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CORRECTION

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR RANDOM MAPPINGS

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The claim "(8) is bounded between

$$Q_n(1) = \sum_{j=0}^{d_n} c_{j,n} \quad \text{and} \quad Q_n(1)\sqrt{n/(n - n^{T'})}\,(1 + 1/n),$$

and the limit of (8) as $n \to \infty$ equals $\lim_{n\to\infty} Q_n(1)$" on page 326 does not follow from the argument given, since the coefficients in the polynomial $Q_n(z) = \sum_{j=0}^{d_n} c_{j,n} z^j$ are not necessarily of the same sign. However, a simpler proof of the convergence of the finite-dimensional distributions in Case 1 (see page 323) can be given, which avoids the problem mentioned above. This proof is given below.

For $0 \le t < t' < 1$, we show that $(\overline{Y}_n(t), \overline{Y}_n(t') - \overline{Y}_n(t))$ converges weakly to $(Z(t), Z(t' - t))$, where $Z(t)$ and $Z(t' - t)$ are independent normal random variables with mean 0 and variance t and $t' - t$, respectively, by showing that for any $a, b \in \mathbb{R}$, $a\overline{Y}_n(t) + b(\overline{Y}_n(t') - \overline{Y}_n(t))$ converges weakly to $aZ(t) + bZ(t' - t)$ (see [1], Theorem 29.4). We do this by using the method of moments, i.e., we show that for any integer $r > 0$,

$$\lim_{n\to\infty} E_n\big(a\overline{Y}_n(t) + b\big(\overline{Y}_n(t') - \overline{Y}_n(t)\big)\big)^r = E\big(aZ(t) + b\big(Z(t') - Z(t)\big)\big)^r.$$

Let r be fixed but arbitrary, and let

$$\mu_n(z) = \sum_{k=1}^{n^t} \left(\frac{A_k}{k!}\right)\left(\frac{z}{e}\right)^k$$

and

$$\tilde{\mu}_n(z) = \sum_{k>n^t}^{n^{t'}} \left(\frac{A_k}{k!}\right)\left(\frac{z}{e}\right)^k.$$

It follows from (6), page 322, that

$$\begin{aligned}
&E_n\big(a\overline{Y}_n(t) + b\big(\overline{Y}_n(t') - \overline{Y}_n(t)\big)\big)^r \\
&\quad = \big[(z_n)^n\big]\frac{n!\,e^n}{n^n} S\left(\frac{z_n}{e}\right) E_{z_n}\big(a\overline{Y}_n(t) + b\big(\overline{Y}_n(t') - \overline{Y}_n(t)\big)\big)^r \\
&\quad = \big[(z_n)^n\big]\frac{n!\,e^n}{n^n} S\left(\frac{z_n}{e}\right) \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k} E_{z_n}\big(\overline{Y}_n(t)\big)^k E_{z_n}\big(\overline{Y}_n(t') - \overline{Y}_n(t)\big)^{r-k}.
\end{aligned}$$

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The above equality holds since

$$\overline{Y}_n(t) = \sum_{j>0}^{n^t} \frac{m_j - \mu_n(z_n)}{\sqrt{\frac{1}{2}\ln n}}$$

and

$$\overline{Y}_n(t') - \overline{Y}_n(t) = \sum_{j>n^t}^{n^{t'}} \frac{m_j - \tilde{\mu}_n(z_n)}{\sqrt{\frac{1}{2}\ln n}}$$

on Ω_{z_n}, and so $\overline{Y}_n(t)$ and $\overline{Y}_n(t') - \overline{Y}_n(t)$ are independent with respect to the product measure P_{z_n} on Ω_{z_n}.

By definition of P_{z_n}, the sums $\sum_{j>0}^{n^t} m_j$ and $\sum_{j>n^t}^{n^{t'}} m_j$ are independent Poisson variables with parameters $\mu_n(z_n)$ and $\tilde{\mu}_n(z_n)$, respectively. For any Poisson variable V_λ, with parameter λ, and integer $m \geq 0$, we have $E(V_\lambda - \lambda)^m = f_m(\lambda)$, where f_m is a certain polynomial of degree at most m. Thus,

$$E_n\big(a\overline{Y}_n(t) + b(\overline{Y}_n(t') - \overline{Y}_n(t))\big)^r = \frac{[(z_n)^n]\, n!\, e^n}{(\frac{1}{2}\ln n)^{r/2} n^n} S\left(\frac{z_n}{e}\right) L_n(z_n),$$

where

$$L_n(z) = \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k} f_k(\mu_n(z))\, f_{r-k}(\tilde{\mu}_n(z)).$$

