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LARGE DEVIATIONS FOR DIFFUSION PROCESSES WITH HOMOGENIZATION AND APPLICATIONS[1]

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We consider a family of periodic diffusion processes in R^n with homogenization and a small parameter multiplying the diffusion coefficient. We establish a large deviations principle and as an application we derive an iterated logarithm law for periodic diffusions.

1. Introduction. Let σ be a periodic matrix field on $\mathbb{R}^d$ and h, g: $\mathbb{R}^+ \to \mathbb{R}^+$ functions such that $\lim_{\alpha\to\infty} h(\alpha) = \lim_{\alpha\to\infty} g(\alpha) = +\infty$. It is well known ([4], Section 3.4) that if $\tilde{y}^\alpha$ is the solution of the stochastic differential equation (SDE),

$$d\tilde{y}_t^\alpha = \sigma(h(\alpha)\tilde{y}_t^\alpha)\,dw_t, \qquad \tilde{y}_0^\alpha = x,$$

then $\tilde{y}^\alpha$ converges in law as $\alpha \to \infty$ to the Gaussian process z which is the solution of

$$dz_t = \sqrt{q}\,dw_t, \qquad z_0 = x,$$

where q is a suitable (constant) matrix. Now let y^α be the solution of

$$dy_t^\alpha = \frac{1}{g(\alpha)}\sigma(h(\alpha)y_t^\alpha)\,dw_t, \qquad y_0^\alpha = x.$$

In this paper we study the large deviations for y^α and prove that if $\lim_{\alpha\to\infty}(h(\alpha)/g(\alpha)^2) = +\infty$, then

$$\limsup_{\alpha\to\infty} \frac{1}{g(\alpha)^2} \log P\{y^\alpha \in F\} \le -\Lambda_x(F),$$

$$\liminf_{\alpha\to\infty} \frac{1}{g(\alpha)^2} \log P\{y^\alpha \in G\} \ge -\Lambda_x(G)$$

for every F and G, respectively, closed and open sets of paths, where

$$\Lambda_x(A) = \inf_{\gamma\in A} L_x(\gamma) \tag{1.1}$$

and

$$L_x(\gamma) = \tfrac{1}{2}\int_0^1 \langle q^{-1}\gamma_s', \gamma_s'\rangle\,ds \tag{1.2}$$

if γ is absolutely continuous and $\gamma(0) = x$, $L_x(\gamma) = +\infty$ otherwise.

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In the last section we give the following application of the above estimates (Strassen law for periodic diffusions). Let z be the diffusion process which is the solution of

$$dz_t = \sigma(z_t)\, dw_t,$$

where σ is periodic as above; for every $\alpha \geq 0$, let ζ_α be the r.v. taking values in $\mathscr{C} = \mathscr{C}([0,1], \mathbb{R}^d)$ defined by

$$\zeta_\alpha(\omega) = \left(t \to \frac{z_{\alpha t}}{\sqrt{\alpha \log\log \alpha}}\right). \tag{1.3}$$

Then the limit set of ζ_α as $\alpha \to \infty$ is given by

$$\mathscr{K} = \{\gamma \in \mathscr{C}; L_0(\gamma) \leq 1\}.$$

The idea is that ζ^α is a solution of

$$d\zeta_t^\alpha = \frac{1}{\sqrt{\log\log \alpha}} \sigma\left(\sqrt{\alpha \log\log \alpha}\, \zeta_t^\alpha\right) dw_t,$$

so that the previous large deviations estimates can be applied to ζ^α.

2. The main statements. Throughout this paper σ will denote a $d \times d$ matrix-valued field on $\mathbb{R}^d$ and we shall suppose:

ASSUMPTION A. (i) σ is Lipschitz continuous.

(ii) $\langle \sigma(x)\xi, \xi\rangle \geq \lambda_0 |\xi|^2$ for every $x, \xi \in \mathbb{R}^d$, where λ_0 is some positive constant.

(iii) σ is periodic in each of its coordinates.

If we denote by V the lattice of the periods of σ, then σ is well defined on the torus $T = \mathbb{R}^d/V$. Let us denote by $\pi: \mathbb{R}^d \to T$ the canonical projection; if z is a solution of the SDE on $\mathbb{R}^d$,

$$dz_t = \sigma(z_t)\, dw_t, \qquad z_0 = x,$$

then it is easy to check that $\bar{z}_t = \pi(z_t)$ is a solution of

$$d\bar{z}_t = \sigma(\bar{z}_t)\, dw_t, \qquad \bar{z}_0 = \pi(x) = \bar{x} \tag{2.1}$$

on T. Under the previous assumptions, $\bar{z}$ has a unique invariant probability m on T such that

$$\lim_{t\to\infty} E[f(\bar{z}_t)] = \int_T f\, dm$$

for every continuous function $f: T \to \mathbb{R}$ and for every starting point $\bar{x} \in T$. Of course this implies that also

$$\lim_{t\to\infty} E[f(z_t)] = \int_T f\, dm$$

for every continuous function $f: \mathbb{R}^d \to \mathbb{R}$ which is periodic and has the same periods as σ and for every starting point $x \in \mathbb{R}^d$.

Now let $h, g: \mathbb{R}^+ \to \mathbb{R}^+$ be functions such that

$$\lim_{\alpha\to\infty} h(\alpha) = \lim_{\alpha\to\infty} g(\alpha) = +\infty.$$

Then it is well known that the solution of

$$d\tilde{y}_t^\alpha = \sigma(h(\alpha)\tilde{y}_t^\alpha)\, dw_t, \qquad \tilde{y}_0^\alpha = x, \tag{2.2}$$

converges in law to the solution of

$$dz_t = \sqrt{q}\, dw_t, \qquad y_0 = x,$$

where q is the matrix defined by

$$q = \int_T a(x) m(dx) \tag{2.3}$$

and $a = \sigma\sigma^*$ [beware: the matrix q in (2.3) is not the same as in [4], (4.7), page 485; it differs by a factor $\sqrt{2}$]. It is clear that q is invertible and that

$$\langle q^{-1}\xi, \xi\rangle \geq \frac{1}{\|\sigma\|_\infty^2}|\xi|^2. \tag{2.4}$$

Let us consider now the family of SDE's

$$dy_t^\alpha = \frac{1}{g(\alpha)}\sigma(h(\alpha)y_t^\alpha)\, dw_t, \qquad y_0^\alpha = x, \tag{2.5}$$

and denote by $\mathscr{C} = \mathscr{C}([0,1], \mathbb{R}^d)$ the space of all continuous paths $\gamma: [0,1] \to \mathbb{R}^d$ endowed with the uniform norm $\| \ \|_\infty$. $\mathscr{C}_x$ will denote the set of all paths such that $\gamma(0) = x$; for every $t > 0$, X_t will be the mapping $X_t: \mathscr{C} \to \mathbb{R}^d$ defined by $X_t(\gamma) = \gamma(t)$.

