The Annals of Probability
1991, Vol. 19, No. 3, 1145–1156

MULTIPLE STOCHASTIC INTEGRALS WITH RESPECT TO SYMMETRIC INFINITELY DIVISIBLE RANDOM MEASURES[1]

BY JERZY SZULGA

Auburn University

Multiple stochastic integrals with respect to an infinitely divisible symmetric random measure are constructed for integrands taking values in a Banach space.

1. Introduction. Consider a Polish space T equipped with the Borel σ-field $\mathscr{B}$ and a σ-additive measure μ on $\mathscr{B}$. Let $d \geq 1$ and

$$\underline{Z}(A) = (Z_1(A), \dots, Z_d(A))$$

denote an infinitely divisible independently scattered symmetric random measure with values in $\mathbb{R}^d$, defined on a δ-ring $\mathscr{R} \subset \mathscr{B}$.

The aim of this paper is a construction of a multiple stochastic integral

$$(1)\qquad Zf = Z_1 \otimes \cdots \otimes Z_d f = \int \underset{T^d}{\cdots} \int f(t_1, \dots, t_d)\, dZ_1(t_1) \cdots dZ_d(t_d)$$

for integrands f with values in a Banach space E. Throughout the paper we assume that functions f are symmetric with respect to permutations of their arguments and vanish on diagonal hyperplanes (i.e., whenever two or more arguments are equal). In some special cases, for example, for independent coordinate processes $Z_1, \dots, Z_d$, this assumption is not necessary.

Many authors have investigated real one-dimensional stochastic integrals under various restrictions on the measure $\underline{Z} = Z_1$ [Prekopa (1957), random measures with finite variation; Urbanik and Woyczyński (1967), symmetry; Kwapień and Woyczyński (1988); etc.]. In all of the previously mentioned papers a deterministic finite control measure played a crucial role. Rosiński (1987) studied the case of Banach space valued integrands and of stationary processes without this limitation.

The origins of the multidimensional case were connected with Wiener (1930) who heuristically introduced a notion of Gaussian random chaoses (all components Z_i were equal to a Brownian motion). His theory was formalized and then extended to processes with finite variance by Itô (1951). Itô's approach was used by Engel (1982) for a construction of the multiple integral (1) for such processes and for $T = [0, t]$. Banach space valued integrands and

Received January 1989; revised January 1990.

[1]This research was initiated during author's visit to Case Western Reserve University, Cleveland, Ohio, under support from NSF Grant DMS-87-13103.

AMS 1980 *subject classifications*. Primary 60H05, 60G57.

Key words and phrases. Stochastic integral, multiple stochastic integral, integral, infinitely divisible random measure, Poisson process, multiple Wiener integral, Hilbert space, Banach space.

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the special case of strictly stable processes Z_i (either identical or independent) were developed by Krakowiak and Szulga (1988a). Real multiple stochastic integrals of type (1) with respect to symmetric and positive pure jump processes were investigated by Kallenberg and Szulga (1989) (even for functions with supports of infinite measure μ).

The list is far from complete and we refer for additional information to results quoted in the mentioned papers.

2. Notation. All random variables appearing in this paper are defined on a probability space (Ω, A, P) which is assumed to be sufficiently rich.

Throughout the paper T denotes a Polish space and $\mathscr{B}$ stands for the class of Borel sets. Let $\mathscr{R} \subset \mathscr{B}$ be a δ-ring. A random measure M with values in $\mathbb{R}^d$ defined on $\mathscr{R}$ is a map

$$M\colon \Omega \to L_0(\mathbb{R}^d)$$

such that for $A = \bigcup_i A_i \in \mathscr{R}$, A_i disjoint, $M(A) = \sum_i M(A_i)$ a.s., where the series converges unconditionally in probability. In other words, M is a vector measure with values in $L_0(\mathbb{R}^d)$.

Consider two random measures M_i on $\mathscr{R}_i$, $i = 1, 2$. If M is a random measure on a δ-ring $\mathscr{R}_{1,2}$ such that $\mathscr{R}_1 \times \mathscr{R}_2 \subset \mathscr{R}_{1,2}$ and

$$M(A_1 \times A_2) = M_1(A_1) \cdot M(A_2), \qquad A_i \in \mathscr{R}_i, \qquad i = 1, 2,$$

we call M a product random measure and write $M = M_1 \otimes M_2$. If both measures are the same, we write M_1^2.

