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ASYMPTOTIC TAIL BEHAVIOR OF UNIFORM MULTIVARIATE EMPIRICAL PROCESSES[1]

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Let α_n be the empirical process of independent uniformly distributed random vectors on the unit square I^2. We study the asymptotic distribution of the random variable $\sup|\alpha_n(s,t)|/(s^\nu t^\mu L(s)G(s))$ when sup is taken over various subintervals of I^2. We show that in the case of $-\infty < \mu, \nu < 1/2$ the limit is given in terms of a two-time parameter Wiener process, and for $1/2 < \mu, \nu < \infty$ it is determined by a Poisson process.

1. Introduction. Let $X_1 = (X_1^{(1)}, X_1^{(2)})$, $X_2 = (X_2^{(1)}, X_2^{(2)}), \dots$ be independent random vectors taking values in the unit square $I^2 = [0,1] \times [0,1]$ with distribution function $F(t,s) = ts$, $0 \le t, s \le 1$. We define the nth empirical distribution function F_n by

$$F_n(t,s) = \frac{1}{n} \#\{1 \le i \le n \colon X_i^{(1)} \le t,\ X_i^{(2)} \le s\},$$

and the uniform bivariate empirical process α_n by

$$\alpha_n(t,s) = n^{1/2}\{F_n(t,s) - ts\}, \qquad n = 1, 2, \dots .$$

In recent years there has been considerable interest in the asymptotic behavior of the supremum of weighted empirical processes. The one-dimensional case is essentially solved in the papers by Rényi (1953), Chibisov (1964), O'Reilly (1974), Eicker (1979), Jaeschke (1979), Mason (1985), Csörgő and Mason (1985), Csörgő, Csörgő, Horváth and Mason (1986), Csörgő and Horváth (1986), Csörgő, Horváth and Steinebach (1987) and Csörgő and Horváth (1988), and in the book of Shorack and Wellner (1986).

There are only partial results available in the multivariate case. Using Alexander's inequalities [cf. Alexander (1982, 1984)], it has been proved that there is a sequence of two-time parameter Brownian bridges such that

$$\sup_{0 \le s, t \le 1} |\alpha_n(t,s) - B_n(t,s)|/q(st) \to_P 0, \qquad n \to \infty, \tag{1.1}$$

if $q\colon (0,1] \to (0,\infty)$ is continuous, nondecreasing and $q(x)/(x \log(1/x))^{1/2} \to \infty$, $x \downarrow 0$ [cf. Theorem 3.1 in Einmahl (1987)]. By a two-time parameter Brownian bridge $\{B(t,s);\ 0 \le s, t \le 1\}$, we mean a Gaussian process with mean 0 and covariance function $EB(t_1, s_1)B(t_2, s_2) = (t_1 \wedge t_2)(s_1 \wedge s_2) - t_1 t_2 s_1 s_2$, where

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$a \wedge b = \min(a, b)$. From (1.1) it follows immediately that we have

$$\sup_{0 \le st \le k_n/n} |\alpha_n(t,s)|/(st)^{\nu} \to_P 0 \tag{1.2}$$

if $\nu < 1/2$ and $k_n/n \to 0$, $n \to \infty$. A summary of characterizations of the almost sure behavior of the random variable of (1.2) is summarized in Einmahl (1987), pages 53–62.

In this paper we study the asymptotic distribution of the random variable $\sup|\alpha_n(s,t)|/(s^{\nu}t^{\mu}L(s)G(t))$, where sup is taken over various subintervals of the unit square I^2.

Throughout this exposition we will assume that the functions L and G are slowly varying at 0. We will see that in the case of $-\infty < \mu, \nu < 1/2$ the asymptotic behavior in question is given in terms of a Gaussian process, while it is determined by a Poisson process when $1/2 < \mu, \nu < \infty$. Our theorems on the weighted asymptotic behavior of uniform empirical processes are stated in Section 2. The proofs of the results in Section 2 are given in Section 3. We will assume without loss of generality that all random variables and processes are defined on the same probability space [cf., e.g., de Acosta (1982)].

We state all our results in terms of appropriate two-time parameter stochastic processes. It will be clear from the proofs that similar results can be also proved in the d-time parameter case ($d \ge 2$). However, the notation required for the sake of stating and proving these results would take up quite a bit more space and would also make reading of this exposition somewhat more difficult.

2. Weighted uniform multivariate empirical processes. Let $\{W(t,s); 0 \le s, t < \infty\}$ be a two-time parameter Wiener process. For definition, existence and properties of this process we refer to Csörgő and Révész (1981). We also work with functions slowly varying at 0. We say that a function l is slowly varying at 0 if it is nonnegative, measurable and $\lim_{t \to 0} l(tc)/l(t) = 1$ for all $c > 0$.

THEOREM 2.1. *Let L and G be slowly varying functions at 0 and $-\infty < \mu, \nu < 1/2$. We assume that $\{k_n\}$ and $\{m_n\}$ satisfy*

$$k_n \to \infty, \qquad m_n \to \infty, \qquad n \to \infty, \tag{2.1}$$

$$k_n n^{-1/2} \to 0, \qquad m_n n^{-1/2} \to 0, \qquad n \to \infty. \tag{2.2}$$

Then, as $n \to \infty$, we have

$$r_n \sup_{\substack{0 < t \le k_n n^{-1/2} \\ 0 < s \le m_n n^{-1/2}}} |\alpha_n(t,s)|/(t^{\nu}s^{\mu}L(t)G(t)) \to_{\mathscr{D}} \sup_{\substack{0<t<1 \\ 0<s<1}} |W(t,s)|/(t^{\nu}s^{\mu}), \tag{2.3}$$

where

$$r_n = (k_n n^{-1/2})^{\nu - 1/2}(m_n n^{-1/2})^{\mu - 1/2} L(k_n n^{-1/2}) G(m_n n^{-1/2}). \tag{2.4}$$

Let $Z(B)$ be a homogeneous spatial Poisson process with intensity parameter 1, where B denotes a bounded and measurable set of R^2. For a definition we refer to Karlin and Taylor (1981), page 398. We define $N(t, s) = Z([0, t] \times [0, s])$, $0 \le t, s < \infty$, and we will refer to it as a two-time parameter Poisson process, or simply a Poisson process, with intensity parameter 1.

