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THE RUIN PROBLEM FOR FINITE MARKOV CHAINS

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We derive an asymptotic approximation of the joint distribution $\mathrm{prob}(N(u) - n \in A, S_{N(u)} - u \in B)$ as n and $u \to \infty$. Here $N(u) = \min\{n;\ S_n > u\}$ denotes the first passage time for a random walk of the form $S_n = \sum_{k=1}^n U_k(\xi_{k-1}, \xi_k)$, where $\xi_0, \xi_1, \dots$ is a finite Markov chain and where $\{U_k(i,j)\}_{k=1}^\infty$ is a sequence of independent random variables. The approximation holds for all sets B and a fairly large class of sets A.

1. Introduction and results. Consider a Markovian random walk of the form

$$S_n = \sum_{k=1}^{n} U_k(\xi_{k-1}, \xi_k), \qquad n > 0, \tag{1.1}$$

and the first passage time

$$N(u) = \min\{n > 0;\ S_n > u\}. \tag{1.2}$$

Here $\{\xi_k\}_{k=0}^\infty$ is an irreducible and stationary Markov chain with finite state space $\{1, 2, \dots, r\}$, and $U_k(i,j)$ are integer-valued random variables such that the matrices $\{U_k(i,j)\}_{i,j=1}^r$, $k = 1, 2, \dots,$ have a common distribution and are independent of each other and of the Markov chain.

Let $A \subset \mathbb{Z}$ (the integers) and $B \subset \mathbb{Z}_+$ (the positive integers) and put

$$Z_{ij}(n,u) = \mathrm{prob}\big(N(u) \in n + A, S_{N(u)} \in u + B, \xi_{N(u)} = j \mid \xi_0 = i\big). \tag{1.3}$$

We shall in this paper determine the asymptotic behaviour of $Z_{ij}(n,u)$ as n and u tend to infinity. The approximation is valid for all $B \subset \mathbb{Z}_+$ and for a fairly large class of sets A.

If we let $r = 1$, we get the corresponding result for ordinary random walks (on $\mathbb{Z}$). The other extreme is when $U_1(i,j)$ is a deterministic function of (i,j).

The more general problem of estimating the probability that a two-dimensional random walk $(T_k, S_k)_{k=1}^\infty$ hits the set $(n,u) + A \times B$ the first time $S_k > u$ can be treated by the same method [see Höglund (1990a) for the corresponding problem for independent random variables].

Miller (1962a, b) considered the case $A = \mathbb{Z}$, $B = \mathbb{Z}_+$ and found the asymptotic behaviour of $\mathrm{prob}(N(u) < \infty)$ as $u \to \infty$ using a Wiener–Hopf factorization. Presman (1969) used a similar technique to study the asymptotic behaviour of $\mathrm{prob}(N(u) \le n)$ ($A = \{\dots, -2, -1, 0\}$, $B = \mathbb{Z}_+$). This technique was further developed by Arndt (1981, 1982). He obtained upper bounds for

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prob($N(u) < \infty$), prob($N(u) \le n$) and their difference and he gave conditions under which explicit calculations are possible.

Let (p_{ij}) stand for the transition probability matrix and define the (matrix-valued) probability density of U_k, $P(u) = (P_{ij}(u))$, by

$$P_{ij}(u) = \text{prob}(U_k(\xi_{k-1}, \xi_k) = u, \xi_k = j | \xi_{k-1} = i) \\ = \text{prob}(U_k(i,j) = u) p_{ij}. \tag{1.4}$$

The approximation will be expressed in terms of quantities related to the matrices $\hat{P}(\theta) = (\hat{P}_{ij}(\theta))$, $\theta \in \Theta$, where

$$\hat{P}_{ij}(\theta) = \sum_u e^{\theta u} P_{ij}(u) = p_{ij} E e^{\theta U_1(i,j)} \tag{1.5}$$

and where Θ denotes the interior of the set of $\theta \in \mathbf{R}$ for which this matrix is finite. The set Θ is then an open interval which we shall assume is nonempty.

The matrix $\hat{P}(\theta)$ is thus a positive and irreducible matrix whose coefficients are analytic in θ; and hence $\hat{P}(\theta)$ has a maximal positive eigenvalue $\lambda(\theta)$ corresponding to strictly positive left and right eigenvectors $\sigma(\theta) = \{\sigma_i(\theta)\}$ and $\rho(\theta) = \{\rho_i(\theta)\}$. This eigenvalue is simple and analytic in Θ and $\sigma_i(\theta)$ and $\rho_i(\theta)$ can be chosen to be analytic in Θ. We shall use the normalization $\sigma(\theta) \cdot \rho(\theta) = 1$.

