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EXPONENTIAL DECAY FOR SUBCRITICAL CONTACT AND PERCOLATION PROCESSES

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We study the contact process, together with a version of the percolation process with one continuously varying coordinate. It is proved here that the radius of the infected cluster has an exponentially decaying tail throughout the subcritical phase. The same is true of the Lebesgue measure (in space-time) of this cluster. Certain critical-exponent inequalities are derived and the critical point of the percolation process in two dimensions is determined exactly.

1. Statement of results.

1.1. *Introduction.* The contact process is a stochastic model for the spread of disease amongst the members of a population distributed about d-dimensional space. Individuals inhabit the points of a lattice ($\mathbb{Z}^d$, say) and the process evolves roughly as follows. Infected individuals infect each of their neighbours at rate λ and are cured of the disease at rate δ. When fully formulated, these rules give rise to an infinite particle system which has received much attention [see Liggett (1985), Durrett (1988), Bezuidenhout and Grimmett (1990)]. There is a critical value ρ_c of the ratio $\rho = \lambda/\delta$ such that the probability $\theta^o(\lambda, \delta)$ that the disease survives forever from a single initial infective satisfies

$$\theta^o(\lambda, \delta)\begin{cases} = 0 & \text{if } \lambda/\delta \le \rho_c, \\ > 0 & \text{if } \lambda/\delta > \rho_c. \end{cases}$$

One of the main techniques for studying the contact process is the graphical representation of Harris (1974, 1978). Consider the graph $\mathbb{Z}^d \times [0, \infty)$, in which $\mathbb{Z}^d$ represents the spatial component and $[0, \infty)$ represents time. Along each time line $x \times [0, \infty)$ is positioned a Poisson process of points (with intensity δ) called deaths and between each ordered pair $x_1 \times [0, \infty)$ and $x_2 \times [0, \infty)$ of adjacent time lines, there is a Poisson process (with intensity λ) of crossings oriented in the direction x_1 to x_2. Then $\theta^o(\lambda, \delta)$ is the probability that there is an unbounded directed path from the origin of $\mathbb{Z}^d \times [0, \infty)$, using time lines in the direction of increasing time but crossing no death, together with crossings in the directions of their orientations. This graphical representation enables one to couple together contact processes with all initial configurations.

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As witnessed in the previous paragraph, the contact process has much in common with the percolation model [see Grimmett (1989)] and many arguments developed for the latter model have proved valuable when studying the former. One major difficulty can arise from the fact that the contact process has one coordinate (time) which is *continuously* varying rather than discrete. In loose terms the difficulty is as follows. Where in percolation-theoretic arguments one counts points, the corresponding argument for the contact process may require a counting of intervals. Now intervals may be long or short and so counting *by number* may give a very different result from counting *by Lebesgue measure*. The principal contribution of the present paper is to develop a technique for dealing with this problem. We do this while proving the following concrete result: the radius and Lebesgue measure of the region of $\mathbb{Z}^d \times [0, \infty)$ which is infected from the origin, when the process is subcritical (i.e., when $\rho < \rho_c$), has an exponentially decaying tail. This fills a gap in the theory of the contact process which was referred to but not dealt with by Aizenman and Barsky [(1987), pages 520–522)].

The contact process is a type of oriented percolation process. Its unoriented sibling arises as follows. Suppose for the moment that we are working in two dimensions and consider the anisotropic bond percolation process on the square lattice in which each horizontal bond is open with probability $\varepsilon\lambda$ and each vertical bond with probability $1 - \varepsilon\delta$, where $\varepsilon, \lambda, \delta > 0$. Writing $\theta_\varepsilon(\lambda, \delta)$ for the probability that the origin lies in an infinite cluster, it is established that $\theta_\varepsilon(\lambda, \delta) = 0$ if and only if $\lambda/\delta \leq 1$; proof of this may be distilled from Kesten (1982) and Grimmett (1989). In the limit as $\varepsilon \downarrow 0$, and with a suitable rescaling, the process approaches a certain partially continuous process which we may think of in the following way. Along each vertical line there is a Poisson process of points (with intensity δ) called deaths, while between each pair of adjacent vertical lines there is a Poisson process of horizontal crossings (with intensity λ). In this partially continuous process, let $\theta(\lambda, \delta)$ be the probability that there is an unbounded path from the origin following vertical lines and horizontal crossings but traversing no death. The original motivation for the present work was to verify the natural guess that $\theta(\lambda, \delta) = 0$ if and only if $\lambda/\delta \leq 1$. We note that the above limit, as $\varepsilon \downarrow 0$, arises naturally in work of Grimmett and Newman (1990), who have studied percolation processes on a class of graphs having three distinct phases, corresponding to the existence of 0, 1 and infinitely many open clusters, respectively.

The modern route to establishing the last result claimed above proceeds by extending the results and techniques of Menshikov (1986) and Aizenman and Barsky (1987) and then using special properties of the two-dimensional process in order to derive the exact calculation of the critical surface. In doing so, the same difficulty arises as that sketched above for the contact process. In circumventing this difficulty, we obtain certain results valid for a broader class of processes, namely, oriented and nonoriented percolation models in any number of dimensions, one of whose components is allowed to vary continuously; this class includes the contact process. In addition to demonstrating exponential decay below the critical point, we shall obtain certain critical-exponent inequalities.

1.2. *The models.* We introduce informally the two types of process studied in this paper, namely, a continuous-time percolation process and the contact process. We shall need later to discretize these processes for the purposes of analysis and therefore postpone until Section 2 a rigorous discussion of their properties.

Continuous-time percolation. Let $\mathbb{Z}^d$ be the set of d-vectors with integral components, as usual; for $x \in \mathbb{Z}^d$, we write $x = (x_1, \dots, x_d)$. We consider the set $\mathbb{Z}^d \times \mathbb{R}$ and think of it as comprising vertical copies of $\mathbb{R}$ laid out in the manner of a grid. Let $\lambda, \delta \geq 0$. On each vertical line $x \times \mathbb{R}$ (we often abbreviate $\{x\}$ to x), as x varies over $\mathbb{Z}^d$, we position a Poisson process of deaths with intensity δ, independently of all other vertical lines. Between each pair $x_1 \times \mathbb{R}$, $x_2 \times \mathbb{R}$ of adjacent vertical lines (i.e., x_1 and x_2 are neighbours in $\mathbb{Z}^d$), we place horizontal crossings or bonds in such a way that their centres form a Poisson process on $\frac{1}{2}(x_1 + x_2) \times \mathbb{R}$ with intensity λ, independently of all other Poisson processes in the construction. We write $P_{\lambda,\delta}$ and $E_{\lambda,\delta}$ for the corresponding probability measure and expectation operator. For $(x,t) \in \mathbb{Z}^d \times \mathbb{R}$, let $C(x,t)$ be the (random) set of points in $\mathbb{Z}^d \times \mathbb{R}$ which are joined to (x,t) by paths which follow vertical line segments and horizontal bonds but traverse no death. We write $C = C(0,0)$ for the corresponding set of points joined to the origin $(0,0)$ of $\mathbb{Z}^d \times \mathbb{R}$; we write 0 for this origin.

We are interested primarily in the function

$$\theta(\lambda, \theta) = P_{\lambda,\delta}(C \text{ is unbounded}). \tag{1.1}$$

It is easily seen, by rescaling vertically, that $\theta(\lambda, \delta) = \theta(\eta\lambda, \eta\delta)$ for any $\eta > 0$, and therefore $\theta(\lambda, \delta)$ is a function of λ/δ only. For $\rho > 0$, we define $\theta(\rho) = \theta(\rho\delta, \delta)$, where $\delta > 0$. It is clear that $\theta(0) = 0$ and that $\theta(\rho)$ is a nondecreasing function of ρ and therefore we may define the critical value

$$\rho_c = \sup\{\rho\colon \theta(\rho) = 0\}. \tag{1.2}$$

It is not difficult to see that $0 < \rho_c < \infty$.

Let $|C|$ be the (one-dimensional) Lebesgue measure of C and let

$$\chi(\lambda, \delta) = E_{\lambda,\delta}|C|. \tag{1.3}$$

One of our main steps will be to show that $\chi(\lambda, \delta) < \infty$ if $\rho = \lambda/\delta < \rho_c$.

There is nothing vital about the choice of $\mathbb{Z}^d$. For example, techniques similar to those used below are valid if $\mathbb{Z}^d$ is replaced by any vertex-transitive lattice and probably the requirement of vertex-transitivity is not crucial.

We note that continuous-time percolation may be a misnomer, since we allow paths which proceed backward in time.

Contact process. The percolation structure of the contact process is very similar to the process defined above; see Harris (1974, 1978), Durrett (1988), Griffeath (1979, 1981) and Liggett (1985) for accounts of the contact process. Starting with $\mathbb{Z}^d \times \mathbb{R}$ as before, we position deaths in the same way. Horizontal bonds, however, are directed. Each Poisson process of horizontal bonds is replaced by two independent processes, each with intensity λ, the first marking the centres of crossings with one orientation and the second marking crossings with the opposite orientation. It is well known that the ensuing random directed graph may be used to generate realizations of the contact process on $\mathbb{Z}^d$ with death rate δ and aggregate infection rate $2d\lambda$ from each site. For $t \geq 0$, let ξ_t^0 be the set of points x ($\in \mathbb{Z}^d$) such that there is a path in $\mathbb{Z}^d \times \mathbb{R}$ from the origin $(0, 0)$ to (x, t) using vertical line segments traversed in the upward (i.e., increasing time) direction and oriented horizontal crossings and traversing no deaths. Let

$$\theta^o(\lambda, \delta) = P_{\lambda,\delta}^o(\xi_t^0 \neq \varnothing \text{ for all } t), \tag{1.4}$$

where $P_{\lambda,\delta}^o$ (and $E_{\lambda,\delta}^o$) are the appropriate measure (and expectation operator) and $\lambda, \delta \geq 0$; the superscript o stands for oriented. As before, $\theta^o(\lambda, \delta) = \theta^o(\eta\lambda, \eta\delta)$ for $\eta > 0$ and we write $\theta^o(\rho) = \theta^o(\rho\delta, \delta)$ for $\delta > 0$. The critical value of ρ is

$$\rho_c^o = \sup\{\rho\colon \theta^o(\rho) = 0\} \tag{1.5}$$

and it is well known that $0 < \rho_c^o < \infty$.

