The Annals of Probability
1991, Vol. 19, No. 3, 935–959

L_2 RATES OF CONVERGENCE FOR ATTRACTIVE REVERSIBLE NEAREST PARTICLE SYSTEMS: THE CRITICAL CASE[1]

By Thomas M. Liggett

University of California, Los Angeles

Reversible nearest particle systems are certain one-dimensional interacting particle systems whose transition rates are determined by a probability density $\beta(n)$ with finite mean on the positive integers. The reversible measure for such a system is the distribution v of the stationary renewal process corresponding to this density. In an earlier paper, we proved under certain regularity conditions that the system converges exponentially rapidly in $L_2(v)$ if and only if the system is supercritical. This in turn is equivalent to $\beta(n)$ having exponential tails. In this paper, we consider the critical case, and give moment conditions on $\beta(n)$ which are separately necessary and sufficient for the convergence of the process in $L_2(v)$ at a specified algebraic rate. In order to do so, we develop conditions on the generator of a general Markov process which correspond to algebraic L_2 convergence of the process. The use of these conditions is also illustrated in the context of birth and death chains on the positive integers.

1. Introduction. Exponential convergence to equilibrium in the L_2 sense has been proved for a number of interacting particle systems in recent years. Examples are the stochastic Ising model [see, e.g., Holley (1985) and Aizenman and Holley (1987)] and attractive reversible nearest particle systems [Liggett (1989a)]. The result for nearest particle systems says roughly that exponential L_2 convergence occurs if and only if the process is supercritical. In this paper, we examine the sense in which one can say that the convergence occurs at an algebraic rate if the process is critical.

Nearest particle systems were introduced by Spitzer (1977). They are Markov processes on the set of configurations $\eta \in \{0,1\}^Z$ which satisfy

$$\sum_{x\geq 0}\eta(x) = \sum_{x\leq 0}\eta(x) = \infty.$$

The process makes transitions $1 \to 0$ at rate one and $0 \to 1$ at site x at a rate $\beta(l,r)$, where l and r are the distance from x to the nearest sites to the left and right, respectively, at which there is a one. The birth rate $\beta(l,r)$ is assumed to be a bounded, nonnegative symmetric function defined for positive integers l and r. The process is attractive if $\beta(l,r)$ is a decreasing function of l and r. It is said to survive if it has an invariant probability measure. A nearest particle system which survives is said to be supercritical if there is a

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Received June 1989; revised April 1990.
[1]Research supported in part by NSF Grant DMS-86-01800.
AMS 1980 *subject classification*. Primary 60K35.
Key words and phrases. Interacting particle systems, nearest particle systems, algebraic rates of convergence for semigroups, renewal theory.

$\lambda < 1$ so that the system with birth rates $\lambda\beta(l, r)$ also survives. Otherwise, it is said to be critical.

Spitzer's theorem asserts that a nearest particle system with strictly positive birth rates is reversible with respect to some probability measure v if and only if

$$\beta(l, r) = \frac{\beta(l)\beta(r)}{\beta(l+r)} \tag{1.1}$$

for some positive probability density $\beta(n)$ on the positive integers with finite mean M. The reversible measure v is then the stationary renewal measure corresponding to this density. A reversible nearest particle system is attractive if and only if the density is logconvex and we then let

$$\rho = \lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)}, \tag{1.2}$$

which exists by monotonicity and satisfies $0 < \rho \le 1$. An easy consequence of Spitzer's theorem is that an attractive reversible nearest particle system is supercritical if $\rho < 1$ and critical if $\rho = 1$. For these and other basic results about nearest particle system, see Chapter VII of Liggett (1985).

A Markov process with semigroup $S(t)$ and invariant probability measure v is said to converge exponentially rapidly in $L_2(v)$ if there exists an $\varepsilon > 0$ so that

$$\left\| S(t)f - \int f\,dv \right\|_2 \le e^{-\varepsilon t} \left\| f - \int f\,dv \right\|_2 \tag{1.3}$$

for all $f \in L_2(v)$. In Liggett (1989a), we proved under some additional regularity assumptions on the density $\beta(n)$ that an attractive reversible nearest particle system converges exponentially in $L_2(v)$, where v is the stationary renewal measure corresponding to β, if and only if the process is supercritical. Furthermore, the largest ε for which (1.3) holds for all $f \in L_2(v)$ satisfies the bounds

$$\frac{\beta(1)}{4M\rho}(1-\rho)^2 \le \varepsilon \le 4(1-\rho)^2 \sum_{n=1}^{\infty} n^2\beta(n)\rho^{-n}.$$

This suggests that some sort of algebraic L_2 convergence should occur in the critical case, with the algebraic rate depending on the size of the tails of $\beta(n)$.

It is natural to say that a Markov process with semigroup $S(t)$ and invariant probability measure v converges algebraically rapidly in $L_2(v)$ if there exists an $\alpha > 0$ so that

$$\left\| S(t)f - \int f\,dv \right\|_2^2 \le \frac{V(f)}{t^\alpha} \tag{1.4}$$

for all $t > 0$ and some $V(f)$ which is finite for sufficiently many $f \in L_2(v)$. An immediate difficulty arises in deciding what $V(f)$ should be and for which f's it should be finite. We will see in Section 2 that for a reversible process, if (1.4) holds with $V(f) = \|f\|^2$ or with $V(f)$ equalling the Dirichlet form of f, then the process actually converges exponentially rapidly in $L_2(v)$. On the other side of the picture, it is an easy consequence of the spectral theorem that any process for which v is an extremal reversible measure satisfies (1.4) with a $V(f)$ which is finite for a dense set of f's. The problem is that neither this dense set nor the $V(f)$ can be determined unless the spectral decomposition of the process is known explicitly.

A further difficulty comes from the fact that the exponent α in (1.4) will in general depend on the choice of $V(f)$. For example, J. D. Deuschel has shown (private communication) that the critical Ornstein–Uhlenbeck process which he considered in Deuschel (1989) has the following property: If $d > 2$ and $1 \le q < d/2$, then there are two positive constants c_1 and c_2 so that

$$\frac{c_1}{t^{(d/2q)-1}} \le \sup\left\{\left\|S(t)f - \int f\,dv\right\|_2^2 : \sum_{j\in Z^d} \sup_{x\in R^{Z^d}} \left|\frac{\partial f}{\partial x(j)}\right|^{2q/(q+1)} = 1\right\} \le \frac{c_2}{t^{(d/2q)-1}}$$

for $t \ge 1$. Here v is one of the reversible Gaussian measures identified in Section 2 of Deuschel (1989). We conclude that in order for an algebraic convergence result to be meaningful, the $V(f)$ in (1.4) should be given explicitly. [See also Deuschel (1991).]