Let $\underline{Z}$ admit the Lévy representation

$$E\exp\{i\langle \underline{u}, \underline{Z}(A)\rangle\}$$

$$= \exp\left\{-\tfrac{1}{2}\langle Q(A)\underline{u}, \underline{u}\rangle + \int_{\mathbb{R}^d}\int_A (\cos\langle \underline{u}, \underline{x}\rangle - 1)\nu(d\underline{s}, d\underline{x})\right\},$$

where $Q(A) = (q_{ij}(A))$ is a covariance matrix and ν is a Lévy measure. We shall write shortly

$$\underline{Z} =_D [0, Q, \nu].$$

Let $K \subset \{1, \dots, d\}$ and K^* denote its complement. For any operation o involving d coordinates we shall write o_K for its restriction to coordinates indexed by K. For example, if $\underline{z} = (z_1, \dots, z_d)$, then $\underline{z}_K = (z_i\colon i \in K)$, if $f\colon T^k \to \mathbb{R}$ is a function of d variables, then $\underline{t}_K \to f_K(\underline{t}_K, \underline{t}_{K^*})$ determines a family of functions on T^K labeled by $\underline{t}_{K^*}$ and so on.

If μ is a deterministic positive measure on B, we shall write $\mathscr{B}_\mu = \{B \in \mathscr{B}\colon \mu(B) < \infty\}$.

Let $\mathscr{B}_d$ denote the σ-field of Borel sets in T^d symmetric with respect to all permutations of axes. Let E be a Banach space. Define $\mathscr{F}(E) = \mathscr{F}_d(E) = \{f\colon f$ is a Borel E-valued function on T^d, symmetric and vanishing on diagonal hyperplanes$\}$. In the sequel we shall omit E if $E = \mathbb{R}$.

Let $\mathscr{F}_{\mu,0} \subset \mathscr{F}$ be a family of $\mathscr{B}_\mu \times \cdots \times \mathscr{B}_\mu$-measurable functions from $\mathscr{F}$, that is, admitting a representation

$$f = \sum_{\underline{i} \in \mathbb{N}^d} f(\underline{i}) \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_d}}, \tag{2}$$

where $A_i \in \mathscr{B}_\mu$ are disjoint sets and denote

$$\mathscr{F}_\mu = \mathscr{F}_{d,\mu} = \left\{ f \in \mathscr{F} \colon \operatorname{supp} f \subset A^d \text{ for some } A \in \mathscr{B}_\mu \right\}.$$

3. Control measure. We say that a positive countably additive deterministic measure μ on $(T, \mathscr{B})$ is a control measure of M if $\mathscr{B}_\mu \subset \mathscr{R}$ and

$$M(A_n) \to_{\mathrm{P}} 0 \quad \Leftrightarrow \quad \mu(A_n) \to 0, \qquad A_n \in \mathscr{B}_\mu.$$

As a control measure μ of $\underline{Z}$, one may choose

$$\mu(A) = \sum_{i=1}^{d} q_{ii}(A) + \int_{\mathbb{R}^d} \{\|\underline{x}\|^2 \wedge 1\} \nu(A, d\underline{x}), \qquad A \in \mathscr{B}.$$

The following result is widely known.

PROPOSITION 3.1. *Let $\underline{Z}$ be a symmetric, independently scattered infinitely divisible random measure on $(T, \mathscr{B})$ taking values in $\mathbb{R}^d$. Let μ be its control measure. Then the integral* $\mathbf{Zf}$ *converges for every $f \in \mathscr{F}_\mu \cap L_2(T^d, \mathscr{B}_\mu^d, \mu^d)$.*

PROOF. The random measure $\underline{Z}$, restricted to a set of finite measure μ, can be decomposed into a sum of two independent random $\mathbb{R}^d$-valued measures, the first of which possesses all moments and the second has a finite support. So, for a fixed $A \in \mathscr{B}_\mu$, let us write

$$\underline{Z}(B) = \underline{M}(B) + \underline{N}(B), \qquad B \in \mathscr{B}, B \subset A.$$

For each $K \subset \{1, \dots, d\}$, the integrals $\mathbf{N}_{K^*} f \mathbf{1}_{A^d}(\underline{t}_K, \cdot\,)$ are well defined for μ^K-almost all $\underline{t}_K$ since they are finite sums a.s. $\mathbf{M}_K f \mathbf{1}_{A^d}(\cdot\,, \underline{t}_{K^*})$ can be derived by a standard Itô approach using the L_2-theory [see Engel (1982), Theorem 4.5].

