The Annals of Probability
1990, Vol. 18, No. 4, 1784–1789

ESTIMATES OF THE LARGEST DISC COVERED BY A RANDOM WALK[1]

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Let $R(n)$ be the largest integer for which the disc of radius $R(n)$ around the origin is covered by the first n steps of a random walk. The main objective of the present paper is to obtain better estimates for the upper tail of the distribution of $R(n)$. For example, we show that there are constants $0 < \lambda_2 < \lambda_1 < \infty$ such that

$$\exp(-\lambda_1 z) \le \liminf_{n\to\infty} \mathbf{P}\left\{\frac{(\log R(n))^2}{\log n} > z\right\}$$

$$\le \limsup_{n\to\infty} \mathbf{P}\left\{\frac{(\log R(n))^2}{\log n} > z\right\} \le \exp(-\lambda_2 z).$$

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of independent, identically distributed random vectors taking values from R^2 with distribution

$$\mathbf{P}\{X_1 = (0,1)\} = \mathbf{P}\{X_1 = (0,-1)\} = \mathbf{P}\{X_1 = (1,0)\}$$
$$= \mathbf{P}\{X_1 = (-1,0)\} = \tfrac{1}{4},$$

and let

$$S_0 = 0 = (0,0) \quad \text{and} \quad S(n) = S_n = X_1 + X_2 + \cdots + X_n, \qquad n = 1,2,\dots,$$

i.e., $\{S_n\}$ is the simple symmetric random walk on the plane. Further, let

$$\xi(x,n) = \#\{k: 0 < k \le n,\ S_k = x\},$$

$n = 1,2,\dots,$ $x = (i,j)$, $i,j = 0, \pm 1, \pm 2, \dots,$ be the local time of the random walk. We say that the circle

$$Q(N) = \left\{x = (i,j): \|x\| = \left(i^2 + j^2\right)^{1/2} \le N\right\}$$

is covered by the random walk in time n if

$$\xi(x,n) > 0 \quad \text{for every } x \in Q(N).$$

Let $R(n)$ be the largest integer for which $Q(R(n))$ is covered in n.

We quote two previous results on the properties of $R(n)$.

Received March 1989; revised October 1989.

[1]Research supported by Fonds zur Förderung der Wissenschaftlichen Forschung Project Nr. P6076.

AMS 1980 *subject classifications*. Primary 60J15; secondary 60F05.

Key words and phrases. Random walk, covered disc, local time, limit distributions, strong theorems.

THEOREM A [Erdös and Révész (1988) and Auer and Révész (1989)]. *For any $\varepsilon > 0$, we have*

$$\exp\left(\frac{(\log n)^{1/2}}{(\log\log n)^{1/2+\varepsilon}}\right) \le R(n) \le \exp\bigl(2(\log n)^{1/2}\log_3 n\bigr) \quad a.s.,$$

for all but finitely many n, where $\log_p$ is the p-th iterate of log.

THEOREM B [Révész (1989)]. *For any $z > 0$ and $n = 2, 3, \dots,$ we have*

$$\mathbf{P}\left\{\frac{(\log R(n))^2}{\log n} > z\right\} \le \exp\left(-\frac{z}{4}\right).$$

In the present paper we prove the following.

THEOREM 1. *For any $\varepsilon > 0$,*

$$R(n) \ge \exp\left(\frac{1-\varepsilon}{(120)^{1/2}}(\log n \log_3 n)^{1/2}\right) \quad i.o.\ a.s.$$

THEOREM 2. *For any $z > 0$, we have*

$$\liminf_{n\to\infty} \mathbf{P}\left\{\frac{(\log R(n))^2}{\log n} > z\right\} \ge \exp(-120z).$$

In fact, instead of Theorem 1 we prove the following stronger theorem.