A subscript θ on probabilities and expectations indicates that the underlying probability measure is given by the initial distribution

$$\text{prob}_\theta(\xi_0 = i) = \sigma_i(\theta)\rho_i(\theta), \tag{1.6}$$

and the cylinder set probabilities

$$\text{prob}_\theta(U_k(\xi_{k-1}, \xi_k) = u_k, \xi_k = i_k, k = 1, \dots, n | \xi_0 = i_0) \\ = \prod_{k=1}^{n} \frac{e^{\theta u_k} P_{i_{k-1} i_k}(u_k) \rho_{i_k}(\theta)}{\lambda(\theta) \rho_{i_{k-1}}(\theta)}. \tag{1.7}$$

We shall write $m(\theta)$ for the expectation of $U_1(\xi_0, \xi_1)$ with respect to this measure:

$$m(\theta) = E_\theta U_1(\xi_0, \xi_1) = \sigma(\theta) \cdot \hat{P}'(\theta)\rho(\theta)/\lambda(\theta). \tag{1.8}$$

LEMMA 1.1. *The function m satisfies $m(\theta) = \lambda'(\theta)/\lambda(\theta)$ and $m'(\theta) > 0$ for all θ unless* prob($U_1(i,j) = a + v(j) - v(i)) = 1$ *for some integer a and some sequence $v(1), \dots, v(r)$.*

PROOF. Differentiating the identity

$$\hat{P}(\theta)\rho(\theta) = \lambda(\theta)\rho(\theta), \tag{1.9}$$

we get

$$\hat{P}'(\theta)\rho(\theta) + \hat{P}(\theta)\rho'(\theta) = \lambda'(\theta)\rho(\theta) + \lambda(\theta)\rho'(\theta). \tag{1.10}$$

Multiply (scalar product) by $\sigma(\theta)$ and observe that

$$\sigma(\theta)\cdot\hat{P}(\theta)\rho'(\theta) = \lambda(\theta)\sigma(\theta)\cdot\rho'(\theta). \tag{1.11}$$

The equality of the lemma follows. The inequality is Theorem 1.2 of Keilson and Wishart (1964). □

The approximation will depend on the support of the distribution, $L(n,u) = (L_{ij}(n,u))$, of the first strict ascending ladder point (N, S_N):

$$L_{ij}(n,s) = \text{prob}(N = n, S_N = s, \xi_N = j \mid \xi_0 = i). \tag{1.12}$$

Here and below $N = N(0)$. This distribution defines a certain group $G(L) \subset \mathbb{Z}^2$. A complication here is that L need not be irreducible even though P is. We shall avoid this complication simply by assuming irreducibility. Define with $t = (n,u)$,

$$L^{k*}(t) = \sum_{t_1+\cdots+t_k=t} L(t_1)\cdots L(t_k), \tag{1.13}$$

that is, L^{k*} is the distribution of the kth strict ascending ladder point. Put

$$S_{ij}(L) = \bigcup_{k=1}^{\infty}\{(n,s); L_{ij}^{k*}(n,s) > 0\} \tag{1.14}$$

and let $G_{ij}(L)$ denote the smallest subgroup of $\mathbb{Z}^2$ that contains all the differences $t_1 - t_2$, $t_1 \in S_{ij}(L)$, $t_2 \in S_{ij}(L)$. It turns out that if L is irreducible [i.e., the matrix $\Sigma_t L(t)$ is irreducible], then the groups $G_{ij}(L) = G(L)$ do not depend on i, j and $S_{ij}(L) \subset v(i) - v(j) + G(L)$ for some sequence $v(1),\dots,v(r)$ in $\mathbb{Z}^2$ [see Lemma 1.1 in Höglund (1990b)].

Let

$$\omega_j(b,\theta) = e^{-\theta b}\frac{\text{prob}_\theta^*(N<\infty, S_N \ge b, \xi_N = j)}{E_\theta^* S_N}\rho_j(\theta)^{-1}, \tag{1.15}$$

where the star indicates that the underlying probability distribution is given by the transition probabilities (1.7), but the initial distribution (1.6) which is stationary for $\text{prob}_\theta(\xi_1 = j \mid \xi_0 = i)$ is replaced by the initial distribution which is stationary for $\text{prob}_\theta(N < \infty, \xi_N = j \mid \xi_0 = i)$:

$$\text{prob}_\theta^*(\xi_0 = i) = \sigma_i^*(\theta)\rho_i(\theta). \tag{1.16}$$

Here $\sigma^*(\theta)$ is the unique left positive eigenvector of the matrix $\Sigma_n\Sigma_u e^{\theta u}L(n,u)/\lambda(\theta)^n$, that satisfies $\sigma^*(\theta)\cdot\rho(\theta) = 1$.

