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SAMPLE AND ERGODIC PROPERTIES OF SOME MIN-STABLE PROCESSES

BY KEITH STEVEN WEINTRAUB

CITICORP Investment Bank, New York

A random vector is *min-stable* (or jointly negative exponential) if any weighted minimum of its components has a negative exponential distribution. The vectors can be subordinated to a two-dimensional homogeneous Poisson point process through positive $\mathscr{L}_1$ functions called *spectral functions*. A critical feature of this representation is the point of the Poisson process, called the location, that defines a min-stable random variable.

A measure of association between min-stable random variables is used to define mixing conditions for min-stable processes. The association between two min-stable random variables X_1 and X_2 is defined as the probability that they share the same location and is denoted by $q(X_1, X_2)$. Mixing criteria for a min-stable process $X(t)$ are defined by how fast the association between $X(t)$ and $X(t+s)$ goes to zero as $s \to \infty$.

For some stationary processes (including the moving-minimum process), conditions on the spectral functions are derived in order that the processes satisfy mixing conditions.

1. Introduction. Let $\mathbf{X} = (X_1, \dots, X_k)$ be a random vector. Then X is min-stable (multivariate negative exponential) if and only if the weighted minimum $\bigwedge_{i=1}^{k} X_i/a_i$ has a negative exponential distribution for all $(a_1, \dots, a_k)$ with $0 \le a_i < \infty$ and at least one $a_i > 0$. The symbol $\wedge$ means minimum. This definition implies that $\mathbf{X}$ has negative exponential marginals. One of the most important aspects of min-stable theory is the relationship of min-stable variables, vectors and processes to a two-dimensional Poisson process on the strip $[0,1] \times \mathscr{R}_+$. This relationship is determined by *spectral functions* and *pistons*.

In the following sections we show how a univariate negative exponential random variable and then a min-stable random vector are related to the Poisson process via *spectral* functions from $[0,1]$ into $\mathscr{R}_+$. For random vectors, these functions characterize the joint distributions just as the variance-covariance matrix does in the Gaussian case.

The main goal of this paper is to describe the spectra, and hence the dependence structure, of min-stable *processes*. A min-stable process is a continuous time stochastic process with finite dimensional distributions which are min-stable random vectors. In general, the sample paths of min-stable

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processes are not continuous but are continuous at any given point with probability 1.

A min-stable vector may be subordinated to a two-dimensional Poisson point process by *spectral* functions and a group of $\mathscr{L}_1$ operators called *pistons*. This relationship determines the multivariate dependence structure. The basics of min-stable theory as described in de Haan and Pickands (1986) are reviewed briefly in Section 2. Section 2.1 shows how the *pistons* and *spectral* functions must behave in order for a min-stable process to be stationary.

The ergodic properties of moving minimum processes, a special class of stationary min-stable processes, are discussed in Section 3. A measure of association is defined, and through this measure of association three mixing conditions are proposed. For a min-stable process $X(t)$, $t \in \mathscr{R}$, the three mixing criteria measures are:

0. The association between two separated marginal values of the process $X(t)$ and $X(t+s)$.
1. The maximum association between a single value $X(t)$ and a weighted combination of the future of the process after time $t+s$.
2. The maximum association between a weighted past of the process up to time t and a weighted future after time $t+s$.

The point of the two-dimensional Poisson process that defines a min-stable random variable is called the *location*. The location of a moving-minimum process is intimately related to mixing behaviour. A moving-minimum process $X(t)$ can be mixing only when the horizontal coordinate of the location goes to ∞ as $t \to \infty$. Section 3.1 shows the conditional distribution for X_1 given X_2 for a pair of jointly min-stable random variables. This conditional distribution is applied to part of the Kaplan–Meier double censoring problem.

2. Min-stable theory. A two-dimensional homogeneous Poisson process on the strip $[0,1] \times \mathscr{R}_+$ is a random array of points $\mathbf{\Pi} := \{(U_l, Y_l) | l = 1, 2, \dots\}$ defining a homogeneous Poisson random measure $\mathbf{M}$ with unit intensity. In effect, $\mathbf{\Pi}$ is the cross product of a sequence of independent uniform random variables on $(0,1)$ with a homogeneous Poisson process $\{Y_l\}_{l=1,2,\dots}$. All the U_l are independent of all the Y_l.

Certain measure preserving transformations from the strip $[0,1] \times \mathscr{R}_+$ to the upper half-plane $\mathscr{R} \times \mathscr{R}_+$ preserve the homogeneous Poisson process. The upper half-plane is the most convenient space to work in when examining min-stable processes. An example of a transformation to the upper half-plane can be found in de Haan and Pickands (1986). No new notation will be introduced when operating in these spaces. We will still use $\mathbf{\Pi}$ to represent the Poisson process and (U_l, Y_l) the points of $\mathbf{\Pi}$ regardless of whether the space is $[0,1] \times \mathscr{R}_+$ or $\mathscr{R} \times \mathscr{R}_+$.

A single negative exponential random variable X can be represented in terms of the strip. Let f: $[0,1] \to \mathscr{R}_+$ with $\|f\| = \int f(u)\,du = \lambda < \infty$, and define

$$X = \bigwedge_{l=1}^{\infty} \frac{Y_l}{f(U_l)}. \tag{2.1}$$

Then X has a negative exponential distribution with $EX = 1/\lambda$:

$$\begin{aligned} P(X \geq x) &= P\left(\bigwedge_{l=1}^{\infty} \frac{Y_l}{f(U_l)} \geq x \right) \\ &= P(Y_l \geq xf(U_l), \forall\, l) \\ &= P(\exists \text{ no points of } \mathbf{\Pi} \text{ below } xf(u)) \\ &= \exp(-\text{area below } xf(u)) \\ &= \exp\left(-\int_0^1 xf(u)\, du \right) \\ &= e^{-\lambda x}. \end{aligned}$$

Thus a random variable with the same distribution as X can be obtained by replacing f in the denominator of 2.1 with any $g > 0$ such that $\|g\| = \|f\|$.