THEOREM 3. *For any $0 < \theta < (\pi/120)^{1/2}$, $9/10 < \delta^2 < 1$ and $\varepsilon > 0$, we have*

$$\inf_{\|x\| \le \exp[(1-\varepsilon)\theta/\sqrt{\pi}](\log n \log_3 n)^{1/2}} \xi(x,n) \ge (1-\delta)\frac{\sqrt{\pi}}{12\theta}(\log n \log_3 n)^{1/2} \quad i.o.\ a.s.$$

2. Notation and lemmas. Introduce the following notation:

$$\begin{aligned} p(0 \rightsquigarrow x) &= \mathbf{P}\{\inf\{n\colon n \ge 1,\ S_n = 0\} > \inf\{n\colon n \ge 1,\ S_n = x\}\} \\ &= \mathbf{P}\{\{S_n\} \text{ reaches } x \text{ before returning to } 0\}. \end{aligned}$$

Let $\rho_1(0 \rightsquigarrow x), \rho_2(0 \rightsquigarrow x), \dots,$ resp., $\rho_1(x \rightsquigarrow 0), \rho_2(x \rightsquigarrow 0), \dots$ be the first, second, ... waiting times to reach x from 0, resp., to reach 0 from x, i.e.,

$$\begin{aligned} \rho_1(0 \rightsquigarrow x) &= \inf\{n\colon n \ge 1,\ S_n = x\}, \\ \rho_1(x \rightsquigarrow 0) &= \inf\{n\colon n \ge \rho_1(0 \rightsquigarrow x),\ S_n = 0\} - \rho_1(0 \rightsquigarrow x), \\ \rho_2(0 \rightsquigarrow x) &= \inf\{n\colon n \ge \rho_1(0 \rightsquigarrow x) + \rho_1(x \rightsquigarrow 0),\ S_n = x\} \\ &\quad -(\rho_1(0 \rightsquigarrow x) + \rho_1(x \rightsquigarrow 0)), \\ \rho_2(x \rightsquigarrow 0) &= \inf\{n\colon n \ge \rho_1(0 \rightsquigarrow x) + \rho_1(x \rightsquigarrow 0) + \rho_2(0 \rightsquigarrow x),\ S_n = 0\} \\ &\quad -(\rho_1(0 \rightsquigarrow x) + \rho_1(x \rightsquigarrow 0) + \rho_2(0 \rightsquigarrow x)), \dots . \end{aligned}$$

Let $\tau(0 \rightsquigarrow x, n)$ be the number of $0 \rightsquigarrow x$ excursions completed before n, i.e.,

$$\tau(0 \rightsquigarrow x, n) = \max\left\{i: \sum_{j=1}^{i-1} \big(\rho_j(0 \rightsquigarrow x) + \rho_j(x \rightsquigarrow 0)\big) + \rho_i(0 \rightsquigarrow x) \le n\right\}.$$

Put $\rho_1 = \min\{k: k > 0, S_k = 0\}$, $\rho_2 = \min\{k: k > \rho_1, S_k = 0\}, \dots, \rho_n = \min\{k: k > \rho_{n-1}, S_k = 0\}$.

Let $\alpha(r)$ be the probability that the random walk $\{S_n\}$ exits from the open disc of radius r before returning to $0 = (0,0)$, i.e.,

$$\alpha(r) = \mathbf{P}\{\inf\{n: \|S_n\| \ge r\} < \inf\{n: n \ge 1, S_n = 0\}\}.$$

LEMMA A [Erdös and Révész (1988)].

$$\lim_{r \to \infty} \alpha(r) \log r = \frac{\pi}{2}.$$

LEMMA 1. *There exists a positive constant C such that*

$$p(0 \rightsquigarrow x) \ge \frac{C}{\log\|x\|}, \tag{1}$$

for any $x \in Z^2$ with $\|x\| \ge 2$. Further,

$$\liminf_{\|x\| \to \infty} p(0 \rightsquigarrow x) \log\|x\| \ge \frac{\pi}{12}. \tag{2}$$

PROOF. Let $x = \|x\| e^{i\varphi}$. Then, by Lemma A for any $\varepsilon > 0$, there exists an $R_0 = R_0(\varepsilon) > 0$ such that the probability that the particle crosses the arc $\|x\| e^{i\psi}$, $\varphi - \pi/3 < \psi < \varphi + \pi/3$, before returning to 0 is larger than $(1-\varepsilon)(\pi/6)(\log\|x\|)^{-1}$. Since starting from any lattice point within unit distance of the arc $\|x\| e^{i\psi}$, $\varphi - \pi/3 < \psi < \varphi + \pi/3$, the probability that the particle hits x before 0 is larger than $\frac{1}{2}$ we obtain (2). (1) is a trivial consequence of (2). □

Spitzer [(1964), pages 117, 124 and 125] obtained the exact order of $p(0 \rightsquigarrow x)$. He proved the following.