The next theorem is concerned with heavily weighted empirical processes on the tails of the unit box.

THEOREM 2.2. *Let L and G be slowly varying functions at* 0 *and* $1/2 < \mu, \nu < 1$. *We assume that* $\{k_n\}$ *and* $\{m_n\}$ *satisfy* (2.1) *and* (2.2). *Then, as* $n \to \infty$, *we have*

$$q_n \sup_{\substack{0<t\le k_n n^{-1/2}\\ 0<s\le m_n n^{-1/2}}} |\alpha_n(t,s)|/(t^\nu s^\mu L(t)G(s)) \to_{\mathscr{D}} \sup_{0<t,s<\infty} |N(t,s) - st|/(t^\nu s^\mu), \tag{2.5}$$

where $\{N(t, s);\ 0 < t, s < \infty\}$ *is a two-time parameter Poisson process with intensity parameter* 1, *and*

$$q_n = n^{(1/2-\nu)/2} n^{(1/2-\mu)/2} L(n^{-1/2}) G(n^{-1/2}). \tag{2.6}$$

REMARK 2.1. If ν and/or $\mu > 1$, then the limiting random variable in (2.5) is infinite with probability 1.

REMARK 2.2. The proof of Theorem 2.2 will show that for all $\lambda_1, \lambda_2 > 0$,

$$q_n \sup_{\substack{\lambda_1 n^{-1/2}\le t\le k_n n^{-1/2}\\ \lambda_2 n^{-1/2}\le s\le m_n n^{-1/2}}} |\alpha_n(t,s)|/(t^\nu s^\mu L(t)G(s)) \to_{\mathscr{D}} \sup_{\substack{\lambda_1\le t<\infty\\ \lambda_2\le s<\infty}} |N(t,s) - st|/(t^\nu s^\mu) \tag{2.7}$$

if $1/2 < \mu, \nu < \infty$. It also follows from Remark 2.1 that λ_1 and λ_2 in (2.7) cannot, in general, be replaced with sequences tending to 0.

Our next theorem is concerned with the asymptotics of heavily weighted empirical processes in the middle of the unit box.

THEOREM 2.3. *Let L and G be slowly varying functions at* 0 *and* $1/2 < \mu, \nu < \infty$. *We assume that* $\{k_n\}$ *and* $\{m_n\}$ *satisfy* (2.1) *and* (2.2). *Then, as* $n \to \infty$, *we have*

$$r_n \sup_{\substack{k_n n^{-1/2}\le t\le 1\\ m_n n^{-1/2}\le s\le 1}} |\alpha_n(t,s)|/(t^\nu s^\mu L(t)G(s)) \to_{\mathscr{D}} \sup_{1\le t,s<\infty} |W(t,s)|/(t^\nu s^\mu), \tag{2.8}$$

where r_n is defined in (2.4).

REMARK 2.3. We note

$$\sup_{1\le t,s<\infty} |W(t,s)|/(t^\nu s^\mu) =_{\mathscr{D}} \sup_{0<t,s\le 1} |W(t,s)|/(t^{1-\nu} s^{1-\mu}).$$

REMARK 2.4. In the statements of Theorems 2.1–2.3 we can take off the absolute value signs and the thus appearing asymptotic results hold true.

The proofs of our results are based on the well-known Kac representation of empirical processes [cf. Kac (1949), Bretagnolle and Massart (1989) and Csörgő and Révész (1981), Chapter 7, for further references]. Let $\eta(n)$ be a Poisson random variable with $E\eta(n) = n$, independent of $(X_i,\, i \ge 1)$. We write

$$\alpha_n(t,s) = \frac{1}{n^{1/2}}\left(\sum_{i=1}^{\eta(n)} \mathbb{1}\{X_i^{(1)} \le t,\, X_i^{(2)} \le s\} - nts\right) + \frac{1}{n^{1/2}}\left(\sum_{i=1}^{n} \mathbb{1}\{X_i^{(1)} \le t,\, X_i^{(2)} \le s\} - \sum_{i=1}^{\eta(n)} \mathbb{1}\{X_i^{(1)} \le t,\, X_i^{(2)} \le s\}\right).$$

Kac observed that $\sum_{i=1}^{\eta(n)} \mathbb{1}\{X_i^{(1)} \le t,\, X_i^{(2)} \le s\}$ is a Poisson process with mean nts for each $n \ge 1$. We prove that the asymptotic behavior of $|\alpha_n(t,s)|/(t^{\nu}s^{\mu}L(t)G(s))$ is determined by a weighted Poisson process while the weighted random sum $(\sum_{i=\eta+1}^{n} \mathbb{1}\{X_i^{(1)} \le t,\, X_i^{(2)} \le s\})/(t^{\nu}s^{\mu}L(t)G(s))$, where $\sum_{i=j}^{k} = -\sum_{i=k}^{j}$ if $k < j$, does not play any role in the limit. For further remarks on, and versions of, the Kac representation we refer to Shorack and Wellner (1986), pages 339, 556 and 578). In particular, the conditional representation of α_n may be also used to construct new proofs of our Theorems 2.1, 2.2 and 2.3.

3. Proofs of Theorems 2.1–2.3.

We prove our results when $L = G = 1$. Using de Haan (1975), one can establish that the special case of $L = G = 1$ implies the more general cases stated in Theorems 2.1–2.3.

First we need a simple lemma which can be easily proved by elementary calculations.

LEMMA 3.1. *Let $\eta(n)$ be a Poisson random variable with mean value n, and let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables of mean 0 and variance σ^2 which are also independent of $\eta(n)$. Then there is a constant $c_{3,1}$ such that we have*

$$E\left(\sum_{i=n}^{\eta(n)} \xi_i\right)^2 \le c_{3,1}\sigma^2 n^{1/2}, \tag{3.1}$$

where $\sum_{i=n}^{k} = -\sum_{i=k}^{n}$ if $k < n$.

