that in the right-hand side $\sup_{\mu(\overline{U})=1}$ may be replaced by $\sup_{\operatorname{supp}\mu \subseteq U}$. Let $U \in \mathscr{K}(\delta)$, $v \in W$. It is loss to assume $W \subseteq U$. We have

$$P_v(\eta_\delta \wedge \hat{e} > t) \geq P_v(\eta_U > t), \tag{6.2}$$

where $\eta_U = \inf\{t > 0\colon V(t) \notin U\}$. Thus we are only concerned with the behavior of $V(\cdot)$ on a neighborhood of $\overline{U}$. Hence the complicating factor of the possibility of $V(\cdot)$ exploding does not really enter into the scheme of things. So outside of $\overline{U}$ we modify the coefficients of the differential operator L associated to $V(\cdot)$ in a convenient manner, and then apply the Donsker–Varadhan theory. Now the details.

Modify the coefficients of L outside a neighborhood of $\overline{U}$ in such a way that a new operator $\tilde{L}$ is obtained with the following properties. First, $\tilde{L}$ is associated to a process $\tilde{V}(t)$ with state space $\mathbb{R}_+^2$ if $\tilde{V}(0) \in \mathbb{R}_+^2$. Second, the law of $\tilde{V}(\cdot)$ on $C([0,\infty), \mathbb{R}_+^2)$ is the unique nonexploding solution to the $(\tilde{L}, C_0^2(\mathbb{R}_+^2))$-martingale problem and $V(\cdot) = \tilde{V}(\cdot)$ in law up to the first exit time from U. We will write $\tilde{\eta}_U = \inf\{t > 0\colon \tilde{V}(t) \notin U\}$. Of course, $\eta_U = \tilde{\eta}_U$ in law, so our problem is reduced to obtaining lower bounds on $\inf_{v\in W} P_v(\tilde{\eta}_U > t)$ [see (6.2)].

Define a random measure $\tilde{L}_t \in \mathscr{M}$ by

$$\tilde{L}_t(A) = t^{-1}\int_0^t I_A(\tilde{V}(s))\,ds,$$

where A is a Borel set in $\mathbb{R}_+^2$. Thus $\tilde{L}_t(A)$ is just the proportion of time up to t spent by $\tilde{V}$ in A. Observe $\tilde{L}_t$ induces a probability measure $\tilde{Q}_{v,t}$ on $\mathscr{M}$ defined by

$$\tilde{Q}_{v,t}(\mathscr{O}) = P_v(\tilde{L}_t(\cdot) \in \mathscr{O}),$$

where $\mathscr{O}$ is a Borel set in $\mathscr{M}$. The semigroup

$$\tilde{T}_t\colon B(\mathbb{R}_+^2) \to B(\mathbb{R}_+^2)$$

defined by $\tilde{V}$ [i.e., $\tilde{T}_t f(v) = E_v f(\tilde{V}(t))$, $f \in B(\mathbb{R}_+^2)$] is actually Feller:

$$\tilde{T}_t\colon C_b(\mathbb{R}_+^2) \to C_b(\mathbb{R}_+^2). \tag{6.3}$$

This is because the law of $\tilde{V}(\cdot)$ on $C([0,\infty), \mathbb{R}_+^2)$ is the unique nonexploding solution to the $(\tilde{L}, C_0^2(\mathbb{R}_+^2))$-martingale problem [cf. Stroock and Varadhan (1979), proof of Corollary 6.3.3, pages 151 and 152].

Let $\tilde{L}$ be the strong infinitesimal generator, with domain $\tilde{\mathscr{D}} \subseteq C_b(\mathbb{R}_+^2)$, of the semigroup $\{\tilde{T}_t\}$. We write $\tilde{\mathscr{D}}^+ = \{f \in \tilde{\mathscr{D}}\colon \inf f > 0\}$. For each $\mu \in \mathscr{M}$ with

$\operatorname{supp}\mu \subseteq U$, define

$$I(\mu) = -\inf_{f\in\tilde{\mathscr{D}}^+} \int \frac{\tilde{L}f}{f}\,d\mu.$$

Apparently there is a conflict of notation with (6.1). However, since $\tilde{V}(\cdot)$ gives rise to the unique solution of the $(\tilde{L}, C_0^2(\mathbb{R}_+^2))$-martingale problem, by Pinsky's theorem [Pinsky (1985), Theorem 1.4 and Section 4, pages 344, 361 and 362] for $\mu \in \mathscr{M}$ with $\operatorname{supp}\mu \subseteq U$ we have

$$\inf_{f\in\tilde{\mathscr{D}}^+} \int \frac{\tilde{L}f}{f}\,d\mu = \inf_{f\in\mathscr{C}^+} \int \frac{\tilde{L}f}{f}\,d\mu.$$

Since $\operatorname{supp}\mu \subseteq U$, the latter is $\inf_{f\in\mathscr{C}^+} \int (Lf/f)\,d\mu$ and there is no ambiguity.

The following hypotheses are required in the Donsker–Varadhan theory. Define

$$\tilde{T}_t^U f(v) = E_v\big[f(\tilde{V}(t))I(\tilde{\eta}_U > t)\big], \qquad f \in B(\mathbb{R}_+^2), \tag{6.4}$$

$$\tilde{R}_U^\lambda(v, A) = \int_0^\infty e^{-\lambda t}\big(\tilde{T}_t^U I_A\big)(v)\,dt, \qquad \lambda > 0,\ A \subseteq \mathbb{R}_+^2. \tag{6.5}$$

Let m be Lebesgue measure on $\mathbb{R}^2$. Let

$$\begin{aligned} B_0 &= \Big\{f \in C_b(\mathbb{R}_+^2)\colon \lim_{t\to 0}\sup_v |\tilde{T}_t f - f| = 0\Big\},\\ B_{00} &= \Big\{f \in C_b(\mathbb{R}_+^2)\colon \lim_{t\to 0}\sup_v \big[\tilde{T}_t|f - f(v)|\big](v) = 0\Big\}. \end{aligned} \tag{6.6}$$

An argument similar to that in the proof of Lemma 5.4 shows

$$\{f \in C_b^2(\mathbb{R}_+^2)\colon f \equiv \text{constant outside some compact subset of } \mathbb{R}_+^2\} \subseteq B_{00}.$$

HYPOTHESIS H_1. *For each $v \in U$, $\tilde{R}_U^\lambda(v, dy) \ll m(dy)$.*

HYPOTHESIS H_2. *Let $\alpha \in \mathscr{M}$ with $I(\alpha) < \infty$. Then every neighborhood of α in $\mathscr{M}$ contains a neighborhood of the form*

$$\Big\{\mu \in \mathscr{M}\colon \Big|\int f_j[d\mu - d\alpha]\Big| < \varepsilon, 1 \le j \le k\Big\}, \tag{6.7}$$

where $f_1, \dots, f_k \in B_{00}$.

HYPOTHESIS H_3. *Let $\mu \in \mathscr{M}$ with $I(\mu) < \infty$ and $\operatorname{supp}\mu \subseteq U$. Then for any $f \in B(\mathbb{R}_+^2)$ there is a sequence $\{f_n\} \subseteq B_0$ such that $\sup|f_n| \le \sup|f|$ and $f_n \to f$ a.e. (μ).*

HYPOTHESIS H_4. *If $v \in U$ and $E \subseteq U$ with $m(E) > 0$, then $\tilde{R}_U^\lambda(v, E) > 0$.*

HYPOTHESIS H_5. *For each $E \subseteq U$ and $\lambda > 0$, $v \in U \to \tilde{R}_U^\lambda(v, E)$ is continuous.*

Now we can state the lower bound of Donsker and Varadhan (1976), Theorem 8.1, page 446.

THEOREM 6.2. *Let $\mu \in \mathscr{M}$ satisfy $I(\mu) < \infty$ and* $\operatorname{supp}\mu \subseteq U$. *Suppose N is a neighborhood of μ in $\mathscr{M}$ and $\mathscr{M}(U)$ is the set of probability measures in $\mathscr{M}$ with support contained in U. Under Hypotheses* H_1–H_5,

$$\liminf_{t\to\infty} t^{-1}\log \tilde{Q}_{v,t}(N \cap \mathscr{M}(U)) \geq -I(\mu),$$

uniformly for v in compact subsets of U.

REMARK. For H_1 Donsker and Varadhan (1976) actually require the existence of a reference measure β on $\mathbb{R}_+^2$ such that $P_v(V(t) \in dy) = p(t, v, y)\beta(dy)$. However, examination of the proof shows that this is needed only to assert $\tilde{R}_U^\lambda(v, dy) \ll \beta(dy)$—our Hypothesis H_1. Also, their Hypotheses H_3 and H_4 are only required for the measure μ in the statement of their Theorem 8.1.