Let $P^{\alpha,x}$ be the law of the solution y^α of (2.5). $P^{\alpha,x}$ is a probability law on $\mathscr{C}$ which lives in $\mathscr{C}_x$; $E^{\alpha,x}$ will denote the expectation with respect to $P^{\alpha,x}$.

Throughout this paper, w will be a Brownian motion (not always the same) on a suitable probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$; E will be the expectation with respect to P. For every Borel subset $A \subset \mathscr{C}$, let $\Lambda_x(A)$ be the set function defined in (1.1). Our main result is the following.

THEOREM 2.1. *If σ satisfies Assumption A and $\lim_{\alpha\to\infty}(h(\alpha)/g(\alpha)^2) = \infty$, then*

$$\limsup_{\alpha\to\infty} \frac{1}{g(\alpha)^2} \log P^{\alpha,x}(F) \leq -\Lambda_x(F),$$

$$\limsup_{\alpha\to\infty} \frac{1}{g(\alpha)^2} \log P^{\alpha,x}(G) \geq -\Lambda_x(G)$$

for every closed $F \subset \mathscr{C}$ and for every open $G \subset \mathscr{C}$.

In order to prove Theorem 2.1 we make use of the following general large deviations result (Theorem 1.1 of [3]), which is an infinite dimensional extension of a theorem of Gärtner [8].

Let $\{\mu_\alpha\}_{\alpha>0}$ be a family of probability laws on the topological vector space X and define for $\xi \in X'$ their Laplace transforms

$$\hat{\mu}_\alpha(\xi) = \int_X \exp\langle \xi, x\rangle \mu_\alpha(dx).$$

ASSUMPTION B. The family $\{\mu_\alpha\}_{\alpha>0}$ is said to satisfy this assumption if there exists a function $\lambda\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(i) $\lim_{\alpha\to\infty} \lambda(\alpha) = +\infty$.

(ii) $\lim_{\alpha\to\infty}(1/\lambda(\alpha))\log \hat{\mu}_\alpha(\lambda(\alpha)\xi) = H(\xi)$, where $H\colon X' \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and finite in a neighborhood of the origin.

(iii) The Legendre transform L of H, defined by

$$L(x) = \sup_{\xi\in X'} (\langle \xi, x\rangle - H(\xi)),$$

is strictly convex at each point $x \in X$ such that $L(x) < +\infty$.

(iv) For every $R > 0$, there exists a compact set $K_R \subset X$ such that

$$\limsup_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha(K_R^C) \le -R.$$

THEOREM 2.2 (See [2]). *Under Assumption B, for every Borel subset $A \subset X$,*

$$-\Lambda(\mathring{A}) \le \liminf_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha(A) \le \limsup_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha(A) \le -\Lambda(\bar{A}),$$

where $\Lambda(A) := \inf_{x\in A} L(x)$.

Theorem 2.1 will follow from Theorem 2.2 applied to $\mu_\alpha = P^{\alpha,x}$, $X = \mathscr{C}$, $\lambda(\alpha) = g(\alpha)^2$. In this case the topological dual X' coincides with the space $\mathscr{C}'$ of all $\nu = (\nu_1, \dots, \nu_d)$, each ν_i being a signed measure on $([0,1], \mathscr{B}([0,1]))$ with finite variation.

In the next section we check that Assumption B is satisfied and compute H and L. The main points to look at are (ii) and (iv), (i) being obvious and the strict convexity of L being immediate once L is explicitly given.

It might be interesting to compare this approach with previous large deviation results for the occupation time of Freidlin and Gärtner (see, for instance, [7]).

3. Proof of the main result. We start by proving (iv) of Assumption B. Let $c = \{c_k\}_k$ be a sequence of positive real numbers decreasing to 0 and $K \subset \mathbb{R}^d$; let us define

$$A_{c,k} = \left\{\gamma \in \mathscr{C}, w_\gamma(c_k) \le \frac{1}{k}, \gamma(0) \in K\right\},$$

$$A = \bigcap_{k=N}^{+\infty} A_{c,k}$$

for some positive integer N, w_γ being the modulus of continuity of γ defined by

$$w_\gamma(s) = \sup_{\substack{0 \le t_1 \le t_2 \le 1 \\ t_2 - t_1 \le s}} |\gamma(t_1) - \gamma(t_2)|.$$

Then, by the Ascoli–Arzelà theorem, A is a compact subset of $\mathscr{C}$. We shall now prove that for every $R > 0$ there exist $c = \{c_k\}_k$ and N such that

$$\limsup_{\alpha \to \infty} \frac{1}{g(\alpha)^2} \log P^{\alpha, x}(A^C) \le -R. \tag{3.1}$$