A $(k \times 1)$ multivariate negative exponential vector $\mathbf{X}$ can be subordinated to the Poisson process on the strip by having a function f_i for each of the components X_i of $\mathbf{X}$. As shown by de Haan and Pickands (1986),

$$\mathbf{X} \text{ min-stable} \Leftrightarrow X_i = \bigwedge_{l} \frac{Y_l}{f_i(U_l)} \quad \text{with} \tag{2.2}$$
$$f_i\colon [0,1] \to \mathscr{R}_+,\ \|f_i\| < \infty,\ i = 1, \dots, k.$$

The $f_i \in \mathscr{L}_1[0,1]_+$ are the *spectral functions* of $\mathbf{X}$ and determine the dependence structure of $\mathbf{X}$. de Haan and Pickands (1986) also show that an equivalent representation of $\mathbf{X}$ can be obtained by applying a unitary operator to the spectral functions f_i. They call these operators *pistons* and show that equivalent distributions can be obtained *only* through the use of pistons. This equivalence is important for finding the spectral representation of stationary processes.

If $\mathbf{Y}$ is a $(k \times 1)$ min-stable random variable with spectral functions g_i, then by Theorem (4.2) of de Haan and Pickands (1986),

$$\mathbf{X} =_d \mathbf{Y} \Leftrightarrow g_i = \Gamma f_i, \qquad i = 1, \dots, k, \tag{2.3}$$

for some piston Γ. The symbol $=_d$ means equal in distribution. One easy observation is that since a piston is norm preserving, the expected values of the marginals of $\mathbf{X}$ and $\mathbf{Y}$ in 2.3 are equal.

2.1. *Min-stable processes.* For a process $X(t)$ to be min-stable it must have finite dimensional distributions which are min-stable. This characterization [de Haan and Pickands (1986)] shows that

$$X(t) = \bigwedge_{l=1}^{\infty} \frac{Y_l}{f_t(U_l)}. \tag{2.4}$$

For the process $X(t)$ to be separable, we need continuity in probability which occurs if and only if the functions f_t are $\mathscr{L}_1$ continuous in t.

Equivalent processes can be obtained in the same manner as in the vector case. Let $W(t)$ be a min-stable process with spectral functions g_t. Then $\mathbf{W} =_d \mathbf{X}$ if and only if there exists a piston Γ such that $g_t = \Gamma f_t$ for all t.

The moving-minimum process is a min-stable analog to a Gaussian moving-average process:

$$X(t) = \bigwedge_{l=1}^{\infty} \frac{Y_l}{f(U_l - t)}.$$

Here we have a weighted minimum of the underlying Poisson process with weighting function f.

Distributional equivalence via pistons gives a simple characterization of stationarity for min-stable processes. Let the min-stable process $X(t)$, $t \in \mathscr{R}$, with spectral functions f_t be subordinate to a two-dimensional Poisson process $\mathbf{\Pi}$ in the upper half-plane. By definition, $X(\cdot)$ is stationary if and only if $X(\cdot) =_d X(\cdot + s)$ for each fixed s. This equivalence occurs if and only if the spectral functions f_t of $X(\cdot)$ are related to the spectral functions f_{t+s} of $X(\cdot + s)$ by a piston, that is, $f_{t+s} = \Gamma^s f_t$. The superscript s on Γ denotes the fact that a different piston is needed for each fixed s.

Stationary min-stable processes are therefore defined by *one* spectral function and a power group $\{\Gamma^t\}$ of pistons satisfying

$$\Gamma^{t+s} = \Gamma^t \Gamma^s = \Gamma^s \Gamma^t,$$

$$\Gamma^0 = \text{identity operator},$$

$$(\Gamma^t)^{-1} = \Gamma^{-t}.$$

3. Association, mixing and the moving-minimum processes.

Definition 3.1. The location of a min-stable random variable X with spectral function f is the point $(U_f^*, Y_f^*) \in \mathbf{\Pi}$ such that

$$X = \bigwedge_l \frac{Y_l}{f(U_l)} = \frac{Y_f^*}{f(U_f^*)}.$$

The location of X is the point of $\mathbf{\Pi}$ that defines X. The location of a min-stable random variable leads to a measure of association between two jointly min-stable random variables.

Suppose that X_f and X_g are min-stable random variables defined via the homogeneous Poisson process $\mathbf{\Pi}$ in the upper half-plane with spectral functions f and g, respectively. Define the *association* between X_f and X_g as

$$q(X_f, X_g) = P\big((U_f^*, Y_f^*) = (U_g^*, Y_g^*)\big).$$

The association is the probability that X_f and X_g share the same location. It is shown below that $q(X_f, X_g)$ can be defined in terms of integrals of the spectral functions f and g. The association is analogous to a squared correlation and has the following desirable properties [Weintraub (1987)]:

(3.1) $$0 \le q \le 1.$$

(3.2) $$q = 1 \quad \text{if and only if } X_g = cX_f \text{ for some constant } c > 0.$$

(3.3) $$q(aX_f, bX_g) = q(X_f, X_g) \quad \text{for all constants } a, b > 0.$$

(3.4) $$q(X_f, X_g) = 0 \quad \text{if and only if } X_f \text{ and } X_g \text{ are independent}.$$

(3.1) is obvious since q is a probability. (3.2) and (3.3) are properties of correlation for random variables, except that translation is not allowed. This is no great loss since the negative exponential distribution is not translation invariant. Note also that (3.4) is not true of general random variables but is true in the min-stable case, which mimics the property of the squared correlation in Gaussian random variables.