THEOREM 6.3. *In the present context, Hypotheses* H_1–H_5 *hold.*

PROOF. H_1: Let E be a Borel set in U with $m(E) = 0$. Then for each $t > 0$,

$$\begin{aligned}
&\int_0^t e^{-\lambda s} P_v\big(\tilde{\eta}_U > s, \tilde{V}(s) \in E\big)\,ds\\
&\quad= \int_0^t e^{-\lambda s} P_v(\eta_U > s, V(s) \in E)\,ds\\
&\quad= \int_0^t e^{-\lambda s} P_v\big(\eta_U^1 > \tau_1(s), V_1(\tau_1(s)) \in E\big)\,ds \quad [\text{by } (3.14)]\\
&\quad= E_v \int_0^{\tau_1(t)} e^{-\lambda\tau_1^{-1}(s)} I\big(\eta_U^1 > s, V_1(s) \in E\big) \frac{2\,ds}{\big[V_1^{(1)}(s) + V_1^{(2)}(s)^2 + 1\big]^2}.
\end{aligned}$$

But

$$\begin{aligned}
P_v(V_1(s) \in E) &= E_v \overline{R}_1^1(s) I\big(\overline{V}_1(s) \in E\big) \qquad [\text{by } (3.11)]\\
&\leq e^{cs}\big[P_v\big(\overline{V}_1(s) \in E\big)\big]^{1/2} \qquad (\text{by Lemma 3.1})\\
&= 0,
\end{aligned}$$

since $\overline{V}_1(\cdot)$ has a density with respect to m (cf. the proof of Lemma 5.1). Thus

$$\tilde{R}_U^\lambda(v, E) = \lim_{t\to\infty} \int_0^t e^{-\lambda s} P_v\big(\tilde{\eta}_U > s, \tilde{V}(s) \in E\big)\,ds = 0$$

as desired.

H_2: Let $\alpha \in \mathscr{M}$ and consider any neighborhood N of α. It is no loss to assume that for some $\varepsilon > 0$ and $h_1, \dots, h_p \in C_b^2(\mathbb{R}^2)$,

$$N = \left\{\mu \in \mathscr{M}: \left|\int h_j\,d(\alpha - u)\right| < \varepsilon, 1 \leq j \leq p\right\}.$$

Choose $M > 0$ such that

$$\alpha\big(\mathbb{R}^2_+ \setminus B_{M-1}(0)\big) < \varepsilon\Big[16 \max_{1 \le j \le p} \sup|h_j|\Big]^{-1}.$$

Let $f \in C_b^2(\mathbb{R})$ satisfy $0 \le f \le 1$, $f \equiv 1$ on $[M, \infty)$ and $f \equiv 0$ on $(-\infty, M-1]$. Set $\tilde{h}_{p+1}(x) := f(|x|)$, and observe it is in B_{00}. For $K := \mathbb{R}^2_+ \cap B_M(0)$ extend $h_j|_K$, $1 \le j \le p$, by $\tilde{h}_j \in C_0^2(\mathbb{R}^2_+)$ with $|\tilde{h}_j| \le h_j|$ pointwise. Thus $\tilde{h}_j \in B_{00}$, $1 \le j \le p$. All that remains is to show for $\gamma = (\frac{1}{2} \wedge [16 \max_{1 \le j \le p} \sup|h_j|]^{-1})\varepsilon$,

$$\Big\{\mu \in \mathscr{M}\colon \Big|\int \tilde{h}_j \, d(\alpha - \mu)\Big| < \gamma, 1 \le j \le p+1\Big\} \subseteq N.$$

Indeed, if μ satisfies $|\int \tilde{h}_j \, d(\alpha - \mu)| < \gamma$, $1 \le j \le p+1$, then for $K^c = \mathbb{R}^2_+ \setminus K$,

$$\begin{aligned}\mu(K^c) &\le \int \tilde{h}_{p+1} \, d\mu \le \gamma + \int \tilde{h}_{p+1} \, d\alpha \\ &\le \gamma + \alpha\big(\mathbb{R}^2_+ \setminus B_{M-1}(0)\big) \\ &\le \varepsilon\Big[8 \max_{1 \le j \le p} \sup|h_j|\Big]^{-1}.\end{aligned}$$

Thus for $1 \le j \le p$ we have

$$\begin{aligned}\Big|\int h_j \, d(\alpha - \mu)\Big| &\le \Big|\int \tilde{h}_j \, d(\alpha - \mu)\Big| + \Big|\int \big(h_j - \tilde{h}_j\big) \, d(\alpha - \mu)\Big| \\ &\le \gamma + \Big|\int_{K^c} \big(h_j - \tilde{h}_j\big) \, d(\alpha - \mu)\Big| \\ &\le \frac{\varepsilon}{2} + \Big[2 \max_{1 \le j \le p} \sup|h_j|\Big]\big[\alpha(K^c) + \mu(K^c)\big] \\ &\le \frac{\varepsilon}{2} + \Big[2 \max_{1 \le j \le p} \sup|h_j|\Big]\Big[\varepsilon\Big[16 \max_{1 \le j \le p} \sup|h_j|\Big]^{-1} \\ &\qquad + \varepsilon\Big[8 \max_{1 \le j \le p} \sup|h_j|\Big]^{-1}\Big] \\ &< \varepsilon.\end{aligned}$$

H_3: Since $C_0^2(U)$ is dense in $L^1(U, d\mu)$, we can choose $f_n \in C_0^2(U)$ with $f_n \to f|_U$ in $L^1(U, d\mu)$ and $\sup_U |f_n| \le \sup_U |f|$. Extract a subsequence $f_{n_k} \to f|_U$ a.e. (μ). Since $f_{n_k} \in C_0^2(U) \subseteq C_0^2(\mathbb{R}^2_+) \subseteq B_{00} \subseteq B_0$, H_3 follows.

H_4: Let $E \subseteq U$ with $m(E) > 0$ and $v \in U$. First assume $v = (v_1, v_2)$ satisfies $v_1 > 0$. Then we can choose a bounded open (in $\mathbb{R}^2$) set $D \subseteq \overline{D} \subseteq (0, \infty) \times \mathbb{R}$ with $v \in D$ and $m(D \cap E) > 0$. Since the diffusion $\tilde{V}$ is very well behaved on $\overline{D}$,

$$\begin{aligned}0 &< \int_0^\infty e^{-\lambda t} P_v\big(\tilde{V}(s) \in \overline{D} \text{ for all } 0 \le s \le t \text{ and } \tilde{V}(t) \in E \cap D\big) \, dt \\ &\le \int_0^\infty e^{-\lambda t} P_v\big(\tilde{\eta}_U > t, \tilde{V}(t) \in E\big) \, dt \\ &= \tilde{R}_U^\lambda(v, E) \quad \text{as desired.}\end{aligned}$$

An easy stopping time argument handles the case when $v_1 = 0$.

H_5: This follows immediately from Lemma 5.1 and the fact that $T_t^U = \tilde{T}_t^U$. This completes the proof of Theorem 6.3. □

Now we complete the proof of Theorem 6.1. We have

$$\begin{aligned}\liminf_{t\to\infty} t^{-1}\log\inf_{v\in W} P_v(\eta_U > t) &= \liminf_{t\to\infty} t^{-1}\log\inf_{v\in W} P_v(\tilde{\eta}_U > t)\\ &= \liminf_{t\to\infty} t^{-1}\log\inf_{v\in W} \tilde{Q}_{v,t}(\mathscr{M}(U))\\ &\geq \sup_{\operatorname{supp}\mu\subseteq U}[-I(\mu)]\end{aligned}$$

by Theorems 6.2 and 6.3. Since $U\in\mathscr{K}(\delta)$ was arbitrary, by (6.2) the desired conclusion holds. □

7. Upper bounds. For $\partial_+ H_\delta = \{0\}\times\{\mathbb{R}\setminus(-\delta,\delta)\}$ (as before) define

$$\mathscr{G}(\delta) = \Big\{ f\in C^2(H_\delta) \mid \text{for some constants } c_1 \text{ and } c_2,\ \lim_{v\to\partial_+ H(\delta)} f(v) = c_1,$$
$$\lim_{v\to\infty} f(v) = c_2,\ -\infty < \inf L\log|f|,\ \sup\frac{Lf}{f} < \infty\Big\}.$$

Write $\mathscr{G}(\delta)^+$ for those elements f of $\mathscr{G}(\delta)$ with $\inf f > 0$. Our upper bound is given in the following theorem.

THEOREM 7.1.

$$\begin{aligned}&\limsup_{t\to\infty} t^{-1}\log\sup_{H(\delta)} P_v(\eta_\delta\wedge\hat{e} > t)\\ &\quad\leq \inf\Big\{\sup_{H(\delta)}\frac{Lf}{f} : f\in\mathscr{G}(\delta)^+,\ \sup_{H(\delta)}\frac{Lf}{f} < 1\Big\}.\end{aligned}$$

For the proof we need the following results.

Let $H(\delta)^{**} = H_\delta\cup\{\pm\infty\}$ be a two-point metrizable compactification of H_δ obtained by identifying $\partial_+ H_\delta$ to $-\infty$ and then performing the usual one-point compactification procedure on the result by adjoining $+\infty$. Thus a neighborhood of $+\infty$ is of the form $(H_\delta\cup\{-\infty\})\setminus K$ for some compact subset K of $H_\delta\cup\{-\infty\}$ and a neighborhood of $-\infty$ is of the form $\mathbb{R}_+^2\cap U$ for some open set $U\subseteq\mathbb{R}^2$ with $\{0\}\times\{\mathbb{R}\setminus(-\delta,\delta)\}\subseteq U$.

Define the semigroup $T_t^{**}: B(H_\delta^{**})\to B(H_\delta^{**})$ by

$$T_t^{**}f(v) = E_v f(V(t\wedge\eta_\delta\wedge\hat{e}))$$

$$(7.1)\qquad = \begin{cases} T_t^{H(\delta)}f(v) + f(-\infty)P_v(\eta_\delta\wedge\hat{e}\leq t,\ \eta_\delta<\hat{e})\\ \qquad +f(\infty)P_v(\eta_\delta\wedge\hat{e}\leq t,\ \eta_\delta\geq\hat{e}), & v\in H_\delta,\\ f(-\infty), & v=-\infty,\\ f(\infty), & v=\infty.\end{cases}$$

Let $\mathscr{D}(\delta) = \mathscr{D}_\delta$ be the domain of the strong generator $\mathscr{L}_\delta$ of T_t^{**}. Then for any $f \in \mathscr{D}_\delta$,

$$T_t^{**}f - f = \int_0^t T_s^{**}\mathscr{L}_\delta f\,ds = \int_0^t \mathscr{L}_\delta T_s^{**} f\,ds \tag{7.2}$$

[Dynkin (1965), volume 1, page 23, 1.3.C].