Therefore, using the decomposition

$$\mathbf{Z} = \sum_{K \subset \{1, \dots, d\}} \mathbf{M}_K \otimes \mathbf{N}_K$$

and applying Fubini's theorem, we can complete the construction of the integral. □

COROLLARY 1. *The product measure $\mathbf{Z}(A^d \cap \cdot\,)$ is well defined for every $A \in \mathscr{B}_\mu$.*

We refer for details to Rosiński and Woyczyński (1987) (their argument must be completed by the assumption of the finiteness of the control measure).

However, the extension of the product measure to sets not covered by regular rectangles required additional techniques. A major obstacle is the lack of a control measure of Z. A natural candidate μ^d is not suitable, except for some special cases, like Gaussian or stable processes.

EXAMPLE 1. Let $Z_1 = Z_2$ be a unit intensity Poisson processes. Then $Z_1 \otimes Z_2\{(x, y) \in [0, \infty)^2 : xy \leq 1\} < \infty$ even though the underlying set has infinite measure. Moreover, $Z_1 \otimes Z_2$ does not allow any deterministic control measure [see Kallenberg and Szulga (1989)].

EXAMPLE 2. Let $T = \mathbb{R}$ and $Z_1 = Z_2 = G + X$, where G is a Gaussian measure with $q(A) = E|G(A)|^2$ and X is generated by an α-stable stationary motion with intensity λ, $0 < \alpha < 2$. We can pick $\mu = q + \lambda$ as a control measure.

Choosing appropriately $A = \bigcup_i B_i \times B_{i+1}$, one can design either a set of infinite measure μ^2 such that $\mathbf{Z}A = \sum(G + X)(A_i)(G + X)(A_{i+1})$ converges a.s., or a set for which $\mathbf{Z}A$ does not exist even though it is of finite measure μ^2. Notice that the necessary and sufficient condition for convergence of $\mathbf{Z}A$ is finiteness of the following numerical series, where $q_i = q(B_i)$, $\lambda_i = \lambda(B_i)$:

$$\sum \lambda_i q_{i+1}^{\alpha/2}, \qquad \sum \lambda_{i+1} q_i^{\alpha/2},$$

$$\sum q_i q_{i+1}, \qquad \sum \lambda_i \lambda_{i+1}\left(\log_+ \frac{1}{\lambda_i \lambda_{i+1}} + 1\right).$$

4. Marginal Gaussian and Poisson multiple integrals. We decompose $\underline{Z}$ into a sum of independent Gaussian and Poissonian components

$$\underline{Z} = \underline{G} + \underline{X},$$

where

$$\underline{G} =_D [0, Q, 0] \quad \text{and} \quad \underline{X} =_D [0, 0, \nu].$$

We need some facts from Kallenberg and Szulga (1988). Although in that paper statements are formulated for real integrands, all results needed here immediately carry over to the Hilbert space context.

Let $k \leq d$. Let ξ be a Poisson process on a separable measure space (S, Σ, λ), where λ is its intensity measure. Let $\{\tau_i\}$ be a collection of all atoms of ξ and (ε_i) be a Rademacher sequence independent of ξ. We introduce a symmetrized Poisson process $\tilde{\xi}A = \sum_i \varepsilon_i 1_A(\tau_i)$.

Let $f: S^k \to H$ be a function with values in a Hilbert space H vanishing on diagonal hyperplanes. The integral $\tilde{\xi}^k f$ exists if and only if the integral $\xi^k \|f\|^2 < \infty$ a.s. The latter integrals can be interpreted as random multiple series.

On the other hand, natural Poisson processes ξ and $\tilde{\xi}$ on $S = T \times (\mathbb{R}^k \setminus \{0\})$ can be associated with the given Lévy process $\underline{X}$ so that

$$\xi = |\tilde{\xi}|, \qquad \tilde{\xi} = \sum_i \varepsilon_i \mathbf{1}.(\tau_i),$$

$$X_i(A) = \int_A \int_{\mathbb{R}^k} x_i \tilde{\xi}(ds, dx_1, \dots, dx_k).$$

Observe that

$$E\left[\|\tilde{\xi}^k f\|^2 | \xi\right] = \xi^2 \|f\|^2 \quad \text{a.s.} \tag{3}$$

REMARK. The random variable under the conditional expectation is not necessarily integrable. Nevertheless, the conditional expectation is well defined.

An operator acting from the space of functions $f\colon T^k \to H$ into the space of functions $f'\colon T^k \times \mathbb{R}^k \to H$ defined by the formula

$$L_k f(t_1, \dots, t_k; x_1, \dots, x_k) = x_1 \cdots x_k \cdot f(t_1, \dots, t_k),$$

yields a Poisson representation of the Lévy multiple integral

$$Xf = X_1 \otimes \cdots \otimes X_k = \tilde{\xi}^k L_k f.$$

The following result is due to Kallenberg and Szulga (1989).