PROOF OF THEOREM 2.1. Let $\eta(n)$ be a Poisson random variable with mean value n, independent of $\{X_i,\, i \ge 1\}$. We have

$$\alpha_n(t,s) = \alpha_n^{(1)}(t,s) - \alpha_n^{(2)}(t,s) - \alpha_n^{(3)}(t,s), \tag{3.2}$$

where

$$\alpha_n^{(1)}(t,s) = n^{-1/2}\left\{\sum_{j=1}^{\eta(n)} \mathbb{1}\left(X_j^{(1)} \le t,\, X_j^{(2)} \le s\right) - nts\right\}, \tag{3.3}$$

$$\alpha_n^{(2)}(t,s) = n^{-1/2}\sum_{j=n}^{\eta(n)} \left(\mathbb{1}\left(X_j^{(1)} \le t,\, X_j^{(2)} \le s\right) - ts\right) \tag{3.4}$$

and

$$\alpha_n^{(3)}(t,s) = n^{-1/2}(\eta(n) - n)ts, \tag{3.5}$$

where $\mathbb{1}(\cdot)$ is the indicator function.

First we show

$$r_n^{(1)} \sup_{\substack{n^{-1} \le t \le k_n n^{-1/2} \\ n^{-1} \le s \le m_n n^{-1/2}}} \left|\alpha_n^{(3)}(t,s)\right|/(t^\nu s^\mu) = o_P(1), \tag{3.6}$$

where $r_n^{(1)} = (k_n n^{-1/2})^{\nu-1/2}(m_n n^{-1/2})^{\mu-1/2}$. By the central limit theorem $n^{-1/2}(\eta(n) - n) = O_P(1)$, which gives (3.6) immediately.

Next we show

$$r_n^{(1)} \sup_{\substack{n^{-1} \le t \le k_n n^{-1/2} \\ n^{-1} \le s \le m_n n^{-1/2}}} \left|\alpha_n^{(2)}(t,s)\right|/(t^\nu s^\mu) = o_P(1). \tag{3.7}$$

Let $-\infty < \gamma, \delta < \infty$, and $t_0 < t_1 < \cdots < t_K$, $s_0 < s_1 < \cdots < s_M$. We have

$$\begin{aligned}
&\sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} \left|\frac{\alpha_n^{(2)}(t,s)}{t^\gamma s^\delta} - \frac{\alpha_n^{(2)}(t_j,s_i)}{t_j^\gamma s_i^\delta}\right| \\
&\quad\le \frac{\left|\alpha_n^{(2)}(t_j,s_i)\right|}{t_j^\gamma s_i^\delta} \sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} \left|\left(\frac{t_j}{t}\right)^\gamma \left(\frac{s_i}{s}\right)^\delta - 1\right| \\
&\qquad + \sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} \left|\alpha_n^{(2)}(t,s) - \alpha_n^{(2)}(t_j,s_i)\right| \sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} t^{-\gamma}s^{-\delta} \\
&\quad\le \frac{\left|\alpha_n^{(2)}(t_j,s_i)\right|}{t_j^\gamma s_i^\delta} \sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} \left|\left(\frac{t_j}{t}\right)^\gamma \left(\frac{s_i}{s}\right)^\delta - 1\right| \\
&\qquad + \sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} t^{-\gamma}s^{-\delta}\left\{\frac{2|\eta(n) - n|}{n^{1/2}}(t_{j+1}s_{i+1} - t_j s_i)\right. \\
&\qquad\qquad + n^{-1/2}\sum_{l=1}^{\eta(n)} \left(\mathbb{1}\left(X_l^{(1)} \le t_{j+1},\, X_l^{(2)} \le s_{i+1}\right)\right. \\
&\qquad\qquad \left.\left. - \mathbb{1}\left(X_l^{(1)} \le t_j,\, X_l^{(2)} \le s_i\right) - (t_{j+1}s_{i+1} - t_j s_i)\right)\right\}.
\end{aligned} \tag{3.8}$$

By Lemma 3.1 we obtain

$$E\left(\frac{\alpha_n^{(2)}(t_j, s_i)}{t_j^{\gamma} s_i^{\delta}}\right)^2 \le c_{3,2} n^{-1/2} t_j^{1-2\gamma} s_i^{1-2\delta} \tag{3.9}$$

and also

$$\begin{aligned} E\Bigg\{ n^{-1/2} \sum_{l=n}^{\eta(n)} \Big(\mathbb{1}\big(X_l^{(1)} \le t_{j+1}, X_l^{(2)} \le s_{i+1}\big) - \mathbb{1}\big(X_l^{(1)} \le t_j, X_l^{(2)} \le s_i\big) \\ - \big(t_{j+1}s_{i+1} - t_j s_i\big)\Big)\Bigg\}^2 \\ \le c_{3,3} n^{-1/2}\big(t_{j+1}s_{i+1} - t_j s_i\big). \end{aligned} \tag{3.10}$$

We now consider (3.7) with $0 < \nu, \mu < 1/2$. We specify the points of subdivision t_j and s_i as follows: $t_0 = n^{-1}$, $t_1 = n^{-3/4}$, $t_j = e^j n^{-1/2}$, $j = 2, \dots, j_0 = [\log k_n] + 1$, and $s_0 = n^{-1}$, $s_1 = n^{-3/4}$, $s_i = e^i n^{-1/2}$, $i = 2, \dots, i_0 = [\log m_n] + 1$. Then

$$\sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} \left| \left(\frac{t_j}{t}\right)^{\nu} \left(\frac{s_i}{s}\right)^{\mu} - 1 \right| \le 1 \tag{3.11}$$

and

$$\sup_{\substack{t_j \le t \le t_{j+1} \\ s_i \le s \le s_{i+1}}} t^{-\nu} s^{-\mu} \le t_j^{-\nu} s_i^{-\mu}. \tag{3.12}$$

By (3.9) and Chebyshev's inequality

$$\begin{aligned} P\left\{ \left(\frac{k_n}{n^{1/2}}\right)^{\nu - 1/2} \left(\frac{m_n}{n^{1/2}}\right)^{\mu - 1/2} \max_{\substack{0 \le j \le j_0 \\ 0 \le i \le i_0}} \frac{\left|\alpha_n^{(2)}(t_j, s_i)\right|}{t_j^{\nu} s_i^{\mu}} > \varepsilon \right\} \\ \le c_{3,2} \varepsilon^{-2} \big(k_n n^{-1/2}\big)^{2\nu - 1} \big(m_n n^{-1/2}\big)^{2\mu - 1} n^{-1/2} \sum_{\substack{0 \le j \le j_0 \\ 0 \le i \le i_0}} t_j^{1-2\nu} s_i^{1-2\mu}. \end{aligned} \tag{3.13}$$