In this paper, we will prove the following result, which clarifies this situation for one class of interacting particle systems. The two parts of the theorem are proved in Sections 3 and 4, respectively. Parts of the proofs draw heavily on the approach used in Liggett (1989a) for the supercritical case.

In the following statement, let

$$B(n) = \sum_{k=n}^{\infty} \beta(k)$$

be the tail probabilities for β and

$$V(f) = \left\{\sum_x \sup_\eta |f(\eta_x) - f(\eta)|\right\}^2$$

be the square of the triple norm used in Chapter I of Liggett (1985). As usual, η_x is the configuration obtained from η by flipping the value of the coordinate $\eta(x)$. Note that $V(f) < \infty$ for any function which depends on finitely many coordinates and that our choice of $V(f)$ is analogous to the one used by Deuschel if his q is one.

THEOREM 1.5. *Let $S(t)$ be the semigroup for the critical attractive reversible nearest particle system determined by the density β and let v be the corresponding renewal measure. Fix $q > 1$.*

(i) *Suppose that* $\beta(m+n)$ *is totally positive of order three and* $B(n)$ *satisfies*

$$\sup_N \frac{\sum_{n \ge N} B(n)}{NB(N)} < \infty \tag{1.6}$$

and

$$\sum_n \frac{B^2(n)}{B(2n)} < \infty. \tag{1.7}$$

If

$$\sum_{k=1}^{\infty} k^{2q+2}\beta(k) < \infty, \tag{1.8}$$

then there is a constant C *so that*

$$\left\| S(t)f - \int f\,dv \right\|_2^2 \le C\frac{V(f)}{t^{q-1}} \tag{1.9}$$

for all continuous functions f.

(ii) *Suppose that*

$$\inf_n \frac{B(n)}{n\beta(n)} > 0 \tag{1.10}$$

and

$$\sum_n \frac{\beta^2(n)}{\beta(2n)} < \infty. \tag{1.11}$$

If there is a constant C *so that* (1.9) *holds for all* f *which depend on finitely many coordinates, then*

$$\sum_{k=1}^{\infty} k^{\alpha}\beta(k) < \infty$$

for all $\alpha < q - 2$.

REMARKS. (i) Note that assumptions (1.6), (1.7), (1.10) and (1.11) are all satisfied if

$$\beta(n) \sim \frac{C}{n^{\alpha}}$$

for some $\alpha > 2$.

(ii) For more on total positivity, see Karlin (1968). For connections with renewal theory, see Liggett (1989b). The logconvexity of $\beta(n)$ is equivalent to total positivity of order two of $\beta(n+m)$. Thus the total positivity assumption in part (i) of the theorem is slightly stronger than the logconvexity assumption. It is automatically satisfied if $\beta(n) = Cn^{-\alpha}$ for some $\alpha > 2$.

(iii) There is a substantial gap between the moment conditions appearing in the two parts of Theorem 1.5. It would be of considerable interest to close this gap.

(iv) It would also be of interest to determine whether (1.9) ever holds in the critical case with $V(f) = \sup_{\eta,\zeta}|f(\eta) - f(\zeta)|^2$.

The first step in the proof of Theorem 1.5 is to establish a characterization of algebraic L_2 convergence in terms of the generator of the process. This is carried out for general Markov processes in Section 2. In that section, we also illustrate the use of this characterization in the context of birth and death chains on the nonnegative integers. A special case of the result proved there is the following. Suppose the chain goes from k to $k-1$ at rate one and from $k-1$ to k at rate $1 - c/k$ for $k \geq 1$. Then (1.4) holds with $V(f) = C\sup_k|f(k+1) - f(k)|^2$ for some constant C if $2\alpha < c - 3$, but not if $2\alpha > c - 3$.

During a recent visit to UCLA, Marc Yor asked about potential applications of results such as Theorem 1.5 which give rates of convergence to equilibrium in the L_2 sense. We conclude this section by giving two simple applications. The first provides a central limit theorem. Its proof is modeled after the proof of the central limit theorem for the contact process given by Schonmann (1986). The second is an ergodic theorem which resulted from communications with Jeffrey Steif.

THEOREM 1.12. *Let η_t be an attractive reversible nearest particle system with birth rates given by* (1.1) *in terms of the density β and take η_0 to be distributed according to the invariant renewal measure v, so that η_t is a stationary process. Let f be a continuous increasing function on $\{0,1\}^Z$ which satisfies $V(f) < \infty$, $\int f\,dv = 0$ and $\int f^2\,dv > 0$. Assume that β is totally positive of order three and that* (1.6) *and* (1.7) *are satisfied. If β has a finite moment of order greater than eight, then*

$$\frac{1}{\sqrt{t}}\int_0^t f(\eta_s)\,ds$$

converges in distribution as $t \to \infty$ to a nondegenerate normal law.

PROOF. By Harris' theorem on the preservation of positive correlations [see Theorem 2.14 and Corollary 2.21 of Chapter II of Liggett (1985), for example], the random variables $\{f(\eta_t),\ t \geq 0\}$ are associated. Therefore by Newman's central limit theorem for stationary associated processes [see Newman and Wright (1981) or Newman (1984), for example], in order to prove our result, it is enough to prove that

$$\int_0^\infty Ef(\eta_0)f(\eta_t)\,dt = \int_0^\infty \int f(\eta)S(t)f(\eta)\,dv\,dt < \infty.$$

But this is an immediate consequence of Theorem 5.7 of Liggett (1989a) and

Theorem 2 of Liggett (1989b) in the supercritical case and of Theorem 1.5 (applied to an appropriate $q > 3$) in the critical case. □

REMARK. An alternative approach to the proof of this theorem is to use the Kipnis–Varadhan (1986) central limit theorem for functionals of reversible Markov processes. This approach is potentially applicable to processes which do not satisfy the assumptions of Harris' theorem (e.g., exclusion processes).