THEOREM 4.1. *The integral* $\mathbf{X} f$ *exists* ($\mathbf{X} f_n \to_{\mathrm{P}} 0$, *respectively*) *if and only if*

$$\xi^k \|L_k f\|^2 < \infty \quad a.s. \qquad \left(\xi^k \|L_k f_n\|^2 \to_{\mathrm{P}} 0, \quad respectively\right). \tag{4}$$

Moreover,

(i) $\mathbf{X} f$ *exists for every bounded function* $f \in \mathscr{F}_{k,\mu}$;

(ii) Xf *exists for every function* $f(t_1, \dots, t_d) = f_1(t_1) \cdots f_d(t_d)$ *such that* $X_i f_i$ *exists*, $i = 1, \dots, k$, *and*

$$\mathbf{X} f = X_1 f_1 \cdots X_k f_k.$$

(iii) *If* $f_n \to \mu^k$ *a.e.*, $\sup_n |f_n| \le g$ *and* $\mathbf{X} g$ *exists, then* $\mathbf{X} f$ *and* $\mathbf{X} f_n$ *exist and* $\mathbf{X} f_n \to_{\mathrm{P}} \mathbf{X} f$.

Moreover, the integral $\mathbf{X}$ *is maximal in the sense that any linear operator* $\hat{\mathbf{X}}$ *satisfying* (i) *and* (iii) (*with* $\hat{\mathbf{X}}$ *replacing* $\mathbf{X}$) *and the condition*

(ii′) *for every* $A_1, \dots, A_k \in \mathscr{B}_\mu$,

$$\hat{\mathbf{X}}(A_1 \times \cdots \times A_k) = X_1(A_1) \cdots X_k(A_k)$$

must agree with X, *and corresponding integrable functions fulfill* (4).

The question concerning classes of integrable functions is rather delicate, especially if one wishes to go beyond the symmetric case [cf. a discussion in Kallenberg (1989)].

On the other hand, the Gaussian integral $\mathbf{G}f$ is much easier to characterize.

PROPOSITION 4.2. *Let $dq = dq_{11} \otimes \cdots \otimes dq_{kk}$ be a measure generated by the covariance matrix Q. Gf exists if and only if $\int_{T^k} |f|^2 \, dq < \infty$.*

PROOF. Observe that $\mathbf{G}f$ defines the isometry between $\mathscr{F}_{q,0}$ and $L_2(\Omega, \mathscr{A}, P)$, which can be extended onto $L_2(d_q)$. The identity

$$E|Gf|^2 = \int_{T^k} |f|^2 \, dq, \qquad f \in \mathscr{F}, \tag{5}$$

is easy to check. □

Let us point out that the Hilbert space $H = L(G) = L_2(T^k, \mathscr{F}, dq)$ appears in a natural way and will play the important role in the future construction.

5. Multiple stochastic integral. Let $K \subset \{1, \dots, d\}$, K^* be its complement and $k = \#K$. Define

$$\|f\|_{K^*} = \left(\int_{T^{K^*}} \|f\|^2 \, dq_{K^*} \right)^{1/2}, \qquad f \colon T^K \to \mathbb{R},$$

$$H_{K^*} = \{ f \in \mathscr{F} \colon \|f(\underline{t}_K, \cdot)\|_{K^*} < \infty \ \mu^K \text{ a.e.} \},$$

$$\mathscr{L}_K = \{ g \in \mathscr{F}_K \colon \xi^K (L_K g)^2 < \infty \},$$

where $L_K g(\underline{x}_K, \underline{t}_K) = \prod_{i \in K} x_i g(\underline{t}_K)$, according to our convention.

By virtue of Fubini's theorem, for every $f \in \mathscr{F}$ such that

$$\text{the function } \underline{t}_K \to \|f(\underline{t}_K, \cdot)\|_{K^*} \in \mathscr{L}_K, \tag{6}$$

we can define a mixed multiple integral

$$\mathbf{Y}_K f = \mathbf{X}_K(\mathbf{G}_{K^*} f) = \tilde{\xi}^K L_K(\mathbf{G}_{K^*} f). \tag{7}$$

PROPOSITION 5.1. *The mixed integral $\mathbf{Y}_K = \mathbf{X}_K \mathbf{G}_{K^*} f$ has the following properties*:

