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L^P ESTIMATES ON ITERATED STOCHASTIC INTEGRALS

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For a continuous martingale M, let $\langle M, M\rangle$ denote the increasing process. Let $I_0, I_1, \dots$ denote the iterated stochastic integrals of M. We prove the inequalities of Burkholder–Davis–Gundy type,

$$A_{p,n}\left\|\langle M, M\rangle_t^{1/2}\right\|_{np}^n \le \|I_n(t)\|_p \le B_{p,n}\left\|\langle M, M\rangle_t^{1/2}\right\|_{np}^n,$$

where $\ln A_{p,n} \sim \ln B_{p,n} \sim -(n/2)\ln n$ as $n \to \infty$. Our proof requires the sharp constant b_p in Burkholder–Davis–Gundy inequalities $\|M\|_p \le b_p\|\langle M, M\rangle^{1/2}\|_p$.

In the Appendix we prove $\sup_{p\ge 1}(b_p/\sqrt{p}) = 2$. We apply our inequality to the study of the L^p convergence of the Neuman series $\Sigma I_n(t)$ for exponential martingales.

1. Introduction. Let $(M_t, \mathscr{F}_t, P)$ be a bounded and continuous martingale with $M_0 = 0$ and with increasing process $\langle M, M\rangle_t$. Consider the corresponding sequence of iterated stochastic integrals defined inductively by

$$I_n(t) = \int_0^t I_{n-1}(s)\, dM(s), \tag{1}$$

with $I_0(t) = 1$ and $I_1(t) = M_t$.

Our main result, which we prove in Section 2, is an analog for iterated stochastic integrals of the Burkholder–Davis–Gundy (B.D.G.) inequalities

$$a_p\left\|\langle M, M\rangle_t^{1/2}\right\|_p \le \|M_t\|_p \le b_p\left\|\langle M, M\rangle^{1/2}\right\|_p, \tag{2}$$

where the right side holds for $p \ge 1$ and the left side for $p > 1$.

Theorem 1. *For all $n \ge 1$, all $t > 0$ and all continuous and bounded martingales,*

$$A_{p,n}\left\|\langle M, M\rangle_t^{1/2}\right\|_{pn}^n \le \|I_n(t)\|_p \le B_{p,n}\left\|\langle M, M\rangle_t^{1/2}\right\|_{pn}^n, \tag{3}$$

where the right side holds for $p \ge 1$ and the left side for $p > 1$. Putting

$$\alpha_p = 1 + \sqrt{1 + p^{-1}}$$

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we have

$$B_{p,n} = \frac{1}{n!}(pn)^{n/2}\alpha_p^n, \tag{4}$$

$$A_{p,n} = \left(1 + \sqrt{1 + \frac{\sqrt{2}}{a_2}}\right)^{-1} \quad \text{for } n = 2, \tag{5}$$

$$= B_{p,n}(\alpha_p)^{-n}\left[\frac{2(np)^n}{(n-2)!}\frac{np}{p-1}\alpha_p^n + \frac{2\sqrt{n-1}}{n!a_p}(pn)^{1/2}\right]^{-1} \quad \text{for } n > 2. \tag{6}$$

Remark 1. Note that since (3) holds with the same constants for all bounded martingales, the usual truncation argument can be used to show these inequalities hold for all continuous martingales with well defined increasing processes such that the right sides of these inequalities are finite.

Remark 2. Note that for $n = 1$, the right side of (2) follows from the sharp B.D.G. inequalities

$$\|M\|_p \le 2\sqrt{p}\,\|\langle M, M\rangle^{1/2}\|_p, \qquad p \ge 1. \tag{7}$$

It is known [2] that (7) holds for $p \ge 2$ for some C in place of 2. Using a result of Davis [3], we prove in the Appendix that 2 is the best possible p-independent choice of C.

Remark 3. Consider how the constants $A_{p,n}$ and $B_{p,n}$ depend on n. By Stirling's formula,

$$B_{p,n} \sim \frac{\left(\alpha_p\sqrt{ep}\right)^n}{\sqrt{n!}}(2\pi n)^{-1/4}.$$

Again by Stirling's formula,

$$\frac{n^{n+1}}{(n-2)!} \sim \frac{n^{5/2}e^n}{\sqrt{2\pi}}.$$

Therefore, comparing (5) and (6), our bounds are the best possible up to a geometric factor. Put differently,

$$\ln A_{n,p} \sim \ln B_{n,p} \sim -\frac{n}{2}\ln n \quad \text{as } n \to \infty. \tag{8}$$

The rapid decay in n of the constants $B_{p,n}$ is crucial for our applications discussed in Section 3. For example, the Neumann series for

$$Z_t = 1 + \int_0^t Z_s\, dM_s \tag{9}$$

has the sum

$$\sum_{n=0}^{\infty} I_n(t) = \exp\left(M_t - \tfrac{1}{2}\langle M, M\rangle_t\right) \tag{10}$$

and converges a.e. by a theorem of Doleans-Dade [4]. We will give sufficient conditions for the L^p convergence of this series. We also give necessary conditions of essentially the same form: see Theorems 2 and 3. In fact, one motivation of this work was to develop effective methods for the direct treatment of the Neumann series.