THEOREM 1.13. *Let η_t be an attractive reversible nearest particle system with birth rates given by* (1.1) *in terms of the density β. Let $\mu_{\eta,t}$ denote the distribution at time t if the initial configuration is η. Assume that β is totally positive of order three and that* (1.6) *and* (1.7) *are satisfied. If β has a finite moment of order greater than six, then for almost every η* (*with respect to the invariant renewal measure v*), *$\mu_{\eta,t}$ converges weakly to v.*

PROOF. If f satisfies $V(f) < \infty$, then Theorem 5.7 of Liggett (1989a) and Theorem 2 of Liggett (1989b) in the supercritical case and Theorem 1.5 (applied to an appropriate $q > 2$) in the critical case imply that

$$\int_0^\infty \int \left[S(t) f - \int f dv \right]^2 dv\, dt < \infty.$$

Since

$$\frac{d}{dt} S(t) f(\eta)$$

is uniformly bounded in t and η, it follows that

$$\lim_{t \to \infty} S(t) f(\eta) = \int f dv$$

for almost every η. The result follows by applying this statement of a countable dense set of functions. □

2. A criterion for algebraic convergence. Consider a Markov process on a complete separable metric space with finite invariant measure v, strongly continuous semigroup $S(t)$ on $L_2(v)$ and generator Ω with domain $D(\Omega)$. A necessary and sufficient condition for exponential convergence in $L_2(v)$ is that there exists a constant C so that

$$\left\| f - \int f dv \right\|_2^2 \le -C \int f \Omega f dv \tag{2.1}$$

for all $f \in D(\Omega)$; see Theorem 2.3 of Liggett (1989a), for example. The main result in this section provides an analogous condition on the generator which characterizes algebraic L_2 convergence. It is based on Nash's lemma and the related results which have been proved by Stroock, Varopoulos and others in recent years; see, for example, Carlen, Kusuoka and Stroock (1987) and the references given there. I am grateful to Dan Stroock for having provided me with the original version of the following result and its proof.

THEOREM 2.2. *Take $1 < p, q < \infty$ such that $p^{-1} + q^{-1} = 1$ and a function V on $L_2(v)$ satisfying $0 \le V(f) \le \infty$ and $V(cf + d) = c^2V(f)$ for all constants c and d. Consider the following two statements*:

(a) *There exists a constant C so that*

$$\left\| f - \int f\,dv \right\|_2^2 \le C\left[-\int f\Omega f\,dv \right]^{1/p} [V(f)]^{1/q} \tag{2.3}$$

for all $f \in D(\Omega)$.

(b) *There exists a constant C so that*

$$\left\| S(t)f - \int f\,dv \right\|_2^2 \le C\frac{V(f)}{t^{q-1}} \tag{2.4}$$

for all $f \in L_2(v)$ and all $t > 0$.

(i) *If* (a) *holds and V satisfies $V[S(t)f] \le V(f)$ for all $f \in L_2(v)$ and all $t > 0$, then* (b) *holds.*

(ii) *If* (b) *holds and the process is reversible with respect to v, then* (a) *holds.*

REMARKS. (i) If $p = 1$, then (2.3) becomes (2.1).

(ii) If (a) is satisfied with

$$V(f) = \left\| f - \int f\,dv \right\|_2^2 \quad \text{or} \quad V(f) = -\int f\Omega f\,dv,$$

then (2.1) is satisfied and hence the rate of L_2 convergence is in fact exponential. Thus neither of these choices for V is useful.

(iii) The monotonicity assumption in the first part of the theorem is rather natural in view of the following observation. If for some q we define

$$V(f) = \sup_{t \ge 0} t^{q-1} \left\| S(t)f - \int f\,dv \right\|_2^2,$$

then $V(f)$ is automatically monotone.

(iv) In searching for V's which satisfy the monotonicity assumption, one might first think about squares of L_p norms. These appear not to be particularly useful in our context. Other potential choices are squares of the Besov norms described in Chapter V of Stein (1970) and Chapter 2 of Triebel (1983), for example. These are defined in terms of the semigroup as follows, for appropriate r and s:

$$\left\{ \int_0^\infty \left\| \frac{\partial}{\partial t} S(t)f \right\|_2^r t^s\,dt \right\}^{1/r}.$$

This choice merely recasts the problem in another form, since (2.4) is then immediate for some q (depending on r and s). In our applications, we will force the monotonicity property by applying Theorem 2.2, not to the V we are interested in, which is not monotone, but to another V which is defined by taking suprema over t. See the proofs of Theorem 2.10 and 1.5, for example.

PROOF. (i) Take an $f \in D(\Omega)$ which satisfies $\int f\,d\nu = 0$ and let $f_t = S(t)f$. Apply (2.3) to the function f_t to get

$$\frac{d}{dt}\int f_t^2\,d\nu = 2\int f_t \Omega f_t\,d\nu \le -2C^{-p}\left\{\int f_t^2\,d\nu\right\}^p \{V(f_t)\}^{-p/q}.$$

Letting

$$F(t) = \int f_t^2\,d\nu$$

and using the monotonicity of V under the semigroup, we obtain

$$F'(t) \le -2C^{-p}[F(t)]^p[V(f)]^{-p/q}.$$

Therefore

$$\frac{d}{dt}[F(t)]^{-p/q} = -\frac{p}{q}[F(t)]^{-p}F'(t) \ge \frac{2p}{q}C^{-p}[V(f)]^{-p/q}.$$

Integrating this inequality from 0 to t and discarding the boundary term at 0 gives

$$F(t) \le V(f)C^q\left\{\frac{q}{2pt}\right\}^{q/p},$$

which is the required result since $q/p = q - 1$.

(ii) Take an $f \in D(\Omega)$ which satisfies $\int f\,d\nu = 0$ and let $f_t = S(t)f$. Then

$$\int f f_t\,d\nu \le \|f\|_2\|f_t\|_2 \le \|f\|_2\sqrt{CV(f)t^{1-q}}, \tag{2.5}$$

by Hölder's inequality and (2.4). On the other hand, since the process is reversible,

$$\frac{d}{ds}\int f\Omega f_s\,d\nu = \frac{d}{ds}\int f_s\Omega f\,d\nu = \int(\Omega f)(S(s)\Omega f)\,d\nu$$
$$= \left\|S\left(\frac{s}{2}\right)\Omega f\right\|_2^2 \ge 0,$$

so that

$$\int f f_t\,d\nu = \int f^2\,d\nu + \int_0^t \int f\Omega f_s\,d\nu\,ds \ge \|f\|_2^2 + t\int f\Omega f\,d\nu. \tag{2.6}$$

Combining (2.5) and (2.6), we see that

$$\|f\|_2^2 + t\int f\Omega f\,d\nu \le \|f\|_2\sqrt{CV(f)t^{1-q}} \tag{2.7}$$

for all $t > 0$. The choice of t which provides the best inequality is

$$t = \left\{ \frac{-2\int f\Omega f\,dv}{\|f\|\sqrt{CV(f)}\,(q-1)} \right\}^{-2/(q+1)}.$$

Using this choice in (2.7) gives (2.3). □

Next we prove a simple result which will be useful in applying Theorem 2.2 to both birth and death chains and nearest particle systems.