We write

$$\chi^o(\lambda, \delta) = E_{\lambda,\delta}^o\left\{\int_0^\infty |\xi_t^0|\, dt\right\}, \tag{1.6}$$

and it is one of our targets to show that $\chi^o(\lambda, \delta) < \infty$ if $\rho = \lambda/\delta < \rho_c^o$.

In a companion paper [Bezuidenhout and Grimmett (1990)], we prove that $\theta^o(\rho_c^o) = 0$. The corresponding question is open for the continuous-time percolation model, except in the case $d = 1$, for which the continuity of θ at ρ_c follows as for two-dimensional percolation [Theorem (1.12)].

1.3. *Results for continuous-time percolation.* It is elementary that the critical value ρ_c in (1.2) satisfies $0 < \rho_c < \infty$. The inequality $\rho_c < \infty$ follows by a comparison with the contact process. The inequality $\rho_c > 0$ may be obtained in any of several ways, one of which is to show that the number of crossings in C is stochastically smaller than a branching process with mean family-size $4d\lambda/\delta$.

Turning to the question of exponential decay, we define the distance function

$$\delta((x, t), (y, s)) = |t - s| + \sum_{i=1}^{d} |x_i - y_i|$$

for $(x, t), (y, s) \in \mathbb{Z}^d \times \mathbb{R}$. For $r > 0$, we define the ball

$$S(r) = \{\pi \in \mathbb{Z}^d \times \mathbb{R}\colon \delta(0, \pi) \leq r\}$$

and its surface, the sphere

$$\partial S(r) = \{\pi \in \mathbb{Z}^d \times \mathbb{R}: \delta(0, \pi) = r\}.$$

If U and V are closed subsets of $\mathbb{Z}^d \times \mathbb{R}$, we write $\{U \leftrightarrow V\}$ for the event that some point $u \in U$ is joined to some point $v \in V$ by a path comprising vertical line segments and horizontal bonds but traversing no death; we write $A_r = \{0 \leftrightarrow \partial S(r)\}$.

1.7 THEOREM. *If $\lambda < \rho_c \delta$, then there exists $\psi(\lambda, \delta) > 0$ such that*

$$P_{\lambda, \delta}(A_r) \le e^{-r\psi(\lambda, \delta)} \quad \textit{for all } r \ge 0.$$

It is actually the case that the existence of such a $\psi(\lambda, \delta)$ is equivalent to the finiteness of $\chi(\lambda, \delta)$; the first part of the next result is a consequence.

1.8 THEOREM. (a) *$\chi(\lambda, \delta) < \infty$ if and only if $\lambda < \rho_c \delta$.*
(b) *If $\lambda < \rho_c \delta$, then there exists $\alpha(\lambda, \delta) > 0$ such that*

$$P_{\lambda, \delta}(|C| \ge s) \le e^{-s\alpha(\lambda, \delta)} \quad \textit{for all } s \ge 0. \tag{1.9}$$

We shall give proofs of neither Theorem 1.8 nor the statement preceding it. These facts may be proved by adaptations of the arguments of Hammersley (1957) and Aizenman and Newman (1984), in the forms given in Grimmett [(1989), pages 83 and 96].

Next we discuss critical exponents. It may be proved without great difficulty that either $\theta(\rho_c) \ge \frac{1}{2}$ or there exist $a, b > 0$ such that

$$\theta(\rho) - \theta(\rho_c) \ge a(\rho - \rho_c) \quad \text{for } 0 \le \rho - \rho_c \le b;$$

this may be obtained in very much the same way as was Theorem (3.8) of Grimmett (1989). Alternatively, one can imitate the proof of the main result of Chayes and Chayes (1987), using the comparison lemma of Section 2.4; similar arguments appear in Section 3.2. More demanding is the following, proved in Section 3.2. Define

$$\theta(\lambda, \delta; \gamma) = 1 - E_{\lambda, \delta}(e^{-\gamma|C|}) \quad \text{for } \gamma > 0. \tag{1.10}$$

1.11 THEOREM. *Either $\theta(\rho_c) > 0$ or there exist $a, b > 0$ such that*

$$\theta(\rho_c \delta, \delta; \gamma) \ge a\gamma^{1/2} \quad \textit{for } 0 < \gamma < b.$$

By Tauberian theory, this is tantamount to a critical-exponent inequality for $P_{\rho_c \delta, \delta}(|C| \ge s)$ as $s \to \infty$.

We now turn to the case $d = 1$, for which we have an exact determination of ρ_c.

1.12 THEOREM. *If $d = 1$, then $\rho_c = 1$ and $\theta(\rho_c) = 0$.*

The proof may be found in Section 3.3.

To write out complete proofs of the results stated above is a laborious task and involves the reproduction of large quantities of material which is already in the literature. Some new ideas are needed and the principal such idea may be found in Section 2.4. Our strategy in this paper has been to indicate with care the places where new ideas are needed to adapt known technical arguments, but we have omitted sizeable portions of standard material. For the important Theorem 1.7, we give a complete proof in Section 3.1.

1.4. *Results for the contact process.* Broadly similar results are valid for the contact process as for continuous-time percolation. We shall not give separate proofs of these, which may be obtained in exactly analogous ways. The two situations are so similar that the reader will have no difficulty in reworking the proofs for the contact process. We have two main reasons for using this approach. The first of these is that the notation is easier this way, since there is no complication arising from the orientations. For example, for the contact process, we would have to introduce the idea of both left-pivotal and right-pivotal intervals in the discussion of Russo's formula (see Section 2.3) and the statement and proof of that formula would be correspondingly more complicated. Our second reason is that we have an extra exact calculation (Theorem 1.12) for the percolation case.

For our first result, we have as in Theorem 1.7 that

$$P^o_{\lambda,\delta}(0 \to \partial S(r)) \le e^{-r\psi(\lambda,\delta)} \quad \text{for all } r \tag{1.13}$$

for some ψ satisfying $\psi(\lambda, \delta) > 0$ when $\lambda/\delta < \rho^o_c$; here $A \to B$ means that there exist $a \in A$ and $b \in B$ such that a is joined to b by a directed path of the process. Hence $\chi^o(\lambda, \delta) < \infty$ throughout the subcritical phase. The argument of Aizenman and Newman (1984) may be adapted to show that

$$P^o_{\lambda,\delta}(|C^o| \ge s) \le e^{-s\alpha(\lambda,\delta)} \quad \text{for all } s \tag{1.14}$$

for some α satisfying $\alpha(\lambda, \delta) > 0$ when $\lambda/\delta < \rho^o_c$; here

$$|C^o| = \int_0^\infty |\xi^0_t|\, dt,$$

the Lebesgue measure of the set of points in space-time infectable from the origin.

The argument leading to (1.13) may be adapted and extended to obtain a critical-exponent inequality for $\theta^o(\rho)$. When combined with the result of Bezuidenhout and Grimmett (1990), this may be phrased as follows: There exist positive constants a and b such that

$$\theta^o(\rho) \ge a(\rho - \rho^o_c) \quad \text{if } 0 \le \rho - \rho^o_c \le b. \tag{1.15}$$

As in Theorem 1.11, one obtains also that

$$\theta^o(\lambda, \delta; \gamma) = 1 - E^o_{\lambda,\delta}(e^{-\gamma|C^o|})$$

satisfies

$$\theta^o(\rho^o_c\delta, \delta; \gamma) \geq a\gamma^{1/2}$$

for some $a > 0$ and all small positive values of γ. As in the unoriented case (see the comment following Theorem 1.11), one can derive from this a critical-exponent inequality for $P^o_{\rho^o_c\delta,\delta}(|C| \geq s)$. Note that when $d = 1$, this critical-exponent inequality follows from the known fact [see Durrett (1988), page 73] that $\sqrt{t}\, P^o_{\rho^o_c\delta,\delta}(\xi^0_t \neq \varnothing)$ tends to infinity as $t \to \infty$.

Similar results have been reported by Aizenman and Barsky (1987) for oriented percolation. They have conjectured that such conclusions are valid for the contact process, but they lacked the technique of the forthcoming Section 2.4, designed to enable us to deal successfully with continuously varying time.

2. Technicalities. This section contains technical details of the topology we use in approximating continuous-time models by discrete-time ones, as well as some technical lemmas needed later. The reader may skip Sections 2.1–2.2, referring back to them later as necessary.

We begin with a sketch of the measure-theoretic and topological details associated with the discretization of percolation processes in continuous time. This is followed by an account of the FKG and BKF inequalities in such a context. Russo's formula appears in Section 2.3. The principal estimate of the paper is contained in Section 2.4, in which we prove the necessary lemma linking the number of pivotal bonds for an event A to the number and length of subintervals of vertical lines which are pivotal for A, for certain events A.

2.1. *The space of configurations.* We shall describe configurations as collections of points (deaths and crossings) on vertical (time) lines. In order to say when two configurations are close to each other, we introduce the natural topology on the space of such configurations: Two configurations are close in this topology if they have the same numbers of deaths and crossings in some large space-time box and if the positions of corresponding deaths and crossings are approximately the same. This topology is familiar in various guises. For example, if we restrict to a single interval on a vertical line and identify a configuration of points with the increasing right-continuous integer-valued step function which is 0 at time $t = 0$ and jumps up by one unit at every point in the configuration, then our topology is the restriction of the Skorohod topology on the space of real-valued right-continuous functions defined on the real line. There is some discussion of the relation between this topology and the weak topology following (2.3).

Let $e_1, \dots, e_d$ be the unit vectors generating the lattice $\mathbb{Z}^d$ and define

$$S = \mathbb{Z}^d \cup \left\{ \bigcup_{i=1}^{d} \left(\mathbb{Z}^d + \tfrac{1}{2}e_i\right) \right\}; \tag{2.1}$$

we write Ω for the collection of locally finite subsets of $S \times \mathbb{R}$, that is those whose intersection with any bounded subset of $\mathbb{R}^{d+1}$ is finite.