Lemma 3.2 gives $q(X_f, X_g)$ in terms of the spectral functions.

LEMMA 3.2. *The association between the min-stable random variables X_f and X_g, defined on the Poisson process in the upper half-plane, with spectra f and g, respectively, is*

(3.5) $$q(X_f, X_g) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{f(v)}{f(u)} \vee \frac{g(v)}{g(u)} \, dv \right]^{-1} du.$$

PROOF. Denote the conditional probability that the min-stable random variables X_f and X_g are defined by the point $(u, y) \in \mathscr{R} \times \mathscr{R}_+$, given there is a point of $\mathbf{\Pi}$ at (u, y) by $P_{f,g}(u, y)$. The probability that there is a point of $\mathbf{\Pi}$ at (u, y) is $du\,dy$ by the definition of the homogeneous Poisson process in the upper half-plane. Then,

(3.6) $$q(X_f, X_g) = \int_{u=-\infty}^{\infty} \int_{y=0}^{\infty} P_{f,g}(u, y) \, du \, dy.$$

Given that there is a point of $\mathbf{\Pi}$ at (u, y), it is the location of both X_f and X_g if and only if

$$\bigwedge_l \frac{Y_l}{f(U_l)} = \frac{y}{f(u)}$$

and

$$\bigwedge_l \frac{Y_l}{g(U_l)} = \frac{y}{g(u)},$$

which implies that for all l,

$$Y_l \ge \frac{y}{f(u)} f(U_l) \vee \frac{y}{g(u)} g(U_l),$$

so that the area below

$$y \left[\frac{f(v)}{f(u)} \vee \frac{g(v)}{g(u)} \right]$$

as a function of v is empty. $P_{f,g}(u,y)$ is the probability of the area being empty so that for all points $(u,y) \in \mathscr{R} \times \mathscr{R}_+$,

$$P_{f,g}(u,y) = \exp\left[-y\int_{v=-\infty}^{\infty} \frac{f(v)}{f(u)} \vee \frac{g(v)}{g(u)}\, dv\right].$$

Substituting in (3.6) for $P_{f,g}(u,y)$ and integrating over y gives the desired result. □

3.1. In this section the conditional distributions for bivariate min-stable random variables and the distributions for the horizontal coordinate of the location of a min-stable variable are shown. We first show that a min-stable random variable is independent of the horizontal coordinate of its location and that the marginal distribution of the horizontal coordinate takes on a nice form. The marginal distribution of the horizontal coordinate of the location is used in the sequel where mixing conditions for moving-minimum processes are discussed.

LEMMA 3.3. *Let X_f be a min-stable random variable with spectral function f. Also let U^* be the horizontal coordinate of the location of X_f. Then the joint density of U^* and X_f is given by*

$$e^{-x\|f\|} f(u)\, dx\, du, \tag{3.7}$$

which implies that U^* is independent of X_f.

PROOF. If $X_f \in [a, a+da)$, then the set of points C below the function $a \cdot f(u)$ contains no points of $\mathbf{\Pi}$ and there is a point of $\mathbf{\Pi}$ in the set $A = \{(u,y) | af(u) \le y \le (a+da) f(u)\}$, since $X_f \in [a, a+da)$ implies that $X_f \ge y$ for some $y \in [a, a+da)$. The nature of the Poisson process allows us to make da small enough so that we can ignore the probability of there being more than one point of $\mathbf{\Pi}$ in A. Now suppose that simultaneously with $X_f \in [a, a+da)$ that $U^* \in [b, b+db)$. Therefore the location must be in the set $B = A \cap \{(u,y) | b \le u \le b+db\}$. Notice that B and C are disjoint and that

$$\text{area}(B) = db\int_a^{a+da} f(u)\, du.$$

Letting $\mathbf{M}$ represent the Poisson random measure defined by $\mathbf{\Pi}$, we get

$$\begin{aligned} P\big(X_f \in [a, a+da), U^* \in [b, b+db)\big) &= P(\mathbf{M}(B) = 1, \mathbf{M}(C) = 0) \\ &= P(\mathbf{M}(B) = 1) P(\mathbf{M}(C) = 0) \\ &= \text{area}(B) \exp(-\text{area}(B)) e^{-a\|f\|} \\ &\propto e^{-a\|f\|} f(b)\, da\, db, \end{aligned} \tag{3.8}$$

as $da, db \to 0$. This last expression gives the joint density of X_f and U^* as shown in (3.7). If we integrate over b we get the marginal density for X_f

which is negative exponential with mean $\|f\|^{-1}$. If we integrate over a we get $f(b)\|f\|^{-1}\,db$, the marginal density for U^*. The product of these marginals is the joint density (3.8) and therefore by the factorization theorem the result follows. □

For any class of multivariate random variables, the conditional distributions are important objects of study, especially in terms of prediction. In the rest of this section the derivation of the distribution of X_1 given X_2 for a min-stable pair is established.