(i) *$\mathbf{Y}_K f$ exists for every bounded function with support contained in a rectangle $B \times C \in \mathscr{B}_K \times \mathscr{B}_{K^*}$ such that $\nu_K(B) \cdot q_{K^*}(C) < \infty$;*

(ii) *$\mathbf{Y}_k$ exists for every function $f(\underline{t}_K, \underline{t}_{K^*}) = g(\underline{t}_K) \cdot h(\underline{t}_{K^*})$ such that $\mathbf{G}_{K^*} h$ and $\mathbf{X}_K g$ exist and*

$$\mathbf{Y}_K f = \mathbf{X}_K g \cdot \mathbf{G}_{K^*} h.$$

(iii) *If $f_n \to f$ a.e., $|f_n| \le g$ and $\mathbf{Y}_K g$ exists, then $Y_K f$ exists and $Y_K f_n \to_{\mathrm{P}} Y_K f$. Moreover, any integral $\hat{\mathbf{Y}}_K$ satisfying (i) and (iii) (with $\mathbf{Y}_K$ replaced by $\hat{\mathbf{Y}}_K$) and the condition*

(ii′) *for every B and C such that $\nu_K(B) \cdot q_{K^*}(C) < \infty$*

$$\hat{\mathbf{Y}}_K B \times C = \mathbf{X}_K B \cdot G_{K^*} C$$

must agree with $\mathbf{Y}_K$, and corresponding integrable functions satisfy (6).

PROOF. By definition, we can view $Y_K f$ as a multiple Poisson integral $\tilde{\xi}^K F_K$ of a Hilbert space-valued function $F_K = L_K \mathbf{G}_{K^*} f\colon T^K \to H_{K^*}$. This is a consequence of Theorem 4.1 and the definition of $\mathbf{Y}_K$. More precisely, we need a part of the multiple integration theory which can be carried over to the Hilbert space case. As has been noted in Kallenberg and Szulga (1989), the condition for the existence of the multiple Poisson integral $\tilde{\xi}^K F_K$ is that $\xi \|F_K\|^2_{H_{K^*}} < \infty$. Similarly, the sequence $(\tilde{\xi} F_n) \to_{\mathrm{P}} 0$ if and only if the sequence of Poisson integrals $(\xi \|F_n\|^2_{H_{K^*}}) \to_{\mathrm{P}} 0$. Therefore, the conditions (i), (ii), (iii) follow immediately.

For the additional statement, we do not need the Hilbert space context. We apply a standard procedure: First, (ii′) implies that $\mathbf{Y} f$ and $\hat{\mathbf{Y}} f$ agree for simple functions f and then (iii) is used to prove the final statement. □

Now, we define

$$Zf = \sum_K \mathbf{Y}_K f$$

for $f \in \mathscr{F}_d$ with the property

(8) the function $\underline{t}_K \mapsto \|f(\underline{t}_K, \cdot)\|_{K^*} \in \mathscr{L}_K$ for every $K \subset \{1, \dots, d\}$.

$\mathscr{L}_Z$ will denote the space of functions satisfying (8).

THEOREM 5.2. *The properties* (i)–(iii) *are carried over to the integral* $\mathbf{Z}$. *That is,*

(i) $\mathbf{Z} f$ *exists for every bounded function with support contained in a cube* B^d *for some* $\mathscr{B} \in \mathscr{B}_\mu$;

(ii) $\mathbf{Z} f$ *exists for every function* $f(t_1, \dots, t_d) = f_1(t_1) \cdots f_d(t_d)$ *such that* $Z_i f_i$ *exists*, $i = 1, \dots, d$ *and*

$$\mathbf{Z} f = \prod_{i=1}^{d} Z_i f_i.$$

(iii) *If* $f_n \to f$ *a.e.*, $|f_n| \le g$ *and* $\mathbf{Z} g$ *exists, then* $\mathbf{Z} f$, $\mathbf{Z} f_n$ *exist and* $\mathbf{Z} f_n \to_{\mathrm{P}} \mathbf{Z} f$; *Any integral* $\hat{\mathbf{Z}}$ *satisfying* (i), (iii) (*with* $\mathbf{Z}$ *replaced by* $\hat{\mathbf{Z}}$) *and the condition*

(ii′) *for every* $A_1, \dots, A_d \in \mathscr{B}_\mu$,

$$\hat{\mathbf{Z}} A_1 \times \cdots \times A_d = Z_1(A_1) \cdots Z_d(A_d)$$

must agree with $\mathbf{Z}$, *and the class of* $\hat{\mathbf{Z}}$*-integrable functions is contained in* $\mathscr{L}_Z$.