Iterated stochastic integrals for continuous martingales are closely related to Hermite polynomials and Wiener's chaos decomposition, which we will briefly discuss in Section 4.

2. Proof of Theorem 1. We first inductively prove the right side starting with $n = 1$, which is just sharp B.D.G. (7). We will use the Kailath–Segall identity [9]

$$I_n = \frac{1}{n}\{I_{n-1}M - I_{n-2}\langle M, M\rangle\}, \tag{11}$$

valid for $n > 1$ and also for $n = 1$ with the convention $I_{-1} = 0$, which can be established inductively by making two stochastic integrations by parts. First, by Hölder,

$$\|I_{n-1}M\|_p \le \|I_{n-1}\|_{pn/(n-1)}\|M\|_{pn}. \tag{12}$$

The choice of Hölder exponents here is crucial: $(I_{n-1})^{pn/n-1}$ and M^{pn} have the same homogeneity in M. Sharp B.D.G. (7) gives

$$\|M\|_{pn} \le 2\sqrt{pn}\,\|\langle M, M\rangle^{1/2}\|_{pn}. \tag{13}$$

Therefore,

$$\|I_{n-1}M\|_p \le 2\sqrt{pn}\,\|I_{n-1}\|_{pn/(n-1)}\|\langle M, M\rangle^{1/2}\|_{pn}. \tag{14}$$

Next apply Hölder to the other term in (11):

$$\|I_{n-2}\langle M, M\rangle\|_p \le \|I_{n-2}\|_{pn/(n-2)}\|\langle M, M\rangle\|_{pn/2}. \tag{15}$$

Again, appropriate Hölder exponents are determined by homogeneity. Combining (14) and (15) we have

$$\begin{aligned}\|I_n\|_p \le \frac{1}{n}\Big\{&\|I_{n-1}\|_{pn/(n-1)}\|\langle M, M\rangle^{1/2}\|_{pn}2\sqrt{pn} \\ &+\|I_{n-2}\|_{pn/(n-2)}\|\langle M, M\rangle^{1/2}\|_{pn}^2\Big\}.\end{aligned} \tag{16}$$

Now we are ready to prove inductively that for $n > 1$,

$$\|I_n\|_p \le \frac{1}{n!}(pn)^{n/2}\alpha_p^n\|\langle M, M\rangle^{1/2}\|_{pn}^n.$$

Supposing it has been established for all Lebesgue classes L^p and all powers of iteration less than n, we show it holds for all p at n. First,

$$
\begin{aligned}
&\frac{1}{n} 2\sqrt{pn}\, \|I_{n-1}\|_{pn/(n-1)} \\
(17)\qquad &\le \frac{1}{n} 2\sqrt{pn}\, \frac{1}{(n-1)!}\left(p\frac{n}{n-1}(n-1)\right)^{(n-1)/2} \alpha_{pn/(n-1)}^{n-1} \\
&\times \|\langle M, M\rangle^{1/2}\|_{pn}^{n-1} \\
&= \frac{1}{n!}(pn)^{n/2} 2\alpha_p^{n-1} \|\langle M, M\rangle^{1/2}\|_{pn}^{n-1},
\end{aligned}
$$

where we have used the fact that α_p is monotone decreasing in p. Second,

$$\|I_{n-2}\|_{pn/(n-2)} \le \frac{1}{(n-2)!}(pn)^{(n-2)/2} \alpha_p^{n-2} \|\langle M, M\rangle^{1/2}\|_{pn}^{n-2}.$$

Therefore, inserting the last inequality and (17) into (16),

$$\|I_n\|_p \le \frac{1}{n!}(pn)^{n/2} \alpha_p^n \left(\frac{2}{\alpha_p} + \frac{n-1}{n}\frac{1}{p\alpha_p^2}\right) \|\langle M, M\rangle\|_{pn/2}^{n/2}.$$