Indeed, $P^{\alpha,x}(A^C) \le \Sigma_{k=N}^{\infty} P^{\alpha,x}(A_k^C)$ and if we set

$$B_t^k = \left\{ \gamma \in \mathscr{C};\ \sup_{t \le s \le t + c_k} |\gamma(t) - \gamma(s)| > \frac{1}{3k} \right\},$$

then by the triangle inequality

$$A_k^C \subset \bigcup_{i \le c_k^{-1}} B_{ic_k}^k.$$

By the exponential inequality (see, e.g., [9], Theorem I.18),

$$P^{\alpha,x}(B_0^k) = P\left\{ \sup_{0 \le s \le c_k} |y_s^\alpha - y_0^\alpha| > \frac{1}{3k} \right\}$$
$$\le 2d \exp\left[-\frac{1}{18\,dM} \frac{g(\alpha)^2}{k^2 c_k} \right],$$

where $M = \|\sigma\|_\infty^2$; also by the Markov property,

$$P^{\alpha,x}\left(B_{ic_k}^k\right) = E^{\alpha,x}\left[P^{\alpha, X(ic_k)}\left(B_0^k\right)\right] \le 2d \exp\left[-\frac{1}{18\,dM} \frac{g(\alpha)^2}{k^2 c_k} \right].$$

Thus

$$P^{\alpha,x}\left(A_k^C\right) \le \frac{2d}{c_k} \exp\left[-\frac{1}{18\,dM} \frac{g(\alpha)^2}{k^2 c_k} \right].$$

Choosing $c_k = 1/k^3$,

$$P^{\alpha,x}\left(A_k^C\right) \le 2dk^3 \exp\left[-\frac{1}{18\,dM} k g(\alpha)^2 \right]$$

and for k larger than some N,

$$P^{\alpha,x}\left(A_k^C\right) \le \frac{1}{k^2} \exp\left[-\frac{1}{36\,dM} k g(\alpha)^2 \right]$$

and for N possibly larger,

$$P^{\alpha,x}\left(A_k^C\right) \le \frac{1}{k^2} \exp\left[-R g(\alpha)^2 \right]$$

so that

$$P^{\alpha,x}(A^C) \le \exp\left[-Rg(\alpha)^2\right] \sum_{k=N}^{\infty} \frac{1}{k^2} \le \text{const.} \exp\left[-Rg(\alpha)^2\right]$$

and (3.1) holds. This proves (iv) of Assumption B already. We wish, however, to point out that a straightforward extension of the previous arguments gives the following more precise result.

PROPOSITION 3.1. *Let x^ε be the solution of the SDE on $\mathbb{R}^d$,*

$$dx_t^\varepsilon = b(x_t^\varepsilon)\, dt + \varepsilon\sigma(x_t^\varepsilon)\, dw_t,$$

and suppose b and σ to be locally Lipschitz continuous and bounded in the uniform norm by a constant M. Let us denote by $P^{\varepsilon,x}$ the law of x^ε starting at $x \in \mathbb{R}^d$. Then for every compact set $K \subset \mathbb{R}^d$ and $R > 0$, there exists a universal compact set $\mathscr{K} = \mathscr{K}_{R,C} \subset \mathscr{C}$ (depending on M but not on b and σ) such that

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log P^{\varepsilon,x}(\mathscr{K}^C) \le -R$$

for every $x \in K$.

We turn now to the proof of (ii) of Assumption B, that is, to the computation of the limit

$$\lim_{\alpha\to\infty} \frac{1}{g(\alpha)^2} \log E^{\alpha,x}\left[\exp\left(g(\alpha)^2 \int_0^1 \langle X_s, d\nu(s)\rangle\right)\right]$$
$$= \lim_{\alpha\to\infty} \frac{1}{g(\alpha)^2} \log E\left[\exp\left(g(\alpha)^2 \int_0^1 \langle y_s^\alpha, d\nu(s)\rangle\right)\right].$$

We begin with some preliminary results.

LEMMA 3.2. *Let z be the solution of the SDE,*

$$dz_t = \sigma(z_t)\, dw_t, \qquad z_0 = x, \tag{3.2}$$

on the torus T and P^x its law on $\mathscr{C}(\mathbb{R}^+, T)$ and $m \in \mathscr{M}_1(T)$ its invariant probability. Then for every continuous function $f\colon T \to \mathbb{R}$ and $\eta > 0$, there exists $\delta > 0$ such that

$$\sup_{x\in T} P^x\left\{\left|\frac{1}{t}\int_0^t f(z_s)\, ds - \int_T f\, dm\right| \ge \eta\right\} \le e^{-\delta t}.$$

PROOF. Let $L_t = (1/t)\int_0^t \delta_{z_s}\, ds \in \mathscr{M}_1(T)$ be the occupation time of z. By the ergodic theorem, $L_t \to m$, P^x a.s. for every $x \in T$ and by the Donsker–Varadhan large deviations estimates ([5] or [10]), for every closed set $F \subset \mathscr{M}_1(T)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in T} P^x\{L_t \in F\} \le -\Lambda(F),$$

where $\Lambda(F) = \inf_{\nu \in F} I(\nu)$, the functional I being defined in [5], (1.12). The only properties that we need to know are that I is lower semicontinuous, $I(\nu) \geq 0$ and $I(\nu) = 0$ only if $\nu = m$ (see [5], Lemma 2.5 and the corollary to Lemma 4.2). This implies that $\Lambda(F) = 2\delta > 0$ for every closed set $F \subset \mathscr{M}_1(T)$ not containing m. In particular, this is true for

$$F = \left\{\nu; \left|\int_T f\,d\nu - \int_T f\,dm\right| \geq \eta\right\},$$

which ends the proof. □

The same argument and the Markov property give the following.

COROLLARY 3.3. *With the notation of Lemma* 3.2, *there exists* $\delta > 0$ *such that for every* $1 \geq \beta \geq \alpha \geq 0$,

$$\sup_{x \in T} P^x\left\{\left|\frac{1}{t}\int_{\alpha t}^{\beta t} f(z_s)\,ds - (\beta - \alpha)\int_T f\,dm\right| > \eta\right\} \leq \exp[-(\beta - \alpha)\,\delta t].$$

Let us remark now that if ν is an element in the topological dual $\mathscr{C}'$ of $\mathscr{C}$, that is, $\nu = (\nu_1, \dots, \nu_d)$ where the ν_i's are signed measures of finite variation on $[0,1]$ and we set $\nu_s = (\nu_1(]s,1]), \dots, \nu_d(]s,1]))$, then

$$\begin{aligned}
\int_0^1 \langle y_s^\alpha, \nu(ds)\rangle &= \int_0^1 \left\langle x + \frac{1}{g(\alpha)}\int_0^s \sigma(h(\alpha)y_u^\alpha)\,dw_u, \nu(ds)\right\rangle \\
&= \langle x, \nu([0,1])\rangle + \frac{1}{g(\alpha)}\int_0^1 \left\langle \sigma(h(\alpha)y_s^\alpha)\,dw_s, \int_s^1 \nu(du)\right\rangle \qquad (3.3)\\
&= \langle x, \nu([0,1])\rangle + \frac{1}{g(\alpha)}\int_0^1 \langle \nu_s, \sigma(h(\alpha)y_s^\alpha)\rangle\,dw_s
\end{aligned}$$

(recall that x is the starting point of y^α).