We shall determine the asymptotic behaviour of $Z(n,u)$ when (n,u) tends to infinity in the cone

$$\{(\tau, \tau m(\theta)); \tau > 0, \theta \in \Theta, m(\theta) > 0\}. \tag{1.17}$$

It follows from Lemma 1.1 that the equation $m(\theta) = u/n$ has a unique solution $\theta = \hat{\theta}(u/n)$ when (n,u) belongs to this cone. We shall also need the set

$$\hat{A} = \left\{\theta \in \Theta;\ \sum_{a \in A} \lambda(\theta)^a < \infty\right\}$$

$$(1.18)\qquad = \begin{cases} \{\theta \in \Theta; \lambda(\theta) > 1\} & \text{if } \inf A = -\infty, \sup A < \infty, \\ \Theta & \text{if } \inf A > -\infty, \sup A < \infty, \\ \{\theta \in \Theta; \lambda(\theta) < 1\} & \text{if } \inf A > -\infty, \sup A = \infty. \end{cases}$$

Note that $\hat{A}^c$ does not stand for the complement of $\hat{A}$ but

$$\hat{A}^c = \left\{\theta \in \Theta;\ \sum_{a \in A^c} \lambda(\theta)^a < \infty\right\}.$$

THEOREM 1.2. *Suppose that L is irreducible and that $G(L) = \mathbb{Z}^2$. If $m(\theta) > 0$, then $\text{prob}_\theta(N < \infty) = 1$ and the expressions* (1.15) *are well-defined.*

Let n and $u \to \infty$ in the cone (1.17) *in such a way that $\hat{\theta} = \hat{\theta}(u/n)$ stays within but away from the boundary of the set $\{\theta \in \Theta; m(\theta) > 0, \theta \in \hat{A} \cup \hat{A}^c\}$.*

(i) *If $\hat{\theta}(u/n) \in \hat{A}$, then*

$$(1.19)\qquad \begin{aligned} Z_{ij}(n,u) &= \lambda(\hat{\theta})^n e^{-\hat{\theta}u}\big(2\pi n Q(\hat{\theta})\big)^{-1/2} \rho_i(\hat{\theta}) \\ &\quad \times \left(\sum_{a \in A} \lambda(\hat{\theta})^a \sum_{b \in B} \omega_j(b, \hat{\theta}) + o(1) \right), \end{aligned}$$

uniformly in $\hat{\theta}(u/n)$. Here $0 < Q(\theta) < \infty$ is as in (3.18).

(ii) *If $\hat{\theta}(u/n) \in \hat{A}^c$ and if there is a $\kappa \in \Theta$ such that $\lambda(\kappa) = 1$ and $m(\kappa) > 0$, then*

$$(1.20)\qquad Z_{ij}(n,u) = e^{-\kappa u} \rho_i(\kappa) \left(\sum_{b \in B} \omega_j(b, \kappa) + o(1) \right),$$

uniformly in $\hat{\theta}(u/n)$.

The theorem will be proved in the following sections.

Write I for the interval $I = \{m(\theta); \theta \in \Theta, m(\theta) > 0\}$. The function $m(\theta)$ is strictly increasing and hence the approximation (i) holds for $u/n \in m(\hat{A}) \cap I$ and the approximation (ii) for $u/n \in m(\hat{A}^c) \cap I$. Furthermore, $\log \lambda(\theta)$ is convex and $\lambda(0) = 1$. So if $m(0) > 0$, then $\kappa = 0$, and if $m(0) < 0$, then there is a possibility that $\lambda(\kappa) = 1$ for some $\kappa > 0$, and then $m(\kappa)$ is necessarily positive.

Thus, for example, if we let $A = \{\dots, -2, -1, 0\}$, $B = \mathbb{Z}_+$, we get an approximation for

$$\text{prob}\left(\max_{k \le n} S_k > u \right).$$

The approximation (i) holds when $m(\kappa) < u/n \in I$ and (ii) holds when $m(\kappa) > u/n \in I$. Here we assumed the existence of κ.

The condition $G(L) = \mathbb{Z}^2$ is just a normalization; see the comment around formula (1.14) in Höglund (1990b).

Theorem 1.2(ii) is applicable to the case $A = \mathbb{Z}$.

COROLLARY 1.3. *Suppose that* L *is irreducible and that* $G(L) = \mathbb{Z}^2$. *If* $0 \in \Theta$ *and if there is a* $\kappa \in \Theta$ *such that* $\lambda(\kappa) = 1$ *and* $m(\kappa) > 0$, *then* $\text{prob}_\kappa(N < \infty) = 1$, $\omega_j(b, \kappa)$ *are well defined and*

$$\text{prob}\big(N(u) < \infty, S_{N(u)} \in u + B, \xi_{N(u)} = j | \xi_0 = i\big)$$

$$= e^{-\kappa u} \rho_i(\kappa) \Big(\sum_{b \in B} \omega_j(b, \kappa) + o(1) \Big) \tag{1.21}$$

as $u \to \infty$.

Thus in particular,

$$\text{prob}\big(S_{N(u)} = u + s | \xi_0 = i\big) \to \text{prob}_0^*(S_N \geq s)/E_0^* S_N \tag{1.22}$$

for $s = 1, 2, \dots$ when $m(0) > 0$. Here we used the fact that we can take $\rho_i(0) = 1$ for all i.

It follows from Proposition 2.3 that the corollary holds as it stands under the slightly weaker assumption $\{u \in \mathbb{Z}; (n, u) \in G(L)\} = \mathbb{Z}$.