Lemma 3.4. *Suppose that X_i, $i = 1, 2$ are jointly min-stable random variables with spectral functions f_i. Then*

$$P(X_2 \geq x_2 | X_1 = x_1) = \exp\left(-\int [x_2 f_2(u) - x_1 f_1(u)]_+ \,du\right) \frac{1}{\|f_1\|} \int_A f_1(u)\,du,$$

where

$$A = \{u | x_1 f_1(u) > x_2 f_2(u)\},$$

and $[y]_+ = y$ if y is positive and equals 0 otherwise.

Proof. Notice that the limits on the integrals have been deliberately left ambiguous, since the proof is the same for all underlying spaces for u.

Taking the ratio of infinitessimals we get

$$P(X_2 \geq x_2 | X_1 = x_1) = \frac{P(X_2 \geq x_2,\, X_1 = x_1)}{P(X_1 = x_1)}. \tag{3.9}$$

For X_1 to be equal to x_1 we must have no points of Π strictly below $x_1 f_1(u)$ and one point of Π on the curve $x_1 f_1(u)$. Given that there is a point of Π on the curve, the probability of there being another on the curve is 0 so that we can ignore this possibility. For $X_2 \geq x_2$, the area below $x_2 f_2(u)$ must contain no points of Π. This implies that in order for the event in the numerator of (3.9), to be true, the point on the curve $x_1 f_1(u)$ must be at a point of A so that it will not be below $x_2 f_2(u)$. The conditional probability is

$$\frac{P(\text{Empty under } x_1 f_1(u) \vee x_2 f_2(u),\, \exists\, (U_l, Y_l) \in \Pi \text{ on } x_1 f_1(u) \ni U_l \in A)}{P(\text{Empty under } x_1 f_1(u),\, \exists\, (U_l, Y_l) \in \Pi \text{ on } x_1 f_1(u))}.$$

The event that the area strictly below a curve is empty and the event that a point of Π lies somewhere on the curve are independent since they are statements about disjoint sets. This shows that the conditional probability is the product of

$$\frac{P(\text{Empty under } x_1 f_1(u) \vee x_2 f_2(u))}{P(\text{Empty under } x_1 f_1(u))}$$

and

$$\frac{P(\exists\, (U_l, Y_l) \in \Pi \text{ on } x_1 f_1(u) \ni U_l \in A)}{P(\exists\, (U_l, Y_l) \in \Pi \text{ on } x_1 f_1(u))}.$$

The first of these probability ratios is

$$\frac{\exp(-\int x_1 f_1(u) \vee x_2 f_2(u)\,du)}{\exp(-\int x_1 f_1(u)\,du)} = \exp\left(-\int [x_2 f_2(u) - x_1 f_1(u)]_+\,du\right).$$

Let $B = \{(u, y) | y \in [x_1 f_1(u), (x_1 + \Delta) f_1(u))\}$. Then the second ratio is

$$\lim_{\Delta \to 0} \frac{P(\text{One point in } A \cap B)}{P(\text{One point in } B)} = \lim_{\Delta \to 0} \frac{\exp(-\Delta \int_A f_1(u)\,du) \Delta \int_A f_1(u)\,du}{\exp(-\Delta \int f_1(u)\,du) \Delta \int f_1(u)\,du}$$
$$= \frac{1}{\|f_1\|} \int_A f_1(u)\,du,$$

which gives the result. □

Suppose that X_1 and X_2 are jointly min-stable as above and that one can observe $Z = X_1 \wedge X_2$ and $I \equiv I(X_1 \le X_2)$. The following lemma shows that Z and I are independent and therefore that knowing which variable produces the minimum adds no new information for the prediction of Z or the estimation of its distribution. This is the standard Kaplan–Meier (1958) random censoring problem in a min-stable setting.

LEMMA 3.5. *Let X_1, X_2, Z and I be defined as above. Then Z and I are independent.*

PROOF. Since Z is min-stable with spectral function $f_1(u) \vee f_2(u)$, $P(Z \ge z) = \exp(-z\|f_1 \vee f_2\|)$. Let $A = \{u | f_1(u) \ge f_2(u)\}$. Applying Lemma 3.4 implies

$$P(I = 1) = P(X_1 \le X_2)$$
$$= \int_{x_1=0}^{\infty} P(X_2 \ge x_2 | X_1 = x_1)\,dP(\{X_1 \le x_1\})$$
$$= \int_{x_1=0}^{\infty} \frac{\int_{f_1 \ge f_2} f_1(u)\,du}{\|f_1\|} \exp\left(-x_1 \int [f_2(u) - f_1(u)]_+\,du\right)$$
$$\times \|f_1\| e^{-x_1 \|f_1\|}\,dx_1$$
$$= \left(\int_A f_1(u)\,du\right) \int_{x_1=0}^{\infty} \exp\left(-x_1 \int_{A^c} (f_2(u) - f_1(u))\,du\right.$$
$$\left. - x_1 \int f_1(u)\,du\right) dx_1$$
$$= \left(\int_A f_1(u)\,du\right) \int_{x_1=0}^{\infty} e^{-x_1 \|f_1 \vee f_2\|}\,dx_1$$
$$\times \frac{1}{\|f_1 \vee f_2\|} \int_A f_1(u)\,du,$$

which implies that

$$P(I=0)=\frac{1}{\|f_1\vee f_2\|}\int_{A^c} f_2(u)\,du.$$

Calculating the joint probability $P(Z\ge z,\ I=0)$ and showing that it factors proves the lemma:

$$\begin{aligned}P(Z\ge z,\ I=0)&=P(X_2\ge z,\ X_2\le X_1)\\&=\int_{v=z}^{\infty}P(X_1\ge v|X_2=v)\,dP(\{X_2\le v\}).\end{aligned}$$