Let $\mathscr{I}$ be the set of bounded open intervals in $\mathbb{R}$ with rational endpoints. For $\omega \in \Omega$, $I \in \mathscr{I}$ and $x \in S$, let

$$N(x, I; \omega) = |\omega \cap (x \times I)|, \tag{2.2}$$

where $|Y|$ is the cardinality of Y. For $\omega_\infty, \omega_1, \omega_2, \dots \in \Omega$, we write $\omega_n \to \omega_\infty$ as $n \to \infty$ if, whenever $x \in S$, $I \in \mathscr{I}$ and $\omega_\infty \cap (x \times \partial I) = \varnothing$,

$$N(x, I; \omega_n) = N(x, \bar{I}; \omega_n) = N(x, I; \omega_\infty) \tag{2.3}$$

for all large n. This defines a separable topology τ on Ω.

Note that an element ω of Ω can be identified with the counting measure on $S \times \mathbb{R}$ with atoms at the points in ω. Therefore another way of topologizing Ω is to declare two of its element to be close if the corresponding counting measures are close in the weak topology on the set of measures on some large bounded subset of $S \times \mathbb{R}$. We shall refer to this topology on $S \times \mathbb{R}$ as the weak topology and to ours as the Skorohod topology because if one restricts our topology to a finite interval in $S \times \mathbb{R}$, it is indeed the Skorohod topology applied to the distribution functions of the corresponding counting measures. Ethier and Kurtz (1990) use the term weak atomic topology instead of Skorohod topology. Note that the Skorohod topology is (strictly) stronger than the weak topology (any sequence that converges in the Skorohod topology also does so in the weak topology, but not vice versa). We use the stronger topology because the set of point locations (counting measures) is not closed in the weak topology—for example the weak topology allows atoms to coalesce.

If $\omega \in \Omega$ and $t > 0$, let $f_s(x, t, \omega)$, $-t \le s \le t$, be the right-continuous increasing step function defined on $[-t, t]$ with jumps of size 1 at the points in $\omega \cap (x \times [-t, t])$ which takes the value 0 when $s = -t$ [or 1 if $(x, -t) \in \omega$]. Let d_t be a complete metric bounded by 1 generating the Skorohod topology on $D([-t, t])$. Then the metric on Ω given by

$$d(\omega, \omega') = \sum_{x \in S} 2^{-\sum_{i=1}^d |x_i|} \int_0^\infty d_t(f_\cdot(x, t; \omega), f_\cdot(x, t; \omega')) e^{-t}\, dt$$

is complete and generates τ. See Ethier and Kurtz [(1986), page 117]. Write $\mathscr{B}(\Omega)$ for the Borel σ-field generated by τ.

For $\omega \in \Omega$, we write

$$\mathscr{D}(\omega) = \omega \cap (\mathbb{Z}^d \times \mathbb{R}), \tag{2.4}$$

$$\mathscr{C}(\omega) = \omega \cap (\{S - \mathbb{Z}^d\} \times \mathbb{R}); \tag{2.5}$$

we call points in $\mathscr{D}(\omega)$ *deaths* and in $\mathscr{C}(\omega)$ *bonds* (or *crossings*). For $\omega \in \Omega$, let $G(\omega)$ be the subset of $\mathbb{R}^{d+1}$ comprising $(\mathbb{Z}^d \times \mathbb{R}) - \mathscr{D}(\omega)$ together with all horizontal line segments of unit length centered at points in $\mathscr{C}(\omega)$, that is, all line segments in the collection $\{x + se_i\colon 0 \le s \le 1\} \times \{\mathscr{C}(\omega) \cap [(x + \frac{1}{2}e_i) \times \mathbb{R}]\}$ for $i = 1, 2, \dots, d$ and $x \in \mathbb{Z}^d$; each point in $\mathscr{C}(\omega)$ corresponds to exactly one of the latter line segments which we call *bonds* (or *crossings*). (Note that we use this terminology both for these intervals and their centres.) We may think of $G(\omega)$ as a (random) graph whose edges are bonds together with connected vertical line segments. There is a one-to-one correspondence between configurations ω and their graphs $G(\omega)$.

For $\omega \in \Omega$, closed subsets U and V of $\mathbb{Z}^d \times \mathbb{R}$ and subsets Γ of $\mathbb{R}^{d+1}$, we write $U \leftrightarrow_\Gamma V$ if there exist $u \in U$ and $v \in V$ such that u and v are in the same connected component of $G(\omega) \cap \Gamma$ [i.e., there exists a path from u to v in $G(\omega)$ lying entirely within Γ]; if $\Gamma = \mathbb{R}^{d+1}$ then we suppress explicit reference to Γ.

We say that an event $A \in \mathscr{B}(\Omega)$ is *determined by the configuration inside* the open subset Γ of $\mathbb{R}^{d+1}$ if $1_A(\omega) = 1_A(\omega')$ whenever $\omega \cap \Gamma = \omega' \cap \Gamma$, where 1_A is the indicator function of A.

There is a natural partial order on Ω: We write $\omega \le \omega'$ if $\mathscr{C}(\omega) \subseteq \mathscr{C}(\omega')$ and $\mathscr{D}(\omega) \supseteq \mathscr{D}(\omega')$. For $A \in \mathscr{B}(\Omega)$, A is called *increasing* if $1_A(\omega) \le 1_A(\omega')$ whenever $\omega \le \omega'$ and called *decreasing* if A^c is increasing.

Let $\lambda, \delta \ge 0$ and let $\mathscr{P} = \{\mathscr{P}(x)\colon x \in S\}$ be a collection of independent Poisson point processes on $\mathbb{R}$, having intensity δ if $x \in \mathbb{Z}^d$ and λ otherwise. Almost every realization of $\mathscr{P}$ corresponds naturally to a configuration $\omega \in \Omega$ and we denote by $P_{\lambda,\delta}$ the measure induced on $(\Omega, \mathscr{B}(\Omega))$ by $\mathscr{P}$.

We may approximate $P_{\lambda,\delta}$ (in the sense of weak convergence) by measures corresponding to certain processes in discrete space. Suppose that ε is positive and satisfies $\varepsilon \max\{\lambda, \delta\} < 1$ and let $\{X_\varepsilon(x,k)\colon (x,k) \in \mathbb{Z}^{d+1}\}$ and $\{Y_\varepsilon(x,k,i)\colon (x,k) \in \mathbb{Z}^{d+1}, 1 \le i \le d\}$ be independent 0–1 valued random variables with

$$P\big(X_\varepsilon(x,k) = 0\big) = \varepsilon\delta, \qquad P\big(Y_\varepsilon(x,k,i) = 1\big) = \varepsilon\lambda.$$

Let

$$\omega_\varepsilon = \big\{(x, \varepsilon k)\colon X_\varepsilon(x,k) = 0\big\} \cup \big\{\big(x + \tfrac{1}{2}e_i, \varepsilon k + \tfrac{1}{2}\varepsilon\big)\colon Y_\varepsilon(x,k,i) = 1\big\}$$

and write $P^\varepsilon_{\lambda,\delta}$ for the measure induced by ω_ε on $(\Omega, \mathscr{B}(\Omega))$. It is not difficult to show that

$$P^\varepsilon_{\lambda,\delta} \Rightarrow P_{\lambda,\delta} \quad \text{as } \varepsilon \downarrow 0. \tag{2.6}$$

We omit the proof of this, but note that it follows from the facts that: (i) $\{P^\varepsilon_{\lambda,\delta}\colon \varepsilon > 0\}$ is a tight family of measures [see Billingsley (1968), pages 35–37], and so the sequence has at least one weak limit point; and (ii) all weak limits of this sequence must be products of Poisson point locations, and therefore there is a unique weak limit point.

For future reference, we record the following consequence of weak convergence:

2.7 PROPOSITION. *If $A \in \mathscr{B}(\Omega)$ satisfies $P_{\lambda,\delta}(\partial A) = 0$, then*

$$P^\varepsilon_{\lambda,\delta}(A) \to P_{\lambda,\delta}(A) \quad \textit{as } \varepsilon \downarrow 0.$$

In applying this fact, the following observation will be useful. Let Ω' be the set of configurations in which there are no simultaneous points:

$$\Omega' = \{\omega \in \Omega : |\omega \cap (S \times t)| \le 1 \text{ for all } t \in \mathbb{R}\};$$

then clearly $P_{\lambda,\delta}(\Omega') = 1$.

2.2. *Two fundamental inequalities.* We shall make later use of analogues of the BK inequality [see van den Berg and Kesten (1985), van den Berg and Fiebig (1987)] and the FKG inequality [see Harris (1960), Fortuin, Kasteleyn and Ginibre (1971)] and a sketch of their proofs is contained in this section. The only extra difficulty [over and above the usual proofs—see Grimmett (1989), Chapter 2] is due to the nature of the measure $P_{\lambda,\delta}$. Our strategy is simple—we check that the inequalities are valid for the measures $P^{\varepsilon}_{\lambda,\delta}$ and then we take the weak limit as $\varepsilon \downarrow 0$.

If $A, B \in \mathscr{B}(\Omega)$, we define $A \square B$ to be the event that A and B occur disjointly, which is to say that there exist disjoint measurable subsets K and L of $S \times \mathbb{R}$ such that the cylinders ω_K and ω_L are contained in A and B, respectively [$\omega_\Gamma = \omega_\Gamma(\omega)$ is defined to be $\{\omega' \in \Omega: \omega' \cap \Gamma = \omega \cap \Gamma\}$].

We name the following lemma after van den Berg, Kesten and Fiebig.

2.8 BKF INEQUALITY. *Suppose that $A, B \in \mathscr{B}(\Omega)$ and each is the intersection of an increasing and a decreasing event, both of which are determined by the configuration inside the bounded region Γ of $\mathbb{R}^{d+1}$. If*

$$P_{\lambda,\delta}(\partial A) = P_{\lambda,\delta}(\partial B) = P_{\lambda,\delta}(\partial(A \square B)) = 0, \tag{2.9}$$

then

$$P_{\lambda,\delta}(A \square B) \le P_{\lambda,\delta}(A) P_{\lambda,\delta}(B). \tag{2.10}$$

It is straightforward to check in the present context that the BKF inequality may be applied to events A and B which are finite intersections of events of the form $\{U \leftrightarrow_\Gamma V\}$ and complements of such events, for any bounded region Γ of $\mathbb{R}^{d+1}$.