In Section 2 we shall express the approximations in quantities that are only implicitly determined by P via L. In Section 3 we relate some of these quantities to the corresponding ones for P, which I think is preferable.

The above approximations are of the form $e^{-nH} n^{-1/2} D$ or $e^{-nH} D$. The exponent H is directly given by P and u/n, but D only implicitly. I think that it is possible to express even this directly via an analogue, Spitzer's formula when $B = \mathbb{Z}_+$, but I have not done this. See formula (2.13) in Höglund (1990a) for the corresponding expressions for ordinary random walks.

2. The implicit solution. The idea of the proof is to show that Z is of the form (2.9) for some suitably chosen F and then simply apply a renewal theorem to the right-hand side of this identity to get an approximation for Z.

Let $Z_{ij}^s(n, u)$, $u \geq 0$, denote the probability (1.3) in the special case when $A = \{0\}$, $B = \{s\}$ and define $Z_{ij}^s(n, u) = 0$ for $u < 0$.

The event $N(u) = n$, $S_{N(u)} = u + s$, $\xi_{N(u)} = j$ occurs if and only if either $N(0) = n$, $S_{N(0)} = u + s$, $\xi_{N(0)} = j$ or else there is an $m \leq n$, $v \leq u$ and a $k \in \{1, \dots, r\}$ such that $N(0) = m$, $S_{N(0)} = v$, $\xi_{N(0)} = k$ and then $N(u) = n$, $S_{N(u)} = u + s$, $\xi_{N(u)} = j$.

Therefore

$$Z^s(n, u) = F^s(n, u) + L * Z^s(n, u). \tag{2.1}$$

Here

$$F_{ij}^s(n, u) = \begin{cases} \text{prob}(N = n, S_N = u + s, \xi_N = j | \xi_0 = i) & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases} \tag{2.2}$$

and the star denotes convolution

$$L * Z^s(n,u) = \sum_{m,v} L(m,v) Z^s(n-m, u-v). \tag{2.3}$$

The equation (2.1) implies

$$Z^s = \sum_{k=0}^{h-1} L^{k*} * F^s + L^{h*} * Z^s \tag{2.4}$$

for $h = 1, 2, \dots$. Observe that $\operatorname{supp}(L) \subset [1,\infty) \times [1,\infty)$ and hence that $\operatorname{supp}(L^{h*}) \subset [h,\infty) \times [h,\infty)$; use the fact that $Z^s(n,u) = 0$ when $n \le 0$ or $u < 0$ and conclude that $L^{h*} * Z^s(n,u) = 0$ for all $h \ge \min(n, u+1)$.

Therefore

$$Z^s = R * F^s, \tag{2.5}$$

where

$$R = \sum_{n=0}^{\infty} L^{n*}. \tag{2.6}$$

Let A, B and Z be as in (1.3) and note that

$$\begin{aligned} &\sum_{k-n \in A} \sum_{s \in B} Z^s(k,u) = Z(n,u), \\ &\sum_{k-n \in A} \sum_{s \in B} F^s(k,u) \\ &\quad = \operatorname{prob}(N - n \in A, S_N - u \in B, \xi_N = j \mid \xi_0 = i) \end{aligned} \tag{2.7}$$

for $u \ge 0$. The identity (2.9) therefore follows from (2.5) with

$$F_{ij}(n,u) = \begin{cases} \operatorname{prob}(N - n \in A, S_N - u \in B, \xi_N = j \mid \xi_0 = i) & \text{for } u \ge 0, \\ 0 & \text{for } u < 0. \end{cases} \tag{2.8}$$

LEMMA 2.1. *The probabilities* (1.3) *satisfy*

$$Z = R * F \tag{2.9}$$

for all $A \subset \mathbb{Z}$, $B \subset \mathbb{Z}_+$.

First some notation: Let

$$\hat{L}_{ij}(\eta,\theta) = \sum_n \sum_u e^{\eta n + \theta u} L_{ij}(n,u) = E\big(e^{\eta N + \theta S_N} \delta_j(\xi_N) \mid \xi_0 = i\big), \tag{2.10}$$

where $\delta_j(\xi) = 1$ if $\xi = j$ and $\delta_j(\xi) = 0$ if $\xi \ne j$. Let furthermore Θ^L denote the interior of the set of $\zeta = (\eta, \theta) \in \mathbf{R}^2$ for which $\hat{L}(\zeta) < \infty$. (We do not exclude the possibility $\Theta^L = \varnothing$ in this section.) Assume that $\hat{L}(\zeta)$ is irreducible and write $\lambda(\zeta)$ for its maximal positive eigenvalue and $\sigma(\zeta)$ and $\rho(\zeta)$ for the corresponding left and, respectively, right positive eigenvectors, normalized in such a way that $\sigma(\zeta) \cdot \rho(\zeta) = 1$. The symbols λ, ρ and σ thus have another meaning in this section than that in the introduction.