PROOF (The first part). The properties (i)–(iii) follow immediately from Proposition 5.1. For the remaining part of the proof we need an auxiliary result. Notice that the conditional expectation appearing in the formulation is well-defined regardless of the integrability of the underlying random variable.

PROPOSITION 5.3. *Let f, f_n satisfy* (8). *Then*

(a) $$E\left[(\mathbf{Z}f)^2|\xi\right] = \sum_K \xi^K \|L_K f\|^2_{K^*},$$

(b) $$\mathbf{Z}f_n \to_{\mathrm{P}} 0 \quad \Leftrightarrow \quad \mathbf{Y}_K f_n \to_{\mathrm{P}} 0 \quad \textit{for every } K \subset \{1,\dots,d\}.$$

PROOF. The statement (a) can be easily proved for $f \in \mathscr{F}_0$ by virtue of Fubini's theorem and by applying (5) and (3). In the general case, it follows classically by the already proven part (ii) of Theorem 5.2 and the dominated convergence theorem.

The implication $\Leftarrow$ in (b) follows from the construction of the integral $\mathbf{Z}$. To prove $\Rightarrow$, we may assume that G, $\underline{\varepsilon}$ and ξ are defined on a product probability space $(\Omega_g \otimes \Omega_\varepsilon \otimes \Omega_p, \mathscr{A}_g \otimes \mathscr{A}_\varepsilon \otimes \mathscr{A}_p, \mathrm{P}_g \otimes \mathrm{P}_\varepsilon \otimes \mathrm{P}_p)$ and the mentioned processes are projection mappings. Denoting corresponding expectations by E_g and E_ε, we infer that

$$E\left[(Zf)^2|\xi\right] = E_g E_\varepsilon (\mathbf{Z}f)^2 \qquad \xi \text{ a.s.}$$

The latter identity can be easily seen by using the routine approach, first deriving it for simple functions, then passing to the limit. The first, already proven part of Theorem 5.2 can be used here.

If, in a given class of random variables, the convergence in probability and in L_2 are equivalent, then the property continues to hold for the closure in probability of this class [Krakowiak and Szulga (1988b), Proposition 2.2 and Corollary 2.3].

This observation applies to any sequence of integrals $\mathbf{Z}f_n$ since each of these integrals, conditioned on ξ, can be viewed as an a.s. limit of multivariate polynomials in independent Gaussian random variables. For such polynomials, the convergence in L_2 and in probability coincide [Krakowiak and Szulga (1988b), Corollary 2.8 and 2.6].

Thus the mapping $\mathbf{Z}f \to E[(\mathbf{Z}f)^2|\xi]$ is continuous by means of the convergence in probability. To see this, we switch to a.s. convergence by passing to subsequences and use Fubini's theorem.

An application of (4) of Theorem 4.1 and (a) of this Proposition completes the proof. □

PROOF (Theorem 5.2, the conclusion). If $\hat{\mathbf{Z}}$ is an integral operator satisfying (i), (ii′) and (iii), then by (i), $\hat{\mathbf{Z}}$ agrees with $\hat{\mathbf{Z}}$ on $\mathscr{F}_0$ and by (ii′) and (iii), on the class of bounded functions with rectangular supports.

Let $\hat{\mathbf{Z}}f$ exist. There is a sequence (f_n) of bounded functions with rectangular supports such that $f_n \to f$ and $\sup_n |f_n| \le f$. Then $\hat{\mathbf{Z}}f_n \to_{\mathrm{P}} \hat{\mathbf{Z}}f$ by (iii). By Proposition 5.3(b), $\mathbf{Y}_K f_n$ is a Cauchy sequence for each K and thus $\xi^K(L_K f)^2 = \lim_n \xi^K(L_K f_n)^2 < \infty$ a.s. Hence, by Proposition 5.1(iii), $\mathbf{Y}_K f$ exists and $\mathbf{Y}_K f_n \to_{\mathrm{P}} \mathbf{Y}_K f$. Since K is arbitrary, this completes the proof of Theorem 5.2. □

REMARK. There is an alternative way of constructing the multiple stochastic integral of the type discussed in this paper. We can begin with the definition of a mixed multiple integral

$$\hat{\mathbf{Y}}_K f = \mathbf{G}_{K^*}(\mathbf{X}_K f),$$

which, by Fubini's theorem, is well defined for the functions $f: T^K \times T^{K^*} \to \mathbb{R}$ satisfying the condition

$$(9) \qquad \int_{T^{K^*}} \xi^K \big(L_K f(\underline{t}_K, \underline{t}_{K^*})\big)^2 q_{K^*}(dt_{K^*}) < \infty \quad \text{for } \mu^K\text{-almost all } \underline{t}_K.$$

We readily recognize the condition (6).