This proves the right inequality in (3) since the quantity in the parentheses is dominated by 1. We now pass to the left. We will use the following identity for iterated stochastic integrals of a continuous martingale:

$$(18)\qquad I_n I_{n-2} = I_{n-1}^2 - \sum_{m=1}^{n} \frac{(n-m)!}{n!} I_{n-m}^2 \langle M, M\rangle^{m-1}, \qquad \forall\, n \ge 2.$$

Note that each term in the sum above is positive. Retaining only the first and the last term in the sum, we immediately obtain

$$(19)\qquad \frac{\langle M, M\rangle^{n-1}}{(n-1)!} \le (n-1) I_{n-1}^2 - n I_n I_{n-2}, \qquad \forall\, n \ge 2.$$

To prove (18) inductively, we first note that it reduces to the Kailath–Segall identity (11) when $n = 2$.

Next, multiply (11) through by I_{n-2} to obtain

$$I_n I_{n-2} = \frac{1}{n}\{I_{n-1} M I_{n-2} - I_{n-2}^2 \langle M, M\rangle\}.$$

But again by (11), $MI_{n-2} = (n-1)I_{n-1} + I_{n-3}\langle M, M\rangle$, so that

$$(20)\qquad I_n I_{n-2} = \frac{1}{n}\{(n-1) I_{n-1}^2 - I_{n-2}^2 \langle M, M\rangle\} + \frac{1}{n} I_{n-1} I_{n-3} \langle M, M\rangle.$$

Now suppose (18) is established with n replaced by $n-1$. Applying this to the last term in (20) yields

$$I_n I_{n-2} = \frac{1}{n}\{(n-1)I_{n-1}^2 - I_{n-2}^2\langle M, M\rangle\}$$

$$+\frac{1}{n}\left\{I_{n-2}^2 - \sum_{m=1}^{n-1}\frac{(n-m-1)!}{(n-1)!}I_{n-m-1}^2\langle M, M\rangle^{m-1}\right\}\langle M, M\rangle$$

$$= I_{n-1}^2 - \frac{1}{n}I_{n-1}^2 - \sum_{m=1}^{n-1}\frac{(n-m-1)!}{n!}I_{n-m-1}^2\langle M, M\rangle^m,$$

which is clearly equivalent to (18).

Having established (19), we integrate both sides against $d\langle M, M\rangle$ to obtain

$$\frac{\langle M, M\rangle_t^n}{n!} \le (n-1)\langle I_n, I_n\rangle_t - n\int_0^t I_n(s)I_{n-2}(s)\,d\langle M, M\rangle_s.$$

Hence

$$\text{(21)}\quad \frac{\langle M, M\rangle_t^{n/2}}{\sqrt{n!}} \le \sqrt{n-1}\,\langle I_n, I_n\rangle_t^{1/2} + \sqrt{n}\left(I_n^*(t)I_{n-2}^*(t)\langle M, M\rangle_t\right)^{1/2},$$

where of course the asterisk indicates the maximal process.

We may now estimate L^p norms using the Minkowski and B.D.G. inequalities:

$$\text{(22)}\quad \frac{1}{\sqrt{n!}}\left\|\langle M, M\rangle^{1/2}\right\|_{np}^n \le \sqrt{n-1}\,\frac{1}{a_p}\|I_n\|_p + \sqrt{n}\left\|\left(I_n^* I_{n-2}^*\langle M, M\rangle\right)^{1/2}\right\|_p.$$

Next, by Schwarz,

$$E\left(I_n^* I_{n-2}^*\langle M, M\rangle\right)^{p/2} \le \left(EI_n^{*p}\right)^{1/2}\left(E\left(I_{n-2}^*\langle M, M\rangle\right)^p\right)^{1/2}.$$

So

$$\text{(23)}\quad \left\|\left(I_n^* I_{n-2}^*\langle M, M\rangle\right)^{1/2}\right\|_p \le \left(\frac{p}{p-1}\right)^{1/2}\|I_n\|_p^{1/2}\left\|I_{n-2}^*\langle M, M\rangle\right\|_p^{1/2}.$$

For $n = 2$, this gives us the quadratic inequality

$$\frac{1}{\sqrt{2}}\left\|\langle M, M\rangle^{1/2}\right\|_{2p}^2 \le \frac{1}{a_2}\|I_2\|_p + \sqrt{2}\|I_2\|_p^{1/2}\|\langle M, M\rangle\|_{2p},$$

which implies (6) for $n = 2$.