Substituting the result for the conditional distribution from Lemma 3.4 yields

$$\int_{v=z}^{\infty}\frac{\int_{A^c}f_2(u)\,du}{\|f_2\|}\exp\left(-v\int[f_1(u)-f_2(u)]_+\,du\right)\|f_2\|e^{-v\|f_2\|}\,dv.$$

The integrals in the exponents combine so that

$$\begin{aligned}P(Z\ge z,\ I=0)&=\left(\int_{A^c}f_2(u)\,du\right)\int_{v=z}^{\infty}e^{-v\|f_1\vee f_2\|}\,dv\\&=\frac{1}{\|f_1\vee f_2\|}\left(\int_{A^c}f_2(u)\,du\right)e^{-z\|f_1\vee f_2\|}\\&=P(I=0)P(Z\ge z).\end{aligned}$$

□

3.2. The definition and properties of the association for min-stable random variables are a basis for building mixing conditions for min-stable processes. For some properties of moving-minima in discrete time, see Deheuvels (1983).

Three mixing criteria for min-stable processes based on the measure of association q of the previous section are

$$\begin{aligned}q_0(s)&\equiv q(X(t),X(t+s)),\\q_1(s)&\equiv\bigvee_g q\left(X(t),\ \bigwedge_{r\ge t+s}\frac{X(r)}{g(r)}\right),\\q_2(s)&\equiv\bigvee_{g,h} q\left(\bigwedge_{r\le t}\frac{X(r)}{h(r)},\ \bigwedge_{v\ge s+t}\frac{X(v)}{g(v)}\right),\end{aligned}$$

where g and h are positive integrable functions. The subscript i in q_i, $i=0,1,2$, represents the number of tails of the process which are being weighted before comparison. A min-stable process $X(t)$ is said to be *i-mixing* if

$$\lim_{s\to\infty}q_i(s)=0.$$

The first two of these criteria are applied to moving-minimum processes to show which spectral functions yield processes which are mixing.

If the spectral function f of $X(t)$, a moving-minimum process, has finite support, then $X(t)$ is defined on a strip in $\mathscr{R}\times\mathscr{R}_+$. It follows that $X(t)$ is

i-mixing for $i = 0, 1, 2$, since when

$$s > \sup\{u|f(u) > 0\} - \inf\{u|f(u) > 0\}$$

(the diameter of the support of f), $X(r)$ and $X(t+w)$ are independent for all $r \le t$ and $w \ge s$. That is, $X(r)$ and $X(t+w)$ are defined on nonintersecting portions of the Poisson process. Not only are these processes i-mixing, but for large enough s, the past and future of the processes are strictly independent. This is analogous to a well-known result for moving-average processes.

The next result is that *all* moving-minimum processes are 0-mixing.

THEOREM 3.6. *If $X(t)$ is a moving-minimum process, then it is* 0*-mixing.*

PROOF. Moving-minimum processes are stationary so that

$$q_0(s) = q(X(0), X(s)).$$

The spectral function for $X(0)$ is $f(u)$ and the spectral function for $X(s)$ is $f(u-s)$. Let $A_\delta = \{u|f(u) \ge \delta\}$. Then

$$\begin{aligned} q_0(s) &= \int_{u=-\infty}^{\infty}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)} \vee \frac{f(v-s)}{f(u-s)}\,dv\right]^{-1} du \\ &= \int_{A_\delta^c}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)} \vee \frac{f(v-s)}{f(u-s)}\,dv\right]^{-1} du \\ &\quad + \int_{A_\delta}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)} \vee \frac{f(v-s)}{f(u-s)}\,dv\right]^{-1} du \qquad (3.10) \\ &\le \int_{A_\delta^c}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)}\,dv\right]^{-1} du + \int_{A_\delta}\left[\int_{v=-\infty}^{\infty}\frac{f(v-s)}{f(u-s)}\,dv\right]^{-1} du \\ &= \frac{1}{\|f\|}\int_{A_\delta^c} f(u)\,du + \frac{1}{\|f\|}\int_{A_\delta} f(u-s)\,du. \end{aligned}$$

The integral in the first term above goes to zero as $\delta \to 0$ by the Lebesgue dominated convergence theorem. The second integral in (3.10) can be split into two pieces:

$$\int_{A_\delta \cap [m,\infty)} f(u-s)\,du + \int_{A_\delta \cap (-\infty, m]} f(u-s)\,du. \qquad (3.11)$$

Since f is integrable, the Lebesgue measure $\nu(A_\delta) < \infty$ and

$$\lim_{m\to\infty} \nu(A_\delta \cap [m,\infty)) = 0,$$

which further entails that the first integral in (3.11) goes to zero as $m \to \infty$. As for the second integral in (3.11),

$$\int_{A_\delta \cap (-\infty, m]} f(u-s)\,du \le \int_{u=-\infty}^{m} f(u-s)\,du$$
$$= \int_{u=-\infty}^{m-s} f(u)\,du,$$

which goes to zero as $s \to \infty$. By taking limits in the correct order we have an upper bound for $q_0(s)$ which goes to zero and the result follows. □

DEFINITION 3.7. If $U(s)$, the horizontal coordinate of the location of $X(s)$, goes to $\pm\infty$ w.p.1 as $s \to \pm\infty$, we say that the location is well-behaved.