Let

$$(2.11)\quad \lambda'(\zeta) = (\partial_1\lambda(\zeta), \partial_2\lambda(\zeta)) \quad \text{and} \quad \lambda''(\zeta) = \begin{pmatrix} \dfrac{\partial^2\lambda(\zeta)}{\partial\eta^2} & \dfrac{\partial^2\lambda(\zeta)}{\partial\eta\,\partial\theta} \\ \dfrac{\partial^2\lambda(\zeta)}{\partial\theta\,\partial\eta} & \dfrac{\partial^2\lambda(\zeta)}{\partial\theta^2} \end{pmatrix}$$

denote the gradient and the second order derivative matrix.

We shall determine the asymptotic behaviour of $Z(n, u)$ when (n, u) tends to infinity in the cone

$$(2.12)\quad \{\tau\lambda'(\zeta); \tau > 0, \zeta \in \Theta^L, \lambda(\zeta) = 1\}.$$

The probability measure corresponding to that given by (1.6) and (1.7) is

$$(2.13)\quad \begin{aligned} \text{prob}_\zeta(\xi_0 = i) &= \sigma_i(\zeta)\rho_i(\zeta), \\ \text{prob}_\zeta(N = n, S_N = u, \xi_N = j|\xi_0 = i) &= \frac{e^{\eta n + \theta u} L_{ij}(n, u)\rho_j(\zeta)}{\lambda(\zeta)\rho_i(\zeta)}. \end{aligned}$$

It is seen in the same way as in the proof of Lemma 1.1 that $\lambda'(\zeta) = (E_\zeta N, E_\zeta S_N)$, where E_ζ stands for expectation with respect to the previous probability measure. The probabilistic intepretation of $\lambda''(\zeta)$ is, however, more complicated.

The set that corresponds to $\hat{A}$ is

$$(2.14)\quad \check{A} = \left\{\eta; \sum_{a \in A} e^{-\eta a} < \infty\right\}$$

and instead of $\hat{\theta}(u/n)$ we shall consider $\check{\zeta}(n, u)$ which is the unique $\zeta = \check{\zeta}(n, u)$ for which $\lambda(\zeta) = 1$ and $\lambda'(\zeta)$ and (n, u) have the same direction. See Lemma 1.4 in Höglund (1990b).

PROPOSITION 2.2. *Assume that L is irreducible and that $G(L) = \mathbb{Z}^2$. Put*

$$(2.15)\quad c_{ij}(b, \zeta) = e^{-\theta b}\rho_i(\zeta)\text{prob}_\zeta(N < \infty, S_N \geq b, \xi_N = j)/\rho_j(\zeta).$$

Let u and $n \to \infty$ in the cone (2.12) in such a way that $\check{\zeta} = \check{\zeta}(n, u) = (\check{\eta}(n, u), \check{\theta}(n, u))$ stays within but away from the boundary of the set of $\zeta = (\eta, \theta)$ determined by

$$(2.16)\quad \zeta \in \Theta^L, \qquad (\eta, 0) \in \Theta^L, \qquad \lambda'(\zeta) \neq 0, \qquad \eta \in \check{A} \cup \check{A}^c.$$

(i) *If $\check{\eta}(n, u) \in \check{A}$, then*

$$(2.17)\quad \begin{aligned} Z_{ij}(n, u) &= e^{-\check{\eta} n - \check{\theta} u}\left(2\pi T(\check{\zeta})C(\check{\zeta})\right)^{-1/2} \\ &\quad \times\left(\sum_{a \in A} e^{-\check{\eta} a} \sum_{b \in B} c_{ij}(b, \check{\zeta}) + o(1)\right) \end{aligned}$$

uniformly in $\check{\zeta}(n,u)$. *Here* $T(\check{\zeta})$ *is determined by* $(n,u) = T\lambda'(\check{\zeta})$ *and* $C(\zeta) = \lambda'(\zeta)\cdot\lambda''(\zeta)^{-1}\lambda'(\zeta)$. *The existence of* $\lambda''(\zeta)^{-1}$ *is part of the conclusion.*

(ii) *If* $\check{\eta}(n,u) \in \check{A}^c$ *and if there is a* $\kappa \geq 0$ *such that* $\lambda(0,\kappa) = 1$ *and* $\partial_2\lambda_L(0,\kappa) > 0$, *then*

$$Z_{ij}(n,u) = e^{-\kappa u}\,\partial_2\lambda(0,\kappa)^{-1}\Big(\sum_{b\in B} c_{ij}(b,(0,\kappa)) + o(1)\Big) \tag{2.18}$$

uniformly in $\check{\zeta}(n,u)$.

The case $A = \mathbb{Z}$ will be needed in the proof. In this case the functions Z and F will depend only on the variable u and we can replace L in the identities (2.6) and (2.9) by $L_2(u) = \Sigma_n L(n,u)$, the marginal distribution for S_N.