LEMMA 3.4. *For every* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $\alpha_0 > 0$ *such that*

$$\sup_{x \in \mathbb{R}^d} P\left\{\left|\int_0^1 |\langle \nu_s, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds - \tfrac{1}{2}\int_0^1 \langle q\nu_s, \nu_s\rangle\,ds\right| > \varepsilon\right\} \leq e^{-\delta(h(\alpha)^2/g(\alpha)^2)}$$

for every $\alpha > \alpha_0$.

PROOF. Let $\bar{\nu} = \sum_{i=1}^m \beta_i 1_{]t_i, t_{i+1}]}$ be a piecewise constant function $\bar{\nu}: [0,1] \to \mathbb{R}^d$ such that $\|\nu - \bar{\nu}\|_\infty < \eta$ for some $\eta > 0$ to be specified later. Then it is easy to obtain that for every $s \in [0,1]$,

$$\left| |\langle \nu_s, \sigma(h(\alpha)y_s^\alpha)\rangle|^2 - |\langle \bar{\nu}_s, \sigma(h(\alpha)y_s^\alpha)\rangle|^2 \right| \leq (2\|\nu\|_\infty + 1)\|\sigma\|_\infty^2 \eta,$$

$$|\langle q\nu_s, \nu_s\rangle - \langle q\bar{\nu}_s, \bar{\nu}_s\rangle| \leq (2\|\nu\|_\infty + 1)\|\sigma\|_\infty^2 \eta.$$

Thus for small η and for every starting point x,

$$P\left\{\left|\int_0^1 |\langle \nu_s, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds - \int_0^1 \langle q\nu_s, \nu_s\rangle\,ds\right| > \varepsilon\right\}$$

$$\le P\left\{\left|\int_0^1 |\langle \bar\nu_s, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds - \int_0^1 \langle q\bar\nu_s, \bar\nu_s\rangle\,ds\right| > \frac{\varepsilon}{2}\right\}$$

$$\le P\left\{\sum_{i=1}^m \left|\int_{t_i}^{t_{i+1}} |\langle \beta_i, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds - (t_{i+1}-t_i)\langle q\beta_i, \beta_i\rangle\right| > \frac{\varepsilon}{2}\right\}$$

$$\le \sum_{i=1}^m P\left\{\left|\int_{t_i}^{t_{i+1}} |\langle \beta_i, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds - (t_i-t_{i+1})\langle q\beta_i, \beta_i\rangle\right| > \frac{\varepsilon}{2m}\right\}.$$

If we now set

$$z_t^\alpha = h(\alpha)y^\alpha\left(t\frac{g(\alpha)^2}{h(\alpha)^2}\right),$$

it is easily checked by time change and Itô's formula that z^α solves

$$dz_t^\alpha = \sigma(z_t^\alpha)\,dw_t, \qquad z_0^\alpha = h(\alpha)x.$$

Thus

$$\int_{t_i}^{t_{i+1}} |\langle \beta_i, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds = \int_{t_i}^{t_{i+1}} \left|\left\langle \beta_i, \sigma\left(z^\alpha\left(s\frac{h(\alpha)^2}{g(\alpha)^2}\right)\right)\right\rangle\right|^2 ds$$

$$= \frac{g(\alpha)^2}{h(\alpha)^2}\int_{t_i(h(\alpha)^2/g(\alpha)^2)}^{t_{i+1}(h(\alpha)^2/g(\alpha)^2)} |\langle \beta_i, \sigma(z_s^\alpha)\rangle|^2\,ds.$$

Thus for α large there exists $\delta > 0$ such that

$$P\left\{\left|\int_0^1 |\langle \nu_s, \sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds - \int_0^1 \langle q\nu_s, \nu_s\rangle\,ds\right| > \varepsilon\right\}$$

$$\le \sum_{i=1}^m \exp\left[-(t_{i+1}-t_i)\delta\frac{h(\alpha)^2}{g(\alpha)^2}\right] \le m\exp\left[-\delta_1\frac{h(\alpha)^2}{g(\alpha)^2}\right],$$

where $\delta_1 = \delta\min_i(t_{i+1}-t_i)$ which allows us easily to conclude the proof. □

PROPOSITION 3.5.

$$\lim_{\alpha\to\infty}\frac{1}{g(\alpha)^2}\log E\left[\exp\left(g(\alpha)^2\int_0^1\langle y_s^\alpha, d\nu(s)\rangle\right)\right]$$

$$= \frac{1}{2}\int_0^1 \langle q\nu_s, \nu_s\rangle\,ds + \langle x, \nu([0,1])\rangle$$

for every $\nu \in \mathscr{C}'$, *where* $\nu_s = (\nu_1(]s,1]), \dots, \nu_d(]s,1]))$.