PROPOSITION 2.8. *Suppose that π is a strictly positive probability density on $\{0, 1, 2, \dots\}$ and let*

$$\sigma(n) = \frac{1}{\pi(n)} \sum_{k=n}^{\infty} \pi(k). \tag{2.9}$$

For a function f on $\{0, 1, 2, \dots\}$, let

$$\operatorname{var}_{\pi}(f) = \sum_{n=1}^{\infty} \left\{ f(n) - \sum_{k=1}^{\infty} f(k)\pi(k) \right\}^2 \pi(n)$$

be the variance of f relative to π. Then

$$\operatorname{var}_{\pi}(f) \le 4 \sum_{k=0}^{\infty} |f(k+1) - f(k)|^2 [\sigma(k)]^2 \pi(k).$$

PROOF. Square out the following sum and use the Schwarz inequality to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \pi(n) \left\{ \sum_{k=0}^{n} |f(k+1) - f(k)| \right\}^2 \\
&\quad \le 2 \sum_{n=0}^{\infty} \pi(n) \sum_{0 \le j \le k \le n} |f(j+1) - f(j)|\,|f(k+1) - f(k)| \\
&\quad = 2 \sum_{k=0}^{\infty} |f(k+1) - f(k)| \sigma(k)\pi(k) \sum_{j=0}^{k} |f(j+1) - f(j)| \\
&\quad \le 2 \left(\sum_{k=0}^{\infty} |f(k+1) - f(k)|^2 [\sigma(k)]^2 \pi(k) \right. \\
&\qquad \left. \times \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} |f(j+1) - f(j)| \right\}^2 \pi(k) \right)^{1/2}.
\end{aligned}$$

Squaring this and dividing gives

$$\sum_{n=0}^{\infty} \pi(n)\left\{\sum_{k=0}^{n} |f(k+1) - f(k)|\right\}^2 \le 4\sum_{k=0}^{\infty} |f(k+1) - f(k)|^2[\sigma(k)]^2\pi(k).$$

This gives the required result when combined with

$$\begin{aligned}\operatorname{var}_\pi(f) &= \tfrac{1}{2}\sum_{j,k=0}^{\infty} \pi(j)\pi(k)[f(k) - f(j)]^2\\ &\le \sum_{0\le j<k} \pi(j)\pi(k)\left\{\sum_{i=j}^{k-1} |f(i+1) - f(i)|\right\}^2. \qquad \square\end{aligned}$$

The following theorem illustrates the use of Theorem 2.2 in the context of birth and death chains. Rates of convergence in a different sense for birth and death chains have been proved by Lindvall (1979). Versions of those results for one-dimensional diffusions are contained in Lindvall (1983). In Section 3 of Liggett (1989a), we proved that a positive recurrent birth and death chain whose transition rates are bounded above and below by positive constants converges exponentially rapidly in L_2 if and only if its stationary distribution has exponential tails in the sense that $\sigma(n)$ is uniformly bounded. (Note that the sufficiency of this condition for exponential convergence is an immediate consequence of Proposition 2.8.) We see next that algebraic L_2 convergence is associated with algebraic decay of the stationary distribution.

THEOREM 2.10. *Consider a positive recurrent birth and death chain X_t on $\{0, 1, 2, \dots\}$ with transition rates $q(i, j)$ and invariant measure $\pi(n)$. Define $V(f) = \sup_k |f(k+1) - f(k)|^2$ and fix q with $1 < q < \infty$.*

(i) *Suppose that $\inf_i q(i, i+1) > 0$, that the σ defined in (2.9) satisfies $\sup_n \sigma(n)/n < \infty$ and that*

$$\sup_{t\ge 0}\sum_{k=0}^{\infty} \left|E^{k+1}X_t - E^kX_t\right|^2 k^{2q}\pi(k) < \infty. \tag{2.11}$$

Then there is a constant C so that

$$\operatorname{var}_\pi(S(t)f) \le CV(f)t^{1-q} \tag{2.12}$$

for all $f \in L_2(\pi)$ and all $t > 0$.

(ii) *Suppose that $\sup_i q(i, i+1) < \infty$ and that there is a constant C so that (2.12) is satisfied for all $f \in L_2(\pi)$ and all $t > 0$. Then*

$$\sum_{k=0}^{\infty} k^\alpha\pi(k) < \infty$$

for all $\alpha < 2q$.

PROOF. (i) The $V(f)$ we are using is not necessarily decreasing under the action of the semigroup of the birth and death chain. Therefore, we define

$$V^*(f) = \sup_{t \geq 0} \sum_{k=0}^{\infty} |S(t)f(k+1) - S(t)f(k)|^2 k^{2q}\pi(k),$$

which does satisfy this monotonicity property by definition. The result will therefore follow from part (i) of Theorem 2.2 provided that we verify condition (2.3) with V replaced by V^* and then show that

$$V^*(f) \leq CV(f) \tag{2.13}$$

for some constant C. For the first of these, recall that

$$\sum_{k \geq 0} f(k)\Omega f(k)\pi(k) = -\sum_{k \geq 0} [f(k+1) - f(k)]^2 q(k, k+1)\pi(k) \tag{2.14}$$

[see Proposition 3.3 of Liggett (1989a), for example]. By Proposition 2.8 and Hölder's inequality,

$$\operatorname{var}_\pi(f) \leq 4\left\{\sum_{k \geq 0} |f(k+1) - f(k)|^2 \pi(k)\right\}^{1/p} \times \left\{\sum_{k \geq 0} |f(k+1) - f(k)|^2 [\sigma(k)]^{2q}\pi(k)\right\}^{1/q}.$$

Together with (2.14) and the assumptions on $q(i, i+1)$ and $\sigma(n)$, this gives (2.3) with $V(f)$ replaced by $V^*(f)$. In order to check (2.13), write

$$|f(j) - f(k)| \leq |j - k| \sup_i |f(i+1) - f(i)|,$$

so that

$$|S(t)f(k+1) - S(t)f(k)| = |E^{k+1}f(X_t) - E^k f(X_t)| \leq E(Y_t - X_t)\sup_i |f(i+1) - f(i)|,$$

where Y_t and X_t are two copies of the birth and death chain with initial states $k+1$ and k, respectively, which are coupled so that $Y_t \geq X_t$ for all t. Under assumption (2.11), this gives (2.13).