THEOREM 7.2 (Maximum principle). *Assume* $f \in B(H_\delta^{**})$

(i) $$f(t,\cdot) \in \mathscr{D}_\delta \quad \textit{for } t \geq 0,$$

(ii) $$\frac{\partial f}{\partial t}(t,\cdot) \in \mathscr{D}_\delta \quad \textit{for } t \geq 0,$$

(iii) $$\frac{\partial}{\partial t}[\mathscr{L}_\delta(t,\cdot)] = \mathscr{L}_\delta\left[\frac{\partial f}{\partial t}(t,\cdot)\right].$$

Then

$$E_v f(t, V(t \wedge \eta_\delta \wedge \hat{e})) - f(0,v) = E_v \int_0^t \left(\frac{\partial f}{\partial u} + \mathscr{L}_\delta f\right)(u, V(u \wedge \eta_\delta \wedge \hat{e}))\,du.$$

PROOF. Write $\xi(t) = V(t \wedge \eta_\delta \wedge \hat{e})$. Then

$$\begin{aligned}
&E_v f(t,\xi(t)) - f(0,v)\\
&\quad = E_v[f(t,\xi(t)) - f(0,\xi(t)) + f(0,\xi(t)) - f(0,v)]\\
&\quad = E_v \int_0^t \frac{\partial f}{\partial u}(u,\xi(t))\,du + E_v \int_0^t (\mathscr{L}_\delta f(0,\cdot))(\xi(u))\,du\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{by (7.2) and (i)}]\\
&\quad = E_v \int_0^t \frac{\partial f}{\partial u}(u,\xi(u))\,du + E_v \int_0^t \left[\frac{\partial f}{\partial u}(u,\xi(t)) - \frac{\partial f}{\partial u}(u,\xi(u))\right] du\\
&\qquad + E_v \int_0^t [\mathscr{L}_\delta f(u,\cdot)](\xi(u))\,du\\
&\qquad + E_v \int_0^t [[\mathscr{L}_\delta f(0,\cdot)](\xi(u)) - [\mathscr{L}_\delta f(u,\cdot)](\xi(u))]\,du\\
&\quad = E_v \int_0^t \left[\frac{\partial f}{\partial u} + \mathscr{L}_\delta f\right](u,\xi(u))\,du + E_v \int_0^t \int_u^t \left[\mathscr{L}_\delta \frac{\partial f}{\partial u}\right](u,\xi(s))\,ds\,du\\
&\qquad - E_v \int_0^t \int_0^u \frac{\partial}{\partial s}[\mathscr{L}_\delta f(s,\cdot)](\xi(u))\,ds\,du \quad [\text{by (7.2) and (ii)}].
\end{aligned}$$

The proof will be complete once we show that the last two terms cancel.

Indeed, in the last term, by (iii) and an interchange of integrals,

$$E_v\int_0^t\int_0^u \frac{\partial}{\partial s}[\mathscr{L}_\delta f(s,\cdot)](\xi(u))\,ds\,du = E_v\int_0^t\int_s^t \mathscr{L}_\delta\left[\frac{\partial f}{\partial s}(s,\cdot)\right](\xi(u))\,du\,ds,$$

precisely the second-to-last term, as desired. □

PROOF OF THEOREM 7.1. Write

$$l_\delta = \inf\left\{\sup_{H(\delta)}\frac{Lf}{f}: f\in\mathscr{G}(\delta)^+,\ \sup_{H(\delta)}\frac{Lf}{f}<1\right\}.$$

Let $\varepsilon>0$ and choose $f\in\mathscr{G}(\delta)^+$ with $\sup Lf/f\le l_\delta+\varepsilon$. Consider any $g\in C_0^\infty(\mathbb{R}^2)\cap\{\operatorname{supp} g\cap\mathbb{R}_+^2\subseteq H_\delta\}$ with $0\le g\le 1$. Define $w(t,v):=cf(v)e^{(l_\delta+\varepsilon)t}$, where $c=[\inf f]^{-1}$. Notice

$$\begin{aligned} &w\in C^{1,2}([0,\infty)\times H_\delta)\cap C_b([0,\infty)\times H_\delta),\\ (7.3)\qquad &w(0,v)=cf(v)\ge 1\ge g(v),\\ &\left(\frac{\partial}{\partial t}-L\right)w=\left(l_\delta+\varepsilon-\frac{Lf}{f}\right)w\ge 0. \end{aligned}$$

We are going to use the maximum principle (Theorem 7.2) to show $w\ge T_t^{H(\delta)}g$. Indeed, let $T>0$ and for $t\in[0,T]$ define $h(t,v)=T_{T-t}^{H(\delta)}g(v)$ if $v\in H_\delta$ and 0 if $v\in\{\pm\infty\}$.

For the rest of the proof we take $v\in H_\delta$. Notice $T_t^{H(\delta)}g=T_t^{**}g$ and $T_t^{H(\delta)}Lg=T_t^{**}Lg$ on H_δ. Then by Lemma 5.3, Lg and $g\in\mathscr{D}_\delta$ and $\mathscr{L}_\delta g=Lg$. Moreover, $h(t,\cdot)=T_{T-t}^{**}g(\cdot)$ and $h(t,\cdot)\in\mathscr{D}_\delta$ for $t\in[0,T]$. Also,

$$\begin{aligned} (7.4)\qquad \frac{\partial h}{\partial t}(t,\cdot) &= -\frac{\partial}{\partial s}[T_s^{**}g]\Big|_{s=T-t} = -\mathscr{L}_\delta T_{T-t}^{**}g\\ &= -T_{T-t}^{**}Lg\in\mathscr{D}_\delta, \end{aligned}$$

and consequently

$$\begin{aligned} \mathscr{L}_\delta\frac{\partial h}{\partial t}(t,\cdot) &= -\mathscr{L}_\delta T_{T-t}^{**}Lg\\ &= \frac{\partial}{\partial t}[T_{T-t}^{**}Lg] = \frac{\partial}{\partial t}\mathscr{L}_\delta T_{T-t}^{**}g\\ &= \frac{\partial}{\partial t}\mathscr{L}_\delta h(t,\cdot). \end{aligned}$$

Thus (i)–(iii) in Theorem 7.2 hold for $h(t,\cdot)$, $t\in[0,T]$, and hence for $t\in[0,T]$ we have

$$E_v h(t,V(t\wedge\eta_\delta\wedge\hat{e}))-h(0,v)=E_v\int_0^t\left[\frac{\partial h}{\partial v}+\mathscr{L}_\delta h\right](u,V(u\wedge\eta_\delta\wedge\hat{e}))\,du.$$

By (7.4) $\partial h/\partial t = -\mathscr{L}_\delta h$ so this becomes

$$E_v h(t, V(t \wedge \eta_\delta \wedge \hat{e})) - h(0, v) = 0, \qquad t \in [0, T].$$

But $h(t, \pm\infty) = 0$ by definition, hence we get

$$(7.5) \qquad E_v[h(t, V(t)) I(\eta_\delta \wedge \hat{e} > t)] - h(0, v) = 0, \qquad t \in [0, T].$$

Since the law of $V(\cdot)$ on $(\Omega_2, \mathscr{M})$ solves the $(L, C_0^2(\mathbb{R}_+^2))$-martingale problem up to time $\hat{e}$ (Theorem 3.2), by optional stopping we have for any bounded open subset K of H_δ,

$$\begin{aligned} &E_v w(T - t \wedge \eta_K, V(t \wedge \eta_K)) - w(T, v) \\ &\quad = E_v \int_0^{t \wedge \eta_K} \left(Lw - \frac{\partial w}{\partial s} \right)(T - s, V(s))\, ds \\ &\quad \le 0 \quad \text{by } (7.3) \end{aligned}$$

[cf. Stroock and Varadhan (1979), Theorem 4.2.1, page 86]. Letting $K \uparrow H_\delta$ gives

$$E_v w(T - t \wedge \eta_\delta \wedge \hat{e}, V(t \wedge \eta_\delta \wedge \hat{e})) - w(T, v) \le 0, \qquad t \in [0, T].$$

Subtraction of (7.5) yields

$$\begin{aligned} &E_v w(T - t \wedge \eta_\delta \wedge \hat{e}, V(t \wedge \eta_\delta \wedge \hat{e})) - E_v[\{T_{T-t}^{H(\delta)} g(V(t))\} I(\eta_\delta \wedge \hat{e} > t)] \\ &\quad \le w(T, v) - T_T^{H(\delta)} g(v), \end{aligned}$$

where $t \in [0, T]$. Since $w \ge 0$,

$$E_v[w(T - t, V(t)) - T_{T-t}^{H(\delta)} g(V(t))] I(\eta_\delta \wedge \hat{e} > t) \le w(T, v) - T_T^{H(\delta)} g(v),$$

and upon letting $t \uparrow T$,

$$E_v[w(0, V(T)) - g(V(T))] I(\eta_\delta \wedge \hat{e} \ge T) \le w(T, v) - T_T^{H(\delta)} g(v),$$

where we have used Lemma 5.3 to get $T_t^{H(\delta)} g \to g$ as $t \to 0$, uniformly on H_δ. Now the left-hand side is nonnegative by (7.3), hence we end up with $T_T^{H(\delta)} g(v) \le w(T, v)$. Let $g \uparrow 1$ pointwise on H_δ and obtain $P_v(\eta_\delta \wedge \hat{e} > T) \le w(T, v)$. Finally, we get

$$\limsup_{t \to \infty} t^{-1} \log \sup_{H(\delta)} P_v(\eta_\delta \wedge \hat{e} > t) \le \limsup_{t \to \infty} t^{-1} \log \left[\frac{\sup f}{\inf f} e^{(l_\delta + \varepsilon)t} \right] = l_\delta + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof of the theorem is complete. □

8. Equality of the bounds. First we work on the upper bound.
Let $H(\delta)^{**}$ be as in Section 7. For any

$$f \in G(\delta)^+ \cap \left\{ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}$$

define

$$\frac{Lf}{f}(\pm\infty) := \limsup_{v\to\pm\infty} \frac{Lf}{f}(v).$$

For any metric space S define $\mathscr{M}(S)$ to be the space of probability measures on S; endow $\mathscr{M}(S)$ with the topology of weak convergence.