From the definition of the points of subdivision t_j, s_i, it follows that we have

$$\big(k_n n^{-1/2}\big)^{2\nu - 1} t_j^{1-2\nu} = o(1), \qquad j = 0, 1, \tag{3.14}$$

$$\big(m_n n^{-1/2}\big)^{2\mu - 1} s_i^{1-2\mu} = o(1), \qquad i = 0, 1, \tag{3.15}$$

$$\big(k_n n^{-1/2}\big)^{2\nu - 1} n^{-1/4} \sum_{2 \le j \le j_0} t_j^{1-2\nu} = o(1) \tag{3.16}$$

and

$$\big(m_n n^{-1/2}\big)^{2\mu - 1} n^{-1/4} \sum_{2 \le i \le i_0} s_i^{1-2\mu} = o(1). \tag{3.17}$$

We have also

$$(3.18)\qquad \max_{0\le j\le j_0-1}\left(k_n n^{-1/2}\right)^{\nu-1/2} t_j^{1-\nu}\left(t_{j+1}/t_j\right)=o(1)$$

and

$$(3.19)\qquad \max_{0\le i\le i_0-1}\left(m_n n^{-1/2}\right)^{\nu-1/2} s_i^{1-\mu}\left(s_{i+1}/s_i\right)=o(1).$$

Hence and by (3.12) we have

$$(3.20)\qquad \max_{\substack{0\le j<j_0\\ 0\le i<i_0}}\ \sup_{\substack{t_j\le t\le t_{j+1}\\ s_i\le s\le s_{i+1}}} t^{-\nu}s^{-\mu}|\eta(n)-n|n^{-1/2}\left(t_{j+1}s_{i+1}-t_i s_i\right)=o_P(1).$$

Using now Chebyshev's inequality in combination with (3.10) and (3.12), we obtain

$$(3.21)\qquad \begin{aligned} P\Bigg\{ \max_{\substack{0\le j<j_0\\ 0\le i<i_0}}\ \sup_{\substack{t_j\le t\le t_{j+1}\\ s_i\le s\le s_{i+1}}} t^{-\nu}s^{-\mu}n^{-1/2}\Bigg|\sum_{l=1}^{\eta(n)}\Big\{&\mathbb{1}\left(X_l^{(1)}\le t_{j+1}, X_l^{(2)}\le s_{i+1}\right)\\ &-\mathbb{1}\left(X_l^{(1)}\le t_j, X_l^{(2)}\le s_i\right)-\left(t_{j+1}s_{i+1}-t_j s_i\right)\Big\}\Bigg|\\ &> \varepsilon\left(k_n n^{-1/2}\right)^{1/2-\nu}\left(m_n n^{-1/2}\right)^{1/2-\mu}\Bigg\}\\ \le c_{3,3}\varepsilon^{-2}\left(k_n n^{-1/2}\right)^{2\nu-1}\left(m_n n^{-1/2}\right)^{2\mu-1}n^{-1/2}&\sum_{\substack{0\le j<j_0\\ 0\le i<i_0}} t_j^{-2\nu}t_{j+1}s_i^{-2\mu}s_{i+1}. \end{aligned}$$

By definition of t_j, s_i we have

$$(3.22)\qquad \left(k_n n^{-1/2}\right)^{2\nu-1}n^{-1/4}t_{j+1}/t_j^{2\nu}=o(1),\qquad j=0,1,$$

$$(3.23)\qquad \left(m_n n^{-1/2}\right)^{2\mu-1}n^{-1/4}s_{i+1}/s_i^{2}=o(1),\qquad i=0,1.$$

Combining now (3.8) and (3.13)–(3.23), we arrive at (3.7) when $0<\nu,\mu<1/2$. Trivial changes in the above calculations also yield (3.7) with $-\infty<\nu,\mu<1/2$.

It is well known [cf., e.g., Gaenssler (1983), page 7] that we have for each $n=1,2,\dots,$

$$(3.24)\qquad \begin{aligned} &\left\{\alpha_n^{(1)}(t,s),\,0\le t,s\le 1\right\}\\ &\quad =_{\mathscr{D}}\left\{n^{-1/2}\left(N(tn^{1/2},sn^{1/2})-nts\right),\,0\le t,s\le 1\right\}, \end{aligned}$$

where $N(x,y)$ is a Poisson process and $=_{\mathscr{D}}$ indicates equality in distribution of the two processes involved.

It is well known [cf., e.g., Kuelbs (1968) and Wichura (1969)] that

$$\begin{aligned} &\left\{(k_n m_n)^{-1/2}(N(xk_n, ym_n)-xyk_n m_n,\,0\le x,y\le 1\right\}\\ &\quad \to_{D([0,1]\times[0,1])}\left\{W(x,y),\,0\le x,y\le 1\right\}. \end{aligned}$$

Using the Skorohod–Dudley–Wichura representation theorem, for each k_n, m_n we can define a Wiener process $W_n(x, y)$ such that

$$(3.25)\qquad (k_n m_n)^{-1/2} \sup_{\substack{0\le x\le k_n\\ 0\le y\le m_n}} |N(x,y) - xy - W_n(x,y)| = o_P(1).$$