(ii) A birth and death chain is automatically reversible, so part (ii) of Theorem 2.2 applies. Applying (2.3) to the function

$$f(k) = (k \wedge N)^\gamma$$

for a large positive γ and using the assumption that the transition rates are bounded above, we see that there is a constant C (depending on γ) so that

$$\sum_{k=0}^{N} k^{2\gamma}\pi(k) \leq C\left\{\sum_{k=0}^{N} \left[(k+1)^\gamma - k^\gamma\right]^2 \pi(k)\right\}^{1/p} \{N^\gamma - (N-1)^\gamma\}^{2/q},$$

where again $1/p + 1/q = 1$. Apply Hölder's inequality to the first term on the right-hand side, using the fact that $(k+1)^\gamma - k^\gamma$ is asymptotic to a constant multiple of $k^{\gamma-1}$, and then divide to obtain

$$\left\{\sum_{k=0}^{N} k^{2\gamma}\pi(k)\right\}^{1-(2\gamma-2)/2\gamma p} \le CN^{(2\gamma-2)/q}$$

for some new constant C. This implies that

$$\sum_{k=N/2}^{N} k^{\alpha}\pi(k) \le CN^{(2\gamma p(\gamma-1)/q(\gamma p-\gamma+1))-(2\gamma-\alpha)},$$

where again the value of C has been changed. The exponent of N converges to $\alpha - 2q$ as $\gamma \to \infty$. Therefore, if $\alpha < 2q$, we may choose the γ so that this exponent is negative. To finish the proof of (ii), simply replace N by 2^m and sum on m. □

We conclude this section with a result which can be used to check assumption (2.11) in many cases.

PROPOSITION 2.15. *Suppose the birth and death chain X_t with transition rates $q(i,j)$ and invariant measure $\pi(n)$ satisfies* (i) $\sum_n n\pi(n) < \infty$ *and* (ii) $q(i,i+1) \le q(i,i-1)$ *for each* $i > 0$.

(a) *Then* $E^k X_t \le C(k+1)$ *for some constant* C.

(b) *If in addition*

$$\inf_{i>0} q(i,i-1) > 0, \qquad \sup_{i>0} i[q(i,i-1) - q(i,i+1)] < \infty \quad \text{and}$$
$$\sup_n \sigma(n)/n < \infty,$$

then there is a constant C so that

$$0 \le E^{k+1}X_t - E^k X_t \le C[\log(k+2)]^{3/2} \tag{2.16}$$

for all $k \ge 0$ and $t \ge 0$.

PROOF. (a) If $f(k) \equiv k$, then $\Omega f(k) = q(k,k+1) - q(k,k-1) \le 0$ for $k \ge 1$ and

$$X_t - \int_0^t \Omega f(X_s)\,ds \tag{2.17}$$

is a martingale. Let τ be the hitting time of zero. Then

$$E^k(X_t, t<\tau) = E^k(X_{\tau\wedge t}) = k + E^k\int_0^{\tau\wedge t}\Omega f(X_s)\,ds \le k \tag{2.18}$$

and

$$E^k(X_t, t>\tau) \le \sup_s E^0(X_s) \le \sum_{j\ge 0} j\pi(j) < \infty. \tag{2.19}$$

Part (a) follows from (2.18) and (2.19).

(b) The first inequality in (2.16) comes from the fact that X_t is a monotone process [see Definition 2.3 of Chapter II of Liggett (1985), for example]. For the second inequality, let τ_k be the hitting time of k and

$$F_k(t) = E^k \int_0^{\tau \wedge t} r(X_s)\, ds$$

for $t \geq 0$, where $r(k) = q(k, k-1) - q(k, k+1) \geq 0$. Then by the Schwarz inequality,

$$\begin{aligned}
[F_k(t) - F_k(s)]^2 &\leq E^k \left\{ \int_{\tau \wedge s}^{\tau \wedge t} r(X_u)\, du \right\}^2 \leq (t-s) E^k \int_0^{\tau} r^2(X_u)\, du \\
&= (t-s) \sum_{j=1}^{k} E^k \int_{\tau_j}^{\tau_{j-1}} r^2(X_u)\, du \\
&\leq (t-s) \sum_{j=1}^{k} \frac{C}{j} E^k \int_{\tau_j}^{\tau_{j-1}} r(X_u)\, du \\
&\leq C(t-s) \log(k+1).
\end{aligned} \tag{2.20}$$

Here the value of C is possibly different in the last two expressions. In the next to last inequality, we have used the assumption that $r(j)$ is at most a constant multiple of $1/j$, and in the last inequality, we have again used the martingale in (2.17). Now,

$$F_{k+1}(t) \geq E^{k+1} \left\{ \int_{\tau_k}^{\tau \wedge t} r(X_s)\, ds,\, t > \tau_k \right\} = E^{k+1}[F_k(t - \tau_k),\, t > \tau_k]$$

by the nonnegativity of r and the strong Markov property, so that by (2.20),

$$\begin{aligned}
F_k(t) - F_{k+1}(t) &\leq E^{k+1}[F_k(t) - F_k(t - \tau_k),\, t > \tau_k] \\
&\quad + F_k(t) P^{k+1}\{t \leq \tau_k\} \\
&\leq C[\log(k+1)]^{1/2} E^{k+1}[\tau_k]^{1/2}.
\end{aligned} \tag{2.21}$$

Let Z_t be the birth and death chain which from k goes to each of $k-1$ and $k+1$ at rate $q(k, k-1)$. Then X_t and Z_t can be coupled together so that $X_t \leq Z_t$. Z_t is a time change of a simple symmetric random walk and its jump rates are bounded below, so that

$$P^{k+1}[Z_s > k \text{ for all } s \leq t] \leq C/\sqrt{t}$$

for some constant C. Therefore

$$P^{k+1}[X_s > k \text{ for all } s \leq t] \leq C/\sqrt{t} \tag{2.22}$$

as well. An elementary computation gives

$$E^{k+1}[\tau_k] = \sigma(k+1)/q(k+1, k), \tag{2.23}$$

which by assumption is bounded by a constant multiple of k. By (2.22), we have

$$E^{k+1}\sqrt{\tau_k} = \int_0^\infty P^{k+1}\{\tau_k > u^2\}\,du = \frac{1}{2}\int_0^\infty P^{k+1}\{\tau_k > v\}\frac{dv}{\sqrt{v}}$$

$$\le \frac{1}{2}\int_0^{k^2} P^{k+1}\{\tau_k > v\}\frac{dv}{\sqrt{v}} + \frac{1}{2k}\int_{k^2}^\infty P^{k+1}\{\tau_k > v\}\,dv$$

$$\le 1 + C\int_1^{k^2}\frac{dv}{v} + \frac{1}{2k}E^{k+1}\tau_k.$$

This last expression is bounded by a constant multiple of $\log(k+2)$ by (2.23). Using this in (2.21) gives the bound

$$F_k(t) - F_{k+1}(t) \le C[\log(k+1)]^{3/2}$$

for some constant C. To complete the proof, use this estimate in

$$E^{k+1}(X_t, t < \tau) - E^k(X_t, t < \tau) = 1 + F_k(t) - F_{k+1}(t)$$

which comes from (2.18) and then use (2.19). □

REMARKS. (i) Combining Theorem 2.10 with Proposition 2.15, it follows for example that if $q(i, i-1) = 1$ for all $i \ge 1$ and $q(i, i+1) \sim 1 - c/i$ for large i, then (2.12) holds if π has a finite moment of order greater than $2q$ and fails if π has an infinite moment of order less than $2q$.