PROOF. In view of (3.3), we have to prove that

$$\lim_{\alpha\to\infty}\frac{1}{g(\alpha)^2}\log E\left[\exp\left(g(\alpha)\int_0^1\langle\nu_s,\sigma(h(\alpha)y_s^\alpha)\rangle\,dw_s\right)\right] = \frac{1}{2}\int_0^1\langle q\nu_s,\nu_s\rangle\,ds. \tag{3.4}$$

Let us note

$$Z(\alpha) = \int_0^1\langle\nu_s,\sigma(h(\alpha)y_s^\alpha)\rangle\,dw_s,$$

$$Y(\alpha) = \int_0^1|\langle\nu_s,\sigma(h(\alpha)y_s^\alpha)\rangle|^2\,ds,$$

$$X(\alpha) = \exp\left[g(\alpha)Z(\alpha) - \frac{g(\alpha)^2}{2}Y(\alpha)\right],$$

so that $E[X(\alpha)] = 1$ for every α. If

$$A_{\alpha,\varepsilon} = \left\{\left|Y(\alpha) - \int_0^1\langle q\nu_s,\nu_s\rangle\,ds\right| > \varepsilon\right\},$$

then

$$1 = E\left[X(\alpha)1_{A_{\alpha,\varepsilon}}\right] + E\left[X(\alpha)1_{A_{\alpha,\varepsilon}^C}\right].$$

By Lemma 3.4,

$$P(A_{\alpha,\varepsilon}^C) \le \exp\left(-\delta\frac{h(\alpha)^2}{g(\alpha)^2}\right)$$

for α large, so that

$$E\left[X(\alpha)1_{A_{\alpha,\varepsilon}^C}\right] \le E\left[X(\alpha)^2\right]^{1/2}P(A_{\alpha,\varepsilon}^C)^{1/2}$$

and since

$$\begin{aligned}E\left[X(\alpha)^2\right] &= E\left[\exp(2g(\alpha)Z(\alpha) - 2g(\alpha)^2Y(\alpha))\exp(g(\alpha)^2Y(\alpha))\right]\\ &\le E\left[\exp(2g(\alpha)Z(\alpha) - 2g(\alpha)^2Y(\alpha))\right]\exp\left[g(\alpha)^2\|\sigma\|_\infty^2\|\nu_.\|_\infty^2\right]\\ &= \exp\left[g(\alpha)^2\|\sigma\|_\infty^2\|\nu_.\|_\infty^2\right],\end{aligned}$$

$$E\left[X(\alpha)1_{A_{\alpha,\varepsilon}^C}\right] \le \exp\left[\frac{g(\alpha)^2}{2}\|\sigma\|_\infty^2\|\nu_.\|_\infty^2 - \frac{\delta}{2}\frac{h(\alpha)^2}{g(\alpha)^2}\right].$$

But

$$E\left[X(\alpha)1_{A_{\alpha,\varepsilon}^C}\right] \le \exp\left[\frac{g(\alpha)^2}{2}\|\sigma\|_\infty^2\|\nu_.\|_\infty^2\right]E\left[\exp(g(\alpha)Z(\alpha))1_{A_{\alpha,\varepsilon}}\right],$$

from which we get finally

$$E\left[\exp(g(\alpha)Z(\alpha))1_{A^C_{\alpha,\varepsilon}}\right] \le \exp\left[g(\alpha)^2\|\sigma\|^2_\infty\|\nu_\cdot\|^2_\infty - \frac{\delta}{2}\frac{h(\alpha)^2}{g(\alpha)^2}\right],$$

so that since $\lim_{\alpha\to\infty}(h(\alpha)/g(\alpha)^2) = +\infty$,

$$\lim_{\alpha\to\infty}\frac{1}{g(\alpha)^2}\log E\left[\exp(g(\alpha)Z(\alpha))1_{A^C_{\alpha,\varepsilon}}\right] = -\infty. \tag{3.5}$$

On the other hand,

$$E\left[X(\alpha)1_{A_{\alpha,\varepsilon}}\right] \ge \exp\left[-\frac{g(\alpha)^2}{2}\left(\int_0^1\langle q\nu_s,\nu_s\rangle\,ds + \varepsilon\right)\right] \times E\left[\exp(g(\alpha)Z(\alpha))1_{A_{\alpha,\varepsilon}}\right], \tag{3.6}$$

$$E\left[X(\alpha)1_{A_{\alpha,\varepsilon}}\right] \le \exp\left[-\frac{g(\alpha)^2}{2}\left(\int_0^1\langle q\nu_s,\nu_s\rangle\,ds - \varepsilon\right)\right] \times E\left[\exp(g(\alpha)Z(\alpha))1_{A_{\alpha,\varepsilon}}\right]. \tag{3.7}$$

From (3.6), we have

$$E\left[\exp(g(\alpha)Z(\alpha))1_{A_{\alpha,\varepsilon}}\right] \le E\left[X(\alpha)1_{A_{\alpha,\varepsilon}}\right]\exp\left[\frac{g(\alpha)^2}{2}\left(\int_0^1\langle q\nu_s,\nu_s\rangle\,ds + \varepsilon\right)\right]$$
$$\le \exp\left[\frac{g(\alpha)^2}{2}\left(\int_0^1\langle q\nu_s,\nu_s\rangle\,ds + \varepsilon\right)\right],$$

which together with (3.5) implies that

$$\lim_{\alpha\to\infty}\frac{1}{g(\alpha)^2}\log E\left[\exp g(\alpha)\int_0^1\langle\nu_s,\sigma(h(\alpha)y^\alpha_s)\rangle\,ds\right] \le \frac{1}{2}\int_0^1\langle q\nu_s,\nu_s\rangle\,ds + \frac{\varepsilon}{2}.$$

Similar arguments using (3.7) yield that the lim inf of the above expression is larger than $\frac{1}{2}\int_0^1\langle q\nu_s,\nu_s\rangle\,ds - \varepsilon/2$, thus proving (3.4). □

In order to prove Theorem 2.1 we only need the following.

PROPOSITION 3.6. *The Legendre transform*

$$L_x(\gamma) = \sup_{\nu\in\mathscr{C}'}\left(\langle\nu,\gamma\rangle - \tfrac{1}{2}\int_0^1\langle q\nu_s,\nu_s\rangle ds - \langle x,\nu([0,1])\rangle\right)$$

is given by

$$L_x(\gamma) = \tfrac{1}{2}\int_0^1\langle q^{-1}\gamma'_s,\gamma'_s\rangle\,ds,$$

if $\gamma\in\mathscr{C}$ *is absolutely continuous and* $\gamma(0) = x$, $L_x(\gamma) = +\infty$ *otherwise.*

The proof of Proposition 3.6 reduces to the computation of an unconstrained extremum and is easily performed by differentiation.