Note that the degree of L_n is less than $\sum_{k=0}^{r}(kn^t + (r-k)n^{t'})$. So, for large enough n, there exists $0 < T < 1$ such that the degree of L_n is less than n^T. If we write $L_n(z) = \sum_{j=0}^{d_n} b_{j,n} z^j$, where d_n denotes the degree of L_n, then

$$\begin{aligned} &E_n\big(a\overline{Y}_n(t) + b(\overline{Y}_n(t') - \overline{Y}_n(t))\big)^r \\ (1')\qquad &= \frac{1}{(\frac{1}{2}\ln n)^{r/2}} \sum_{j=0}^{d_n} b_{j,n} \frac{(n-j)^{n-j} e^j n!}{(n-j)!\, n^n} \\ &= \frac{L_n(1)}{(\frac{1}{2}\ln n)^{r/2}} + \frac{1}{(\frac{1}{2}\ln n)^{r/2}} \sum_{j=0}^{d_n} b_{j,n}\left(\frac{(n-j)^{n-j} e^j n!}{(n-j)!\, n^n} - 1\right). \end{aligned}$$

The second term on the right-hand side of (1′) goes to 0 as $n \to \infty$. To see this, we first note that for $0 \leq j \leq d_n < n^T$,

$$\begin{aligned} \frac{(n-j)^{n-j} e^j n!}{(n-j)!\, n^n} - 1 &\leq \sqrt{\frac{n}{n-n^T}}\left(1 + \frac{1}{n}\right) - 1 \\ &\leq \left(\frac{n}{n-n^T}\right)\left(1 + \frac{1}{n}\right) - 1 \leq \frac{3}{n^{1-T} - 1}, \end{aligned}$$

by Stirling's formula. We claim that $\sum_{j=0}^{d_n}|b_{j,n}| = O((\ln n)^r)$, so

$$\left|\frac{1}{\left(\frac{1}{2}\ln n\right)^{r/2}}\sum_{j=0}^{d_n} b_{j,n}\left(\frac{(n-j)^{n-j}e^j n!}{(n-j)!\,n^n}-1\right)\right| \le \frac{3}{(n^{1-T}-1)\left(\frac{1}{2}\ln n\right)^{r/2}}\sum_{j=0}^{d_n}|b_{j,n}|$$

$$= O\left(\frac{(\ln n)^{r/2}}{n^{1-T}}\right).$$

Thus, the second term on the right-hand side of (1′) goes to 0 as $n \to \infty$, assuming the above claim is true.

To show that $\sum_{j=0}^{d_n}|b_{j,n}| = O((\ln n)^r)$, recall that the $b_{j,n}$'s are coefficients in the polynomial L_n. We define another polynomial $\hat{L}_n(z) = \sum_{j=0}^{\hat{d}_n}\hat{b}_{j,n}z^j$ such that

(i) the coefficients of $\hat{L}_n$ are positive,
(ii) $\hat{d}_n \ge d_n$ and
(iii) $|b_{j,n}| \le \hat{b}_{j,n}$ for each $0 \le j \le d_n$.

Then

$$\sum_{j=0}^{d_n}|b_{j,n}| \le \sum_{j=0}^{\hat{d}_n}\hat{b}_{j,n} = \hat{L}_n(1)$$

and it follows that $\sum_{j=0}^{d_n}|b_{j,n}| = O((\ln n)^r)$ if $\hat{L}_n(1) = O((\ln n)^r)$. Specifically, we define

$$\hat{L}_n(z) = \sum_{j=0}^{r}\binom{r}{j}|a|^j|b|^{r-j}\hat{f}_j(\mu_n(z))\,\hat{f}_{r-j}(\tilde{\mu}_n(z)),$$

where, for any $i \ge 0$, $\hat{f}_i$ is the polynomial obtained from f_i by replacing each coefficient in f_i by its absolute value. Since the coefficients of μ_n, $\tilde{\mu}_n$ and each f_i are positive, so are the coefficients of $\hat{L}_n$. Also, since the degree of $\hat{f}_i$ is the same as that of f_i, the degree of $\hat{L}_n$ must be at least as large as that of L_n, that is, $d_n \le \hat{d}_n$. Finally, for $0 \le j \le d_n$, note that $b_{j,n}$ can be written as a multinomial in a, b and the coefficients of f_i, μ_n and $\tilde{\mu}_n$. Since $\hat{b}_{j,n}$ is given by the same multinomial, with the variables replaced by their absolute values, we have $|b_{j,n}| \le \hat{b}_{j,n}$ by the triangle inequality. This establishes properties (i)–(iii).