The further steps follow the spirit of Proposition 5.1 and Theorem 5.2. Their counterparts can be stated in a similar manner. It will follow that $\hat{\mathbf{Y}}_K \equiv \mathbf{Y}_K$. Exactly as in the original construction, we can derive the following identity, beginning with simple functions and then passing to limits:

$$E\big[|\hat{\mathbf{Y}}_K f|^2 | \xi\big] = \int_T^{K^*} \xi^K \big(L_K f(\underline{t}_K, \underline{t}_{K^*})\big)^2 q_{K^*}(dt_{K^*}) < \infty.$$

We omit the details since the arguments are very similar to those presented earlier and the new construction is more tedious than the original approach.

EXAMPLE 3. Refining Example 2, it is easily seen that, for a symmetric function f, the integral $\mathbf{Z} f$ exists if and only if the following three integrals converge [see also Cambanis, Rosiński and Woyczyński (1985)]:

$$\int \left(\int |f(x,y)|^2 q(dx) \right)^{\alpha} \lambda(dy), \qquad \int\!\!\int |f(x,y)|^2 q(dx) q(dy),$$

$$\int\!\!\int |f(x,y)|^{\alpha} \left(\log_+ \frac{|f(x,y)|^{\alpha}}{\int |f(x,v)|^{\alpha} \lambda(dv) \int |f(u,y)|^{\alpha} \lambda(du)} + 1 \right) \lambda(dx) \lambda(dy).$$

Existence of a general integral is reduced to 2^{d-1} Poisson integrals and finiteness of $L_2(dq)$-norm. Although explicit characterizations of the former integrals are rather tangled, one may employ a recursive procedure [Kallenberg and Szulga (1989), Theorem 3.7] to describe integrable functions by means of a number of deterministic integrals.

The following result does not resolve the characterization problem but may be used to view $\mathbf{Z} f$ as an iterated integral.

PROPOSITION 5.4 (Decoupling principle). *Let $\tilde{\mathbf{Z}}$ be a random measure whose coordinates are independent copies of the coordinates of $\mathbf{Z}$. Then $\mathbf{Z} f$ and $\tilde{\mathbf{Z}} f$ converge or diverge simultaneously.*

PROOF. We can apply the decoupling principles for Poisson and Gaussian multiple integrals. The one for Poisson integrals can be found in Kallenberg and Szulga (1989). Its Gaussian counterpart is easily seen because the underlying isometry (5) involves only diagonal measures from the covariance matrix Q. □

Define a functional

$$\phi f = E1 \wedge \left(E\left[(Zf)^2|\xi\right]\right)^{1/2}.$$

THEOREM 5.5. *The functional ϕ is an F-norm turning the linear space $\mathscr{L}_Z$ of functions satisfying* (8) *into a Fréchet space. Moreover, the natural embedding of $\mathscr{L}_Z$ into $L_0(\Omega, \mathscr{A}, P)$ is continuous and the set $\mathscr{F}_0$ is dense in $\mathscr{L}_Z$.*

PROOF. Metric properties of ϕ are evident. Continuity of the embedding map follows by an argument similar to that used in the proof of Proposition 5.3(b). To prove completeness of the metric ϕ, we can reduce the problem to Poisson integrals ξ^K and then apply Theorem 4.5 in Kallenberg and Szulga (1989). Finally, Theorem 5.2(iii) shows $\mathscr{F}_0$ to be dense in $\mathscr{L}_z$. □

PROPOSITION 5.6 (Contraction principle). *Let $\mathbf{Z}f$ exist and $g \in \mathscr{F}$ be a bounded function. Then $\mathbf{Z}fg$ exists. Moreover,*

$$\phi fg \le (1 \vee \|g\|_\infty)\phi f. \tag{10}$$

PROOF. Apply an elementary inequality $1 \wedge ab \le (1 \vee a)(1 \wedge b)$, $a, b \ge 0$ and Proposition 5.3(a). □

COROLLARY 2. *$\mathbf{Z}A$ is a random measure on*

$$\mathscr{R} = \left\{A \in \mathscr{B}_d \colon \mathsf{E}\left[(Zf)^2|\xi\right] < \infty \text{ a.s.}\right\}.$$

For a fixed $B \in \mathscr{B}_\mu$, $\mu^d(B^d \cap \cdot\,)$ can serve as a control measure for $Z(B^d \cap \cdot\,)$.