LEMMA 8.1. *For any compact set $C \subseteq H(\delta)^{**}$,*

(a)
$$\inf\left\{\sup_C \frac{Lf}{f} : f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1\right\}$$
$$= \sup_{\mu\in\mathscr{M}(C)} \inf\left\{\int \frac{Lf}{f}\, d\mu : f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1\right\};$$

(b)
$$\inf_{f\in\mathscr{G}(\delta)^+} \sup_C \frac{Lf}{f} = \sup_{\mu\in\mathscr{M}(C)} \inf_{f\in\mathscr{G}(\delta)^+} \int \frac{Lf}{f}\, d\mu.$$

PROOF. The proofs of (a) and (b) are quite similar, so we only furnish that for (a). Let

$$\mathscr{S} = \left\{\log f : f \in \mathscr{G}(\delta)^+ \text{ and } \sup_{H(\delta)} \frac{Lf}{f} < 1\right\}$$

and observe that for $h \in \mathscr{S}$ with

$$u = e^h \in \mathscr{G}(\delta)^+ \cap \left\{\sup_{H(\delta)} \frac{Lf}{f} < 1\right\}$$

we have

$$\frac{Lu}{u} = Lh + \frac{1}{2}\langle a\nabla h, \nabla h\rangle = \frac{Le^h}{e^h}, \tag{8.1}$$

where

$$a(v) = \frac{1}{2}\left(v_1 + v_2^2 + 1\right)^2 \begin{pmatrix} 4v_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\mathscr{S} = \left\{h \in C^2(H_\delta) \cap C_b(H_\delta^{**}) \,\middle|\, -\infty < \inf_{H(\delta)} Lh,\ \sup_{H(\delta)} \frac{Le^h}{e^h} < 1\right\}.$$

For $h \in \mathscr{S}$ define $\mathscr{B}_h$ to be the collection of sets of the form

$$N_h(\{\varepsilon_x\}) := \left\{g \in \mathscr{S} : \left|\frac{Le^g}{e^g}(x) - \frac{Le^h}{e^h}(x)\right| < \varepsilon_x\ \forall\, x \in H_\delta^{**} \text{ and } \sup\left|\frac{Le^g}{e^g}\right| \le \sup\left|\frac{Le^h}{e^h}\right|\right\},$$

where $\{\varepsilon_x\} \in (0,\infty)^{H(\delta)^{**}}$. Then $\{\mathscr{B}_h\colon h \in \mathscr{S}\}$ induces a topology on $\mathscr{S}$ in which $\mathscr{B}_h$ is a neighborhood base at h for each $h \in \mathscr{S}$. Moreover, if $h_n \to h$ in this topology then

$$\frac{Le^{h_n}}{e^{h_n}}(x) \to \frac{Le^h}{e^h}(x)$$

for each $x \in H_\delta^{**}$ and $\sup_{n,x}|Le^{h_n}/e^{h_n}(x)| < \infty$. Thus, for fixed $\mu \in \mathscr{M}(C)$, the function

$$h \in \mathscr{S} \to \int \frac{Le^h}{e^h}\, d\mu$$

is lower semicontinuous (by Fatou's lemma) and convex [by (8.1)]. Since Le^h/e^h is upper semicontinuous on H_δ^{**} for each $h \in \mathscr{S}$, $\mu \in \mathscr{M}(C) \to \int (Le^h/e^h)\, d\mu$ is upper semicontinuous and linear. Now $\mathscr{M}(C)$ is convex and compact and $\mathscr{S}$ is convex, so by Sion's minimax theorem [Sion (1958), Corollary 3.3, page 174] we have

$$\begin{aligned}
\mathrm{RHS}(a) &= \sup_{\mu \in \mathscr{M}(C)} \inf_{h \in \mathscr{S}} \int \frac{Le^h}{e^h}\, d\mu \\
&= \inf_{h \in \mathscr{S}} \sup_{\mu \in \mathscr{M}(C)} \int \frac{Le^h}{e^h}\, d\mu \\
&= \inf_{h \in \mathscr{S}} \sup_{C} \frac{Le^h}{e^h} \\
&= \inf\left\{ \sup_C \frac{Lf}{f} \colon f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}
\end{aligned}$$

(by definition of $\mathscr{S}$). □

LEMMA 8.2. *For $\mathscr{M}_{**} = \mathscr{M}(H(\delta)^{**})$, we have*

$$\begin{aligned}
-\lambda_\delta &:= \sup_{\mu \in \mathscr{M}_{**}} \inf_{f \in \mathscr{G}(\delta)^+} \int \frac{Lf}{f}\, d\mu \\
&= \sup_{\mu \in M_{**}} \inf\left\{ \int \frac{Lf}{f}\, d\mu \colon f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}.
\end{aligned}$$

PROOF. Clearly $\leq$ holds. For the opposite inequality, let $-\lambda_\delta$ be the left-hand side. By Lemma 8.1(b), $-\lambda_\delta \leq 0$ and moreover, given $0 < \varepsilon < 1$ we can choose $f \in \mathscr{G}(\delta)^+$ such that

$$\sup_{H(\delta)^{**}} \frac{Lf}{f} \leq -\lambda_\delta + \varepsilon < 1.$$

Thus

$$f \in \mathscr{S}(\delta)^+ \cap \left\{ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}$$

and

$$\sup_{\mu \in \mathscr{M}_{**}} \inf \left\{ \int \frac{Lf}{f} \, d\mu \colon f \in \mathscr{S}(\delta)^+, \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\} \le -\lambda_\delta + \varepsilon.$$

Letting $\varepsilon \to 0$ gives the desired inequality. □

LEMMA 8.3. *For λ_δ as in Lemma 8.2, there is $\mu_0 \in \mathscr{M}_{**}$ such that $\mu_0\{\pm\infty\} = 0$ and*

$$-\lambda_\delta = \inf \left\{ \int \frac{Lf}{f} \, d\mu_0 \colon f \in \mathscr{S}(\delta)^+, \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}.$$

PROOF. By upper semicontinuity, the first supremum in Lemma 8.2 is taken on, say at μ_0, and hence

$$\begin{aligned} -\lambda_\delta &= \inf_{f \in \mathscr{S}(\delta)^+} \int \frac{Lf}{f} \, d\mu_0 = \sup_{\mu \in \mathscr{M}_{**}} \inf \left\{ \int \frac{Lf}{f} \, d\mu \colon f \in \mathscr{S}(\delta)^+, \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\} \\ &\ge \inf \left\{ \int \frac{Lf}{f} \, d\mu_0 \colon f \in \mathscr{S}(\delta)^+, \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\} \\ &\ge \inf_{f \in \mathscr{S}(\delta)^+} \int \frac{Lf}{f} \, d\mu_0 = -\lambda_\delta. \end{aligned}$$

Thus

$$-\lambda_\delta = \inf \left\{ \int \frac{Lf}{f} \, d\mu_0 \colon f \in \mathscr{S}(\delta)^+, \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}. \tag{8.2}$$

It remains to show $\mu_0\{\pm\infty\} = 0$.

First assume $\mu_0\{-\infty\} > 0$. Consider the ordinary differential equation

$$g''(x) + \frac{2-\alpha}{2x} g'(x) + \frac{M^2}{2x} g(x) = 0,$$

where $M > 1$. A solution is given by [see Gradshteyn and Ryzhik (1980), page 971, 8.491.3]

$$g(x) = C x^{\alpha/4} J_{-\alpha/2}(\sqrt{2} M x^{1/2}),$$

where J_ν is Bessel's function and C is chosen so that

$$g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k M^{2k} x^k}{2^k k! \Gamma(k+1-\alpha/2)}.$$

Let $g_M(v) = g(v_1)$ for $v \in \mathbb{R}_+^2$. Then for v_1 small, $g_M(v) > 0$, and hence there is a neighborhood N of $\partial_+ H_\delta = \{0\} \times \{\mathbb{R} \setminus (-\delta, \delta)\}$ in $\mathbb{R}_+^2$ such that $\inf_N g_M > 0$ and for $v \in N$,

$$
\begin{aligned}
Lg_M(v) &= L[g(v_1)] \\
&= \frac{(v_1 + v_2^2 + 1)^2}{4}\left[4v_1 g''(v_1) + 2\left(2 - \alpha - \frac{2(1-\alpha)v_1}{v_1 + v_2^2 + 1}\right)g'(v_1)\right] \\
&= \frac{(v_1 + v_2^2 + 1)^2}{4}\left[4v_1\left(-\frac{2-\alpha}{2v_1}g'(v_1) - \frac{M^2}{2v_1}g(v_1)\right)\right. \\
&\qquad \left. + 2\left(2 - \alpha - \frac{2(1-\alpha)v_1}{v_1 + v_2^2 + 1}\right)g'(v_1)\right] \qquad (8.3)\\
&= \frac{(v_1 + v_2^2 + 1)^2}{4}\left\{-2M^2 g(v_1) - \frac{4(1-\alpha)v_1}{(v_1 + v_2^2 + 1)}g'(v_1)\right\} \\
&\le \frac{(v_1 + v_2^2 + 1)^2}{4}\{-2M^2 g(v_1) + g(v_1)\} \\
&\le \frac{1}{4}[1 - 2M^2]g_M(v).
\end{aligned}
$$

By changing g outside a small neighborhood of 0, we can assume $g_M \in \mathscr{S}(\delta)^+$.