For $n > 2$, we will again obtain a quadratic inequality, but first we must apply the right inequality of (3), after first applying the inequalities of Hölder and Doob:

$$\begin{aligned}
\| I_{n-2}^{*}\langle M, M\rangle \|_p &\le \|I_{n-2}^{*}\|_{np/(n-2)} \|\langle M, M\rangle^{1/2}\|_{np}^2 \\
&\le \frac{p}{p-1}\|I_{n-2}\|_{np/(n-2)}\|\langle M, M\rangle^{1/2}\|_{np}^2 \\
(24)\qquad &\le \frac{p}{p-1}\frac{1}{(n-2)!}(np)^{(n-2)/2} \\
&\quad \times\left(\sqrt{1+p^{-1}}+1\right)^{n-2}\|\langle M, M\rangle^{1/2}\|_{np}^n,
\end{aligned}$$

where we have used the fact that the constants $p/(p-1)$ arising in Doob's inequality and, again, the constants α_p are monotone decreasing in p.

Let us use D_{np} to denote the constant in the right side of (24). From (22), (23) and (24) we have the inequality

$$\begin{aligned}
\frac{1}{\sqrt{n!}}\|\langle M, M\rangle^{1/2}\|_{np}^n &\le \sqrt{n-1}\,\frac{1}{\alpha_p}\|I_n\|_p \\
(25)\qquad &\quad + \sqrt{n}\left(\frac{p}{p-1}\right)^{1/2} D_{np}\|I_n\|_p^{1/2}\|\langle M, M\rangle^{1/2}\|_{np}^{n/2}.
\end{aligned}$$

Solving this quadratic inequality leads, with simple estimates, to (6).

3. Application to exponential martingales. We use the following notations. For any integer k and any $t > 0$, $Q(k,t)$ denotes the chronological domain

$$Q(k,t) = \{(t_1 \cdots t_k) \in \mathbb{R}^k | 0 < t_1 < \cdots < t_k < t\}$$

and $J(k,t)$ denotes the following Stieltjes iterated integral on that domain:

$$(26)\qquad J(k,t) = \int \cdots \int_{Q(k,t)} d\langle M, M\rangle_{t_1} \cdots d\langle M, M\rangle_{t_k}.$$

Note that

$$\|\langle M, M\rangle_t^{1/2}\|_{2k}^{2k} = k!E[J(k,t)].$$

In this way, Theorem 1 enables us to dominate L^p norms of iterated stochastic integrals in terms of expectations of iterated Stieltjes integrals. To do this, for any real s, we denote by $[s]]$ the smallest integer greater than or equal to s. We then have the following theorem.

THEOREM 2. *Let M be a continuous martingale such that for some $p \ge 1$ and some finite $t > 0$,*

$$(27)\qquad \limsup\left\{E\left[J\left(\left[\frac{pn}{2}\right]\right], t\right)\right]\right\}^{[pn/2]]^{-1}} < k_p,$$

where

$$k_p = \frac{2}{e}(p\alpha_p)^{-2}. \tag{28}$$

Then the Neumann series (10) *converges in* L^p.

PROOF. Since $\|I_n(t)\|_p$ is an increasing function of the time $t > 0$, we need only show that $\|I_n(t)\|_p$ is dominated for large n by the terms q^n of a convergent geometric series. In fact, using Theorem 1,

$$\begin{aligned}\|I_n(t)\|_p &\le B_{p,n}\|\langle M, M\rangle^{1/2}\|_{pn}^n = B_{p,n}\|\langle M, M\rangle\|_{pn/2}^{n/2}\\ &\le B_{p,n}\left(\|\langle M, M\rangle\|_{[pn/2]]}^{[pn/2]]}\right)^{n/2[p(n/2)]]^{-1}}\\ &= B_{pn}\left\{\left[\left[\frac{pn}{2}\right]\right]!E\left[J\left(\left[\left[\frac{pn}{2}\right]\right],t\right)\right\}^{n/2}[pn/2]]^{-1}.\end{aligned}$$

Using (8) and Stirling formula we have for arbitrary $\varepsilon > 0$ that for n large enough,

$$\|I_n(t)\|_p \le (1+\varepsilon)^n\left(\frac{E[J([pn/2]],t)]^{[pn/2]]^{-1}}}{k_p}\right)^{n/2}.$$

Then by (27), for ε sufficiently small we have domination by a convergent geometric series. □

Following the same pattern, we obtain the following converse to Theorem 2 for $p > 1$. For any real s we denote by $[[s]$ the largest integer less than or equal to s.

THEOREM 3. *Let M be a continuous martingale such that for some $p > 1$ and some finite time $t > 0$,*

$$\limsup_{n\to\infty}\left\{E\left[J\left(\left[\left[\frac{pn}{2}\right],t\right)\right]\right\}^{[[pn/2]^{-1}} > K_p, \tag{29}$$

where

$$K_p = 2e\alpha_p^2. \tag{30}$$

Then the Neumann series (10) *diverges in* L^p.