LEMMA B.

$$p(0 \rightsquigarrow x) = \frac{\pi + o(1)}{4 \log\|x\|} \quad \text{as } \|x\| \to \infty.$$

LEMMA 2. *For any $0 < \theta < (\pi/120)^{1/2}$, $9/10 < \delta^2 < 1$ and n big enough, we have*

$$\mathbf{P}\left\{n^{-1/2} \inf_{\|x\| \le e^{\theta\sqrt{n}}} \tau(0 \rightsquigarrow x, \rho_n) \le (1-\delta)\frac{\pi}{12\theta}\right\} \le \exp\left(-\frac{\pi}{60}\frac{n^{1/2}}{\theta}\right). \tag{3}$$

PROOF. Let $q = 1 - p = 1 - p(0 \rightsquigarrow x)$. Then, applying the Bernstein inequality, we obtain

$$\mathbf{P}\left\{\left|\frac{\tau(0 \rightsquigarrow x, \rho_n)}{n} - p\right| \geq \delta p\right\} \leq 2\exp\left(-\frac{n\delta^2 p}{2q(1 + (\delta/2q))^2}\right)$$
$$\leq \exp\left(-\frac{\pi}{60}\frac{n}{\log\|x\|}\right),$$

provided that $\|x\|$ is big enough.

Hence,

$$\mathbf{P}\left\{\inf_{\|x\| \leq e^{\theta\sqrt{n}}} n^{-1/2}\tau(0 \rightsquigarrow x, \rho_n) \leq (1-\delta)\frac{\pi}{12\theta}\right\}$$
$$\leq \mathbf{P}\left\{\inf_{\|x\| \leq e^{\theta\sqrt{n}}} \frac{\tau(0 \rightsquigarrow x, \rho_n)}{np} \leq 1 - \delta\right\}$$
$$= \mathbf{P}\left\{\sum_{\|x\| \leq e^{\theta\sqrt{n}}} \left\{\frac{\tau(0 \rightsquigarrow x, \rho_n)}{np} \leq 1 - \delta\right\}\right\}$$
$$\leq e^{2\theta\sqrt{n}}\,\pi\exp\left(-\frac{\pi}{60}\frac{n^{1/2}}{\theta}\right),$$

which implies (3). □

LEMMA C [Erdös and Taylor (1960)]. *For any $\varepsilon > 0$, we have*

$$\xi(0,n) \leq \frac{1+\varepsilon}{\pi}\log n \log_3 n \quad a.s.,$$

for all but finitely many n,

$$\xi(0,n) \geq \frac{1-\varepsilon}{\pi}\log n \log_3 n \quad i.o.\ a.s.,$$

and

$$\lim_{n\to\infty} \mathbf{P}\{\xi(0,n) < z\log n\} = 1 - e^{-\pi z}.$$

Consequently,

$$\rho_n \geq \exp\left(\frac{(1-\varepsilon)\pi n}{\log_2 n}\right) \quad a.s.,$$

for all but finitely many n,

$$\rho_n \leq \exp\left(\frac{(1+\varepsilon)\pi n}{\log_2 n}\right) \quad i.o.\ a.s., \tag{4}$$

and

$$\lim_{n\to\infty} \mathbf{P}\left\{\rho_n < \exp\left(\frac{n}{z}\right)\right\} = \exp(-\pi z). \tag{5}$$

3. Proof of Theorem 3. (3) clearly implies that

$$\liminf_{n\to\infty} n^{-1/2} \inf_{\|x\|\le e^{\theta}\sqrt{n}} \tau(0 \rightsquigarrow x, \rho_n) \ge (1-\delta)\frac{\pi}{12\theta} \quad \text{a.s.,} \tag{6}$$

for any $0 < \theta < (\pi/120)^{1/2}$ and $9/10 < \delta^2 < 1$.