Set $\mu_1(B) = \mu_0(B \cap \{-\infty\})/\mu_0\{-\infty\}$, an element of $\mathscr{M}(\{-\infty\})$. Then by (8.2)

$$-\lambda_\delta \le \mu_0\{-\infty\}\inf\left\{\int \frac{Lf}{f}\, d\mu_1 : f \in \mathscr{S}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1\right\} + 1.$$

Since $\mu_1 \in \mathscr{M}(\{-\infty\})$ we see the infimum on the right only depends on the behavior of Lf/f, $f \in \mathscr{S}(\delta)^+$, in a neighborhood of $\{-\infty\}$. Thus by (8.3)

$$
\begin{aligned}
-\lambda_\delta &\le \mu_0\{-\infty\}\int \frac{Lg_M}{g_M}\, d\mu_1 + 1 \\
&\le \mu_0\{-\infty\}\frac{1}{4}[1 - 2M^2] + 1,
\end{aligned}
$$

and upon letting $M \to \infty$ we get $-\lambda_\delta = -\infty$, a contradiction. Hence $\mu_0\{-\infty\} = 0$.

Next assume $\mu_0\{+\infty\} > 0$. If for each $M > 1$ we can find $f_M \in \mathscr{S}(\delta)^+$ and $K(M) > 0$ such that $Lf_M \le -K(M)f_M$ on a neighborhood of $+\infty$ and $K(M) \to \infty$, then a simple modification of the argument above yields $-\lambda_\delta = -\infty$, a contradiction. Thus $\mu_0\{+\infty\} = 0$. So let us find f_M.

By Corollary 4.3, L is $\tilde{L}$ expressed in the coordinates $(v_1, v_2) = (u_1^2, u_2)$, $u \in \mathbb{R}_+^2$. Thus it suffices to find $f_M \in C^2(H_\delta)$ such that f_M has limits at $\pm\infty$, $\tilde{L}f_M \le -K(M)f_M$ on a neighborhood of $+\infty$, $\inf f_M > 0$, and $-\infty < \inf \tilde{L}(\log f_M)$, $\sup \tilde{L}f_M/f_M < \infty$.

Expressing $\tilde{L}$ in polar coordinates (ρ, θ), where $|u|^2 = \rho^2$ and $\tan\theta = u_2/u_1$, we see its radial part is

$$D_\rho = \frac{(\rho^2+1)^2}{4}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\left(1 + \frac{(1-\alpha)(1-\rho^2)}{\rho^2+1}\right)\frac{\partial}{\partial\rho}\right].$$

Then it suffices to find $\varepsilon > 0$ and $g_M \in C_b^2((\varepsilon^{-1}, \infty))$ such that $\inf_{(\varepsilon^{-1},\infty)} g_M > 0$, $\lim_{\rho\to\infty} g_M(\rho) = C > 0$ exists and $D_\rho g_M \le -K(M)g_M$ on $(\varepsilon^{-1}, \infty)$, where $K(M) \to \infty$ as $M \to \infty$.

Changing variables $s = \rho^{-1}$, we need to find $\varepsilon > 0$ and $g_M \in C_b^2(0, \varepsilon)$ with $\inf_{(0,\varepsilon)} g_M > 0$, $\lim_{s\to 0} g_M(s) = c$ and $\overline{D}_s g_M \le -K(M)g_M$ on $(0, \varepsilon)$, where

$$\overline{D}_s = \frac{(1+s^2)^2}{4}\left[\frac{d^2}{ds^2} + \frac{1}{s}\left(2 - \alpha - \frac{2s^2(1-\alpha)}{1+s^2}\right)\frac{d}{ds}\right].$$

The differential equation

$$f'' + \frac{2-\alpha}{s}f' + M^2 f = 0$$

has

$$f(s) = Cs^{(\alpha-1)/2}J_{(1-\alpha)/2}(Ms)$$

as a solution, where J_ν is Bessel's function [Gradshteyn and Ryzhik (1980), page 971, 8.491.6] and C is chosen so that

$$f(s) = \sum_{k=0}^{\infty} \frac{(-1)^k (Ms)^{2k}}{4^k k!\,\Gamma((3-\alpha)/2 + k)}.$$

Set $g_M = f$ and see that for $\varepsilon > 0$ sufficiently small, $g_M > 0$ on $[0, \varepsilon)$, $g_M \in C_b^2(0, \varepsilon)$ and on $(0, \varepsilon)$

$$\begin{aligned}
&\frac{1}{4}(s^2+1)^2\left[g_M'' + \frac{1}{s}\left(2-\alpha-\frac{2s^2(1-\alpha)}{1+s^2}\right)g_M'\right]\\
&\quad = \frac{1}{4}(s^2+1)^2\left[-\left(\frac{2-\alpha}{s}g_M' + M^2 g_M\right) + \frac{1}{s}\left(2-\alpha-\frac{2s^2(1-\alpha)}{1+s^2}\right)g_M'\right]\\
&\quad = \frac{1}{4}(s^2+1)^2\left[-\frac{2s^2(1-\alpha)}{1+s^2}g_M' - M^2 g_M\right]\\
&\quad \le \frac{1}{4}(s^2+1)^2\left[g_M - M^2 g_M\right]\\
&\quad \le -\frac{1}{4}(M^2-1)g_M\\
&\quad = -\frac{1}{4}K(M)g_M,
\end{aligned}$$

where $K(M) \to \infty$ as $M \to \infty$, as desired. □

Now let us work on the lower bound.

LEMMA 8.4. *Let $U \in \mathscr{K}(\delta)$ (as in Theorem 6.1). Then*

$$\sup_{\mu(\overline{U})=1} [-I(\mu)] = \sup_{\mu(\overline{U})=1} \inf_{f \in \mathscr{C}^+ \cap \{\sup Lf/f < 1\}} \int \frac{Lf}{f}\, d\mu.$$

PROOF. By (6.1)

$$\sup_{\mu(\overline{U})=1} [-I(\mu)] = \sup_{\mu(\overline{U})=1} \inf_{f \in \mathscr{C}^+} \int \frac{Lf}{f}\, d\mu.$$

For any $\mu \in \mathscr{M}$ with $\mu(\overline{U}) = 1$, we see that $\inf_{f \in \mathscr{C}^+ \cap \{\sup Lf/f < 1\}} \int (Lf/f)\, d\mu$ is unaffected by the behavior of Lf/f outside a neighborhood of $\overline{U}$; consequently

$$\sup_{\mu(\overline{U})=1} \inf_{f \in \mathscr{C}^+ \cap \{\sup Lf/f < 1\}} \int \frac{Lf}{f}\, d\mu = \sup_{\mu(\overline{U})=1} \inf_{f \in \mathscr{C}^+ \cap \{\sup_{\overline{U}} Lf/f < 1\}} \int \frac{Lf}{f}\, d\mu.$$

Thus the lemma comes down to showing

$$\sup_{\mu(\overline{U})=1} \inf_{f \in \mathscr{C}^+} \int \frac{Lf}{f}\, d\mu = \sup_{\mu(\overline{U})=1} \inf_{f \in \mathscr{C}^+ \cap \{\sup_{\overline{U}} Lf/f < 1\}} \int \frac{Lf}{f}\, d\mu.$$

Just as in Lemma 8.1 we can use Sion's minimax theorem to interchange inf and sup on both sides [since $\{\mu: \mu(\overline{U}) = 1\}$ is compact]. Then it suffices to show

$$\inf_{f \in \mathscr{C}^+} \sup_{\overline{U}} \frac{Lf}{f} = \inf_{f \in \mathscr{C}^+ \cap \{\sup_{\overline{U}} Lf/f < 1\}} \sup_{\overline{U}} \frac{Lf}{f}.$$

But this is easy (cf. Lemma 8.2). □

We can now prove equality of the bounds.

THEOREM 8.5. *The upper and lower bounds are equal*:

$$\sup_{U \in \mathscr{K}(\delta)} \sup_{\mu(\overline{U})=1} [-I(\mu)] = \inf\left\{ \sup_{H(\delta)} \frac{Lf}{f} : f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\}.$$

Here $\mathscr{K}(\delta)$ is as in Theorem 6.1.