REMARK. Since the probability laws $P^{\alpha,\varepsilon}$ live in $\mathscr{C}_x$ and $L_x \equiv +\infty$ outside of $\mathscr{C}_x$, Theorem 2.1 still holds if $F \subset \mathscr{C}_x$ and $G \subset \mathscr{C}_x$ are, respectively, closed and open in the induced topology of $\mathscr{C}_x$ (Theorem 1.2 of [3]).

4. Applications: Iterated logarithm law for periodic diffusions. In this section, z will be the solution of the SDE,

$$dz_t = \sigma(z_t)\,dw_t, \qquad z_0 = 0,$$

where σ still satisfies Assumption A of Section 2. Let us denote

$$h(\alpha) = \sqrt{\alpha \log\log \alpha}\,, \qquad LL(\alpha) = \log\log \alpha,$$

and define $\zeta_\alpha(t) = z_{\alpha t}/h(\alpha)$. By Itô's formula and time change, ζ_α satisfies

$$d\zeta_\alpha(t) = \frac{1}{\sqrt{LL(\alpha)}}\sigma(h(\alpha)\zeta_\alpha(t))\,dw_t, \qquad \zeta_\alpha(0) = 0.$$

Through (1.3), ζ_α also defines a random variable taking values in $\mathscr{C}$.

THEOREM 4.1. *The family* $\{\zeta_\alpha\}_{\alpha>0}$ *is a.s. relatively compact in* $\mathscr{C}$ *and as* $\alpha \to \infty$ *has a limit set* $\mathscr{K}$ *given by*

$$\mathscr{K} = \left\{\gamma \in \mathscr{C}; \gamma(0) = 0, \tfrac{1}{2}\int_0^1 \langle q^{-1}\gamma_s', \gamma_s'\rangle\,ds \le 1\right\},$$

q being defined in (2.7).

Theorem 4.1 will follow from Propositions 4.5 and 4.7 below. The proof makes use essentially of the large deviations estimates of Section 2. The idea is not far from the one developed for other similar statements (see, e.g., [2], Theorem 2.2). Theorem 2.1 states that

$$\limsup_{\alpha\to\infty} \frac{1}{LL(t)} \log P\{\zeta_\alpha \in F\} \le -\Lambda_0(F), \tag{4.1}$$

$$\liminf_{\alpha\to\infty} \frac{1}{LL(t)} \log P\{\zeta_\alpha \in G\} \ge -\Lambda_0(G), \tag{4.2}$$

where F and G are, respectively, a closed and open subset of $\mathscr{C}$ and Λ_0 is defined in (1.1). In order to simplify the notation we shall write Λ and L instead of Λ_0 and L_0.

REMARK 4.2. We shall often make use of the fact that for every $c > 1$, the quantity

$$a_n = \exp[-kLL(c^n)] = \frac{\text{const.}}{n^k} \tag{4.3}$$

is summable if and only if $k > 1$.

If $\gamma \in \mathscr{C}$ and $A \subset \mathscr{C}$, we shall write $d(\gamma, A) := \inf_{\eta \in A} \|\gamma - \eta\|_\infty$.

LEMMA 4.3. *For every $c > 1$ and every $\varepsilon > 0$, there exists a.s. a positive integer $j_0 = j_0(\omega)$ such that for every $j > j_0$,*

$$d(\zeta_{c^j}, \mathscr{K}) < \varepsilon.$$

PROOF. Let $\mathscr{K}_\varepsilon' = \{\gamma; d(\gamma, \mathscr{K}) \geq \varepsilon\}$. Since L is lower semicontinuous and the level sets $\{\gamma; L(\gamma) \leq k\}$ are compact in $\mathscr{K}$ for every finite k, there exists $\delta > 0$ such that $\Lambda(\mathscr{K}_\varepsilon') > 1 + 2\delta$. Thus from (4.1) for large j,

$$P\{\zeta_{c^j} \in \mathscr{K}_\varepsilon'\} \leq \exp\left[-(1+\delta)LL(c^j)\right],$$

which is summable by Remark 4.2. The Borel–Cantelli lemma allows us now to conclude the proof. □

For every positive integer j and $c > 1$, let us set

$$Y_j = \sup_{c^{j-1} \leq \alpha \leq c^j} \left\| \zeta_\alpha - \frac{h(c^j)}{h(\alpha)} \zeta_{c^j} \right\|_\infty = \sup_{c^{j-1} \leq \alpha \leq c^j} \left\| \frac{1}{h(\alpha)} z_{\alpha\cdot} - \frac{1}{h(\alpha)} z_{c^j\cdot} \right\|_\infty.$$

LEMMA 4.4. *For every $\varepsilon > 0$, there exists a real number $c_\varepsilon > 0$ such that if $1 < c < c_\varepsilon$,*

$$P\{\textit{there exists } j_0 = j_0(\omega) \textit{ such that } Y_j < \varepsilon \textit{ whenever } j \geq j_0\} = 1.$$

PROOF. We want to prove that

$$P\Big(\limsup_{j \to \infty} \{Y_j \geq \varepsilon\}\Big) = 0.$$

But

$$\begin{aligned}
\{Y_j \geq \varepsilon\} &= \left\{ \sup_{c^{j-1} \leq \alpha \leq c^j} \left\| \frac{1}{h(\alpha)} z_{\alpha\cdot} - \frac{1}{h(\alpha)} z_{c^j\cdot} \right\|_\infty \geq \varepsilon \right\} \\
&\subset \left\{ \sup_{c^{j-1} \leq \alpha \leq c^j} \left\| \frac{1}{h(c^{j-1})} z_{\alpha\cdot} - \frac{1}{h(c^{j-1})} z_{c^j\cdot} \right\|_\infty \geq \varepsilon \right\} \\
&\subset \left\{ \sup_{\substack{0 \leq s \leq 1 \\ s/c \leq t \leq s}} \frac{1}{h(c^{j-1})} |z_{c^{j-1}t} - z_{c^{j-1}s}| \geq \varepsilon \right\} \\
&\subset \left\{ \sup_{\substack{0 \leq s \leq 1 \\ s/c \leq t \leq s}} \frac{h(c^j)}{h(c^{j-1})} |\zeta_{c^{j-1}}(t) - \zeta_{c^{j-1}}(s)| \geq \varepsilon \right\}.
\end{aligned}$$