Let $D = D(c) = \{(t, s)\colon 0 < t \le k_n n^{-1/2}, 0 < s \le m_n n^{-1/2}, c/n \le ts\}$, where $c > 0$. Then we have

$$(3.26)\qquad \begin{aligned} &\sup_{(t,s)\in D} |N(n^{1/2}t, n^{1/2}s) - nts - W_n(n^{1/2}t, n^{1/2}s)|/(t^\nu s^\mu)\\ &\quad = (k_n n^{-1/2})^{-\nu}(m_n n^{-1/2})^{-\mu} \sup_{(t,s)\in D_1} |N(k_n t, m_n s) - k_n m_n ts\\ &\qquad\qquad - W_n(k_n t, m_n s)|/(t^\nu s^\mu)\\ &\quad = (k_n n^{-1/2})^{-\nu}(m_n n^{-1/2})^{-\mu}\, \Delta_n^{(1)}, \end{aligned}$$

where $D_1 = \{(t, s)\colon 0 < t \le 1,\ 0 < s \le 1,\ c/(k_n m_n) \le ts\}$. We show that

$$(3.27)\qquad r^{(2)}(n)\, \Delta_n^{(1)} = o_P(1),$$

where $r^{(2)}(n) = (k_n m_n)^{-1/2}$. Let $\varepsilon > 0$ and define $D_2 = D_1 \cap \{(t, s)\colon \varepsilon \le t \le 1, \varepsilon \le s \le 1\}$, $D_3 = D_1 \cap \{(t, s)\colon 0 < t \le \varepsilon, 0 < s \le 1\}$ and $D_4 = D_1 \cap \{(t, s)\colon 0 < t \le 1, 0 < s \le \varepsilon\}$. Then

$$(3.28)\qquad \begin{aligned} \Delta_n^{(1)} \le &\sup_{(t,s)\in D_2} |N(k_n t, m_n s) - k_n m_n ts - W_n(k_n t, m_n s)|/(t^\nu s^\mu)\\ &+ \sup_{(t,s)\in D_3} |W_n(k_n t, m_n s)|/(t^\nu s^\mu)\\ &+ \sup_{(t,s)\in D_4} |W_n(k_n t, m_n s)|/(t^\nu s^\mu)\\ &+ \sup_{(t,s)\in D_3} |N(k_n t, m_n s) - k_n m_n ts|/(t^\nu s^\mu)\\ &+ \sup_{(t,s)\in D_4} |N(k_n t, m_n s) - k_n m_n ts|/(t^\nu s^\mu)\\ = &\ \Delta_n^{(2)}(\varepsilon) + \cdots + \Delta_n^{(6)}(\varepsilon). \end{aligned}$$

We get immediately from (3.25) that

$$(3.29)\qquad r^{(2)}(n)\, \Delta_n^{(2)}(\varepsilon) = o_P(1)$$

for all $\varepsilon > 0$. The scale transformation of the Wiener process gives

$$(3.30)\qquad r^{(2)}(n)\, \Delta_n^{(3)}(\varepsilon) =_{\mathscr{D}} \sup_{(t,s)\in D_3} |W(t, s)|/(t^\nu s^\mu),$$

and therefore

$$r^{(2)}(n)\, \Delta_n^{(3)}(\varepsilon) \to_{\mathscr{D}} \sup_{\substack{0<t\le\varepsilon\\ 0<s\le 1}} |W(t, s)|/(t^\nu s^\mu).$$

Hence for all $\delta > 0$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P\{r^{(2)}(n)\, \Delta_n^{(3)}(\varepsilon) > \delta\} = 0. \tag{3.31}$$

A similar argument gives

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P\{r^{(2)}(n)\Delta_n^{(3)}(\varepsilon) > \delta\} = 0 \tag{3.32}$$

for all $\delta > 0$. It is easy to see that

$$r^{(2)}(n)\, \Delta_n^{(5)}(\varepsilon) =_{\mathscr{D}} \varepsilon^{1/2-\nu}(\varepsilon k_n m_n)^{\mu-1/2} \times \sup_{(t,s) \in D_5} |N(t,s) - ts|/(t^\nu s^\mu), \tag{3.33}$$

where $D_5 = \{(t,s)\colon 0 \le t \le 1,\ 0 \le s \le \varepsilon k_m m_n,\ c \le ts\}$.

Let $M > 0$ and define $D_6 = \{(t,s)\colon 0 \le t \le 1,\ 0 \le s \le M,\ c \le ts\}$. We show that there is a constant $c_{3,4}$ such that for all γ we can find a K such that

$$\lim_{M \to \infty} P\left\{ \sup_{(t,s) \in D_6} |N(t,s) - ts|/(t^\nu s^\mu) > K \right\} \le \gamma. \tag{3.34}$$

First we note that

$$\sup_{\substack{0<t\le 1\\ 0<s\le 1}} |N(t,s) - ts|/(t^\nu s^\mu) = O_P(1). \tag{3.35}$$

Let $t_j = e^{-j}$, $j = 0,1,2,\dots$, $s_i = e^i$, $i = 1,\dots,[\log M] + 1$, and define $D_7 = \{(t,s)\colon 0 < t \le 1,\ 1 \le s \le M,\ c \le ts\}$. Then

$$\begin{aligned} &P\left\{ M^{\mu-1/2} \sup_{(t,s)\in D_7} |N(t,s) - ts|/(t^\nu s^\mu) > x \right\} \\ &\quad \le P\left\{ M^{\mu-1/2} \max_{0 \le i \le [\log M]} \max_{0 \le j \le i+1-\log c} \sup_{\substack{t_{j+1} \le t \le t_j \\ s_i \le s \le s_{i+1}}} \frac{|N(t,s) - ts|}{t^\nu s^\mu} > x \right\} \\ &\quad \le \sum_{i=0}^{[\log M]} \sum_{j=0}^{i+1-\log c} P\left\{ M^{\mu-1/2} t_{j+1}^{-\nu} s_i^{-\mu} \sup_{\substack{t_{j+1} \le t \le t_j \\ s_i \le s \le s_{i+1}}} |N(t,s) - ts| > x \right\} \\ &\quad \le c_{3,4} x^{-2} M^{2\mu-1} \sum_{i=0}^{[\log M]} \sum_{j=0}^{i+1-\log c} t_{j+1}^{-2\nu} s_i^{-2\mu} (t_j s_{i+1} - t_{j+1} s_i) \\ &\quad \le c_{3,5} x^{-2} M^{2\mu-1} \sum_{i=0}^{[\log M]} \sum_{j=0}^{i+1-\log c} e^{-j(1-2\nu)} e^{i(1-2\mu)} \\ &\quad \le c_{3,6} x^{-2}, \end{aligned} \tag{3.36}$$

where we have used Inequality 2.4 in Einmahl (1987) [cf. also Wichura (1969)]. Now the proof of (3.34) is complete.