PROOF. It suffices to show $\geq$ (because $\leq$ follows immediately from Theorems 7.1 and 6.1). Since $H(\delta)^{**}$ is compact, by Lemmas 8.1(a) and 8.2

$$\begin{aligned}
&\inf\left\{ \sup_{H(\delta)} \frac{Lf}{f} : f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\} \\
&\quad \leq \inf\left\{ \sup_{H(\delta)^{**}} \frac{Lf}{f} : f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\} \\
&\quad = \sup_{\mu \in \mathscr{M}_{**}} \inf\left\{ \int \frac{Lf}{f}\, d\mu : f \in \mathscr{G}(\delta)^+,\ \sup_{H(\delta)} \frac{Lf}{f} < 1 \right\} \quad \text{[by Lemma 8.1(a)]} \\
&\quad = -\lambda_\delta \text{ (by Lemma 8.2)}.
\end{aligned}$$

Hence by Lemmas 8.3 and 8.4, it suffices to show

$$(8.4)\qquad \begin{aligned} &\sup_{U\in\mathscr{K}(\delta)}\ \sup_{\mu(\overline{U})=1}\ \inf_{f\in\mathscr{C}^+\cap\{\sup Lf/f<1\}} \int \frac{Lf}{f}\,d\mu \\ &\geq \inf\left\{\int \frac{Lf}{f}\,d\mu_0 \colon f\in\mathscr{S}(\delta)^+,\ \sup_{H(\delta)}\frac{Lf}{f}<1\right\}, \end{aligned}$$

where μ_0 is from Lemma 8.3 ($\mu_0\{\pm\infty\}=0$).

So consider any $U\in\mathscr{K}(\delta)$ with $\mu_0(\overline{U})>0$ and let

$$f\in C^+\cap\left\{\sup\frac{Lf}{f}<1\right\}.$$

Since $\overline{U}\subseteq H_\delta$ in $\mathbb{R}_2^+$ it is no loss to assume $f\equiv$ constant on ∂_+H_δ and hence

$$f\in\mathscr{S}(\delta)^+\cap\left\{\sup_{H(\delta)}\frac{Lf}{f}<1\right\}.$$

For $\mu_U(B):=\mu_0(\overline{U})^{-1}\mu_0(B\cap\overline{U})$ and $\overline{U}^c=\mathbb{R}_+^2\setminus\overline{U}$ we have

$$\begin{aligned} \text{RHS}(8.4) &\leq \int\frac{Lf}{f}\,d\mu_0=\mu_0(\overline{U})\int_{\overline{U}}\frac{Lf}{f}\,d\mu_U+\int_{\overline{U}^c}\frac{Lf}{f}\,d\mu_0 \\ &\leq \mu_0(\overline{U})\int_{\overline{U}}\frac{Lf}{f}\,d\mu_U+\mu_0(\overline{U}^c) \end{aligned}$$

and taking the infimum over all such f yields

$$\begin{aligned} \text{RHS}(8.4) &\leq \mu_0(\overline{U})\inf_{f\in\mathscr{C}^+\cap\{\sup Lf/f<1\}}\int\frac{Lf}{f}\,d\mu_U+\mu_0\big(\mathbb{R}_+^2\setminus\overline{U}\big) \\ &\leq \mu_0(\overline{U})\sup_{\mu(\overline{U})=1}\ \inf_{f\in\mathscr{C}^+\cap\{\sup Lf/f<1\}}\int\frac{Lf}{f}\,d\mu+\mu_0\big(\mathbb{R}_+^2\setminus\overline{U}\big) \\ &\leq \mu_0(\overline{U})\text{LHS}(8.4)+\mu_0\big(\mathbb{R}_+^2\setminus\overline{U}\big) \\ &\to \text{LHS}(8.4)\quad\text{as } \overline{U}\uparrow H_\delta,\ U\in\mathscr{K}(\delta), \end{aligned}$$

since $\mu=\{\pm\infty\}=0$. □

COROLLARY 8.6. *The quantities in Theorem 8.5 have common value* $-\lambda_\delta$.

PROOF. This follows immediately from the proof of Theorem 8.5. □

As an immediate consequence of Theorems 6.1, 7.1 and 8.5 we have the following result.

THEOREM 8.7. *For any compact set* $W\subseteq\mathbb{R}_+^2$,

$$\lim_{t\to\infty}t^{-1}\log\inf_{v\in W}P_v(\eta_\delta\wedge\hat{e}>t)=\lim_{t\to\infty}t^{-1}\log\sup_{v\in H(\delta)}P_v(\eta_\delta\wedge\hat{e}>t)=-\lambda_\delta.$$

9. Analysis of the ρ-part of Z_t. In this section we study the quantity

$$P_r(A(t)\leq s)=P_r\left(\int_0^t\frac{du}{2R(u)^2}\leq s\right)$$

appearing in (4.7). Our main result is the following theorem.

THEOREM 9.1. *For any $l > 0$ and $r > 0$,*

$$\int_0^\infty e^{-lu}\, d_u P_r\left(\int_0^t \frac{ds}{2R(s)^2} \le u\right) \sim B(l)(r^{-2}t)^{-a(l)/2}$$

as $t \to \infty$, where

$$a(l) = \left\{-(2-\alpha) + \left[(2-\alpha)^2 + 4l\right]^{1/2}\right\}/2,$$
$$B(l) = 2^{-a(l)/2}\Gamma([a(l) + 4 - \alpha]/2)/\Gamma(a(l) + (4-\alpha)/2).$$

PROOF. Consider the diffusion W_t on $\mathbb{R}^3$ whose generator in stereographic coordinates (4.2) is

$$\frac{1}{2}\left\{\frac{\partial^2}{\partial r^2} + \frac{3-\alpha}{r}\frac{\partial}{\partial r} + \frac{(|u|^2+1)^2}{4r^2}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right)\right\}.$$

Then W_t has a skew product representation $(R(t), \theta(A(t)))$, where $R(t)$ is the Bessel process with parameter $4 - \alpha$ and generator G_R [see (4.5)], $\theta(t)$ is Brownian motion on S^2 with generator

$$L_{S^2} = \frac{(|u|^2+1)^2}{4}\left[\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right]$$

[see (4.5)], and

$$A(t) = \frac{1}{2}\int_0^t R(s)^{-2}\, ds.$$

Thus the theorem comes down to proving for each $x \in \mathbb{R}^3 \setminus \{0\}$,

$$\int_0^\infty e^{-lu}\, d_u P_x(A(t) \le u) \sim B(l)\left[|x|^{-2}t\right]^{-a(l)/2} \quad \text{as } t \to \infty. \tag{9.1}$$

Let $\varphi(x)$ be the magnitude of the angle between $x \in \mathbb{R}^3 \setminus \{0\}$ and $(0, 0, 1)$ and for $\theta \in (0, \pi)$ let $C_\theta = \{x \in \mathbb{R}^3 \setminus \{0\}\colon\ 0 \le \varphi(x) < \theta\}$ be the open right circular cone of angle θ and vertex $0 \in \mathbb{R}^3$. Define

$$\tau_\theta^W = \inf\{t > 0\colon W_t \notin C_\theta\},$$
$$\eta_\theta^W = \inf\{t > 0\colon \theta_t \notin C_\theta \cap S^2\}.$$

Then

$$\begin{aligned} P_x(\tau_\theta^W > t) &= P_x(\theta(A(s)) \in C_\theta \cap S^2\ \forall\, s \in [0,t]) \\ &= P_x(\eta_\theta^W > A(t)) \\ &= \int_0^\infty P_x(\eta_\theta^W > u)\, d_u P_x(A(t) \le u). \end{aligned} \tag{9.2}$$

Now

$$P_x(\eta_\theta^W > t) \sim e^{-l_\theta t}\, m_\theta(x/|x|)\int_{C_\theta \cap S^2} m_\theta\, d\sigma \quad \text{as } t \to \infty, \tag{9.3}$$

where (l_θ, m_θ) is the first eigenvalue–eigenfunction pair of L_{S^2} on $C_\theta \cap S^2$ with Dirichlet boundary condition, and $d\sigma$ is normalized Haar measure on

S^2. Moreover, the mapping

$$\theta \in (0, \pi) \to l_\theta$$

is strictly decreasing and continuous, with range $(0, \infty)$.

By the results of DeBlassie (1988) (there the case $\alpha = 1$ was done, but the proof is valid for $0 < \alpha < 2$)

$$(9.4)\quad P_x(\tau_\theta^W > t) \sim B(l_\theta)(|x|^{-2}t)^{-a(l_\theta)/2} m_\theta(x/|x|) \int_{C_\theta \cap S^2} m_\theta \, d\sigma \quad \text{as } t \to \infty,$$

where

$$a(l_\theta) = \left\{-(2-\alpha) + \left[(2-\alpha)^2 + 4l_\theta\right]^{1/2}\right\}\Big/2,$$

$$B(l_\theta) = 2^{-a(l_\theta)/2}\Gamma([a(l_\theta) + 4 - \alpha]/2)/\Gamma(a(l_\theta) + (4-\alpha)/2).$$

Thus

$$\theta \in (0, \pi) \to a(l_\theta)$$

is strictly decreasing and continuous with range $(0, \infty)$. Using the method of DeBlassie (1988), (9.2)–(9.4) imply (9.1). □

10. Properties of λ_δ.

LEMMA 10.1. (i) $0 < \delta_1 < \delta_2 \Rightarrow \lambda_{\delta_1} > \lambda_{\delta_2}$.
(ii) $0 < \delta_1 < \delta_2 \Rightarrow \lambda_{\delta_1} \le (\delta_2/\delta_1)^2 \lambda_{\delta_2}$.

PROOF. Since $0 < \delta_1 < \delta_2$, $\mathscr{K}(\delta_1) \subseteq \mathscr{K}(\delta_2)$. Thus (i) follows from Theorem 8.5 and Corollary 8.6.