Since for every $\delta > 0$ and large j,

$$\frac{h(c^j)}{h(c^{j-1})} < \sqrt{c}\,(1+\delta),$$

if we set

$$A_\varepsilon = \left\{ \gamma \in \mathscr{C};\ \sup_{\substack{0 \le s \le 1 \\ s/c \le t \le s}} |\gamma_t - \gamma_s| \ge \frac{\varepsilon}{2} \right\},$$

then if $\sqrt{c}(1+\delta) \le 2$, for large j,

$$P\{Y_j \ge \varepsilon\} \le P\{\zeta_{c^j} \in A_\varepsilon\}. \tag{4.4}$$

But if $\gamma \in A_\varepsilon$, there exist s, t with $0 \le s \le 1$, $s/c \le t \le s$ such that $|\gamma_s - \gamma_t| \ge \varepsilon/4$; then using (2.4),

$$\begin{aligned} \frac{\varepsilon}{4} &\le \int_t^s \gamma'_u\, du \le |s-t|^{1/2} \left(\int_s^t |\gamma'_u|^2\, du \right)^{1/2} \\ &\le \left(1 - \frac{1}{c}\right)^{1/2} \sqrt{2}\, \|\sigma\|_\infty \left(\frac{1}{2} \int_0^1 \langle q^{-1}\gamma'_s, \gamma'_s \rangle\, ds \right)^{1/2}. \end{aligned}$$

Thus

$$L(\gamma) \ge \frac{\varepsilon^2 c}{32\|\sigma\|_\infty} \frac{1}{c-1}.$$

Thus for $1 \le c \le c_\varepsilon$, $L(\gamma) \ge 3$ and $\Lambda(A_\varepsilon) \ge 3$. From (4.4) and (4.1),

$$P\{Y_j \ge \varepsilon\} \le \exp[-2LL(c^j)], \tag{4.5}$$

which is summable (Remark 4.2) and the Borel–Cantelli lemma concludes the proof. □

PROPOSITION 4.5. *For every $\varepsilon > 0$, there exists a.s. a positive real number $\alpha_0 = \alpha_0(\omega)$ such that for every $\alpha > \alpha_0$,*

$$d(\zeta_\alpha(\omega), \mathscr{K}) \le \varepsilon.$$

In particular, $\{\zeta_\alpha\}_\alpha$ is a.s. relatively compact and all its limit points as $\alpha \to \infty$ are contained in $\mathscr{K}$.

PROOF. For $c > 1$, we have

$$d(\zeta_\alpha, \mathscr{K}) \le d\left(\zeta_\alpha, \frac{h(c^j)}{h(\alpha)}\zeta_{c^j}\right) + d\left(\frac{h(c^j)}{h(\alpha)}\zeta_{c^j}, \zeta_{c^j}\right) + d(\zeta_{c^j}, \mathscr{K}) = I_1 + I_2 + I_3,$$

where j is such that $c^{j-1} \le \alpha \le c^j$. By Lemma 4.3, there exists j_0 such that for $j > j_0$, $I_3 < \varepsilon/3$. As for I_2, since for every $\delta > 0$ and large j,

$$1 \le \frac{h(c^j)}{h(\alpha)} \le \sqrt{c}\,(1+\delta)$$

and $\|\zeta_{c^j}\|_\infty$ is bounded in j by Lemma 4.3, if $c > 1$ is in a neighborhood of 1 and j large then $I_2 < \varepsilon/3$. Also $I_1 < \varepsilon/3$ if c is in a neighborhood (possibly smaller) of 1 by Lemma 4.4. □

In order to complete the proof of Theorem 2.1 we only need to prove that every $\gamma \in \mathscr{K}$ is a limit point of $\{\zeta_\alpha\}_\alpha$ a.s. We shall make use of the following lemma (Lemma 2.14 of [6]).

LEMMA 4.6. *Let (Ω, Σ, μ) be a probability space and $\{\mathscr{F}_n\}_n$ an increasing sequence of σ fields. Let $E_n \in \mathscr{F}_n$ and define $p_{n-1}(\omega) = \mu\{E_n | \mathscr{F}_{n-1}\}$. Then if for almost all ω,*

$$\sum_n p_{n-1}(\omega) = +\infty,$$

we have $P(\limsup_{n\to\infty} E_n) = 1$.

PROPOSITION 4.7. *Let $\gamma \in \mathscr{K}$ be such that*

$$\tfrac{1}{2}\int_0^1 \langle q^{-1}\gamma_s', \gamma_s'\rangle\, ds < 1.$$

Then for every $\delta > 0$, there exists $c > 1$ such that

$$P\Big(\limsup_{n\to\infty} \{\|\zeta_{c^j} - \gamma\|_\infty \le \delta\}\Big) = 1.$$

In particular, every $\gamma \in \mathscr{K}$ is a limit point of $\{\zeta_\alpha\}_\alpha$ as $\alpha \to \infty$ a.s.