Using (3.33) and (3.34), we get

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\{r^{(2)}(n)\,\Delta_n^{(5)}(\varepsilon)\ge\delta\}=0 \tag{3.37}$$

for all $\delta > 0$. Similar arguments give

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\{r^{(2)}(n)\,\Delta_n^{(6)}(\varepsilon)\ge\delta\}=0 \tag{3.38}$$

for all $\delta > 0$. Now (3.27) follows from (3.28), (3.29), (3.31), (3.32), (3.37) and (3.38).

The scale transformation of the Wiener process gives

$$\begin{aligned} &n^{-1/2}r^{(1)}(n)\sup_{(t,s)\in D}\left|W_n(n^{1/2}t,n^{1/2}s)\right|/(t^{\nu}s^{\mu})\\ &\qquad\to_{\mathscr{D}}\sup_{0,t,s\le 1}\left|W(t,s)\right|/(t^{\nu}s^{\mu}). \end{aligned}\tag{3.39}$$

Next for any $\varepsilon > 0$ there is $c > 0$ such that

$$\liminf_{n\to\infty}\left\{\sup_{ts\le cn^{-1}}F_n(t,s)=0\right\}>1-\varepsilon. \tag{3.40}$$

It is easily seen that we have also

$$n^{1/2}r_n^{(1)}\sup_{ts\le cn^{-1}} ts/(t^{\nu}s^{\mu})=o(1). \tag{3.41}$$

Consequently, Theorem 2.1 will be proven in case of $L = G = 1$ if one can show

$$\begin{aligned} &r_n^{(1)}\sup_{(t,s)\in D}\left|\alpha_n(t,s)\right|/(t^{\nu}s^{\mu})\\ &\qquad\to_{\mathscr{D}}\sup_{0\le t,s\le 1}\left|W(t,s)\right|/(t^{\nu}s^{\mu}). \end{aligned}\tag{3.42}$$

This however follows immediately from (3.2), (3.6), (3.7), (3.24), (3.26), (3.27) and (3.39) combined. □

PROOF OF THEOREM 2.2. We again use the representation of (3.2). By the central limit theorem $n^{-1/2}(\eta(n) - n) = O_P(1)$, and therefore

$$q_n^{(1)}\sup_{\substack{0<t\le k_n n^{-1/2}\\ 0<s\le m_n n^{-1/2}}}\left|\alpha_n^{(3)}(t,s)\right|/(t^{\nu}s^{\mu})=o_P(1), \tag{3.43}$$

where $q_n^{(1)} = n^{(1/2-\nu)/2}n^{(1/2-\mu)/2}$.

Next we show that with $D = D(c) = \{(t, s)\colon\ 0 < t \le k_n n^{-1/2},\ 0 < s \le m_n n^{-1/2},\ c/n \le ts\}$ we have

$$q_n^{(1)}\sup_{(t,s)\in D}\left|\alpha_n^{(2)}(t,s)\right|/(t^{\nu}s^{\mu})=o_P(1). \tag{3.44}$$

Here we use (3.8) with

$$t_0 = cm_n^{-1}n^{-1/2}, \qquad t_1 = m_n^{-1+\alpha}n^{-1/2}, \dots, t_{l_0-1} = m_n^{-\alpha}n^{-1/2}, \qquad t_{l_0} = n^{-1/2},$$

$$t_{l_0+j} = e^j n^{-1/2}, \qquad j = 1, \dots, j_0 = [\log k_n] + 1,$$

$$s_0 = ck_n^{-1}n^{-1/2}, \qquad s_1 = k_n^{-1+\alpha}n^{-1/2}, \dots, s_{l_0-1} = k_n^{-\alpha}n^{-1/2}, \qquad s_{l_0} = n^{-1/2},$$

$$s_{l_0+i} = e^i n^{-1/2}, \qquad i = 1, \dots, i_0 = [\log m_n] + 1,$$

where $0 < \alpha < (2/11)((1-\nu) \wedge (1-\mu))$ and $1/\alpha$ is an integer denoted by l_0. Then $D \subset [t_{l_0}, t_{l_0+j_0}] \times [s_{l_0}, s_{l_0+i_0}] \cup [t_0, t_{l_0}] \times [s_{l_0}, s_{l_0+i_0}] \cup [t_{l_0}, t_{l_0+j_0}] \times [s_0, s_{l_0}]$. Using Chebyshev's inequality and (3.10), we get

$$(3.45)\qquad \begin{aligned} P\Bigg\{ \max_{\substack{l_0 \le j < l_0+j_0 \\ l_0 \le i < l_0+i_0}} t_j^{-\nu} s_i^{-\mu} n^{-1/2} \Bigg| \sum_{l=n}^{\eta(n)} \Big\{ \mathbb{1}\Big(X_l^{(1)} \le t_{j+1}, X_l^{(2)} \le s_{i+1}\Big) \\ - \mathbb{1}\Big(X_l^{(1)} \le t_j, X_l^{(2)} \le s_i\Big) - \big(t_{j+1}s_{i+1} - t_j s_i\big)\Big\}\Bigg| \\ > x n^{1/2(\nu-1/2)} n^{1/2(\mu-1/2)} \Bigg\} \\ \le c_{3,3} x^{-2} n^{1/2-\nu} n^{-1/2-\mu} n^{-1/2} \sum_{\substack{l_0 \le j < l_0+j_0 \\ l_0 \le i < l_0+i_0}} t_j^{-2\nu} s_i^{-2\mu} t_{j+1} s_{i+1} \\ \le c_{3,7} x^{-2} n^{-1/2}. \end{aligned}$$