As for (ii), let $0 < \varepsilon < 1$ and note that the components of the solution $\overline{V}_\varepsilon = (\overline{V}_\varepsilon^{(1)}, \overline{V}_\varepsilon^{(2)})$ to the stochastic differential equation (3.4) have the following scaling properties (here $\mathscr{L}$ denotes "law of") for any $c > 0$:

$$\mathscr{L}(\overline{V}_\varepsilon^{(1)}(\cdot)|\overline{V}_\varepsilon^{(1)}(0) = v_1) = \mathscr{L}(c^{-2}\overline{V}_\varepsilon^{(1)}(c^2\,\cdot)|\overline{V}_\varepsilon^{(1)}(0) = c^2 v_1),$$

$$\mathscr{L}(\overline{V}_\varepsilon^{(2)}(\cdot)|V_\varepsilon^{(2)}(0) = v_2) = \mathscr{L}(c^{-1}\overline{V}_\varepsilon^{(2)}(c^2\,\cdot)|\overline{V}_\varepsilon^{(2)}(0) = cv_2).$$

Since $cH_\delta = H_{\delta c}$ we have for $\xi_\delta = \inf\{t > 0: \overline{V}_\varepsilon(t) \notin H_\delta\}$,

$$\begin{aligned}
P_v(\xi_\delta > t) &= P(\overline{V}_\varepsilon(s) \in H_\delta \ \forall\, s \le t|\overline{V}_\varepsilon(0) = v) \\
&= P((c^{-2}\overline{V}_\varepsilon^{(1)}(c^2 s), c^{-1}\overline{V}_\varepsilon^{(2)}(c^2 s)) \in H_\delta \ \forall\, s \le t|\overline{V}_\varepsilon(0) = (c^2 v_1, cv_2)) \\
&= P((c^{-1}\overline{V}_\varepsilon^{(1)}(c^2 s), \overline{V}_\varepsilon^{(2)}(c^2 s)) \in H_{\delta c} \ \forall\, s \le t|\overline{V}_\varepsilon(0) = (c^2 v_1, cv_2)) \\
&= P(\overline{V}_\varepsilon(c^2 s) \in H_{\delta c} \ \forall\, s \le t|\overline{V}_\varepsilon(0) = (c^2 v_1, cv_2)) \\
&= P(\overline{V}_\varepsilon(s) \in H_{\delta c} \ \forall\, s \le c^2 t|\overline{V}_\varepsilon(0) = (c^2 v_1, cv_2)) \\
&= P_{(c^2 v_1, cv_2)}(c^{-2}\xi_{\delta c} > t).
\end{aligned}$$

The function

$$\bar{\tau}_\varepsilon^{-1}(t) = \int_0^t 2\varepsilon\left[\overline{V}_\varepsilon^{(1)}(s) + \overline{V}_\varepsilon^{(2)}(s)^2 + 1\right]^{-2} ds$$

is continuous in t and so by Proposition 5.4 of Ikeda and Watanabe (1981), page 24, $\bar{\tau}_\varepsilon^{-1}(\xi_\delta)$ is $\mathscr{F}_{\xi_\delta}$-measurable. Then

$$\{\xi_\delta > \bar{\tau}_\varepsilon(t)\} = \{\bar{\tau}_\varepsilon^{-1}(\xi_\delta) > t\} = \{\bar{\tau}_\varepsilon^{-1}(\xi_\delta) > t\} \cap \left\{\xi_\delta > \frac{t}{2}\right\} \in \mathscr{F}_t, \tag{10.1}$$

where we have used $\bar{\tau}_\varepsilon^{-1}(\xi_\delta) \le 2\xi_\delta$, $\varepsilon < 1$. Thus for $c = \delta_1/\delta_2 < 1$, p and $q > 1$ with $1/p + 1/q = 1$ and $v \in H_{\delta_1}$ we have $\max_{a \ge 0}(a+1)/(a+c^2) = 1/c^2$ since $c < 1$ and

$$\begin{aligned}
&P_v(\xi_{\delta_2} > \bar{\tau}_\varepsilon(t))\\
&\quad = P\left(\int_0^{\xi_{\delta_2}} 2\varepsilon\left[\bar{V}_\varepsilon^{(1)}(s) + \bar{V}_\varepsilon^{(2)}(s)^2 + 1\right]^{-2} ds > t \,|\, \bar{V}_\varepsilon(0) = v\right)\\
&\quad = P\left(\int_0^{c^2\xi_{\delta_2}} 2\varepsilon\left[\bar{V}_\varepsilon^{(1)}(c^{-2}u) + \bar{V}_\varepsilon^{(2)}(c^{-2}u)^2 + 1\right]^{-2} c^{-2}\,du > t \,|\, \bar{V}_\varepsilon(0) = v\right)\\
&\quad = P\left(\int_0^{\xi_{\delta_1}} 2\varepsilon\left[c^{-2}\bar{V}_\varepsilon^{(1)}(u) + c^{-2}\bar{V}_\varepsilon^{(2)}(u)^2 + 1\right]^{-2} c^{-2}\,du\right.\\
&\qquad\qquad \left. > t \,|\, \bar{V}_\varepsilon(0) = (c^2 v_1, c v_2)\right)\\
&\quad = P\left(\int_0^{\xi_{\delta_1}} 2\varepsilon\left[\bar{V}_\varepsilon^{(1)}(u) + \bar{V}_\varepsilon^{(2)}(u)^2 + c^2\right]^{-2} du > c^{-2}t \,|\, \bar{V}_\varepsilon(0) = (c^2 v_1, c v_2)\right) \qquad (10.2)\\
&\quad \le P\left(\left[\max_{a \ge 0} \frac{a+1}{a+c^2}\right]^2 \int_0^{\xi_{\delta_1}} 2\varepsilon\left[\bar{V}_\varepsilon^{(1)}(u) + \bar{V}_\varepsilon^{(2)}(u)^2 + 1\right]^{-2} du\right.\\
&\qquad\qquad \left. > c^{-2}t \,|\, \bar{V}_\varepsilon(0) = (c^2 v_1, c v_2)\right)\\
&\quad = P_{(c^2 v_1, c v_2)}(\xi_{\delta_1} > \bar{\tau}_\varepsilon(c^2 t))\\
&\quad = E_{(c^2 v_1, c v_2)} R_\varepsilon^1(t) I(\eta_{\delta_1}^\varepsilon > \tau_\varepsilon(c^2 t)) \quad [\text{by } (3.11), (10.1) \text{ and that } c < 1]\\
&\quad \le \left(E_{(c^2 v_1, c v_2)}[R_\varepsilon^1(t)]^p\right)^{1/p} \left(P_{(c^2 v_1, c v_2)}(\eta_{\delta_1}^\varepsilon > \tau_\varepsilon(c^2 t))\right)^{1/q}\\
&\quad \le \left(P_{(c^2 v_1, c v_2)}(\eta_{\delta_1} \wedge \hat{e} > c^2 t)\right)^{1/q}
\end{aligned}$$

[by Lemma 3.1 and (3.14)].

Thus

$$\begin{aligned}
P_v(\eta_{\delta_2} \wedge \hat{e} > t) &= P_v(\eta_{\delta_2}^\varepsilon > \tau_\varepsilon(t)) \qquad [\text{by } (3.14)]\\
&= E_v \bar{R}_\varepsilon^1(t) I(\xi_{\delta_2} > \bar{\tau}_\varepsilon(t)) \quad [\text{by } (3.11) \text{ and } (10.1)]\\
&\le \left\{E_v[\bar{R}_\varepsilon^1(t)]^p\right\}^{1/p} \left\{P_v(\xi_{\delta_2} > \bar{\tau}_\varepsilon(t))\right\}^{1/q}\\
&\le \exp\left\{\tfrac{1}{2}(p-1)(1-\alpha)^2 \varepsilon t\right\} \left\{\left[P_{(c^2 v_1, c v_2)}(\eta_{\delta_1} \wedge \hat{e} > c^2 t)\right]^{1/q}\right\}^{1/q}
\end{aligned}$$

[by Lemma 3.1 and (10.2)]. This yields

$$\frac{1}{t}\log P_v(\eta_{\delta_2} \wedge \hat{e} > t) \le \frac{1}{2}(p-1)(1-\alpha)^2\varepsilon + \frac{1}{tq^2}\log P_{(c^2v_1, cv_2)}(\eta_{\delta_1} \wedge \hat{e} > c^2 t).$$

Let $\varepsilon \to 0$, then let $t \to \infty$ and use Theorem 8.7 to get

$$-\lambda_{\delta_2} \le -\left(\frac{c}{q}\right)^2 \lambda_{\delta_1} = -q^{-2}\left(\frac{\delta_1}{\delta_2}\right)^2 \lambda_{\delta_1}.$$

Letting $q \downarrow 1$ gives $-\lambda_{\delta_2} \le -(\delta_1/\delta_2)^2 \lambda_{\delta_1}$ as desired. □

LEMMA 10.2. *For $a(\cdot)$ as in Theorem 9.1, we have $a(\lambda_\delta) < \alpha$ for $\delta > 0$.*

PROOF. If τ is the first time the process Y_t (hence Z_t) hits the plane $x_1 = 0$ then τ is really just the first time a Bessel process with parameter $2 - \alpha$ hits the origin. Thus for $y_1 > 0$,

$$E_y \tau^p = \begin{cases} C_{p,\alpha} y_1^{2p} & \text{if } 0 < p < \dfrac{\alpha}{2}, \\ \infty & \text{if } p \ge \dfrac{\alpha}{2} \end{cases} \tag{10.3}$$

[see (2.10) of DeBlassie (1987b)].