2.11 FKG INEQUALITY. *Suppose that $A, B \in \mathscr{B}(\Omega)$, each being an increasing event determined by the configuration inside the bounded region Γ of $\mathbb{R}^{d+1}$. If $P_{\lambda,\delta}(\partial A) = P_{\lambda,\delta}(\partial B) = 0$, then*

$$P_{\lambda,\delta}(A \cap B) \ge P_{\lambda,\delta}(A) P_{\lambda,\delta}(B). \tag{2.12}$$

We omit formal proofs of these inequalities, giving here only a hint of the proof of the BKF inequality; the FKG inequality may be proved in much the same way. By hypothesis (2.9) and the weak convergence (2.6) of $P^{\varepsilon}_{\lambda,\delta}$ to $P_{\lambda,\delta}$, it suffices to show that

$$P^{\varepsilon}_{\lambda,\delta}(A \square B) \le P^{\varepsilon}_{\lambda,\delta}(A) P^{\varepsilon}_{\lambda,\delta}(B) \quad \text{for } \varepsilon > 0.$$

However, $P^{\varepsilon}_{\lambda,\delta}$ is a product measure and the result of van den Berg and Fiebig (1987) may be applied directly in order to obtain the last inequality.

2.3. *Russo's formula.* The preliminaries being largely complete, we move on to the notion of pivotality.

Let A be an increasing event. For $\omega \in A$ and $X \in \mathscr{E}(\omega)$, we say that X is *pivotal* for A (or that the bond at X is a *pivotal bond* for A) if $\omega - X \notin A$. Suppose $x \in \mathbb{Z}^d$ and I is a nonempty open subinterval of $x \times \mathbb{R}$. For $\omega \in A$, we call I *death-pivotal* (or *pivotal*) for A if: (i) $\omega \cup t \notin A$ for all $t \in I$; and (ii) I is maximal with this property. Such intervals have the property that the addition of a death anywhere within them prevents A from occurring.

Let B be a decreasing event. For $\omega \in B$ and $X \in \mathscr{D}(\omega)$, we say that X is *pivotal* for B (or that the death at X is a *pivotal death* for the event B) if $\omega - X \notin B$. If $x \in \mathbb{Z}^d$, $1 \le i \le d$, $\omega \in \Omega$ and I is a nonempty open subinterval of $(x + \frac{1}{2}e_i) \times \mathbb{R}$, we call I *bond-pivotal* (or *pivotal*) for B if: (i) $\omega \cup t \notin B$ for all $t \in I$; and (ii) I is maximal with this property. Any point y lying in a bond-pivotal interval is called bond-pivotal for B.

Any pivotal bond, death or interval is called a pivotal incident. As a shorthand, we omit explicit mention of the event for which an incident is pivotal, whenever no confusion results.

2.13 Russo's formula. *Let A be an increasing event which depends only on the configuration inside the bounded region Γ. Then*

$$\begin{aligned}(2.14)\qquad \frac{\partial}{\partial\lambda}P_{\lambda,\delta}(A) &= \frac{1}{\lambda}E_{\lambda,\delta}(\textit{number of pivotal bonds};\, A)\\ &= E_{\lambda,\delta}(\textit{total length of bond-pivotal intervals};\, A^c)\end{aligned}$$

and

$$\begin{aligned}(2.15)\qquad -\frac{\partial}{\partial\delta}P_{\lambda,\delta}(A) &= \frac{1}{\delta}E_{\lambda,\delta}(\textit{number of pivotal deaths};\, A^c)\\ &= E_{\lambda,\delta}(\textit{total length of death-pivotal intervals};\, A).\end{aligned}$$

Proof. We prove (2.14) only; (2.15) is proved similarly. Let $\delta \ge 0$ and $0 < h < \lambda$. We construct a random configuration ω_λ according to the measure $P_{\lambda,\delta}$. From ω_λ we construct a random configuration $\omega_{\lambda-h}$ by deleting each bond of ω_λ with probability h/λ, independently of all other bonds; $\omega_{\lambda-h}$ induces the measure $P_{\lambda-h,\delta}$ on Ω. Now

$$\begin{aligned}&-\frac{1}{h}(P_{\lambda-h,\delta}(A) - P_{\lambda,\delta}(A))\\ (2.16)\qquad &= \frac{1}{h}P(\omega_\lambda \in A, \omega_{\lambda-h} \notin A)\\ &= \frac{1}{h}P(\omega_\lambda \in A, \omega_{\lambda-h} \notin A, N = 1) + o(1)\end{aligned}$$

as $h \to 0$, where N is the number of bonds of $\omega_\lambda \cap \Gamma$ which are not in $\omega_{\lambda-h}$. The last probability differs by $o(h)$ from the probability that $\omega_\lambda \in A$, there exist in ω_λ pivotal bonds for A, and exactly one such bond is removed in forming $\omega_{\lambda-h}$. Writing $K(\omega)$ for the number of pivotal bonds for A in ω, we find that the left-hand side of (2.16) equals

$$\frac{1}{h}\sum_{k=0}^{\infty} P_{\lambda,\delta}(K=k,A)k\frac{h}{\lambda}\left[1-\frac{h}{\lambda}\right]^{k-1}+o(1),$$

which, by monotone convergence, approaches $\lambda^{-1}E_{\lambda,\delta}(K;A)$ as $h\to 0$. This deals with the left-hand derivative of $P_{\lambda,\delta}(A)$ in (2.14). Replacing λ by $\lambda+h$, we have that

$$\frac{1}{h}(P_{\lambda+h,\delta}(A)-P_{\lambda,\delta}(A))$$
$$=\frac{1}{\lambda+h}\sum_{k=0}^{\infty} P_{\lambda+h,\delta}(K=k,A)k\left[1-\frac{h}{\lambda+h}\right]^{k-1}+o(1);$$

using dominated convergence and the continuity in λ of $P_{\lambda,\delta}(K=k,A)$, we find that the right-hand derivative equals $\lambda^{-1}E_{\lambda,\delta}(K;A)$ also.

Turning to the second part of (2.14), we choose $\lambda,\delta\ge 0$ and $h>0$. We construct a random configuration ω_λ according to $P_{\lambda,\delta}$ and then add extra bonds as (independent) Poisson processes with intensity h, obtaining thus a configuration $\omega_{\lambda+h}$ drawn according to $P_{\lambda+h,\delta}$. Let C be the set of extra bonds. Then

$$\begin{aligned}&\frac{1}{h}(P_{\lambda+h,\delta}(A)-P_{\lambda,\delta}(A))\\ &\quad=\frac{1}{h}P(\omega_\lambda\notin A,\omega_{\lambda+h}\in A,|C\cap\Lambda|=1)+o(1)\end{aligned}\tag{2.17}$$

as before. The latter probability equals the chance that $|C\cap\Lambda|=\{X\}$ for some bond X in $\omega_{\lambda+h}$ and that, in $\omega_{\lambda+h}$, X is pivotal for A. Now $\Lambda\cap\{(S-\mathbb{Z}^d)\times\mathbb{R}\}=\Lambda'$ is a union of intervals; write m for (one-dimensional) Lebesgue measure on Λ' and $L=m(\Lambda')$. Conditional on $\{|C\cap\Lambda|=1\}$, the measure associated with the pair (ω_λ,X) is $P_{\lambda,\delta}\times(L^{-1}m)$. Now $P(|C\cap\Lambda|=1)=hLe^{-hL}$ and therefore the left-hand side of (2.17) equals

$$\frac{1}{h}\int_{\Lambda'}P_{\lambda,\delta}(x\text{ is bond-pivotal for }A^c;A^c)\frac{1}{L}\,dm(x)hLe^{-hL}+o(1)$$

$$\begin{aligned}&\to\int_{\Lambda'}P_{\lambda,\delta}(x\text{ is bond-pivotal for }A^c;A^c)\,dm(x)\quad\text{as }h\downarrow 0\\ &=E_{\lambda,\delta}\left[\int_{\Lambda'}1_{J(\omega)}(x)\,dm(x)\right]\end{aligned}\tag{2.18}$$

by Fubini's theorem, where $J(\omega)$ is the set of points y in Λ' which are bond-pivotal for A^c. This deals with the right-hand derivative of $P_{\lambda,\delta}(A)$. For the left-hand derivative we argue as before, using dominated convergence and the continuity in λ of the integrand in the centre of (2.18). □

2.4. *Comparisons for pivotal incidents.* The result of this section is vital for the proofs of the main theorems. Suppose that Λ is a bounded region in $\mathbb{R}^{d+1}$ and D is a fixed, closed nonempty subset of Λ; denote by A the event $\{0 \leftrightarrow_{\Lambda} D\}$.

2.19 COMPARISON LEMMA. *It is the case that*

$$\begin{aligned} &E_{\lambda,\delta}(\textit{number of death-pivotal intervals};\, A) \\ &\quad \le P_{\lambda,\delta}(A) \\ &\qquad + eE_{\lambda,\delta}(\textit{number of pivotal bonds};\, A) \\ &\qquad + 2\,d\lambda\, eE_{\lambda,\delta}(\textit{total length of death-pivotal intervals};\, A), \end{aligned} \tag{2.20}$$

where such incidents are pivotal for the event A.

PROOF. We prove the lemma by making local changes at bounded cost, thereby turning one type of pivotal incident into another. This technique has been used by Menshikov (1987) and Aizenman and Grimmett (1989).

Fix η (> 0) and let ν be the number of pivotal bonds, μ_+ the number of pivotal intervals of length greater than η, and μ_- the number of such intervals of length less than η. We shall show that, in the limit as $\varepsilon \downarrow 0$ through the irrational multiples of η,

$$\begin{aligned} E^{\varepsilon}_{\lambda,\delta}(\mu_-; A) \le P^{\varepsilon}_{\lambda,\delta}(A) + \{c(\lambda) + 1\}E^{\varepsilon}_{\lambda,\delta}(\nu; A) \\ + c(\lambda)E^{\varepsilon}_{\lambda,\delta}(\mu_+; A) + o(1), \end{aligned} \tag{2.21}$$

where $c(\lambda) = e^{2d\lambda\eta} - 1$. (We restrict the limit to the irrational multiples of η in order to avoid having deaths or crossings coincident with the endpoint of some fundamental interval of length η.) Inequality (2.20) follows from (2.21), as follows. Suppose that (2.21) holds. The events A, $A \cap \{\nu \ge k\}$, $A \cap \{\mu_+ \ge k\}$ have boundaries in the space Ω of configurations which are contained in the union of the complement of Ω' with the set of configurations which have births or deaths on the boundary of the space-time region Λ. Taking the limit as $\varepsilon \downarrow 0$ (through the irrational multiples of η), it therefore follows from (2.7) and the remark following it that (2.21) is valid with the ε's removed. Now

$$E_{\lambda,\delta}(\text{total length of pivotal intervals};\, A) \ge \eta E_{\lambda,\delta}(\mu_+; A), \tag{2.22}$$

so that

$$\begin{aligned} E_{\lambda,\delta}(\mu; A) \le P_{\lambda,\delta}(A) + e^{2d\lambda\eta}E_{\lambda,\delta}(\nu; A) \\ + \frac{1}{\eta}e^{2d\lambda\eta}E_{\lambda,\delta}(\text{total length of pivotal intervals};\, A), \end{aligned} \tag{2.23}$$

where $\mu = \mu_+ + \mu_-$. We have a free choice of η here and we choose $\eta = 1/(2d\lambda)$, obtaining (2.20).