PROPOSITION 2.3. *Assume that L is irreducible and that* $G(L_2) = \mathbb{Z}$. *If* $(0,0) \in \Theta^L$ *and if there is a* $(0,\kappa) \in \Theta^L$ *such that* $\lambda(0,\kappa) = 1$ *and* $\partial_2\lambda(0,\kappa) > 0$, *then*

$$\begin{aligned}&\operatorname{prob}\big(N(u) < \infty, S_{N(u)} \in u + B, \xi_{N(u)} = j \mid \xi_0 = i\big)\\ &\quad = e^{-\kappa u}\,\partial_2\lambda(0,\kappa)^{-1}\Big(\sum_{b\in B} c_{ij}(b,(0,\kappa)) + o(1)\Big)\end{aligned} \tag{2.19}$$

as $u \to \infty$.

PROOF OF PROPOSITION 2.3. It follows from Lemma 2.1 and the one-dimensional version of Theorem 1.5 in Höglund (1990b) that the expression to the left in (2.19) equals

$$e^{-\kappa u}\big(\partial_2\lambda(0,\kappa)\big)^{-1}\Big(\sum_{\nu} \rho_i(0,\kappa)\sigma_\nu(0,\kappa)\hat{F}_{\nu j}(\kappa) + o(1)\Big), \tag{2.20}$$

provided the sums

$$\hat{F}_{ij}(\theta) = \sum_u e^{\theta u} F_{ij}(u) = E\big(e^{\theta S_N}\delta_j(\xi_N) \mid \xi_0 = i\big) \tag{2.21}$$

are convergent for θ in a neighbourhood of κ. Here $F_{ij}(u) = F_{ij}(n,u)$ as defined in (2.8) but with $A = \mathbb{Z}$.

We shall use a superscript to indicate the dependence on the set B. A change of the order of summation yields

$$\begin{aligned}\hat{F}_{ij}^B(\theta) &= \sum_{b\in B} e^{-\theta b}\sum_{v\geq b} e^{\theta v}\sum_m L_{ij}(m,v)\\ &\leq \hat{F}_{ij}^{\mathbb{Z}_+}(\theta) = \begin{cases}\displaystyle\sum_m\sum_v (e^{\theta v} - 1)L_{ij}(m,v)/(e^\theta - 1) & \text{if } \theta \neq 0,\\ \displaystyle\sum_m\sum_v vL_{ij}(m,v) & \text{if } \theta = 0,\end{cases}\end{aligned} \tag{2.22}$$

and these sums are convergent for all $(0,\theta)$ in Θ^L since this set is open and since $(0,0) \in \Theta^L$.

That the sum in (2.20) equals the sum in (2.19) follows from the second identity in (2.13) with $\zeta = (0, \kappa)$. □

PROOF OF PROPOSITION 2.2. The proof of part (i) is similar to the proof of Proposition 2.3. We use the two-dimensional version of the above mentioned theorem and have to verify that the sums

$$\hat{F}_{ij}(\zeta) = \sum_n \sum_u e^{\eta n + \theta u} F_{ij}(n, u) = E\left(e^{\eta N + \theta S_N} \delta_j(\xi_N) | \xi_0 = i\right) \tag{2.23}$$

are convergent for ζ in the set determined by (2.16).

In this case,

$$\begin{aligned} \hat{F}_{ij}^B(\zeta) &= \sum_{a \in A} e^{-\eta a} \sum_{b \in B} e^{-\theta b} \sum_m \sum_{v \geq b} e^{\eta m + \theta v} L_{ij}(m, v) \\ &\leq \hat{F}_{ij}^{\mathbb{Z}^+}(\zeta) \\ &= \sum_{a \in A} e^{-\eta a} \times \begin{cases} \sum_m \sum_v e^{\eta m}(e^{\theta v} - 1) L_{ij}(m, v)/(e^\theta - 1) & \text{if } \theta \neq 0, \\ \sum_m \sum_v e^{\eta m} v L_{ij}(m, v) & \text{if } \theta = 0, \end{cases} \end{aligned} \tag{2.24}$$

and these sums are convergent for all ζ in the set (2.16) since this set is open.

The remainder of part (i) follows from the second identity in (2.13).

In order to prove part (ii), note that

$$Z^A(n, u) + Z^{A^c}(n, u) = Z^{\mathbb{Z}}(n, u). \tag{2.25}$$

Here we used a superscript to indicate the dependence on A. We can apply part (i) to $Z^{A^c}(n, u)$ and Proposition 2.3 to $Z^{\mathbb{Z}}(n, u)$. Furthermore,

$$\check{\eta}(n, u) n + \check{\theta}(n, u) u = \max_{\zeta} \{\eta n + \theta u ; \lambda(\zeta) = 1\} \geq \kappa u \tag{2.26}$$

and hence $Z^A(n, u) = Z^{\mathbb{Z}}(n, u)(1 + O(n^{-1/2}))$.

3. Proof of the explicit version. We shall need an auxiliary function $G(n, u) = (G_{ij}(n, u))$, where for $n \geq 0$,

$$G_{ij}(n, u) = \text{prob}(N > n, S_n = u, \xi_n = j | \xi_0 = i). \tag{3.1}$$

Here $S_0 = 0$ and hence $G(0, u) = \delta(0, u)$, where $\delta(n, u) \neq 0$ only when $(n, u) = (0, 0)$ and equals I, the identity matrix, in that case. Let us agree that both $G(n, u)$ and $L(n, u)$ equal 0 when $n < 0$.