(ii) Loren Pitt has pointed out that, while the semigroup of a birth and death chain does not necessarily decrease the Lipschitz norm, it does decrease appropriately chosen weighted Lipschitz norms. For example, if the transition rates are uniformly bounded and $H(n)$ is a strictly positive sequence such that

$$H(n+1)q(n+1, n+2) - H(n)q(n+1, n)$$

is nonincreasing, then the semigroup decreases

$$\sup_{n\ge 0}\frac{|f(n+1) - f(n)|}{H(n)}.$$

Using this observation, it is easy to use Theorem 2.2 and Proposition 2.8 to show that (2.12) holds, with $V(f)$ taken to be the square of this weighted norm, provided that $\inf_i q(i, i+1) > 0$, $\sup_n \sigma(n)/n < \infty$ and

$$\sum_{n=0}^\infty H^2(n) n^{2q}\pi(n) < \infty.$$

For more on monotonicity of Lipschitz norms under diffusion semigroups, see Herbst and Pitt (1991).

3. The sufficient condition. This section is devoted to the proof of the first part of Theorem 1.5. The hypotheses of that part will be assumed throughout this section. The proof consists of two steps:

STEP 1. Verify the assumptions of part (i) of Theorem 2.2 with $V(f)$ replaced by

$$V^*(f) = \sup_{t \geq 0} \sum_x \int [S(t) f(\eta^x) - S(t) f(\eta)]^2 N_x^{2q}(\eta)\, dv,$$

where $N_x(\eta)$ is the distance from x to the nearest site to the right of x at which there is a one and η^x is the configuration obtained from η by replacing $\eta(x)$ by 0: $\eta^x(x) = 0$ and $\eta^x(y) = \eta(y)$ for all $y \neq x$. Note the similarity between this V^* and the one used in the proof of Theorem 2.10. The monotonicity of V^* under the semigroup is automatic from its definition. Thus we need only check the estimate in (2.3).

STEP 2. Show that $V^*(f)$ is bounded above by a constant multiple of $V(f)$. It is this step which differs most from the analysis used in the supercritical case.

PROOF OF STEP 1. Following the approach in Sections 4 and 5 of Liggett (1989a), define a probability measure π on $Z^+ = \{0, 1, 2, \dots\}$ by $\pi(0) = g(1)$ and

$$\pi(n) = \frac{g(n+1)}{g(n)} - \frac{g(n)}{g(n-1)} \tag{3.1}$$

for $n \geq 1$, where $g(n)$ is the renewal sequence associated with the density β. Let μ be the product measure on $(Z^+)^Z$ with marginal π on each coordinate. Then define a mapping $T\colon (Z^+)^Z \to \{0,1\}^Z$ by $T(X) = \eta$, where

$$\eta(n) = 1 \Leftrightarrow X(n+k) \leq k \quad \text{for all } k \geq 0.$$

By Theorem 4.6 of Liggett (1989a), the measure induced by μ under T is the renewal measure v. Therefore, if $f \in L_2(v)$ and satisfies $\int f\, dv = 0$ and we define a function F on $(Z^+)^Z$ by $F(X) = f(T(X))$, it follows that $F \in L_2(\mu)$, $\int F\, d\mu = 0$ and the corresponding L_2 norms agree.

Take such an f and write

$$\int f^2(\eta)\, dv = \int F^2(X)\, d\mu \leq \tfrac{1}{2} \sum_u \int \sum_k [F(X_{u,k}) - F(X)]^2 \pi(k)\, d\mu, \tag{3.2}$$

where $X_{u,k}$ is defined by $X_{u,k}(u) = k$ and $X_{u,k}(v) = X(v)$ for $v \neq u$. There are various ways of seeing the inequality in (3.2). One way is to note that the right-hand side is the Dirichlet form for the process on $(Z^+)^Z$ in which each coordinate, at independent exponential times with mean one, changes to a value chosen from the measure π. The inequality is then simply the statement that this process has spectral gap equal to one [i.e., inequality (2.1) with $C = 1$]. See Section 2 of Liggett (1989a), for example. Alternatively, one can obtain it by induction on the number of variables from the fact that if U_1, U_2, V_1 and V_2 are independent random vectors with the U's identically distributed and the V's identically distributed and H is a function of two variables, then

$$E[H(U_1,V_1) - H(U_2,V_2)]^2 \le E[H(U_1,V_1) - H(U_2,V_1)]^2 + E[H(U_2,V_1) - H(U_2,V_2)]^2.$$

To check this inequality, simply square out the following expression:

$$0 \le E[H(U_1,V_1) + H(U_2,V_2) - H(U_1,V_2) - H(U_2,V_1)]^2.$$

Define $\sigma(n)$ as in (2.9) in terms of the $\pi(n)$ in (3.1). Apply Proposition 2.8 to the coordinate $X(u)$ with all other coordinates fixed to conclude that the right side of (3.2) is bounded above by

$$4\sum_u \int [F(X_u) - F(X)]^2 \sigma^2(X(u))\, d\mu,$$

where X_u is defined by $X_u(u) = X(u) + 1$ and $X_u(v) = X(v)$ for $v \ne u$. Lemma 5.5 of Liggett (1989a) asserts that if $\eta = T(X)$, then $T(X_u) = \eta^{u - X(u)}$. So, we can break up the previous integral according to the value of $X(u)$ to obtain

$$\int f^2\, dv \le 4 \sum_{u \in Z} \sum_{k=0}^{\infty} \sigma^2(k) \int_{\{X(u)=k\}} \left[f\left(T(X)^{u-k}\right) - f(T(X))\right]^2 d\mu. \tag{3.3}$$

Letting $x = u - k$, (3.3) can be rewritten as

$$\int f^2\, dv \le \sum_x \int [f(\eta^x) - f(\eta)]^2 Q_x(\eta)\, dv, \tag{3.4}$$

where $Q_x(\eta)$ is the conditional expectation (where X has distribution μ)

$$Q_x(\eta) = 4E\left[\sum_{k=0}^{\infty} \sigma^2(k) 1_{\{X(x+k)=k\}} | T(X) = \eta \right]. \tag{3.5}$$