In proving (2.21), we regard $P^{\varepsilon}_{\lambda,\delta}$ as being product measure on the set of subsets of a finite set (see Section 2.1), writing $P^{\varepsilon}_{\lambda,\delta}(B)$ as the summation of $P^{\varepsilon}_{\lambda,\delta}(\omega)$ for $\omega \in B$. Suppose that the event A occurs and there are pivotal incidents. Then any path from 0 to D in Λ must traverse all pivotal incidents (bonds and death-pivotal intervals) in a given order, each being traversed in a given direction. In any such configuration, each pivotal bond or interval has an endpoint closer to 0 and an endpoint further away from 0. For any such incident, we shall speak of the direction leading away from 0, and so on, as being the direction from the endpoint closer to 0 towards the endpoint further from 0.

Let

$$\Lambda_\varepsilon = \{\mathbb{Z}^d \times \varepsilon\mathbb{Z}\} \cup \left\{ \bigcup_{i=1}^{d} \left(\mathbb{Z}^d + \tfrac{1}{2}e_i\right) \times \left(\varepsilon\mathbb{Z} + \tfrac{1}{2}\varepsilon\right) \right\},$$

the discretized version of the space $S \times \mathbb{R}$. Suppose that $x \in (\mathbb{Z}^d \times \mathbb{R}) \cap \Lambda_\varepsilon$ and introduce the following events.

(a) $A(x)$ is the event that there is a pivotal interval I of length less than η starting at x.

(b) $B(x)$ is the subset of $A(x)$ containing configurations for which there exist somewhere in the configuration two bonds having a common endpoint.

(c) $C(x)$ is the subset of $A(x)$ containing configurations for which the other end of I lies in D.

(d) $D(x)$ is the subset of $A(x)$ containing configurations in which the other endpoint of I is the endpoint of a pivotal bond.

(e) $E(x) = A(x) - (B(x) \cup C(x) \cup D(x))$ is partitioned as $E_1(x) \cup E_2(x) \cup E_3(x)$, where: (i) $E_1(x)$ contains configurations for which there exist deaths between x and y (y denotes the point on the same vertical line as x but a distance η away from x in the direction away from 0); (ii) $E_2(x)$ contains configurations for which no such death exists and between y and the first death beyond y (measured from x in the direction away from 0), there is some point joined to D in $\Gamma - [x, y]$; (iii) $E_3(x)$ contains configurations for which no such death exists and neither is there a point between y and the first death beyond y that is joined to D in $\Gamma - [x, y]$.

We shall define a mapping ψ on $E(x) = E_1(x) \cup E_2(x) \cup E_3(x)$ which sends configurations to configurations. Consider $E_2(x)$ first. For any configuration $\omega \in E_2(x)$, we define $\omega' = \psi(\omega)$ to be the configuration obtained from ω by removing all bonds which intersect $(x, y]$; see Figure 1 for diagrams of ω' and similar later constructions. In ω', x is the starting point of a pivotal interval of length exceeding η. Furthermore, for $\omega' \in \psi(E_2(x))$,

$$\sum_{\omega \in \psi^{-1}(\omega')} P^\varepsilon_{\lambda,\delta}(\omega) \le \alpha_2(\varepsilon) P^\varepsilon_{\lambda,\delta}(\omega'), \tag{2.24}$$

where

$$\alpha_2(\varepsilon) = \frac{P^\varepsilon_{\lambda,\delta}(\text{one or more bonds touching } (x, y])}{P^\varepsilon_{\lambda,\delta}(\text{no bond touching } (x, y])} = c(\lambda) + o(1) \quad \text{as } \varepsilon \downarrow 0,$$

note that $\omega' \notin \psi^{-1}(\omega')$, since $\omega' \notin A(x)$.

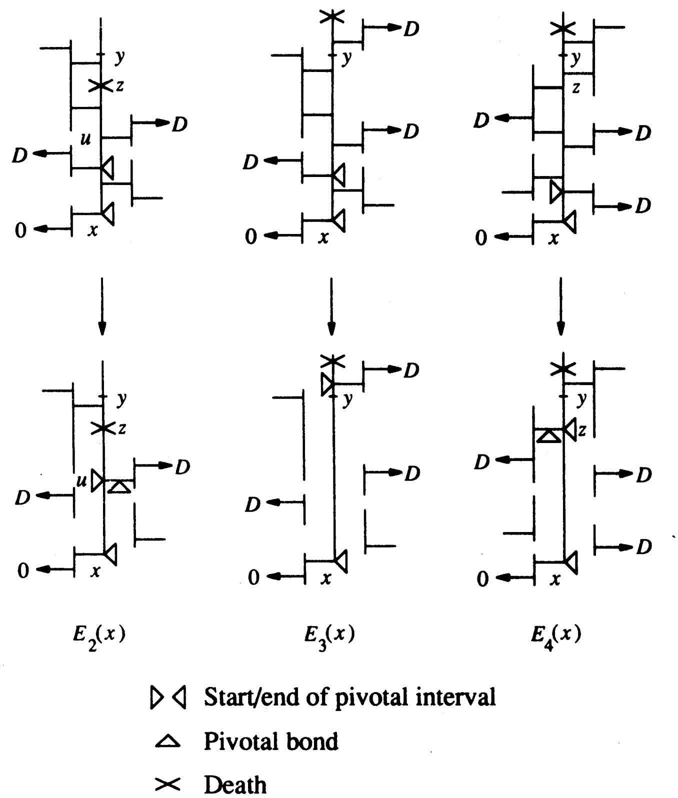

FIG. 1. *Schematic diagrams of the mapping ψ in the cases of the three events $E_2(x)$, $E_3(x)$ and $E_4(x)$.*

We turn next to $E_3(x)$. Suppose $\omega \in E_3(x)$ and let z be the point in $(x, y]$ furthest from x which is joined to D in $\Gamma - [x, y]$; if no such point z exists, then $\omega \in C(x)$, a contradiction. We obtain the configuration $\omega' = \psi(\omega)$ by removing all bonds which intersect $(x, z) \cup (z, y]$. In ω', the interval (x, z) is pivotal and z is the first endpoint of a pivotal bond. Moreover, for $\omega' \in \psi(E_3(x))$, (2.24) holds with $\alpha_2(\varepsilon)$ replaced by

$$\alpha_3(\varepsilon; x, z) = \frac{P^{\varepsilon}_{\lambda,\delta}(\text{one or more bonds touching } (x,z) \cup (z,y])}{P^{\varepsilon}_{\lambda,\delta}(\text{no bonds touching } (x,z) \cup (z,y])}$$

$$= c(\lambda) + o(1) \quad \text{as } \varepsilon \downarrow 0,$$

where the $o(1)$ term is uniform in x and z; note that $\omega' \notin \psi^{-1}(\omega')$ since $\omega' \in D(x)$ and $D(x) \cap E(x) = \varnothing$.

First consider $E_1(x)$. Suppose $\omega \in E_1(x)$ and let z be the death in (x, y) closest to x and let u be the point in (x, z) furthest from x which is joined to D in $\Gamma - [x, z]$. We obtain $\omega' = \psi(\omega)$ by removing all bonds which intersect

$(x,u) \cup (u,z)$. In ω', (x,u) is pivotal and u is the first endpoint of a pivotal bond. For $\omega' \in \psi(E_1(x))$, (2.24) holds with $\alpha_2(\varepsilon)$ replaced by a term $\alpha_1(\varepsilon; x, z, u)$ having order $c(\lambda) + o(1)$, the $o(1)$ term being uniform in x, z and u as before.

Now

$$\mu_- \le \mu_-(c) + \mu_-(d) + \mu_-(e),$$

where $\mu_-(c)$ is the number of pivotal intervals contributing to μ_- that have an endpoint in D, $\mu_-(d)$ is the number leading directly into pivotal bonds and $\mu_-(e)$ is the rest. Thus, with B denoting the set of configurations in which there are two or more bonds in Γ with common endpoints, we have as $\varepsilon \downarrow 0$ that

$$\begin{aligned} E^{\varepsilon}_{\lambda,\delta}(\mu_-; A) &= E^{\varepsilon}_{\lambda,\delta}(\mu_-; A \cap B) + E^{\varepsilon}_{\lambda,\delta}(\mu_-; A - B) \\ &\le o(1) + E^{\varepsilon}_{\lambda,\delta}(\mu_-(c); A - B) + E^{\varepsilon}_{\lambda,\delta}(\mu_-(d); A - B) \\ &\quad + E^{\varepsilon}_{\lambda,\delta}(\mu_-(e); A - B) \\ &\le o(1) + P^{\varepsilon}_{\lambda,\delta}(A) + E^{\varepsilon}_{\lambda,\delta}(\nu; A) + E^{\varepsilon}_{\lambda,\delta}(\mu_-(e); A - B) \end{aligned} \tag{2.25}$$

since $\mu_-(c) \le 1$ and $\mu_-(d) \le \nu$ a.s. Also, as $\varepsilon \downarrow 0$, we have from (2.24) and its analogues that

$$E^{\varepsilon}_{\lambda,\delta}(\mu_-(e); A - B) \le \{c(\lambda) + o(1)\} \sum_{x} \sum_{\omega' \in \Psi} P^{\varepsilon}_{\lambda,\delta}(\omega'),$$

where $\Psi = \psi(E_1(x) \cup E_2(x) \cup E_3(x))$. Hence, as $\varepsilon \downarrow 0$,

$$E^{\varepsilon}_{\lambda,\delta}(\mu_-(e); A - B) \le \{c(\lambda) + o(1)\}\big(E^{\varepsilon}_{\lambda,\delta}(\mu_+; A) + E^{\varepsilon}_{\lambda,\delta}(\nu; A)\big),$$

where we have used the fact that $\psi(E_1(x)) \cap \psi(E_3(x)) = \varnothing$. We combine this inequality with (2.25) to obtain (2.21). □

3. Proofs of main results.

3.1. *Exponential tail of radius distribution.* We prove Theorem 1.7. The basic method is due to Menshikov (1986) [see also Menshikov, Molchanov and Sidorenko (1986)]; we follow Grimmett [(1989), Section 3.2] closely here, making use of the extra estimate of Section 2.4.