It is an interesting question to determine when the expectation of $\mathscr{E}M = \exp(M - \langle M, M\rangle/2)$ is 1. We have just obtained sufficient conditions for the L^p convergence of ΣI_n to $\mathscr{E}M$. Therefore it is interesting to have sufficient conditions such that $EI_n = 0$ for $n > 0$. We have the following result closely related to [8].

PROPOSITION 1. *For any $n \geq 1$ and any $t > 0$, suppose $E[\langle M, M \rangle_t^{n/2}]$ is finite. Then $E(I_n) = 0$. In particular, if for some $\lambda > 0$,*

$$E[\exp(\lambda \langle M, M \rangle_t)] < \infty, \qquad \forall\, t, \tag{31}$$

then

$$E(I_n(t)) = 0, \qquad \forall\, t > 0 \text{ and } n > 0. \tag{32}$$

PROOF. For some constant c_k^n, we have for $0 \leq s \leq t$ [see (37)]

$$I_n(s) = \sum_{k=0}^{n} c_k^n (M_s)^k \langle M, M \rangle_s^{(n-k)/2}$$

$$\Rightarrow \quad |I_n^*(t)| \leq \sum |c_k^n| (M_t^*)^k \langle M, M \rangle_t^{(n-k)/2}.$$

Hence combining Hölder with B.D.G.,

$$E\big[|I_n^*(t)|\big] \leq CE\big[\langle M, M \rangle_t^{n/2}\big].$$

Hence $\{I_n(s);\, 0 \leq s \leq t\}$ is an equiintegrable local martingale, i.e., a martingale and (32) follows. □

REMARKS. Novikov [7] has given examples of continuous martingales M such that $E[\mathscr{E}M] < 1$ but such that (31) holds for all $\lambda < \frac{1}{2}$. For such M, the Neumann series clearly does not converge in L^1, even though $\mathscr{E}M$ and all the terms in the Neumann series belong to L^1. The L^p convergence of the Neumann series implies $E[\mathscr{E}M] = 1$, but not vice versa; hence, the interest of Theorems 2 and 3 giving, respectively, sufficient and necessary condition for this L^p convergence.

Note the following weaker but convenient forms of these criteria.

PROPOSITION 2. (a) *Suppose that for some p $(1 \leq p < \infty)$ and all t,*

$$E[\exp(\lambda \langle M, M \rangle_t)] < \infty, \tag{33}$$

where

$$\lambda > k_p^{-1}. \tag{34}$$

Then we have L^p convergence of the Neumann series for all t and $E[\mathscr{E}M] = 1$.

(b) *Suppose that for some p with $1 \leq p \leq 2$ we have for some $t > 0$,*

$$E[\exp(\lambda \langle M, M \rangle_t)] = \infty,$$

where

$$\lambda < K_p^{-1}.$$

Then the Neumann series $\sum I_n(t)$ is not convergent in L^p.

In fact (33) means

$$\sum_0^\infty \lambda^k E[J(k,t)] < \infty;$$

hence, by the theory of entire series $\lambda \le R$, where

$$R^{-1} = \limsup E[J(k,t)].$$

In view of (34), (27) is satisfied. By Theorem 2 the Neumann series converges in L^p. Hence $E[\mathscr{E}M] = 1$ by Proposition 1. Proof of part (b) follows the same line.

Now we give an example.

PROPOSITION 3. *Let $b = b(x,t)$ be a measurable function on $\mathbb{R}^d \times \mathbb{R}_+$ such that for some $q > d \vee 2$ and all t,*

$$b(x,t) = b'(x,t) + b''(x,t),$$

where b'' is uniformly bounded on $\mathbb{R}^d \times \mathbb{R}_+$ and where $\{b'(\cdot,t),\ t > 0\}$ is a bounded subset of $L^q(\mathbb{R}^d)$. Denoting by $(B_t, \mathscr{F}_t, P)$ an $\mathbb{R}^d$ valued Brownian motion and considering the martingale

$$M_t = \int_0^t b(B_s, s)\, dB_s$$

for any $t > 0$ and any $p \ge 1$, there exists a constant $C = C(t,p,b)$ such that for all n,

$$E[J(n,t)] \le \frac{C^n}{\Gamma(n(1-\alpha)+1)}, \qquad \alpha = \frac{d}{q} \tag{35}$$

and so the Neumann series converges in L^p for $1 \le p < \infty$.