By an obvious modification of the argument [page 599 in Feller (1971)]:

$$\begin{aligned} &\{N = n, S_n = u, \xi_n = j\} \\ &\quad = \bigcup_{k=1}^{r} \bigcup_{t \leq 0} (\{N > n - 1, S_{n-1} = t, \xi_{n-1} = k\} \\ &\qquad \cap \{U(k, \xi_n) = u - t, \xi_n = j\}) \end{aligned} \tag{3.2}$$

for $u > 0$ and hence

$$L(n,u) = \sum_t G(n-1,t)P(u-t), \qquad u > 0. \tag{3.3}$$

In a similar way we obtain

$$G(n,u) = \delta(n,u) + \sum_t G(n-1,t)P(u-t), \qquad u \le 0, \tag{3.4}$$

and hence

$$L(n,u) + G(n,u) = \delta(n,u) + \sum_t G(n-1,t)P(u-t) \tag{3.5}$$

for all u.

Define $\hat{L}(\eta,\theta)$ as in (2.10) and

$$\hat{G}(\eta,\theta) = \sum_{n=0}^{\infty} e^{\eta n}\sum_u e^{\theta u}G(n,u). \tag{3.6}$$

LEMMA 3.1. *If $\theta \in \Theta$, $m(\theta) > 0$ and $e^{\eta}\lambda(\theta) = 1$, then $\hat{G}$ converges in a neighbourhood of the point (η,θ).*

Let $l = \inf_{\theta\in\Theta}\lambda(\theta)$, then $(-\infty, \log(1/l)) \times \Theta \subset \Theta^L$.

PROOF. Let $\theta' < \theta$, then

$$\begin{aligned}\hat{G}(\eta,\theta) &\le \sum_n \sum_{u\le 0} e^{\eta n}e^{\theta u}P^{n*}(u) \le \sum_n\sum_u e^{\eta n}e^{\theta' u}P^{n*}(u)\\ &= \sum_n e^{\eta n}\hat{P}(\theta')^n.\end{aligned} \tag{3.7}$$

The latter sum converges if and only if $\sum_n e^{\eta n}\lambda(\theta')^n < \infty$, that is, if $e^{\eta}\lambda(\theta') < 1$. But $\log\lambda$ has positive derivative $m(\theta)$ at the point θ and hence $\lambda(\theta') < \lambda(\theta)$ when θ' is less than but sufficiently close to θ.

In the same way we see that $\hat{L}(\eta,\theta)$ converges if $e^{\eta n}\lambda(\theta') < 1$ for some $\theta' \ge \theta$. It follows from (3.8) that $\hat{L}(\eta,\theta)$ converges if $\hat{G}(\eta,\theta)$ does. Therefore $\hat{L}(\eta,\theta)$ converges if $e^{\eta}\lambda(\theta') < 1$ for some θ'. □

We get from (3.5),

$$\hat{L}(\eta,\theta) + \hat{G}(\eta,\theta) = I + e^{\eta}\hat{G}(\eta,\theta)\hat{P}(\theta) \tag{3.8}$$

or

$$\hat{G}(\eta,\theta)\big(I - e^{\eta}\hat{P}(\theta)\big) = I - \hat{L}(\eta,\theta). \tag{3.9}$$

We shall compare the two chains defined by (1.7) and (1.6), respectively, with (2.13). In order to keep clear the difference between the two chains we shall add a subscript or superscript P and, respectively, L.

Let $\zeta = (\eta, \theta)$. The mean-value vector $m^L(\zeta) = (E_\zeta^L N, E_\zeta^L S_N)$ is defined by

$$E_\zeta^L N = \sigma^L(\zeta) \cdot \frac{\partial \hat{L}(\zeta)}{\partial \eta} \rho^L(\zeta)/\lambda_L(\zeta), \tag{3.10}$$
$$E_\zeta^L S_N = \sigma^L(\zeta) \cdot \frac{\partial \hat{L}(\zeta)}{\partial \theta} \rho^L(\zeta)/\lambda_L(\zeta).$$

THEOREM 3.2. *Assume that* $m^P(\theta) > 0$.

(i) *The variables* N *and* S_N *are proper with respect to* prob_θ *and* $E_\theta(t^N | \xi_0 = i) < \infty$ *for some* $t > 1$ *and all* $i \in X$.

(ii) $\lambda_L(\eta, \theta) = 1$ *if and only if* $e^\eta \lambda_P(\theta) = 1$.