Suppose $\lambda < \lambda'$, $\delta > \delta'$, $\zeta = \min\{\lambda' - \lambda, \delta - \delta'\}$,

$$(\lambda_t, \delta_t) = (\lambda, \delta) + t(\lambda' - \lambda, \delta' - \delta) \quad \text{for } 0 \le t \le 1,$$

and set $P_t = P_{\lambda_t, \delta_t}$ with expectation operator E_t. We write $g_t(r) = P_t(A_r)$.

We have from Russo's formula [Lemma (2.13)] that

$$\frac{1}{g_t(r)} \frac{d}{dt} g_t(r) = \frac{\lambda' - \lambda}{\lambda_t} E_t(B|A_r) + (\delta - \delta') E_t(L|A_r), \tag{3.1}$$

where, on A_r, B is the number of pivotal bonds (i.e., crossings) for A_r, L is the total length of death-pivotal intervals for A_r and (later) I is the number of pivotal incidents (pivotal bonds and death-pivotal intervals) for A_r. Applying

Lemma 2.19, we find that

$$E_t(I - B; A_r) \le P_t(A_r) + eE_t(B; A_r) + 2\,d\lambda_t\, eE_t(L; A_r).$$

After dividing by $P_t(A_r)$, one obtains from this that

$$\begin{aligned} E_t(I|A_r) + E_t(L|A_r) &\le 1 + (1+e)E_t(B|A_r) \\ &\quad + (1 + 2\,d\lambda_t\, e)E_t(L|A_r), \end{aligned} \tag{3.2}$$

and hence, since $1 + e \le (1 + 2\,d\lambda_t\, e)/\lambda_t$, that the left-hand side of (3.2) is no greater than

$$1 + \frac{1 + 2\,d\lambda_t\, e}{(\delta - \delta') \wedge (\lambda' - \lambda)}\left[\frac{\lambda' - \lambda}{\lambda_t}E_t(B|A_r) + (\delta - \delta')E_t(L|A_r)\right].$$

Hence from (3.1) we have that, for $0 < t < 1$,

$$\frac{d}{dt}\log g_t(r) \ge \frac{\zeta}{2\,d\lambda' e + 1}\{E_t(I|A_r) + E_t(L|A_r)\} - \frac{\lambda' - \lambda}{2\,de\,\lambda + 1}. \tag{3.3}$$

Integrating over t, we obtain

$$g_0(r) \le c_1 g_1(r)\exp\left\{-c_2\zeta\int_0^1 (E_t(I|A_r) + E_t(L|A_r))\,dt\right\} \tag{3.4}$$

for functions $c_1 = c_1(\lambda, \lambda')$, $c_2 = c_2(\lambda')$ which are finite, strictly positive and continuous when $\lambda, \lambda' \in (0, \infty)$.

Suppose that $\omega \in A_r$ and note that $I(\omega)$ is a.s. finite. If $I(\omega) \ge 1$, then the pivotal incidents in ω may be ordered in the usual way, since every path from 0 to $\partial S(r)$ traverses such incidents in a fixed order and a fixed direction. Write x_i and y_i for the initial and final points of the ith pivotal incident in the order of its traversal. With δ the L^1 metric on $\mathbb{Z}^d \times \mathbb{R}$, as in Section 1.3, we define $\rho_1 = \delta(0, x_1)$, and for $i \ge 2$,

$$\rho_i = \begin{cases} \delta(y_{i-1}, x_i) & \text{if } I \ge i, \\ \delta(y_{i-1}, \partial S(r)) & \text{if } I = i - 1, \\ 0 & \text{if } I < i - 1. \end{cases}$$

Let $M = \sup\{s\colon 0 \leftrightarrow \partial S(s)\}$. As in Grimmett [(1989), 3.12 and 3.19], the joint distribution of the ρ_i's, conditional on A_r, is dominated by that of independent copies of M. Thus, proceeding in the usual way (loc. cit.) but with some minor differences, we obtain that, if $s \le r$ and $k \ge 1$, then

$$P_t\left(\sum_{i=1}^{k}\rho_i < s \,\middle|\, A_r\right) \ge P_t\left(\sum_{i=1}^{k} M_i < s\right), \tag{3.5}$$

where $M_1, M_2, \dots$ are independent random variables having the same distribution as M (and we have abused notation by using the measure P_t on the probability space supporting the M_i's). The proof of (3.5) is by induction on k. The inductive step involves a conditioning argument which is technically more complicated than in the discrete case. However, no new ideas are needed and we shall not give the details.

Suppose that A_r occurs and $\rho_1 + \rho_2 + \cdots + \rho_k < r - 2k$. Then, either $I \geq k$ or $I < k$ and

$$r \leq \rho_1 + \rho_2 + \cdots + \rho_k + L + k < r - k + L$$

so that $L \geq k$. Hence

$$(3.6) \qquad P_t\left(\sum_{i=1}^{k} \rho_i < r - 2k \,\middle|\, A_r \right) \leq P_t(I \geq k | A_r) + P_t(L \geq k | A_r),$$

and therefore, by (3.5),

$$(3.7) \qquad P_t\left(\sum_{i=1}^{k} M_i' < r \right) \leq P_t(I \geq k | A_r) + P_t(L \geq k | A_r),$$

where $M_i' = (M_i \wedge r) + 2$. Let $N = \inf\{n\colon \sum_{i=1}^{n} M_i' \geq r\}$. The left-hand side of (3.7) equals $P_t(N \geq k + 1)$, and so, after summing over k, one obtains that $E_t(N) - 1 \leq E_t(I | A_r) + E_t(L | A_r)$. We have from Wald's equation that

$$E_t(N) = \frac{E_t(\sum_{i=1}^{N} M_i')}{E_t(M_1')} \geq \frac{r}{2 + \int_0^r P_t(A_s)\, ds}.$$

Now $P_t(A_s)$ is increasing in t and therefore

$$(3.8) \qquad E_t(I | A_r) + E_t(L | A_r) \geq \frac{r}{2 + \int_0^r P_1(A_s)\, ds} - 1,$$

which may be substituted into (3.4) to yield

$$(3.9) \qquad g_0(r) \leq c_3 g_1(r) \exp\left\{ -\frac{c_2 \zeta r}{2 + \int_0^r P_1(A_s)\, ds} \right\}$$

for some $c_3 = c_3(\lambda, \lambda')$ which is positive, finite and continuous on $(0, \infty)^2$.

It is clear from (3.9) that the theorem is proved once we have shown that

$$\int_0^\infty P_{\lambda, \delta}(A_s)\, ds < \infty \quad \text{if } \lambda < \rho_c \delta.$$

This in turn will follow once we have shown that, if $\lambda < \rho_c \delta$, there exists $\alpha = \alpha(\lambda, \delta) > 0$ such that

$$(3.10) \qquad P_{\lambda, \delta}(A_s) \leq \frac{\alpha}{\sqrt{s}} \quad \text{for } s \geq 0,$$

since this implies by (3.9) that $g_0(r) \leq \beta e^{-\gamma \sqrt{r}}$ for positive constants β, γ depending on λ and δ. The proof of (3.10) follows extremely closely that of Lemma (3.27) of Grimmett (1989). We sketch the details, which are somewhat tiresome but require no new ideas.

Suppose U is a closed ball in $(0, \infty)^2$. Then, by (3.9), there exist constants c_i such that, for all choices of (λ, δ) and (λ', δ') in U with $\lambda' > \lambda$ and $\delta > \delta'$,

$$(3.11) \qquad P_{\lambda, \delta}(A_r) \leq c_3 P_{\lambda', \delta'}(A_r) \exp\left\{ -\frac{c_2 r \min\{\lambda' - \lambda, \delta - \delta'\}}{2 + \int_0^r P_{\lambda', \delta'}(A_s)\, ds} \right\}.$$

Fix (λ_0, δ_0) in the interior of U and suppose that $\lambda_0 < \rho_c\delta_0$. Suppose $(\lambda_1, \delta_1) \in U$ with $\lambda_1 \in (0, \lambda_0)$ and $\delta_1 = \delta_0 + (\lambda_0 - \lambda_1)$. Choose $r_0 \geq 1$ and $r_1 \geq r_0/P_{\lambda_0,\delta_0}(A_{r_0})$. Then, since $P_{\lambda_0,\delta_0}(A_s)$ is decreasing in s, we have that

$$(3.12)\quad 2 + \int_0^{r_1} P_{\lambda_0,\delta_0}(A_s)\,ds \leq 2 + r_0 + r_1 P_{\lambda_0,\delta_0}(A_{r_0}) \leq 4 r_1 P_{\lambda_0,\delta_0}(A_{r_0}).$$

Apply (3.11) with $(\lambda, \delta) = (\lambda_1, \delta_1)$, $(\lambda', \delta') = (\lambda_0, \delta_0)$ and $r = r_1$. Together with (3.12), this gives

$$(3.13)\qquad P_{\lambda_1,\delta_1}(A_{r_1}) \leq c_3 P_{\lambda_0,\delta_0}(A_{r_1}) \exp\left\{-\frac{c_2(\lambda_0 - \lambda_1)}{4P_{\lambda_0,\delta_0}(A_{r_0})}\right\}.$$

Let

$$(3.14)\qquad h(x) = -\frac{4}{c_2}x\log\left(\frac{x}{c_3}\right) \quad \text{for } 0 < x < 1.$$

Suppose we pick λ_1 satisfying

$$(3.15)\qquad \lambda_0 - \lambda_1 = h\big(P_{\lambda_0,\delta_0}(A_{r_0})\big),$$

so that

$$P_{\lambda_0,\delta_0}(A_{r_0}) = c_3 \exp\left\{-\frac{c_2(\lambda_0 - \lambda_1)}{4P_{\lambda_0,\delta_0}(A_{r_0})}\right\}.$$

Then if $(\lambda_1, \delta_1) \in U$, we have from (3.13) that, since $r_1 \geq r_0$,

$$(3.16)\qquad P_{\lambda_1,\delta_1}(A_{r_1}) \leq \left[P_{\lambda_0,\delta_0}(A_{r_0})\right]^2.$$

Now, since $\lambda_0 < \rho_c\delta_0$, $h(P_{\lambda_0,\delta_0}(A_r)) \to 0$ as $r \to \infty$ and therefore, by (3.15), we may be sure that $(\lambda_1, \delta_1) \in U$ so long as r_0 has been chosen sufficiently large.