PROOF. Suppose that $\mathbf{Z}A$ exists and $A = \bigcup_i A_i$, where A_i are disjoint sets. Then, by the Contraction Principle, each $\mathbf{Z}A_i$ exists and the series $\sum_i \mathbf{Z}A_i$ converges unconditionally in probability. The latter statement is easily seen since, for every sequence $(\pm)$ of signs, we have

$$\mathsf{E}1 \wedge \left|\sum_{i=m}^{n} \pm \mathbf{Z}A_i\right| \le \phi \sum_{i=m}^{n} \pm \mathbf{1}_{A_i} \le \phi \bigcup_{i=m}^{n} A_i$$

by (10) and dominated convergence. □

6. Vector integrands. Let E be a separable Banach space (the separability assumption is not essential, though). One of the possible ways of constructing a stochastic integral for functions with values in E is the Dunford–Bartle approach [Bartle (1956)]. However, their generic formula needs to be adjusted to the discussed situation.

We utilize the notation from the previous sections. For functions from $\mathscr{F}_d(E)$, we write $f_n \to f$ [loc μ] if $f_n \to f$ in measure $\mu^d(B^d \cap \cdot\,)$ for every $B \in \mathscr{B}_\mu$.

1. For a simple $\mathscr{R}$-measurable function $f = \sum_B x_B \mathbf{1}_B$, define the operator

$$\mathbf{Z}[f; B] = \sum_B x_B \mathbf{Z} B.$$

2. A B-measurable function f is said to be $\mathbf{Z}$-integrable, $f \in L_{\mathbf{Z}}(E)$ in short, if there exists a sequence of simple $\mathscr{R}$-measurable functions (f_n) such that (a) $f_n \to f$ [loc μ]; (b) $\mathbf{Z}[f_n; B]$ converges in probability for every $B \in \mathscr{R}$.

The limit in 2(b) is then denoted by $\int_B f \, d\mathbf{Z}$, or $\mathbf{Z}[f; B]$, or simply $\mathbf{Z} f$ if $B = T$.

THEOREM 6.1. *The integral* $\mathbf{Z} f$ *is well defined.*

PROOF. We must show that the indefinite integrals $\int_B f \, dZ$ are determined uniquely. By applying functionals and using separability of the underlying Banach space, it is enough to prove the following claim for real simple $\mathscr{R}$-measurable functions and a certain real-valued random set function Z_B:

$$\text{If } \int_B f_n \, d\mathbf{Z} \to_{\mathrm{P}} Z_B \text{ for every } B \in \mathscr{R} \text{ and } f_n \to 0 \text{ [loc } \mu\text{], then } Z_B \equiv 0.$$

Indeed, let $\mathbf{Z} f_n$ converge in probability and $f_n \to 0$ [loc μ]. Since the space of $\mathbf{Z}$-integrable real functions is complete by Theorem 5.5, the limit of $\mathbf{Z} f_n$ is a.s. equal to an integral $\mathbf{Z} f$ for some f. Thus, there is a subsequence $(f_{n'})$ convergent to f μ^d a.e. Hence $f = 0$ μ^d a.e. and the limit must be 0 a.s. □

REMARK. In general, the existence of the integral $\mathbf{Z} f$ does not induce indefinite integrals $\mathbf{Z}[f; B]$ (unlike the real case), for the counterpart of the contraction principle (cf. Proposition 5.6) requires some additional geometric properties of the underlying Banach space. If, for two independent Rademacher sequences (ε_n^1) and (ε_n^2),

$$\sum_{m,n} x_{mn} \varepsilon_m^1 \varepsilon_n^2 \text{ converges a.s.} \quad \Rightarrow \quad \sum_{m,n} a_{mn} x_{mn} \varepsilon_m^1 \varepsilon_n^2 \text{ converges a.s.}$$

for all double arrays $\{x_{mn}\} \subset E$, $\{a_{mn}\} \subset [-1, 1]$, then the required contraction for integrals holds. Every Banach lattice with no finite dimensional subspaces uniformly isomorphic to l_∞^n has this property. For the discussion on contraction principles we refer to Krakowiak and Szulga (1986) [see also Krakowiak and Szulga (1988b)].

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DEPARTMENT OF ALGEBRA,
COMBINATORICS AND ANALYSIS
MATHEMATICS ANNEX 120
AUBURN UNIVERSITY
AUBURN, ALABAMA 37849-3501