(iii) *If* $e^\eta \lambda_P(\theta) = 1$, *then* $E_\zeta^L N < \infty$ *and*

$$E_\zeta^L S_N = m^P(\theta) E_\zeta^L N,$$
$$\rho^L(\zeta) = \rho^P(\theta), \tag{3.11}$$
$$\sigma^L(\zeta) \hat{G}(\zeta) = E_\zeta^L N \sigma^P(\theta).$$

(iv)

$$E\left(\lambda_P(\theta)^{-N} e^{\theta S_N} \frac{\rho_{\xi_N}^P(\theta)}{\rho_{\xi_0}^P(\theta)} \middle| \xi_0 = i \right) = 1. \tag{3.12}$$

The first of the identities (3.11), which is a Markov version of Wald's identity, is certainly related to the generalized Wald's identity of Franken and Lisek (1982). Note that it implies that $m^L(\zeta)$ has the same direction as the vector $(1, m^P(\theta))$ which is what we shall need. The identity (3.12) is a special case of Folgerung 1 of Küchler and Semjonov (1979). We shall not use it but it serves as an intermediate step in the proof.

PROOF OF THEOREM 3.2. Put $\zeta_0 = (\eta(\theta), \theta)$, where $\eta(\theta)$ is defined by $e^{\eta(\theta)}\lambda(\theta) = 1$.

The identity (3.9) yields

$$\big(I - \hat{L}(\zeta_0)\big)\rho^P(\theta) = 0, \tag{3.13}$$

that is, (3.12) holds and hence N and S_N are proper. The remainder of (i) follows from Lemma 3.1 and the observation that $\hat{L}(\zeta)$ converges whenever $\hat{G}(\zeta) < \infty$ [let $t = e^\eta \lambda_P(\theta)$].

Another consequence of (3.9) is that $\rho^L(\zeta_0) = \rho^P(\theta)$ and $\lambda_L(\zeta_0) = 1$ [recall that $\lambda_L(\zeta)$ is the only eigenvalue that has positive eigenvectors]. $\hat{L}(\zeta)$ increases strictly with η and the same must be true for $\lambda_L(\zeta)$ because

$$\lambda_L(\zeta) = \max_{f \geq 0} \min_{i \in X} \frac{\hat{L}(\zeta) f(i)}{f(i)} \tag{3.14}$$

[see Wielandt (1950)]. Therefore $\operatorname{sign}(\lambda_L(\zeta) - 1) = \operatorname{sign}(e^{\eta}\lambda_P(\theta) - 1)$ and hence (ii) follows.

The identity

$$\sigma^L(\zeta_0)\hat{G}(\zeta_0)\big(I - \lambda_P(\theta)^{-1}\hat{P}(\theta)\big) = 0 \tag{3.15}$$

follows from (3.9) and hence there is a positive number c such that

$$\sigma^L(\zeta_0)\hat{G}(\zeta_0) = c\sigma^P(\theta). \tag{3.16}$$

Differentiate (3.9) with respect to η. Multiply by $\sigma^L(\zeta_0)$ and $\rho^L(\zeta_0)$ from the left and right and conclude that

$$c = E^L_{\zeta_0}N. \tag{3.17}$$

The remaining identity follows in the same way after differentiation of (3.9) with respect to θ. □

We shall finally use the results of this section to show how Theorem 1.2 follows from Proposition 2.2.

Proof of Theorem 1.2. Assume $m(\theta) > 0$. Define $\eta(\theta)$ by $e^{\eta(\theta)}\lambda(\theta) = 1$. It follows from Theorem 3.2(ii) that $\lambda^L(\eta, \theta) = 1$ if and only if $\eta = \eta(\theta)$.

Lemma 3.1 implies that $(\eta(\theta), \theta) \in \Theta^L$ and $(\eta(\theta), 0) \in \Theta^L$, for all θ in the open set $\{\theta \in \Theta; m(\theta) > 0\}$. Thus in particular $E_\zeta S_N < \infty$.

Theorem 3.2(iii) implies that $\lambda'_L(\check{\zeta}) \neq 0$, $\check{\theta}(n, u) = \hat{\theta}(u/n)$, $T(\check{\zeta}) = n/E^P_\theta N$ and $\rho^L_i(\check{\zeta}) = \rho^P_i(\hat{\theta})$.

It is clear that $\eta(\theta) \in \check{A}$ if and only if $\theta \in \hat{A}$.

The condition on L implies that $\dim G(L) = 2$ and hence that $\lambda''_L(\zeta)$ is strictly positive definite [see Lemma 1.2 in Höglund (1990b)]. The constant Q is thus given by

$$Q(\theta) = \frac{\lambda'_L(\zeta) \cdot \lambda''_L(\zeta)^{-1}\lambda'_L(\zeta)}{E_\zeta N (E_\zeta S_N)^2}, \tag{3.18}$$

where $\zeta = (\eta(\theta), \theta)$.

The identity

$$c_{ij}(b, \zeta)/E^*_\theta S_N = \rho^P_i(\theta)\omega_j(b, \theta) \tag{3.19}$$

follows from the fact that $\rho^L_i(\zeta) = \rho^P_i(\theta)$ and $\sigma^L_i(\zeta) = \sigma^*_i(\theta)$, when $e^{\eta}\lambda(\theta) = 1$. □

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