Similarly,

$$(3.46)\qquad \begin{aligned} P\Bigg\{ \max_{\substack{0 \le j < l_0 \\ l_0 \le i \le l_0+i_0}} t_j^{-\nu} s_i^{-\mu} n^{-1/2} \Bigg| \sum_{l=n}^{\eta(n)} \Big\{ \mathbb{1}\Big(X_l^{(1)} \le t_{j+1}, X_l^{(2)} \le s_{i+1}\Big) \\ - \mathbb{1}\Big(X_l^{(1)} \le t_j, X_l^{(2)} \le s_i\Big) - \big(t_{j+1}s_{i+1} - t_j s_i\big)\Big\}\Bigg| \\ > x n^{1/2(\nu-1/2)} n^{1/2(\mu-1/2)} \Bigg\} \\ \le c_{3,3} x^{-2} n^{1/2-\nu} n^{1/2-\mu} n^{-1/2} \sum_{\substack{0 \le j < l_0 \\ l_0 \le i < l_0+i_0}} t_j^{-2\nu} s_i^{-2\mu} t_{j+1} s_{i+1} \\ \le c_{3,8} x^{-2} \big(m_n n^{-1/2}\big) m_n^{2\mu-2+11\alpha} \end{aligned}$$

and hence also

$$P\left\{\max_{\substack{l_0<j<l_0+j_0\\0\le i<l_0}} t_j^{-(\nu+\varepsilon)}s_i^{-(\mu+\varepsilon)}n^{-1/2}\left|\sum_{l=n}^{\eta(n)}\left\{\mathbb{1}\left(X_l^{(1)}\le t_{j+1}, X_l^{(2)}\le s_{i+1}\right)\right.\right.\right.$$

$$\left.\left.-\mathbb{1}\left(X_l^{(1)}\le t_j, X^{(2)}\le s_i\right)-\left(t_{j+1}s_{i+1}-t_js_i\right)\right\}\right| \tag{3.47}$$

$$\left.> xn^{1/2(\nu-1/2)}n^{1/2(\mu-1/2)}\right\}$$

$$\le c_{3,9}x^{-2}\left(k_nn^{-1/2}\right)k_n^{2\mu-2+11\alpha}.$$

Continuing with (3.8), we have

$$\begin{aligned}\max_{\substack{0\le j<l_0\\l_0\le i<l_0+i_0}} & t_j^{1-\nu}s_i^{1-\mu}\left(t_{j+1}/t_j\right)\left(s_{i+1}/s_i\right)\\ &\le c_{3,10}t_{l_0-1}^{1-\nu}s_{l_0+i_0-1}^{1-\mu}m_n^{\alpha}\\ &\le c_{3,11}n^{1/2(\nu-1/2)}n^{1/2(\mu-1/2)}\left(m_nn^{-1/2}\right)m_n^{2\alpha-\mu},\end{aligned} \tag{3.48}$$

and similarly

$$n^{(1/2-\nu)/2}n^{(1/2-\mu)/2}\max_{\substack{l_0\le j<l_0+j_0\\0<i<l_0}} t_j^{1-\nu}s_i^{1-\mu}\left(t_{j+1}/t_j\right)\left(s_{i+1}/s_i\right)=o(1), \tag{3.49}$$

as well as

$$n^{(1/2-\nu)/2}n^{(1/2-\mu)/2}\max_{\substack{l_0\le j<l_0+j_0\\0<i<l_0}} t_j^{1-\nu}s_i^{1-\mu}\left(t_{j+1}/t_j\right)\left(s_{i+1}/s_i\right)=o(1). \tag{3.50}$$

Proceeding as in (3.45)–(3.47), we obtain

$$n^{(1/2-\nu)/2}n^{(1/2-\mu)/2}\max_{\substack{l_0\le j\le l_0+j_0\\l_0\le i\le l_0+i_0}}\left|\alpha_n^{(2)}(t_j,s_i)\right|t_j^{-\nu}s_i^{-\mu}=o_P(1), \tag{3.51}$$

$$n^{(1/2-\nu)/2}n^{(1/2-\mu)/2}\max_{\substack{0<j<l_0\\l_0\le i\le l_0+i_0}}\left|\alpha_n^{(2)}(t_j,s_i)\right|t_j^{-\nu}s_i^{-\mu}=o_P(1) \tag{3.52}$$

and

$$n^{(1/2-\nu)/2}n^{(1/2-\mu)/2}\max_{\substack{l_0\le j\le l_0+j_0\\0\le i\le l_0}}\left|\alpha_n^{(2)}(t_j,s_i)\right|t_j^{-\nu}s_i^{-\mu}=o_P(1). \tag{3.53}$$

This also completes the proof of (3.44).

Using (3.24), it can be easily seen that we have

$$q_n^{(1)} \sup_{(t,s)\in D} \left|\alpha_n^{(1)}(t,s)\right|/(t^\nu s^\mu) =_{\mathscr{D}} \sup_{(t,s)\in D_8} \frac{|N(t,s)-ts|}{t^\nu s^\mu}, \tag{3.54}$$

where

$$D_8 = \{(t,s)\colon 0 < t \le k_n, 0 < s \le m_n, c < ts\}. \tag{3.55}$$

We now have

$$q_n^{(1)} \sup_{(t,s)\in D} \left|\alpha_n^{(1)}(t,s)\right|/(t^\nu s^\mu) \to_{\mathscr{D}} \sup_{\substack{0<t,s<\infty \\ c<ts}} |N(t,s)-ts|/(t^\nu s^\mu). \tag{3.56}$$

Using (3.2), (3.33), (3.44) and (3.56), we have for all $c > 0$,

$$q_n^{(1)} \sup_{(t,s)\in D} \left|\alpha_n(t,s)\right|/(t^\nu s^\mu) \to_{\mathscr{D}} \sup_{\substack{0<t,s<\infty \\ c<ts}} |N(t,s)-ts|/(t^\nu s^\mu). \tag{3.57}$$

Also, as $c \to 0$,

$$\sup_{\substack{0<t,s<\infty \\ c<ts}} |N(t,s)-ts|/(t^\nu s^\mu) \to \sup_{0<t,s<\infty} |N(t,s)-ts|/(t^\nu s^\mu). \tag{3.58}$$

It is easy to see that for all $c > 0$,

$$n^{1/2} q_n^{(1)} \sup_{cn^{-1}>ts} ts/(t^\nu s^\mu) = o(1), \tag{3.59}$$

and this also concludes the proof of Theorem 2.2 by (3.40) and (3.57)–(3.59). □

PROOF OF THEOREM 2.3. Using again the representation of (3.2), we first conclude

$$r_n^{(1)} \sup_{\substack{k_n n^{-1/2} \le t \le 1 \\ m_n n^{-1/2} \le s \le 1}} \left|\alpha_n^{(3)}(t,s)\right|/(t^\nu s^\mu) = o_P(1), \tag{3.60}$$

where $r_n^{(1)} = (k_n n^{-1/2})^{\nu-1/2}(mn^{-1/2})^{\mu-1/2}$.