By Theorem 9.1, (4.7) and Theorem 8.7, for any $\theta \in (0, \pi)$ and $y_1 > 0$,

$$E_y \tau_\theta^p \begin{cases} < \infty & \text{if } p < \dfrac{a(\lambda_{\delta(\theta)})}{2}, \\ = \infty & \text{if } p > \dfrac{a(\lambda_{\delta(\theta)})}{2}. \end{cases} \tag{10.4}$$

Roughly speaking, (4.7) and Theorem 8.7 say

$$E_y \tau_\theta^p = \int_0^\infty P_y(\tau_\theta > t^{1/p})\, dt$$

is like

$$\int_1^\infty \int_0^\infty e^{-\lambda_\delta u}\, d_u P_{|y|}\left(\int_0^{t^{1/p}} \frac{ds}{2R(s)^2} \le u\right) dt,$$

and Theorem 9.1 says the latter is like $\int_1^\infty t^{-a(\lambda_\delta)/2p}\, dt$.

Since $\tau \le \tau_\theta$ for $\theta > 0$, by (10.3) and (10.4) we get $a(\lambda_{\delta(\theta)}) \le \alpha$. If $a(\lambda_{\delta(\theta)}) = \alpha$ for some $\theta \in (0, \pi)$, then since $\theta_1 < \theta$ implies $\delta(\theta_1) < \delta(\theta)$ [see (4.6)] implies $\lambda_{\delta(\theta)} < \lambda_{\delta(\theta_1)}$ (see Lemma 10.1), we get $\alpha = a(\lambda_{\delta(\theta)}) < a(\lambda_{\delta(\theta_1)}) \le \alpha$. Contradiction. Thus $a(\lambda_{\delta(\theta)}) < \alpha$ for all $\theta \in (0, \pi)$. □

Next we use a result of Bingham (1973) to explicitly evaluate λ_1.

LEMMA 10.3. *For $\delta = 1$, $\lambda_1 = \alpha(4 - \alpha)/4$.*

PROOF. By (4.6), $\delta = 1$ iff $\theta = \pi/2$. The distribution of the first time $T_{\pi/2}$ the two-dimensional symmetric stable process exits the wedge $W_{\pi/2}$ of angle π is the same as the distribution of the first time $\tilde{T}$ that a one-dimensional symmetric stable process $\tilde{X}$ exits the half-line $(0, \infty)$. Bingham (1973), Theorem 3b, has shown that for $x_2 > 0$,

$$P_{x_2}(\tilde{T} > t) \sim C(x_2)t^{-1/2} \quad \text{as } t \to \infty.$$

Thus for $x \in \mathbb{R}^2$ with $x_2 > 0$, $E_x T_{\pi/2}^{p/\alpha} < \infty$ iff $p < \alpha/2$. Hence by the results of Bass and Cranston (1983), Theorems 3.1 and 3.2, $E_x|X(T_{\pi/2})|^p < \infty$ iff $p < \alpha/2$. By (1.5) we get $E_{(0,x)}\tau_{\pi/2}^{p/2} < \infty$ iff $p < \alpha/2$. Comparing with (10.4), $\alpha/2 = a(\lambda_1)$. Solving for λ_1 [recall $a(l)$ is defined in Theorem 9.1], we get $\lambda_1 = \alpha(4 - \alpha)/4$. □

REMARK 10.4. By Lemmas 10.1 and 10.3, $\lambda_\delta > 0$. Hence by (4.8) when $\alpha = 1$,

$$P_v(\eta_\delta > t) \sim C(v)\exp(-\lambda_\delta t) \quad \text{as } t \to \infty.$$

Thus (10.4) becomes: for $p > 0$ and $\alpha = 1$, $E_y \tau_\theta^p < \infty$ iff $p < a(\lambda_{\delta(\theta)})/2$.

11. Proof of Theorem 1.1.

Define

$$p_{\theta,\alpha} := a(\lambda_{\delta(\theta)})/\alpha, \tag{11.1}$$

where $a(\cdot)$ is as in Theorem 9.1 and $\delta(\theta)$ is as in (4.6).

Observe by (10.4) and (1.5)

$$E_x|X(T_\theta)|^p < \infty \quad \text{if } p < a(\lambda_{\delta(\theta)}),$$

$$E_x|X(T_\theta)|^p = \infty \quad \text{if } p > a(\lambda_{\delta(\theta)}).$$

By Lemma 10.2, $a(\lambda_{\delta(\theta)}) < \alpha$, and hence by the Bass–Cranston results [Bass and Cranston (1983), Theorems 3.1 and 3.2]

$$E_x T_\theta^p < \infty \quad \text{if } p < a(\lambda_{\delta(\theta)})/\alpha = p_{\theta,\alpha},$$

$$E_x T_\theta^p = \infty \quad \text{if } p > a(\lambda_{\delta(\theta)})/\alpha = p_{\theta,\alpha}.$$

Moreover, by Remark 10.4, for $\alpha = 1$, $E_x T_\theta^{p_{\theta,1}} = \infty$. This gives (v).

Part (iv) is an immediate consequence of Lemma 10.2 and (11.1).

As for part (iii), observe for $\theta \in (\pi/2, \pi)$, $\delta(\theta) > \delta(\pi/2) = 1$ [by (4.6)]. Hence by Lemma 10.1(ii), Lemma 10.3 and (4.6),

$$\lambda_{\delta(\theta)} \geq \left[\frac{1}{\delta(\theta)}\right]^2 \lambda_1 = \left[\frac{1 + \cos\theta}{\sin\theta}\right]^2 \frac{\alpha(4 - \alpha)}{4}.$$

Thus

$$a(\lambda_{\delta(\theta)}) \geq a\left(\left[\frac{1 + \cos\theta}{\sin\theta}\right]^2 \frac{\alpha(4 - \alpha)}{4}\right).$$

This yields part (iii).

By Lemma 10.3, $p_{\pi/2,\alpha} = \frac{1}{2}$. Once part (i) is proved we get that $p_{\theta,\alpha} > p_{\pi/2,\alpha} = \frac{1}{2}$ for $\theta \in (0, \pi/2)$. Of course, this is exactly the assertion of part (ii). Thus all that remains is the proof of part (i).

By (4.6) and Lemma 10.1(i), $\theta \to \lambda_{\delta(\theta)}$ is decreasing. Since $l \to a(l)$ is increasing, the monotonicity assertion follows from (11.1). Moreover, the continuity assertion will also follow once we prove $\delta \to \lambda_\delta$ is continuous. For this use Lemma 10.1: Let $\delta_0 > 0$. Then

$$\lambda_{\delta_0} \le \lim_{\delta \uparrow \delta_0} \lambda_\delta \le \lim_{\delta \uparrow \delta_0} (\delta_0/\delta)^2 \lambda_{\delta_0} = \lambda_{\delta_0} \le \lim_{\delta \downarrow \delta_0} (\delta/\delta_0)^2 \lambda_\delta \le \lim_{\delta \downarrow \delta_0} (\delta/\delta_0)^2 \lambda_{\delta_0} = \lambda_{\delta_0}$$

and $\lim_{\delta \to \delta_0} \lambda_\delta = \lambda_{\delta_0}$ as desired. □

APPLICATION. Let $\alpha = 1$ so that Y_t is a three-dimensional Brownian motion. By (1.4) and (1.5), since $Y(\tau_\theta) = Z(\tau_\theta)$, for $x \in W_\theta$,

$$E_{(x,0)}|Y(\tau_\theta)|^{2p} < \infty \quad \text{iff } E_x|X(T_\theta)|^{2p} < \infty$$

and by the Bass–Cranston (1983) results and Theorem 1.1 the latter is finite iff $p < p_{\theta,1}$. By Burkholder's (1979) results we get for $y \in \mathbb{R}^3 \setminus [\{0\} \times W_\theta^c]$,

$$E_y|Y(\tau_\theta)|^{2p} < \infty \quad \text{iff } p < p_{\theta,1}. \tag{11.2}$$

Consequently, we get the following theorem.

THEOREM 11.1. *Consider the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \setminus [\{0\} \times W_\theta^c], \\ u = f & \text{on } \{0\} \times W_\theta^c, \end{cases} \tag{11.3}$$

where f is continuous on $\{0\} \times W_\theta^c$. If $f = O(|x|^p)$ for some $p < 2p_{\theta,1}$ then (11.3) *possesses a solution.*

APPENDIX

THEOREM A.1. *Let l be the half-line $\{(0,0,a)\colon a \ge 0\}$. for $Z_0 \ne 0$, $Z(\cdot)$ never hits 0. For $Z_0 \notin l$, if $0 < \alpha \le 1$, $Z(\cdot)$ never hits l and if $1 < \alpha < 2$, $Z(\cdot)$ can hit l with positive probability.*

PROOF. For $Z_0 \ne 0$, by Itô's formula $\rho(Z_t) = [Z_t^{(1)} + (Z_t^{(2)})^2 + (Z_t^{(3)})^2]^{1/2}$ is a Bessel process with parameter $4 - \alpha > 2$ [cf. (1.9)]. Then $\rho(Z_t)$ never hits 0 and consequently neither does Z_t.

If $Z_0 \notin l$, then by Itô's formula the process $\tilde\rho(Z_t) = [Z_t^{(1)} + (Z_t^{(2)})^2]^{1/2}$ is a Bessel process with parameter $3 - \alpha$. Thus $\tilde\rho(Z_t)$ never hits 0 if $1 \ge \alpha$ and $\tilde\rho(Z_t)$ hits 0 a.s. if $1 < \alpha < 2$. But $\tilde\rho(Z_t)$ hits 0 iff Z_t hits the line $\{(0,0,a)\colon a \in \mathbb{R}\}$ and the desired conclusion follows. □

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