The following two theorems establish the connection between the behavior of the location $U(s)$ and the spectral function $f(u)$ of the moving-minimum process $X(s)$. The two theorems are used to prove the 1-mixing result (Theorem 3.10) for moving minimum processes. Theorem 3.8 gives a necessary and sufficient random condition for $U(s)$ to be well behaved. The second shows some deterministic conditions that imply and are implied by $U(s)$ being well behaved. Some new notation will have to be introduced in order to obtain the results. Let

$$L := \{l | U_l \le 0\},$$
$$R := \{l | U_l > 0\}, \tag{3.12}$$

indicate the index sets of points of Π to the left and right of 0, respectively. Define

$$X_L(t) := \bigwedge_{l \in L} \frac{Y_l}{f(U_l - t)}, \qquad X_R(t) := \bigwedge_{l \in R} \frac{Y_l}{f(U_l - t)},$$

as the left and right parts of $X(t)$. Notice that

$$X(t) = X_L(t) \wedge X_R(t),$$

and that

$$U(t) < 0 \Leftrightarrow X(t) = X_L(t) \Leftrightarrow X_L(t) < X_R(t).$$

THEOREM 3.8. *With the notation introduced above,*

$$\lim_{t \to -\infty} U(t) = -\infty \Leftrightarrow \limsup_{t \to \infty} f(t) X_L(-t) = 0,$$
$$\lim_{t \to \infty} U(t) = \infty \Leftrightarrow \limsup_{t \to -\infty} f(t) X_R(-t) = 0.$$

PROOF. Suppose there is a point of Π at $(t, y(t))$ with $t > 0$. This point generates $X(s)$ if and only if

$$\frac{y(t)}{f(t-s)} \le X_L(s).$$

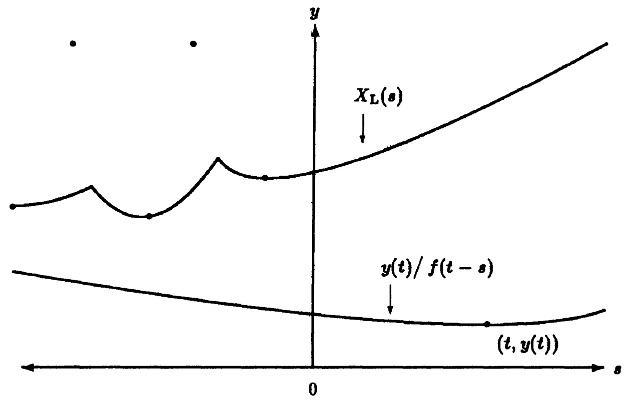

FIG. 1. *$X_L(s)$ and a point in R_0 for which $X_R(s) < X_L(s)$.*

See Figure 1. In this case $U(s)$ is positive. This happens infinitely often as $s \to -\infty$, if and only if

$$\forall\, S < 0, \quad \exists\, s \le S \ni \frac{y(t)}{f(t-s)} \le X_L(s) \Leftrightarrow y(t) \le \limsup_{s \to -\infty} f(t-s) X_L(s).$$

For this to be a probability 0 event we must have for almost all t,

$$\limsup_{s \to -\infty} f(t-s) X_L(s) = 0.$$

This is equivalent to

$$\limsup_{s \to \infty} f(s) X_L(-s) = 0. \tag{3.13}$$

We have shown that $P(U(s) > 0 \text{ i.o. as } s \to \infty) = 0$ if and only if (3.13) holds. That is, $\limsup_{s \to \infty} U(s) \le 0$ if and only if (3.13) holds. Splitting $\mathscr{R} \times \mathscr{R}_+$ around $m \in \mathscr{R}$ (instead of 0) and repeating the above argument shows that $\forall\, m \in \mathscr{R}$, $\limsup_{s \to \infty} U(s) \le m$ if and only if (3.13) holds. This is equivalent to the first double implication in the statement of the theorem.

The second implication follows by an obvious rearrangement of the above argument. □

The integrability of f does not imply that f has zero limit. For example, let f be the *boxcar* function

$$f(u) := \sum_{i=0}^{\infty} I(i \le u \le i + 2^{-i-1}), \tag{3.14}$$

which has integral 1 but does not decrease to 0, so that

$$\limsup_{t \to -\infty} f(-t) X_L(t) = \infty,$$

since $X_L(t) \ge X(t)$ and $\limsup_{t \to -\infty} X(t) = \infty$. By Theorem 3.8, $U(-t)$ is not well behaved. The theorem below shows that $\lim_{u\to\infty} f(u) = 0$ is a necessary condition for $U(-t)$ to be well behaved.

THEOREM 3.9. *Consider the following three statements*:

(i) $f(s)$ *is eventually monotone decreasing to* 0. *That is*, $\exists\ S > 0 \ni f(s+t) \le f(s)\ \forall\ t > 0,\ s \ge S$.

(ii) $U(s)$ *is well behaved as* $s \to \infty$.

(iii) $\lim_{s\to\infty} f(s) = 0$.

Then

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

PROOF. Suppose f is eventually monotone decreasing. This implies that for $u < 0$ and s large enough, $f(u+s) > f(s)$. In other words, $f(s)/f(u+s)$ is eventually bounded by 1. This implies that

$$\limsup_{u\to-\infty} \limsup_{s\to\infty} \frac{f(s)}{f(u+s)} < \infty.$$

Then there exist $C > 0$, $u_0 < 0$ such that for all $u < u_0$,

$$\limsup_{s\to\infty} \frac{f(s)}{f(u+s)} < C.$$

Since $\limsup a(t) = (\liminf 1/a(t))^{-1}$, the following is true for all $u < u_0$:

$$\liminf_{s\to\infty} \frac{f(u+s)}{f(s)} \ge \frac{1}{C}.$$

This implies that the integral

$$\int_{u=-\infty}^{0} \left[\liminf_{s\to\infty} \frac{f(u+s)}{f(s)} \right] du \tag{3.15}$$

is infinite.