Next we show

$$r_n^{(1)} \sup_{\substack{k_n n^{-1/2} \le t \le 1 \\ m_n n^{-1/2} \le s \le 1}} \left|\alpha_n^{(2)}(t,s)\right|/(t^\nu s^\mu) = o_P(1). \tag{3.61}$$

Let $t_j = e^j k_n n^{-1/2}$, $j = 0, 1, \dots, j_0 = [\log n^{1/2} k_n^{-1}] + 1$, and $s_i = e^i m_n n^{-1/2}$, $i = 0, 1, \dots, i_0 = [\log n^{1/2} m_n^{-1}] + 1$. Then it is easily checked that

$$\begin{aligned} &|\eta(n) - n| n^{-1/2} (k_n n^{-1/2})^{\nu-1/2} (m_n n^{-1/2})^{\mu-1/2} \\ &\quad \times \max_{\substack{0\le j\le j_0 \\ 0\le i\le i_0}} t_j^{-\nu} s_i^{-\mu} (t_{j+1}s_{i+1} - t_j s_i) = o_P(1). \end{aligned} \tag{3.62}$$

By Chebyshev's inequality and (3.10) we get

$$(k_n n^{-1/2})^{\nu-1/2}(m_n n^{-1/2})^{\mu-1/2} \times \max_{\substack{0\le j<j_0\\0\le i<i_0}} t_j^{-\nu} s_i^{-\mu} n^{-1/2}\left|\sum_{l=n}^{\eta(n)}\left\{\mathbb{1}\left(X_l^{(1)}\le t_{j+1}, X_l^{(2)}\le s_{i+1}\right) - \mathbb{1}\left(X_l^{(1)}\le t_j, X_l^{(2)}\le s_i\right) - (t_{j+1}s_{i+1} - t_j s_i)\right\}\right| = o_P(1), \tag{3.63}$$

while using (3.9) we obtain

$$(k_n n^{-1/2})^{\nu-1/2}(m_n n^{-1/2})^{\mu-1/2} \times \max_{\substack{0\le j\le j_0\\0\le i\le i_0}} t_j^{-\nu} s_i^{-\mu}\left|\alpha_n^{(2)}(t_j, s_i)\right| = o_P(1). \tag{3.64}$$

This also completes the proof of (3.61).

Concerning $\alpha_n^{(1)}$ of (3.2), by (3.24) it is enough to show

$$r_n^{(1)} n^{-1/2} \sup_{\substack{k_n n^{-1/2}\le t\le 1\\ m_n n^{-1/2}\le s\le 1}} \left|N(tn^{1/2}, sn^{1/2}) - nts\right|/(t^\nu s^\mu) \to_{\mathscr{D}} \sup_{1\le t, s<\infty} |W(t,s)|/(t^\nu s^\mu). \tag{3.65}$$

It is easy to check that

$$r_n^{(1)} n^{-1/2} \sup_{\substack{k_n n^{-1/2}\le t\le 1\\ m_n n^{-1/2}\le s\le 1}} \left|N(n^{1/2}t, n^{1/2}s) - nts\right|/(t^\nu s^\mu) = (k_n m_n)^{-1/2} \sup_{\substack{1\le t\le n^{1/2}/k_n\\ 1\le s\le n^{1/2}/m_n}} \left|N(k_n t, m_n s) - k_n m_n ts\right|/(t^\nu s^\mu). \tag{3.66}$$

Let $T > 1$. The weak convergence of $\{(k_n m_n)^{-1/2}(N(k_n t, m_n s) - k_n m_n ts),\ 0 \le t, s < \infty\}$ implies

$$(k_n m_n)^{-1/2} \sup_{1\le t, s\le T} \left|N(k_n t, m_n s) - k_n m_n ts\right|/(t^\nu s^\mu) \to_{\mathscr{D}} \sup_{1\le t, s\le T} |W(t,s)|/(t^\nu s^\mu). \tag{3.67}$$

The scale transformation of the Wiener process and the law of iterated logarithm give that we have for all $\delta > 0$,

$$\lim_{T\to\infty} \limsup_{N\to\infty} P\left\{\sup_{\substack{T\le t<\infty\\ 1\le s\le n^{1/2}/m_n}} |W(t,s)|/(t^\nu s^\mu) > \delta\right\} = 0 \tag{3.68}$$

and

$$\lim_{T\to\infty}\limsup_{N\to\infty} P\left\{\sup_{\substack{1\le t\le n^{1/2}k_n\\ T\le s<\infty}} |W(t,s)|/(t^\nu s^\mu) > \delta\right\} = 0. \tag{3.69}$$

The proof of (3.65) will be completed if we show that for all $\delta > 0$,

$$\lim_{T\to\infty}\limsup_{N\to\infty} P\left\{(k_n m_n)^{-1/2} \sup_{\substack{T\le t\le n^{1/2}/k_n\\ 1\le s\le n^{1/2}/m_n}} |N(k_n t, m_n s) - k_n m_n ts|/(t^\nu s^\mu) > \delta\right\} = 0 \tag{3.70}$$

and

$$\lim_{T\to\infty}\limsup_{N\to\infty} P\left\{(k_n m_n)^{-1/2} \sup_{\substack{1\le t\le n^{1/2}/k_n\\ T\le s\le n^{1/2}/m_n}} |N(k_n t, m_n s) - k_n m_n ts|/(t^\nu s^\mu) > \delta\right\} = 0. \tag{3.71}$$

Using again Inequality 2.4 in Einmahl (1987) [cf. also Wichura (1969)], (3.70) and (3.71) can be established along the lines of (3.36). The details are omitted.

Now Theorem 2.3 with $L = G = 1$ follows from (3.2), (3.60) and (3.61) and (3.24) and (3.65). □

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