Portenko [8] has proven a related result, however, assuming more regularity on the drift $(q > d+2)$ and only proving $E(\mathscr{E}M) = 1$. Our method for estimating $E[J(n,t)]$ is Portenko's.

PROOF.

$$\begin{aligned} E[J_n(t)] &= \int_{Q_n(t)} E\left[b^2(B_{s_n}, s_1) \cdots b^2(B_{s_n}, s_n)\right] ds_1 \cdots ds_n \\ &= \int_{Q_n(t)} E\left[b^2(B_s, s_1) \cdots b^2(B_{s_{n-1}}, s_{n-1})\right. \\ &\qquad \left. \times E\{b^2(B_{s_n}, s_n) | \mathscr{F}_{s_{n-1}}\}\right] ds_1 \cdots ds_n. \end{aligned} \tag{36}$$

Since $b = b' + b''$ we have

$$b^2(B_{s_n}, s_n) \le 2b'^2(B_{s_n}, s_n) + 2b''^2(B_{s_n}, s_n).$$

Hence

$$\left\| E\{b^2(B_{s_n}, s_n)|\mathscr{F}_{s_{n-1}}\}\right\|_\infty \leq 2\left\| E\{b'^2(B_{s_n}, s_n)\|\mathscr{F}_{s_{n-1}}\}\right\|_\infty + 2\|b''\|_\infty^2.$$

But by Hölder's inequality, the heat kernel p_t on R^d is L^∞-smoothing and for any $f \in L^p(R^d)$,

$$\|p_t * f\|_\infty \leq (2\pi t)^{-d/2p} p'^{-d/2p'} \|f\|_p.$$

Therefore, applying this to $f = b'^2$ with $2p = q$ and putting $\alpha = d/q$, $\tau_j = 2\pi(s_j - s_{j-1})$ for $j = 2, \dots, n$ and $\beta = p'^{-d/2p'}$,

$$\left\| E\{b'^2(B_{s_n}, s_n)\|\mathscr{F}_{s_{n-1}}\}\right\|_\infty \leq \beta \tau_n^{-\alpha}(\|b'\|_q)^2.$$

Since t is given, introducing

$$D = \left(\frac{\|b''\|_\infty}{\|b'\|_{p_0}}\right)^2 t^\alpha,$$

we have

$$(\|b''\|_\infty)^2 \leq D(\|b'\|_{p_0})^2 s^{-\alpha} \quad \text{for } 0 < s \leq t.$$

Hence finally

$$\left\| E\{b^2(B_{s_n}, s_n)|\mathscr{F}_{s_{n-1}}\}\right\|_\infty \leq 2(\beta + D)(\|b'\|_{p_0})^2 \tau_n^{-\alpha}.$$

Hence substituting in (36) and putting $\gamma = 2(\beta + D)$,

$$E[J(n, t)] \leq \gamma \int_{Q_n(t)} E\left[b^2(B_{s_n}, s) \cdots b^2(B_{s_{n-1}}, s_n)\right](\|b'\|_{p_0})^2 \tau_n^{-\alpha}\, d\tau_1 \cdots d\tau_n.$$

Computing the multiple integral with the new variables τ_j, the new domain of integration is $Q_n'(t) = \{(\tau_1 \cdots \tau_n), \sum \tau_j \leq 2\pi\}$. Hence by an induction argument,

$$E[J(n, t)] \leq \left(\gamma(\|b'\|_{p_0})^2\right)^n J_n'(t)$$

with

$$J_n'(t) = \int \cdots \int_{Q_n'(t)} \tau_1^{-\alpha} \cdots \tau_n^{-\alpha}\, d\tau_1 \cdots d\tau_n$$

$$= \frac{\left((2\pi)^{1-\alpha}\Gamma(1-\alpha)\right)^n}{\Gamma(n(1-\alpha)+1)}.$$

This proves (35). But for any $\beta > 0$ and, as is well known in the particular case where $\beta = 1$,

$$\Gamma(n\beta + 1) \sim \left(\frac{n\beta}{e}\right)^{n\beta} \sqrt{2\pi n\beta} \quad \text{as } n \to \infty.$$

Therefore in the present case, Theorem 2 can be applied since the lim sup arising in (27) is zero. Hence the proposition follows from Theorem 2. □

4. Connection with Hermite polynomials. Our proof of Theorem 1 uses identities closely related to Hermite polynomials

$$H_n(x,t) = t^{n/2} h_n(t^{-1/2}x),$$

where

$$h_n(x) = (-1)^n \left[\left(\frac{d}{dx} \right)^n e^{-x^2/2} \right] e^{x^2/2}.$$