The condition in Theorem 3.8 is

$$\limsup_{s\to\infty} f(s) X_L(-s) = 0,$$

and the left-hand side of the previous equation is equal to

$$\limsup_{s\to\infty} \bigwedge_{l\in L} Y_l \frac{f(s)}{f(U_l+s)} \le Y_l \limsup_{s\to\infty} \frac{f(s)}{f(U_l+s)}, \qquad \forall\ l \in L.$$

This implies

$$\limsup_{s\to\infty} f(-s) X_L(-s) \le \bigwedge_{l\in L} Y_l \limsup_{s\to\infty} \frac{f(s)}{f(U_l+s)},$$

and the right-hand side of the above equation is 0 w.p.1 if and only if its expectation is 0. The expectation is 0 if and only if the reciprocal of its expectation is infinite, that is, if and only if

$$\infty = \int_{u=-\infty}^{0} \left(\limsup_{s \to \infty} \frac{f(s)}{f(u+s)} \right)^{-1} du$$
$$= \int_{u=-\infty}^{0} \left[\liminf_{s \to \infty} \frac{f(u+s)}{f(s)} \right] du.$$

Using the fact that the integral 3.15 is infinite, we have that (i) implies (ii).

By the definition of $X_R(s)$, the spectral function of $\bigwedge_{s \geq t} X_R(-s)$ is

$$I(u \geq 0) \bigvee_{s \geq t} f(u+s).$$

Therefore

$$E\left(\bigwedge_{s \geq t} X_R(-s) \right) = \left[\int_{u=0}^{\infty} \left(\bigvee_{s \geq t} f(u+s) \right) du \right]^{-1} \tag{3.16}$$

$$= \left[\int_{u=t}^{\infty} \left(\bigvee_{s \geq u} f(u+s) \right) du \right]^{-1}. \tag{3.17}$$

If the integral in (3.17) is infinite for some value of t, it is infinite for all values of t. This implies that the expectation in (3.16) is 0 for all values of t. Since $\bigwedge_{s \geq t} X_R(-s)$ is a nonnegative random variable, its expectation is 0 if and only if it is 0 with probability 1. If this were the case, we would have

$$\liminf_{s \to \infty} X_R(-s) = 0,$$

which would imply that $U(-s)$ is to the right of 0 infinitely often since as $s \to -\infty$, we would have

$$X_R(-s) \leq X_L(-s)$$

infinitely often. Therefore $U(-s)$ would not be well behaved. We therefore have that (ii) implies

$$\int_{u=0}^{\infty} \left(\bigvee_{s \geq u} f(s) \right) du < \infty. \tag{3.18}$$

Since $\bigvee_{s \geq u} f(s) \geq 0$ is monotone decreasing as a function of u, it must converge to a nonnegative constant. If it did not converge to 0, then the integral in (3.18) would diverge. Also notice that

$$\lim_{s \to \infty} \bigvee_{s \geq u} f(s) \geq \lim_{s \to \infty} f(s),$$

so we have shown that (3.18) implies $\bigvee_{s \geq u} f(s)$ decreases to 0 monotonically as $u \to \infty$ which implies (iii). □

A Dobrushin-like condition for $X(t)$ to be well behaved locally is discussed in de Haan and Pickands (1986). In their paper they show that the realizations of $X(t)$ are arbitrarily close to 0 or bounded away from 0 in every finite interval according as

$$\int_{u=-\infty}^{\infty}\left(\bigvee_{0<s<\delta} f(u+s)\right) du = \text{or} < \infty, \tag{3.19}$$

where $0 < \delta < \infty$. Notice that the integral is finite or infinite for all δ. The proof of one implication is as follows: For $a < b$,

$$\begin{aligned}\bigwedge_{a\le t\le b} X(t) &= \bigwedge_{a\le t\le b}\bigwedge_{l} \frac{Y_l}{f(U_l - t)}\\ &= \bigwedge_{l} \frac{Y_l}{\bigvee_{a\le t\le b} f(U_l - t)}.\end{aligned}$$

If the integral (3.19) is infinite,

$$E\bigwedge_{a\le t\le b} X(t) = \left[\int_{u=-\infty}^{\infty}\left(\bigvee_{a\le t\le b} f(u-t)\right) du\right]^{-1} = 0.$$

Since $X(t)$ is nonnegative, it must be true that $\bigwedge_{a\le t\le b} X(t) = 0$ almost surely. This implies that $X(t)$ comes arbitrarily close to 0 on every finite interval. If the integral in (3.19) is infinite, then so is the integral in (3.18), so that bad local behavior implies bad global behavior of $U(t)$.

The boxcar function of (3.14) is an example of a spectral function which yields a moving minimum process that is badly behaved globally and locally. Substituting the boxcar function into (3.19) gives an infinite integral. This is because for all δ, the integrand in (3.19) is

$$\bigvee_{0\le t\le\delta}\sum_{i=0}^{\infty} I(i \le u - t \le i + 2^{-i-1}),$$

a smeared version of the boxcar function.

The next theorem is a 1-mixing result for moving-minimum processes. A moving-minimum process is 1-mixing if and only if the spectral function f has limit 0. Assuming that f has limit 0, the proof that $X(t)$ is 1-mixing follows in similar fashion to the 0-mixing proof of Theorem 3.6. Proving the result in the other direction is not so simple and requires result (iii) of Theorem 3.9 regarding the location of a moving-minimum process. If $U(s)$ is not well behaved, then the association between the present and the distant future is not going to go to 0. This is the basic idea behind the theorem that follows and the motivation for the results regarding the behavior of $U(s)$.