(4) and (6) combined imply

$$n^{-1/2} \inf_{\|x\|\le e^{\theta}\sqrt{n}} \tau\left(0 \rightsquigarrow x, \exp\left(\frac{(1+\varepsilon)\pi n}{\log_2 n}\right)\right) \ge (1-\delta)\frac{\pi}{12\theta} \quad \text{i.o. a.s.,}$$

that is,

$$\inf_{\|x\|\le \exp[(1-\varepsilon)/\sqrt{\pi}]\theta(\log n \log_3 n)^{1/2}} \tau(0 \rightsquigarrow x, n)$$

$$\ge (1-\delta)\frac{\sqrt{\pi}}{12\theta}(\log n \log_3 n)^{1/2} \qquad \text{i.o. a.s.,}$$

which in turn implies Theorem 3.

4. Proof of Theorem 2. Instead of proving Theorem 2 we prove the following stronger theorem.

THEOREM 4. *For any* $\varepsilon > 0$ *and* $z > 0$ *there exists a positive integer* $N_0 = N_0(\varepsilon, z)$ *such that*

$$\mathbf{P}\left\{\inf_{\|x\|\le \exp\theta(z\log n)^{1/2}} \tau(0 \rightsquigarrow x, n) \ge (1-\delta)\frac{\pi}{12\theta}(z\log n)^{1/2}\right\} \tag{7}$$

$$\ge \exp(-\pi z) - \varepsilon,$$

if $n \ge N_0$, $0 < \theta < (\pi/120)^{1/2}$ *and* $9/10 < \delta^2 < 1$.

PROOF. By (3) and (5), for any $\varepsilon > 0$ there exists a positive integer $N_0 = N_0(\varepsilon)$ such that

$$\mathbf{P}\left\{n^{-1/2} \inf_{\|x\|\le e^{\theta}\sqrt{n}} \tau(0 \rightsquigarrow x, \rho_n) \le (1-\delta)\frac{\pi}{12\theta}\right\} \le \varepsilon$$

and

$$\mathbf{P}\left\{\rho_n < \exp\left(\frac{n}{z}\right)\right\} \ge \exp(-\pi z) - \varepsilon,$$

if $n \geq N_0$. Consequently,

$$\begin{aligned}
&\mathbf{P}\left\{\inf_{\|x\| \leq e^{\theta}\sqrt{n}} \tau\left(0 \rightsquigarrow x, \exp\left(\frac{n}{z}\right)\right) \geq (1-\delta)\frac{\pi}{12\theta} n^{1/2}\right\} \\
&\quad \geq \mathbf{P}\left\{\inf_{\|x\| \leq e^{\theta}\sqrt{n}} \tau\left(0 \rightsquigarrow x, \exp\left(\frac{n}{z}\right)\right) \geq (1-\delta)\frac{\pi}{12\theta} n^{1/2}, \rho_n < \exp\left(\frac{n}{z}\right)\right\} \\
&\quad \geq \mathbf{P}\left\{\inf_{\|x\| \leq e^{\theta}\sqrt{n}} \tau(0 \rightsquigarrow x, \rho_n) \geq (1-\delta)\frac{\pi}{12\theta} n^{1/2}, \rho_n < \exp\left(\frac{n}{z}\right)\right\} \\
&\quad \geq \mathbf{P}\left\{\inf_{\|x\| \leq e^{\theta}\sqrt{n}} \tau(0 \rightsquigarrow x, \rho_n) \geq (1-\delta)\frac{\pi}{12\theta} n^{1/2}\right\} - \mathbf{P}\left\{\rho_n \geq \exp\left(\frac{n}{z}\right)\right\} \\
&\quad \geq 1 - \varepsilon - (1 - \exp(-\pi z)) - \varepsilon \\
&\quad = \exp(-\pi z) - 2\varepsilon.
\end{aligned}$$

Hence, we have Theorem 4. □

Theorems B and 2 suggest the following.

CONJECTURE. *There exists a* $\frac{1}{4} < \lambda < 120$ *such that*

$$\lim_{n \to \infty} \mathbf{P}\left\{\frac{(\log R(n))^2}{\log n} > z\right\} = \exp(-\lambda z).$$

Acknowledgment. Somewhat weaker versions of Theorems B and 2 and the conjecture were found by H. Kesten much earlier.

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