The well-known correspondence is

$$I_n(t) = H_n(M_t, \langle M, M \rangle_t)/n!. \tag{37}$$

For example the Kailath–Segall identity (11) corresponds to the classical recurrence relation

$$\frac{H_n(x,t)}{n!} = \frac{1}{n} \left[x \frac{H_{n-1}(x,t)}{(n-1)!} - t \frac{H_{n-2}(x,t)}{(n-2)!} \right]$$

and (18) corresponds to the identity

$$\frac{H_n(x,t) H_{n-2}(x,t)}{n!(n-2)!} = \frac{H_{n-1}(x,t)^2}{(n-1)!^2} - \sum_{m=1}^{n} \frac{H_{n-m}(x,t)^2}{n!(n-m)!} t^{m-1},$$

which appears to be new. Hence for any continuous and bounded martingale, the Kailath–Segall identity and (11) can be viewed as chaos identities. A sort of chaos decomposition is behind Theorem 1.

In the very particular case of the scalar Brownian motion (B_t), we have

$$I_n(t) = \frac{H_n(B_t, t)}{n!}.$$

Denoting γ the unit Gauss measure on the line, we have

$$\|I_n(1)\|_p = \|h_n\|_p / n! = n!^{-1} \left(\int |h_n(x)|^p \, d\gamma(x) \right)^{1/p}.$$

Since the bracket of B is t, Theorem 1 gives us

$$n! A_{n,p} \le \left(\int |h_n(x)|^p \, d\gamma(x) \right)^{1/p} \le n! B_{n,p}.$$

In fact, Hölder's inequality and Nelson's hypercontractivity inequality [5] provide better upper bounds.

The lower bound may be of interest.

APPENDIX

Let b_p be the best constant in the B.D.G. inequality for a given $p \ge 1$, i.e.,

$$b_p = \sup\{ \|M_t\|_p / \|\langle M, M \rangle_t^{1/2}\|_p \}, \tag{A1}$$

where the supremum is taken over all continuous bounded martingales M and all times t.

THEOREM A.

(A2) $$\sup_{p \geq 1} \frac{b_p}{\sqrt{p}} = 2.$$

This clearly implies (7).

Of course Davis [3] has identified the best constants for all p as certain zeros of certain special functions. Theorem A provides explicit information which is sharp for large p.

Our proof proceeds in two parts. We first show $b_p \leq 2\sqrt{p}$ for $1 \leq p \leq 2$. Using an Itô's formula argument due to Novikov [7] and Zakai [10]. The second part is more substantial and is based on Davis's work. We reduce the estimation of zeros of special functions to an eigenvalue estimation problem which we solve. Let δ and C denote arbitrary positive constants. Fix t and p with $t > 0$ and $1 \leq p \leq 2$. Then by Itô's formula,

$$\begin{aligned}
&\left(\delta + C\langle M, M\rangle_t + M_t^2\right)^{p/2} - \delta^{p/2} \\
&\quad - C\frac{p}{2}\int_0^t \left(\delta + C\langle M, M\rangle_s + M_s^2\right)^{p/2-1} d\langle M, M\rangle_s \\
&\quad - \frac{1}{2}\int_0^t \Big\{p\left(\delta + C\langle M, M\rangle_s + M_s^2\right)^{(p-1)/2} \\
&\qquad\qquad + p(p-2)\left(\delta + C\langle M, M\rangle_s + M_s^2\right)^{p/2-2} M_s^2\Big\}\, d\langle M, M\rangle_s
\end{aligned}$$

is a martingale. Now, take the expectation. Since $p < 2$, the last term in the braces is negative. Therefore

$$\begin{aligned}
&E\left[\left(\delta + C\langle M, M\rangle_t + M_t^2\right)^{p/2}\right] - \delta^{p/2} \\
&\quad \leq \frac{C+1}{C} C\frac{p}{2}\int_0^t \left(\delta + C\langle M, M\rangle_s + M_s^2\right)^{p/2-1} d\langle M, M\rangle_s.
\end{aligned}$$

Now since $p/2 > 0$ and $p/2 - 1 < 0$, we can drop the $C\langle M, M\rangle_t$ term on the left in the parentheses and drop the M_t^2 term on the right. We get

$$E\left[\left(\delta + M_t^2\right)^{p/2}\right] \leq \delta^{p/2} + \frac{C+1}{C} C^{p/2} E\left[\langle M, M\rangle_t^{p/2}\right].$$