THEOREM 3.10. *Suppose $X(t)$ is a moving-minimum process with spectral function $f(u)$. Then*

$$\lim_{s\to\infty} q_1(s) = 0 \Leftrightarrow \lim_{s\to\infty} f(-s) = 0.$$

PROOF. Assume that $\lim_{s\to\infty} f(s) = 0$. Notice that

$$\bigwedge_{r\geq s}\frac{X(r)}{g(r)} = \bigwedge_{r\geq s}\bigwedge_{l}\frac{Y_l}{g(r)f(U_l - r)} = \bigwedge_{l}\frac{Y_l}{\bigvee_{r\geq s} g(r)f(U_l - r)},$$

and since the moving-minimum process $X(s)$ is stationary,

$$q\left(X(t),\ \bigwedge_{r\geq s+t}\frac{X(r)}{g(r)}\right) = q\left(X(0),\ \bigwedge_{r\geq s}\frac{X(r)}{g(r)}\right).$$

This implies that

$$\begin{aligned}
q_1(s) &= \bigvee_g \int_{u=-\infty}^{\infty}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)}\vee\frac{\bigvee_{r_1\geq s} g(r_1)f(v-r_1)}{\bigvee_{r_2\geq s} g(r_2)f(u-r_2)}\,dv\right]^{-1}du\\
&= \bigvee_g \int_{|u|\geq m}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)}\vee\frac{\bigvee_{r_1\geq s} g(r_1)f(v-r_1)}{\bigvee_{r_2\geq s} g(r_2)f(u-r_2)}\,dv\right]^{-1}du\\
&\quad+\bigvee_g \int_{|u|<m}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)}\vee\frac{\bigvee_{r_1\geq s} g(r_1)f(v-r_1)}{\bigvee_{r_2\geq s} g(r_2)f(u-r_2)}\,dv\right]^{-1}du\\
&\leq \bigvee_g \int_{|u|\geq m}\left[\int_{v=-\infty}^{\infty}\frac{f(v)}{f(u)}\,dv\right]^{-1}du\\
&\quad+\bigvee_g \int_{|u|<m}\left[\int_{v=-\infty}^{\infty}\frac{\bigvee_{r_1\geq s} g(r_1)f(v-r_1)}{\bigvee_{r_2\geq s} g(r_2)f(u-r_2)}\,dv\right]^{-1}du\\
&= \frac{1}{\|f\|}\int_{|u|\geq m} f(u)\,du + \bigvee_g \int_{|u|\leq m}\frac{\bigvee_{r_2\geq s} g(r_2)f(u-r_2)}{\int_{-\infty}^{\infty}\left(\bigvee_{r_1\geq s} g(r_1)f(v-r_1)\right)dv}\,du,
\end{aligned}$$

where $m > 0$ is chosen arbitrarily. The integrand in the second term is equal to

$$\begin{aligned}
\bigvee_{r_2\geq s}\frac{g(r_2)f(u-r_2)}{\int_{-\infty}^{\infty}\left(\bigvee_{r_1\geq s} g(r_1)f(v-r_1)\right)dv} &\leq \bigvee_{r_2\geq s}\frac{g(r_2)f(u-r_2)}{\int_{-\infty}^{\infty} g(r_2)f(v-r_2)\,dv}\\
&= \bigvee_{r_2\geq s}\frac{g(r_2)f(u-r_2)}{g(r_2)\|f\|}\\
&= \bigvee_{r_2\geq s}\frac{f(u-r_2)}{\|f\|}.
\end{aligned}$$

When $|u| \leq m$ and s is large, the last expression is less than $\varepsilon_s/\|f\|$, where

$\varepsilon_s \to 0$ as $s \to \infty$. These results lead to an upper bound:

$$q_1(s) \le \frac{1}{\|f\|}\left[\int_{|u|\ge m} f(u)\,du + \bigvee_g \int_{|u|\le m} \varepsilon_s\,du\right]$$
$$= \frac{1}{\|f\|}\left[\int_{|u|\ge m} f(u)\,du + 2m\varepsilon_s\right].$$

Letting $s, m \to \infty$ (in that order) shows that

$$\lim_{s\to\infty} f(-s) = 0 \Rightarrow \lim_{s\to\infty} q_1(s) = 0.$$

To see the implication in the other direction, notice that stationarity of $X(s)$ also implies

$$q\left(X(t),\ \bigwedge_{r\ge s+t} \frac{X(r)}{g(r)}\right) = q\left(X(-s),\ \bigwedge_{r\ge 0} \frac{X(r)}{g(r)}\right), \tag{3.20}$$

and $q_1(s)$ is the maximum value of (3.20) over all g. If a weighting function g can be found so that the association (3.20) is larger than a positive constant δ infinitely often, then $q_1(s)$ will not go to 0. Suppose $U(-s)$, the horizontal coordinate of the location of $X(-s)$, is greater than 0 (or greater than some $N \ge 0$) infinitely often as $s \to \infty$. Let g be a weighting function with

$$\bigvee_{r\ge 0} g(r) f(u-r) > 0,$$

for almost all $u \ge 0$. Then the probability that the weighted minimum

$$\bigvee_{r\ge s} \frac{X(r)}{g(r)}$$

has the same location as $X(-s)$ is positive. Suppose that $\lim_{s\to\infty} f(s) \ne 0$, then by Theorem 3.9 we have $\lim_{s\to\infty} U(-s) \ne -\infty$. This completes the proof. □

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735 KAPPOCK ST., 12C
BRONX, NEW YORK 10463