Recall that each $\hat{f}_i$ has degree at most i, $\mu_n(1) \sim (t/2)\ln n$ and $\tilde{\mu}_n(1) \sim [(t'-t)/2]\ln n$ (see page 322). So

$$\begin{aligned}\hat{L}_n(1) &= \sum_{j=0}^{r}\binom{r}{j}|a|^j|b|^{r-j}\hat{f}_j(\mu_n(1))\,\hat{f}_{r-j}(\tilde{\mu}_n(1))\\ &= \sum_{j=0}^{r}\binom{r}{j}|a|^j|b|^{r-j}O\left((\ln n)^j\right)O\left((\ln n)^{r-j}\right)\\ &= O\left((\ln n)^r\right)\end{aligned}$$

and the claim is proved.

Since the second term on the right-hand side of (1′) goes to 0 as $n \to \infty$, we have

$$\lim_{n\to\infty} E_n\big(a\overline{Y}_n(t) + b(\overline{Y}_n(t') - \overline{Y}_n(t))\big)^r = \lim_{n\to\infty} \frac{L_n(1)}{\left(\frac{1}{2}\ln n\right)^{r/2}}.$$

By definition of the polynomials f_i,

$$\begin{aligned}\frac{L_n(1)}{\left(\frac{1}{2}\ln n\right)^{r/2}} &= \frac{1}{\left(\frac{1}{2}\ln n\right)^{r/2}} \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k} f_k(\mu_n(1)) f_{r-k}(\tilde{\mu}_n(1))\\ &= \frac{1}{\left(\frac{1}{2}\ln n\right)^{r/2}} \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k} E(V_n - \mu_n(1))^k E\big(\tilde{V}_n - \tilde{\mu}_n(1)\big)^{r-k}\\ &= E\left(a\frac{V_n - \mu_n(1)}{\sqrt{\frac{1}{2}\ln n}} + b\frac{\tilde{V}_n - \tilde{\mu}_n(1)}{\sqrt{\frac{1}{2}\ln n}}\right)^r,\end{aligned}$$

where V_n and $\tilde{V}_n$ are independent Poisson random variables with parameters $\mu_n(1)$ and $\tilde{\mu}_n(1)$, respectively. As $n \to \infty$, the normalized variables $(V_n - \mu_n(1))/\sqrt{\mu_n(1)}$ and $(\tilde{V}_n - \tilde{\mu}_n(1))/\sqrt{\tilde{\mu}_n(1)}$ each converge weakly to the normal distribution with mean 0 and variance 1. It follows that the random vector

$$\left(\sqrt{\frac{\mu_n(1)}{\frac{1}{2}\ln n}}\,\frac{V_n - \mu_n(1)}{\sqrt{\mu_n(1)}}, \sqrt{\frac{\tilde{\mu}_n(1)}{\frac{1}{2}\ln n}}\,\frac{\tilde{V}_n - \tilde{\mu}_n(1)}{\sqrt{\tilde{\mu}_n(1)}}\right)$$

converges weakly to the random vector $(Z(t), Z(t' - t))$ as $n \to \infty$, since $\mu_n(1) \sim (t/2)\ln n$ and $\tilde{\mu}_n \sim [(t' - t)/2]\ln n$. Thus

$$\begin{aligned}&\lim_{n\to\infty} E_n\big(a\overline{Y}_n(t) + b(\overline{Y}_n(t') - \overline{Y}_n(t))\big)^r\\ &\quad= \lim_{n\to\infty} \frac{L_n(1)}{\left(\frac{1}{2}\ln n\right)^{r/2}}\\ &\quad= \lim_{n\to\infty} E\left(a\left(\frac{V_n - \mu_n(1)}{\sqrt{\frac{1}{2}\ln n}}\right) + b\left(\frac{\tilde{V}_n - \tilde{\mu}_n(1)}{\sqrt{\frac{1}{2}\ln n}}\right)\right)^r\\ &\quad= E(aZ(t) + bZ(t' - t))^r\end{aligned}$$

as desired.

REFERENCE

[1] Billingsley, P. (1979). *Probability and Measure*. Wiley, New York.

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