We can make δ tend to zero using dominated convergence. The optimal value of C is found to be $(2-p)/p$ at which

$$\frac{C+1}{C} C^{p/2} = \frac{2}{p}\left(\frac{2-p}{p}\right)^{p/2-1}.$$

This function of p decreases from the value 2 at $p = 1$ to the value 1 at $p = 2$. Therefore $b_p \leq 2\sqrt{p}$ for $1 \leq p \leq 2$. Now suppose $p \geq 2$. By the result of Davis

[3], b_p is the largest zero of the solution to

$$-y''(x)+\left(\frac{x^2}{4}-\left(p+\frac{1}{2}\right)\right)y(x)=0 \tag{A3}$$

with rapid decay as x tends to $+\infty$. Let y_p denote this solution normalized so that

$$\int_{b_p}^{\infty} y_p(x)^2\,dx = 1$$

and is positive on $(b_p, +\infty)$. Let $V_p(x)$ denote $x^2/4-(p-1/2)$, i.e., $(x^2-c_p^2)/4$, where $c_p = 2(p+\frac{1}{2})^{1/2}$. Let $W_p(x)$ be the linear function tangent to V_p at the turning point c_p, i.e., $c_p(x-c_p)/2$. Since V_p is convex,

$$W_p(x) \le V_p(x) \quad \text{for all } x. \tag{A4}$$

Consider the Sturm–Liouville operator

$$H_w = -\frac{d^2}{dx^2}+W_p \tag{A5}$$

defined on $(b_p, +\infty)$, with Dirichlet boundary conditions. Let E_p denote the least eigenvalue and let z_p denote the corresponding normalized positive eigenfunction. Of course, z_p is a shifted and scaled Airy function; the scaling depends on c_p and the shift depends on b_p.

By the Rayleigh–Ritz variational principle,

$$E_p \le \int_{b_p}^{\infty}\left(u'(x)^2+W_p(x)u(x)^2\right)dx$$

for any normalized smooth function u of rapid decay with $u(b_p)=0$. In particular, y_n is a function of this type and so

$$\begin{aligned} E_p &\le \int_{b_p}^{\infty}\left(y_p'(x)^2+W_p(x)y_p(x)^2\right)dx \\ &\le \int_{b_p}^{\infty}\left(y_p'(x)^2+V_p(x)y_p(x)^2\right)dx \\ &= \int_{b_p}^{\infty} y_p'(x)\left(-y_p''(x)+V_p(x)y_p(x)\right)dx = 0 \end{aligned}$$

by (A3), (A4) and an integration by parts.

But E_p is directly related to the eigenvalue λ_0 of the Airy differential operator

$$-\frac{d^2}{dx^2}+x \tag{A6}$$

on R_+ with Dirichlet boundary conditions. As is well known [1], $\lambda_0 = 2.3\cdots$, and we will use $\lambda_0 \geq 1$ for example. In fact we will show

$$b_p \leq c_p - \lambda_0\left(p + \tfrac{1}{2}\right)^{-1/6}. \tag{A7}$$

From (A7) the bound $b_p \leq 2\sqrt{p}$ clearly follows. To establish (A7) we use the affine transform

$$x = b_p + \left(p + \tfrac{1}{2}\right)^{-1/6}\xi,$$

taking (A5) into (A6). In detail, let $z_p(\xi)$ also denote $z_p(c_p + (p + \frac{1}{2})^{-1/6}\xi)$. With the new variable ξ, z_p satisfies

$$-\frac{d^2}{d\xi^2}z_p + \xi z_p = \lambda'_p z^p$$

with $\lambda'_p = E_p - 2^{-1}(p + \frac{1}{2})^{-1/3}(b_p - c_p)c_p$.

Since $z_p(x)$ is by definition the ground state of (A5), $z_p(x)$ has no zeros. Therefore, $z_p(\xi)$ also has no zeros, hence is the ground state of (A6). Therefore, $\lambda_p = \lambda'_p$. Since $E_p \leq 0$, this implies (A7).

By switching the roles of z_p and y_p in the variational argument one sees that

$$b_p = 2p^{1/2} + O(p^{-1/6})$$

and hence the equality claimed in (A2) holds. That is, one uses shifted Airy functions as trial functions in the variational calculation of the lowest eigenvalue for

$$-\frac{d^2}{dx^2} + V_p(x)$$

on $[b_p, \infty]$. We know the lowest eigenvalue is zero because we have a positive function which is an eigenfunction with zero eigenvalue. But if a_p is too small, our shifted Airy trial function gives a negative result. We will not provide further details, since the main interest is in the inequality $b_p \leq 2\sqrt{p}$.

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