Next, we need to estimate $Q_x(\eta)$. By Lemma 5.6 of Liggett (1989a), if $0 \le k \le l$, $r \ge 1$, $\eta(u-l) = \eta(u+r) = 1$ and $\eta(j) = 0$ for $u - l < j < u + r$, then

$$P(X(u) = k | T(X) = \eta) \le \frac{\pi(k)}{\beta(r+k)\pi(0)} \sum_{i=0}^{k} \beta(r+i) g(k-i). \tag{3.6}$$

Fix η and x and let $y = \min\{z > x\colon \eta(z) = 1\}$. Note that $\eta(y) = 1$ and $T(X) = \eta$ imply that $X(y+k) \le k$ for all $k \ge 0$, so that $X(x+k) \le$

$x + k - y < k$ for $k \geq y - x$. So we can use (3.6) in (3.5) to obtain

$$
\begin{aligned}
Q_x(\eta) &= 4 \sum_{k=0}^{y-x-1} \sigma^2(k) P(X(x+k) = k \mid T(X) = \eta) \\
&\leq 4 \sum_{k=0}^{y-x-1} \sigma^2(k) \frac{\pi(k)}{\beta(y-x)\pi(0)} \sum_{i=0}^{k} \beta(y-x-k+i) g(k-i).
\end{aligned}
\tag{3.7}
$$

In order to continue, it is necessary to estimate $\pi(k)\sigma^2(k)$. Since $\beta(n+m)$ is totally positive of order three, so is $g(n+m)$, by Theorem 1 of Liggett (1989b). So

$$
\begin{vmatrix} g(n-1) & g(n) & g(\infty) \\ g(n) & g(n+1) & g(\infty) \\ g(\infty) & g(\infty) & g(\infty) \end{vmatrix} \geq 0.
$$

Expanding this determinant and adding and subtracting appropriate terms yields

$$
\begin{aligned}
g(\infty)[g(n+1) - g(n)]^2 \leq{}& [g(n+1) - g(\infty)] \\
&\times [g(n+1)g(n-1) - g^2(n)].
\end{aligned}
$$

Combining this with the definitions of $\pi(n)$ in (3.1) and $\sigma(n)$ in (2.9), we conclude that

$$
\begin{aligned}
\pi(n)\sigma^2(n) &= \frac{g(n)[g(n-1) - g(n)]^2}{g(n-1)[g(n+1)g(n-1) - g^2(n)]} \\
&\leq \frac{g(n)[g(n+1) - g(\infty)][g(n-1) - g(n)]^2}{g(\infty)g(n-1)[g(n+1) - g(n)]^2}.
\end{aligned}
\tag{3.8}
$$

By assumption (1.7) and Theorem 2 of Liggett (1989b),

$$
\lim_{n \to \infty} \frac{g(n) - g(n+1)}{B(n+2)}
$$

exists and is positive. Combining this with (3.7) and (3.8) gives the existence of a constant C so that

$$
Q_x(\eta) \leq C \sum_{k=0}^{y-x-1} \frac{B(y-x-k)}{\beta(y-x)} \sum_{j=k}^{\infty} B(j).
$$

By assumption (1.6), there is a constant C so that

$$(3.9)\quad \begin{aligned} Q_x(\eta) &\le C \sum_{k=0}^{y-x-1} \frac{B(y-x-k)(k+1)B(k+1)}{\beta(y-x)} \\ &= C\frac{y-x+1}{2} \sum_{k=0}^{y-x-1} \frac{B(y-x-k)B(k+1)}{\beta(y-x)}. \end{aligned}$$

Note that the logconvexity of $\beta(n)$ implies the logconvexity of $B(n)$, since

$$(3.10)\quad \begin{aligned} B(n&+1)B(n-1) - B^2(n) \\ &= \beta(n-1)B(n+1) - \beta(n)B(n) \\ &= \sum_{k\ge n} [\beta(n-1)\beta(k+1) - \beta(n)\beta(k)] \ge 0. \end{aligned}$$

Therefore,

$$\frac{B(y-x-k)}{B(y-x+1)} \le \frac{B(k+1)}{B(2k+2)}$$

for $k+1 \le y-x-k$. Using this in (3.9) together with assumption (1.7) gives the existence of a constant C so that

$$(3.11)\quad Q_x(\eta) \le C(y-x+1)\frac{B(y-x)}{\beta(y-x)}.$$

Using the logconvexity of $\beta(n)$ again, note that

$$\begin{aligned} B^2(n) &= \sum_{k,l\ge 0} \beta(n+k)\beta(n+l) \le \beta(n) \sum_{k,l\ge 0} \beta(n+k+l) \\ &= \beta(n) \sum_{k\ge 0} (k+1)\beta(n+k) \\ &= \beta(n) \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \beta(n+k) \\ &= \beta(n) \sum_{l=0}^{\infty} B(n+l). \end{aligned}$$

Therefore, assumption (1.6) implies that $B(n)$ is bounded above by a constant multiple of $n\beta(n)$. Putting this in (3.11) gives our final bound on $Q_x(\eta)$:

$$Q_x(\eta) \le CN_x^2(\eta)$$

for some constant C. Using this bound in (3.4) and then applying Hölder's inequality gives

$$\begin{aligned} \int f^2\,d\nu &\le C\sum_x \int [f(\eta^x) - f(\eta)]^2 N_x^2(\eta)\,d\nu \\ &\le C\left\{\sum_x \int [f(\eta^x) - f(\eta)]^2\,d\nu\right\}^{1/p} \\ &\quad \times \left\{\sum_x \int [f(\eta^x) - f(\eta)]^2 N_x^{2q}(\eta)\,d\nu\right\}^{1/q}. \end{aligned}$$

This is the required estimate (2.3), since the expression in the first set of braces is the Dirichlet form for the nearest particle system and the expression in the second set is bounded above by $V^*(f)$. □

PROOF OF STEP 2. For $x \in Z$, define

$$\Delta_x(f) = \sup_\eta |f(\eta_x) - f(\eta)|.$$

Then for any continuous f,

$$|f(\eta) - f(\zeta)| \le \sum_x |\eta(x) - \zeta(x)| \Delta_x(f). \tag{3.12}$$

Fix x and η and let (η_t, ζ_t) be the basic coupling for two copies of the nearest particle system with initial configurations η and η^x, respectively. Then one can apply (3.12) and the Schwarz inequality to obtain

$$\begin{aligned}[S(t)f(\eta^x) - S(t)f(\eta)]^2 &= [E\{f(\eta_t) - f(\zeta_t)\}]^2 \\ &\le \left\{\sum_u \Delta_u(f) E|\eta_t(u) - \zeta_t(u)|\right\}^2 \\ &\le \sum_u \{E|\eta_t(u) - \zeta_t(u)|\}^2 \Delta_u(f)\{V(f)\}^{1/2}.\end{aligned}$$