To summarize, we have that if (λ_0, δ_0) (with $\lambda_0 < \rho_c\delta_0$) is in the interior of U, if $r_0 = r_0(\lambda_0, \delta_0)$ is large enough to ensure that $(\lambda_1, \delta_1) = (\lambda_0 - h(P_{\lambda_0,\delta_0}(A_{r_0})), \delta_0 + h(P_{\lambda_0,\delta_0}(A_{r_0}))) \in U$ and if $r_1 \geq r_0/P_{\lambda_0,\delta_0}(A_{r_0})$, then (3.16) holds.

Now fix (λ, δ) with $0 < \lambda < \rho_c\delta$. Fix $\varepsilon > 0$ so that $\lambda + \varepsilon < \rho_c(\delta - \varepsilon)$. Let $(\lambda_0, \delta_0) = (\lambda + \varepsilon, \delta - \varepsilon)$ and let U be the closed ball of radius ε centered at (λ_0, δ_0).

Note that the function h defined in (3.14) is increasing on some interval of the form $[0, \eta]$ with $\eta > 0$. Suppose $0 < x_0 < 1$. Let $x_j = x_{j-1}^2$ for $j \geq 1$ and set

$$s(x_0) = \sum_{j=0}^{\infty} h(x_j).$$

Since $x_j \leq x_0^{2j}$, it follows easily from the definition of h that $s(x_0) < \infty$ for x_0 in $(0, 1)$ and that $s(x_0) \to 0$ as $x_0 \to 0$. Choose $x_0 \in (0, 1 \wedge \eta)$ small enough to ensure that $s(x_0) < \lambda_0 - \lambda = \varepsilon$. Pick r_0 large enough to ensure that

$$(3.17)\qquad g_0 = P_{\lambda_0,\delta_0}(A_{r_0}) < x_0.$$

We shall choose $r_1, \lambda_1, \delta_1, r_2, \lambda_2, \delta_2, \dots$ recursively by

$$r_k = r_{k-1}/g_{k-1}, \quad \lambda_k = \lambda_{k-1} - h(g_{k-1}), \quad \delta_k = \delta_{k-1} + h(g_{k-1}) \tag{3.18}$$

for $k \geq 1$, where

$$g_i = P_{\lambda_i, \delta_i}(A_{r_i}). \tag{3.19}$$

We shall prove by induction that

$$\lambda + \varepsilon \geq \lambda_k > \lambda \quad \text{and} \quad \delta - \varepsilon \leq \delta_k < \delta, \text{ so } (\lambda_k, \delta_k) \in U, \tag{3.20}$$

$$g_k \leq g_{k-1}^2, \tag{3.21}$$

$$g_k \leq x_k. \tag{3.22}$$

Assume that (3.20), (3.21) and (3.22) hold with k replaced by j for $j = 1, \dots, k-1$ and choose r_k, λ_k and δ_k according to (3.18). Then we have from the argument leading to (3.16) that if (3.20) holds, then so does (3.21). Now (3.22) follows from (3.21) together with the fact that $g_{k-1} \leq x_{k-1}$ and $x_k = x_{k-1}^2$. To prove (3.20), note that by definition,

$$\lambda_k = \lambda_{k-1} - h(g_{k-1}) = \lambda_0 - \sum_{j=0}^{k-1} h(g_j).$$

Using the fact that h is nonnegative and increasing on $[0, x_0]$ together with the induction hypothesis, we have

$$\lambda + \varepsilon = \lambda_0 \geq \lambda_k \geq \lambda_0 - \sum_{j=0}^{k-1} h(x_j)$$
$$\geq \lambda_0 - s(x_0) > \lambda$$

by the choice of x_0. Similarly $\delta - \varepsilon \leq \delta_k < \delta$. Hence $(\lambda_k, \delta_k) \in U$.

Now if $k \geq 1$, by (3.18) and (3.21),

$$\begin{aligned} g_{k-1}^2 &= g_{k-1} g_{k-1} \leq g_{k-1} g_{k-2}^2 \\ &\leq g_{k-1} g_{k-2} \cdots g_1 g_0^2 \\ &= \frac{r_{k-1} r_{k-2} \cdots r_0}{r_k r_{k-1} \cdots r_1} g_0 \\ &= \frac{r_0 g_0}{r_k} \leq \frac{r_0}{r_k}, \end{aligned}$$

which is to say that

$$P_{\lambda_{k-1}, \delta_{k-1}}(A_{r_{k-1}}) \leq \sqrt{\frac{r_0}{r_k}}.$$

Note that (3.22) implies that $g_k \to 0$ as $k \to \infty$ and hence by (3.18) that $r_k \to \infty$.

Finally, suppose that $r \geq r_0$ and choose k so that $r_{k-1} \leq r < r_k$. Then

$$P_{\lambda,\delta}(A_r) \leq P_{\lambda_{k-1},\delta_{k-1}}(A_r)$$
$$\leq P_{\lambda_{k-1},\delta_{k-1}}(A_{r_{k-1}}) \leq \sqrt{\frac{r_0}{r_k}},$$

since $\lambda < \lambda_{k-1}$, $\delta > \delta_{k-1}$ and $r \geq r_{k-1}$. However $r < r_k$, and so

$$P_{\lambda,\delta}(A_r) \leq 1 \wedge \sqrt{\frac{r_0}{r}} \quad \text{for } r \geq r_0.$$

This inequality is valid also when $0 < r < r_0$ and therefore (3.10) holds with $\alpha(\lambda, \delta) = \sqrt{r_0(\lambda, \delta)}$.

3.2. *Critical-exponent inequalities.* Theorem 1.11 may be proved by an adaptation of the argument of Aizenman and Barsky (1987). Following their method, one obtains a collection of differential inequalities for the finite-volume quantity

$$\theta_\Lambda(\lambda, \delta; \gamma) = 1 - E_{\lambda,\delta}(e^{-\gamma|C_\Lambda|}), \tag{3.23}$$

where $|C_\Lambda|$ is the Lebesgue measure of the set of points reachable from the origin within a closed box Λ with appropriate periodic boundary conditions. There are three such inequalities:

$$\theta_\Lambda \leq 2\theta_\Lambda^2 + \gamma\chi_\Lambda + c_1(\lambda)\theta_\Lambda \frac{\partial \theta_\Lambda}{\partial \lambda} - c_2(\lambda)\theta_\Lambda \frac{\partial \theta_\Lambda}{\partial \delta}, \tag{3.24}$$

$$\frac{\partial \theta_\Lambda}{\partial \lambda} \leq 2\theta_\Lambda \chi_\Lambda, \tag{3.25}$$

$$-\frac{\partial \theta_\Lambda}{\partial \delta} \leq \frac{8\lambda}{\delta}\left\{\frac{1}{4\lambda} + \chi_\Lambda\right\}\theta_\Lambda, \tag{3.26}$$

where $c_1(\lambda) = \lambda(e+1)$, $c_2(\lambda) = 2\,de\,\lambda + 1$ and

$$\chi_\Lambda(\lambda, \delta; \gamma) = E_{\lambda,\delta}\big(|C_\Lambda|e^{-\gamma|C_\Lambda|}\big) = \frac{\partial}{\partial \gamma}\theta_\Lambda(\lambda, \delta; \gamma). \tag{3.27}$$

We sketch the proof of these inequalities. Note that the proof consists basically of discretizing and following Aizenman and Barsky, although Lemma 2.19 is needed in addition in the proof of (3.24). As in Aizenman and Barsky (1987), introduce a random set $\mathscr{G}$ of green sites so that $\{\mathscr{G} \cap (x \times \mathbb{R}) : x \in \mathbb{Z}^d\}$ is an independent family of Poisson point locations with intensity γ, independent of the deaths and crossings. Let $P_{\lambda,\delta,\gamma}$ and $E_{\lambda,\delta,\gamma}$ denote the probability measure and corresponding expectation operator of the resulting configuration of deaths, bonds and green sites. The BKF and FKG inequalities for these measures can be established as before and Lemma 2.19 remains valid with $P_{\lambda,\delta}$ replaced by $P_{\lambda,\delta,\gamma}$, the deterministic set D replaced by $\mathscr{G}$ and A the event $\{0 \leftrightarrow_\Lambda \mathscr{G}\}$.

To prove (3.24), begin as in Aizenman and Barsky (1987), writing

$$\theta_\Lambda(\lambda,\delta;\gamma) = P_{\lambda,\delta,\gamma}(\{0 \leftrightarrow_\Lambda \mathscr{G}\} \square \{0 \leftrightarrow_\Lambda \mathscr{G}\}) + P_{\lambda,\delta,\gamma}(|C_\Lambda \cap \mathscr{G}| = 1)$$
$$+ P_{\lambda,\delta,\gamma}\begin{pmatrix} |C_\Lambda \cap \mathscr{G}| \geq 2 \text{ but there are not} \\ \text{two disjoint connections from 0 to } \mathscr{G} \end{pmatrix}$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}, \quad \text{say.}$$

By the BKF inequality (Section 2.2), $\mathrm{I} \leq \theta_\Lambda(\lambda,\delta;\gamma)^2$. By the independence of the green sites from the deaths and crossings, $\mathrm{II} = \gamma\chi_\Lambda(\lambda,\delta;\gamma)$. The term III is more complicated to handle. Discretize as in Section 2.1, writing $P^\varepsilon_{\lambda,\delta,\gamma}$, $E^\varepsilon_{\lambda,\delta,\gamma}$ and $\theta^\varepsilon_\Lambda$ for the corresponding measure, expectation and analogue of θ_Λ. By the argument used to estimate the probability of F_3 on pages 517–518 of Aizenman and Barsky (1987), the analogue III^ε of III with $P_{\lambda,\delta,\gamma}$ replaced by $P^\varepsilon_{\lambda,\delta,\gamma}$ satisfies

$$\mathrm{III}^\varepsilon \leq \theta^\varepsilon_\Lambda(\lambda,\delta;\gamma)\left[E^\varepsilon_{\lambda,\delta,\gamma}(B;A) + E^\varepsilon_{\lambda,\delta,\gamma}(I;A)\right],$$

where B and I denote the numbers of pivotal bonds and death-pivotal intervals, respectively. To see this, observe that if a site (x,t) in $\mathbb{Z}^d \times \mathbb{R}$ is the last pivotal site for A, then it occurs at the end of a pivotal interval. Let $\varepsilon \downarrow 0$ and use the analogues of (2.7) and Lemma 2.19 to obtain

$$\mathrm{III} \leq \theta_\Lambda(\lambda,\delta;\gamma)\left[\theta_\Lambda(\lambda,\delta;\gamma) + (e+1)E_{\lambda,\delta,\gamma}(B;A) + 2d\lambda eE_{\lambda,\delta,\gamma}(L;A)\right],$$

where L is the total length of pivotal intervals. Inequality (3.24) now follows from Russo's formula; see (2.14) and (2.15).