PROOF. Let $\mathscr{F}_j = \sigma(z_u, u \le c^j) = \sigma(\zeta_{c^j}(s), s \le 1)$. In view of Lemma 4.6, we need to prove that

$$\sum_j P\{\|\zeta_{c^j} - \gamma\|_\infty \le \delta | \mathscr{F}_{j-1}\} = +\infty. \tag{4.6}$$

But

$$\begin{aligned}\{\|\zeta_{c^j} - \gamma\|_\infty \le \delta\} &= \Big\{\sup_{t\le 1/c} |\zeta_{c^j}(t) - \gamma(t)| < \delta\Big\} \cap \Big\{\sup_{1/c\le t\le 1} |\zeta_c^j(t) - \gamma(t)| \le \delta\Big\}\\ &= A_j \cap B_j.\end{aligned}$$

A_j is clearly $\mathscr{F}_j$-measurable and the Markov property gives

$$P\{\|\zeta - \gamma\|_\infty \le \delta | \mathscr{F}_{j-1}\} = 1_{A_j} P^{\zeta_{c^j}(1/c)}\bigg\{\sup_{0\le t\le 1-1/c}\bigg|\zeta_{c^j}(t) - \gamma\Big(t + \frac{1}{c}\Big)\bigg| < \delta\bigg\}. \tag{4.7}$$

Now

$$P^x\bigg\{\sup_{0\le t\le 1-1/c}\bigg|\zeta_{c^j}(t) - \gamma\Big(t + \frac{1}{c}\Big)\bigg| < \delta\bigg\} = P^x\{\zeta_{c^j} \in E_\delta\},$$

where

$$E_\delta = \bigg\{\eta \in \mathscr{C};\ \sup_{0\le t\le 1-1/c}\bigg|\eta(t) - \gamma\Big(t + \frac{1}{c}\Big)\bigg| < \delta\bigg\}.$$

By the Hölder inequality and (2.4),

$$\gamma(t) = \int_0^t \gamma_s'\,ds \le \sqrt{2t}\,\|\sigma\|_\infty \left(\tfrac{1}{2}\int_0^1 \langle q^{-1}\gamma_s', \gamma_s'\rangle\,ds \right)^{1/2},$$

so that $\sup_{t \le 1/c} |\gamma(t)| \le \|\sigma\|_\infty / \sqrt{c}$ and if c is large enough,

$$\sup_{t \le 1/c} |\gamma(t)| \le \frac{\delta}{4}. \tag{4.8}$$

In particular, $\gamma(1/c) \le \delta/4$. Thus if $|x| \le \delta/4$, the path

$$\eta(t) = \begin{cases} x - \gamma(1/c) + \gamma(t + 1/c) & \text{if } t \le 1 - 1/c, \\ \gamma(1) & \text{if } 1 - 1/c \le t \le 1, \end{cases}$$

is in E_δ since $\sup_{0 \le t \le 1 - 1/c} |\eta(t) - \gamma(t + 1/c)| = |x - \gamma(1/c)| \le \delta/2$. Moreover,

$$L_x(\eta) = \tfrac{1}{2}\int_{1/c}^1 \langle q^{-1}\gamma_s', \gamma_s'\rangle\,ds \le L(\gamma) < 1,$$

since $\eta(0) = x$. Thus $\Lambda_x(E_\delta) \le L(\gamma) < 1$ and from (4.2), if $L(\gamma) < c < 1$, $|x| \le \delta/4$ and j is large,

$$P^x\left\{ \sup_{0 \le t \le 1 - 1/c} \left| \zeta_{c^j}(t) - \gamma\left(t + \frac{1}{c}\right) \right| \le \delta \right\} \ge \exp\left[-cLL(c^j)\right]. \tag{4.9}$$

Thus the right term in (4.9) is not summable in j by Remark 4.2. Moreover, for large j,

$$\sup_{t \le 1/c} |\zeta_{c^j}(t)| = \sup_{t \le 1} |\zeta_{c^{j-1}}(t)| \frac{h(c^{j-1})}{h(c^j)} \le \frac{1 + \delta}{\sqrt{c}} \sup_{t \le 1} |\zeta_{c^{j-1}}(t)|. \tag{4.10}$$

By Lemma 4.3, $d(\zeta_{c^{j-1}}, \mathscr{K}) \le \varepsilon$ for $j \ge j_0$, so that $\sup_{t \le 1} |\zeta_{c^{j-1}}(t)| \le \sup\{\|\eta\|_\infty, \eta \in \mathscr{K}\} + \varepsilon$. This bound and (4.10) yield

$$\sup_{t \le 1/c} |\zeta_{c^j}(t)| \le \frac{\delta}{4} \tag{4.11}$$

for $j \ge j_0$ if c is large enough. (4.8) and (4.11) together give that $\sup_{t \le 1/c} |\zeta_{c^j}(t) - \gamma(t)| \le \delta/2$. In conclusion, if c is large enough there exists j_0 such that $\zeta_{c^j} \in A_j$ and $|\zeta_{c^j}(1/c)| \le \delta/4$ for every $j > j_0$. Recalling (4.7) and (4.9), this proves that the sum in (4.6) diverges, thus concluding the proof. □

REMARK. In this section we have considered the process z with initial condition $z_0 = 0$; of course the same result holds for the solution z^x of

$$dz_t^x = \sigma(z_t^x)\,dw_t, \qquad z_0^x = x.$$

Indeed, $z_t^x = \tilde{z}_t + x$ for $\tilde{z}$ the solution of

$$d\tilde{z}_t = \tilde{\sigma}(\tilde{z}_t)\,dw_t, \qquad \tilde{z}_0 = 0,$$

where $\tilde{\sigma}(z) = \sigma(z - x)$. It suffices now to remark that $\tilde{\sigma}$ is still periodic and has the same periods than σ, so that we can apply the results of this section to $\tilde{z}$.

REMARK. Our proof of Theorem 4.1 is relevant only if the dimension m is larger than 1. Indeed, if $m = 1$, there is an easy time change argument at hand since in this case $z_t = W_{A_t}$ for a suitable Brownian motion W, where $A_t = \int_0^t \sigma(z_s)^2\,ds$; this fact enables us to write

$$\zeta_\alpha(t) = \frac{W_{A_{\alpha t}}}{\sqrt{A_{\alpha t} \log\log A_{\alpha t}}} \frac{\sqrt{A_{\alpha t} \log\log A_{\alpha t}}}{\sqrt{\alpha \log\log \alpha}}$$

and one may now use Strassen's law for Brownian motion and the fact that

$$\lim_{\alpha \to \infty} \frac{\sqrt{A_{\alpha t} \log\log A_{\alpha t}}}{\sqrt{\alpha \log\log \alpha}} = \sqrt{q},$$

since by the ergodic theorem

$$\lim_{t \to \infty} \frac{1}{t} A_t = \int_T \sigma(\bar{z})^2\, dm(\bar{z}) = q.$$

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