By Russo's formula (2.14), the left-hand side of (3.25) equals

$$\int_{(x,t)\in\Lambda} P_{\lambda,\delta,\gamma}(B(x,t)\cap A^c)\,dt,$$

where $B(x,t)$ is the event that (x,t) is bond-pivotal for the event A. If $x = y + \frac{1}{2}e_i$ with $y \in \mathbb{Z}^d$, then by the BKF inequality and the periodicity of the boundary conditions,

$$P_{\lambda,\delta,\gamma}(B(x,t)\cap A^c) \leq \theta_\Lambda(\lambda,\delta;\gamma)\big[P_{\lambda,\delta,\gamma}(0 \leftrightarrow_\Lambda (y,t), C_\Lambda \cap \mathscr{G} = \varnothing)$$
$$+P_{\lambda,\delta,\gamma}(0 \leftrightarrow_\Lambda (y+e_i,t), C_\Lambda \cap \mathscr{G} = \varnothing)\big].$$

Inequality (3.25) follows by integrating.

Finally, use the first equation in (2.15) to see that the left-hand side of (3.26) equals $\delta^{-1}E^\varepsilon_{\lambda,\delta,\gamma}(D;A^c)$, where D denotes the number of pivotal deaths. By the argument used to prove 7.1 in Aizenman and Barsky (1987),

$$E^\varepsilon_{\lambda,\delta,\gamma}(D;A^c) \leq \theta^\varepsilon_\Lambda(\lambda,\delta;\gamma)E^\varepsilon_{\lambda,\delta,\gamma}\begin{pmatrix} \text{number of deaths in } \Lambda \text{ joined} \\ \text{to 0 from one side only} \end{pmatrix}; A^c\Bigg)$$
$$\leq 2\theta^\varepsilon_\Lambda(\lambda,\delta;\gamma)\left[1 + E^\varepsilon_{\lambda,\delta,\gamma}(\text{number of bonds in } C_\Lambda;\, A^c)\right].$$

The second inequality follows from the facts that each vertical interval has two endpoints and that every such interval in C_Λ except the one containing the origin requires at least one bond to connect it to C_Λ. At least one of the endpoints of a bond in C_Λ is connected to 0 by a path that does not use the bond itself and so the last expectation in this inequality is no greater than $4\lambda\varepsilon E^{\varepsilon}_{\lambda,\delta,\gamma}$(number of sites in C_Λ; A^c) $\le 4\lambda E^{\varepsilon}_{\lambda,\delta,\gamma}(|C_\Lambda|; A^c)$. Now let $\varepsilon \downarrow 0$ and use the analogue of (2.7) to obtain (3.26).

Substituting (3.25) and (3.26) into (3.24) and using Lemma 4.1 in Aizenman and Barsky (1987), we obtain Theorem 1.11.

3.3. *Exact calculation in two dimensions.* The extra ingredient essential for the proof of Theorem 1.12 is self-duality in two dimensions. Consider the percolation process on $\mathbb{Z} \times \mathbb{R}$ with parameters λ and δ and construct the following dual process on $(\mathbb{Z} + \frac{1}{2}) \times \mathbb{R}$. Each death (x, t) of the first process is mapped onto a bond of the second, joining $(x - \frac{1}{2}, t)$ to $(x + \frac{1}{2}, t)$. Each bond of the first process, with centre at $(x + \frac{1}{2}, t)$ say, is mapped onto a death of the second at $(x + \frac{1}{2}, t)$. Clearly the second process is a percolation process with parameters δ and λ. If $\lambda = \delta$, then the two processes have the same distribution. See Figure 2.

That $\rho_c = 1$ follows now by roughly the same method as used for bond percolation on $\mathbb{Z}^2$ [see Grimmett (1989), Section 9.3]. First we show that $\rho_c \le 1$. Let n be a positive integer and let D_n be the region of $\mathbb{Z} \times \mathbb{R}$ containing all points (x, t) satisfying $|x - \frac{1}{4}| + |t| \le n$; see Figure 2 again. Denote by F_1, F_2, F_3 and F_4 the sides of D_n in clockwise order starting from the lower left side. Suppose that $\lambda = \delta > 0$. Let $A_n = \{F_1 \leftrightarrow_{D_n} F_3\}$ and let A^d_n be the corresponding dual event: A^{d}_n is the event that there exist $(x + \frac{1}{2}, t) \in F_2$, $(y + \frac{1}{2}, s) \in F_4$ such that $(x + \frac{1}{2}, t)$ is joined in D_n to $(y + \frac{1}{2}, s)$ by a path in the dual. By symmetry and the fact that exactly one of A_n and A^{d}_n occurs in any configuration, we have that $P_{\lambda,\delta}(A_n) = P_{\lambda,\delta}(A^{\mathrm{d}}_n) = \frac{1}{2}$. If $\lambda = \delta$ and $\rho_c > 1$,

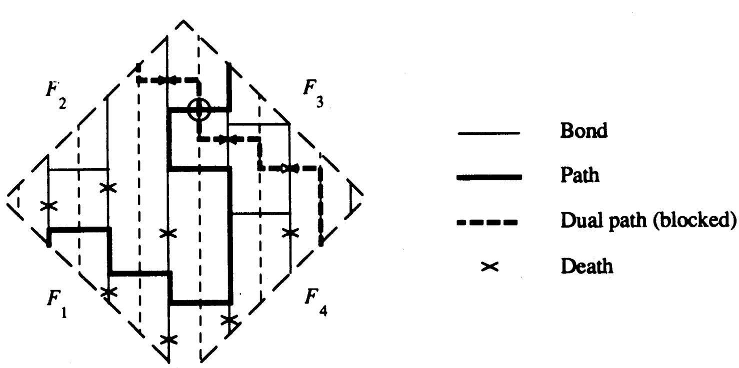

FIG. 2. *Part of the primal and dual processes on the region D_n. Solid horizontal lines indicate crossings in the primal process. Note that the dual death marked with an open circle blocks a path in the dual from F_2 to F_4.*

then

$$P_{\lambda,\delta}(A_n) \leq ne^{-n\psi}$$

for some $\psi = \psi(\lambda, \delta) > 0$, by Theorem (1.7). Thus $P_{\lambda,\delta}(A_n) \to 0$ as $n \to \infty$, a contradiction. Hence $\rho_c \leq 1$.

The first step in the proof that $\rho_c \geq 1$ is to check that the Burton–Keane (1989) proof of the uniqueness of the infinite cluster holds for this situation. This is easy; it is necessary only to replace the notion of an encounter point by that of an encounter interval I, that is an interval of length one, say, which is incident in $(\mathbb{Z} \times \mathbb{R}) - I$ to three or more disjoint infinite clusters. Next we adapt an argument of Zhang [see Grimmett (1989), page 195]. Let n be a positive integer, write $H_n = D_n + (\frac{1}{4}, 0) \subseteq \mathbb{R}^2$ and let $F_1(n)$, $F_2(n)$, $F_3(n)$ and $F_4(n)$ be the sides of H_n taken clockwise and beginning with the lower left side. Let $G_i(n)$ be the event that some point of $F_i(n)$ is the endpoint of an infinite path of $(\mathbb{Z} \times \mathbb{R}) - H_n$ (except for this endpoint). Suppose that $\theta(1) > 0$. Then

$$P_{\lambda,\delta}\left(\bigcup_{i=1}^{4} G_i(n)\right) \to 1 \quad \text{as } n \to \infty,$$

so that, for all large n, $P_{\lambda,\delta}(G_i(n)) > \frac{7}{8}$, by the FKG inequality and symmetry. Moving to the dual, let $H_n^{\mathrm{d}} = D_n - (\frac{1}{4}, 0)$ and define $G_i^{\mathrm{d}}(n)$ to be the events corresponding to $G_i(n)$ but defined in terms of dual clusters and H_n^{d}. If n is such that

$$P_{\lambda,\delta}(G_1(n) \cap G_3(n)) > \tfrac{3}{4} \quad \text{and} \quad P_{\lambda,\delta}(G_2^{\mathrm{d}}(n) \cap G_4^{\mathrm{d}}(n)) > \tfrac{3}{4},$$

then we have a contradiction, since if all four of these events occur, an event with probability at least $\frac{1}{2}$, then there are two disjoint infinite clusters in

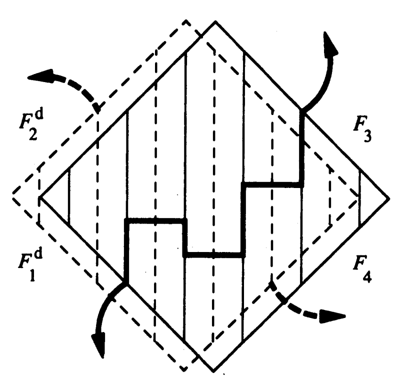

FIG. 3. *The regions H_n and H_n^{d}. The infinite cluster of the primal process is unique and hence there are two or more such clusters in the dual, a contradiction.*

either the primal or the dual process (see Figure 3). This contradicts the uniqueness of the infinite cluster.

This proves that $\theta(1) = 0$ and hence that $\rho_c \geq 1$.

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