Using this estimate in the definition of $V^*(f)$ and using the translation invariance of the process and of v, we see that we need only show that

$$\sup_{t\ge 0} \sum_u \int \left[P^\eta\{\eta_t(u) = 1\} - P^{\eta^0}\{\eta_t(u) = 1\}\right]^2 N_0^{2q}(\eta)\, dv < \infty. \tag{3.13}$$

To do so, define the probability measures μ_i for $i = 1, 2, 3, 4$ by

$$\int f(\eta)\, d\mu_4 = \frac{\int \eta(0) f(\eta)\, dv}{\int \eta(0)\, dv},$$

$$\int f(\eta)\, d\mu_3 = \frac{\int f(\eta)\eta(0) N_0^{2q}(\eta)\, dv}{\int \eta(0) N_0^{2q}(\eta)\, dv},$$

$$\int f(\eta)\, d\mu_2 = \int f(\eta^0)\, d\mu_3$$

and

$$\int f(\eta)\, d\mu_1 = \frac{\int f(\eta)[1 - \eta(0)] N_0^{2q}(\eta)\, dv}{\int [1 - \eta(0)] N_0^{2q}(\eta)\, dv}.$$

Note that the previous denominators are finite because of the moment assumption (1.8). We claim that these measures are stochastically ordered,

$$\mu_1 \le \mu_2 \le \mu_3 \le \mu_4 \tag{3.14}$$

and that

$$\mu_1 S(t)\uparrow \quad \text{and} \quad \mu_4 S(t)\downarrow \tag{3.15}$$

as t increases. The middle inequality in (3.14) is clear, since μ_2 is obtained from μ_3 by replacing $\eta(0)$ by 0. To check the first and last inequalities in (3.14), start by observing that the logconvexity of $\beta(n)$ is equivalent to v satisfying the hypothesis of the FKG Theorem [Corollary 2.12 in Chapter II of Liggett (1985)]. This is easy to check directly and is done in a continuous time context in Burton and Waymire (1986). Using this fact and

$$N_0(\eta \vee \zeta) = N_0(\eta) \wedge N_0(\zeta)$$

and

$$N_0(\eta \wedge \zeta) \geq N_0(\eta) \vee N_0(\zeta),$$

it follows that the hypotheses of Holley's theorem [Theorem 2.9 in Chapter II of Liggett (1985)] are satisfied for each of the pairs μ_1, μ_2 and μ_3, μ_4. This completes the proof of (3.14). To check (3.15), let $S_1(t)$ be the semigroup for the process on $\{\eta\colon \eta(0) = 0\}$ in which the transition rates are modified by multiplying the original birth rates at x by zero if $x = 0$ and by

$$\frac{N_0^{2q}(\eta_x)}{N_0^{2q}(\eta)} \leq 1$$

if $x \neq 0$. Also, let $S_4(t)$ be the semigroup for the process on $\{\eta\colon \eta(0) = 1\}$ in which the transition rates are modified by setting the death rate at zero equal to 0. Then μ_1 is reversible for $S_1(t)$ and μ_4 is reversible for $S_4(t)$ and it follows by coupling that

$$\mu_1 = \mu_1 S_1(t) \leq \mu_1 S(t) \leq \mu_4 S(t) \leq \mu_4 S_4(t) = \mu_4.$$

Statements (3.15) follow from this and the semigroup property of $S(t)$.

Next, we perform the following computation:

$$\begin{aligned}
&\frac{\int \eta(0)\left[P^{\eta}\{\eta_t(u) = 1\} - P^{\eta^0}\{\eta_t(u) = 1\}\right] N_0^{2q}(\eta)\, dv}{\int \eta(0) N_0^{2q}(\eta)\, dv} \\
&\quad = \int P^{\eta}\{\eta_t(u) = 1\}\, d\mu_3 - \int P^{\eta}\{\eta_t(u) = 1\}\, d\mu_2 \\
&\quad \leq \int P^{\eta}\{\eta_t(u) = 1\}\, d\mu_4 - \int P^{\eta}\{\eta_t(u) = 1\}\, d\mu_1 \qquad (3.16)\\
&\quad \leq \mu_4\{\eta\colon \eta(u) = 1\} - \mu_1\{\eta\colon \eta(u) = 1\} \\
&\quad = \frac{\int [g(|u|) - \eta(u)][1 - \eta(0)] N_0^{2q}(\eta)\, dv}{\int [1 - \eta(0)] N_0^{2q}(\eta)\, dv}.
\end{aligned}$$

Here the two equalities follow from the definition of the four μ_i's and the two inequalities follow from (3.14) and (3.15), respectively. So, in order to verify (3.13), it suffices to show that the sum on u of the numerator on the right of

(3.16) is finite. This follows from assumption (1.8), as the following computation shows. In the next equality, we partition the configuration space according to the locations of the nearest ones to the left and right of 0, respectively.

$$\sum_u \int [g(|u|) - \eta(u)][1 - \eta(0)] N_0^{2q}(\eta)\, dv/v\{\eta: \eta(0) = 1\}$$
$$= \sum_{k,l\geq 1} l^{2q}\beta(k+l)\left\{ \sum_{u=-k+1}^{l-1} g(|u|) + \sum_{u=-\infty}^{-k} [g(|u|) - g(-k-u)] + \sum_{u=l}^{\infty} [g(|u|) - g(u-l)]\right\}$$
$$\leq \sum_{k,l\geq 1} l^{2q}(k+l)\beta(k+l) < \infty,$$

since $g(n)$ is decreasing. □

4. The necessary condition. In this section, we will prove the second part of Theorem 1.5. The hypotheses of that part of the theorem will be assumed throughout this section. By part (ii) of Theorem 2.2, there is a constant C so that

$$\left\| f - \int f dv \right\|_2^2 \leq C\left[\sum_x \int [f(\eta^x) - f(\eta)]^2 dv\right]^{1/p} [V(f)]^{1/q} \tag{4.1}$$

for all f which depend on finitely many coordinates. If $a(n)$ is an increasing function on the positive integers which is equal to the constant $a(\infty)$ for all sufficiently large n, define the function $A(\eta)$ on the space of configurations by $A(\eta) = 0$ if $\eta(0) = 1$, $A(\eta) = a(l+r)$ if $\eta(-l) = \eta(r) = 1$ and $\eta(x) = 0$ for all $-l < x < r$. By Proposition 6.3 of Liggett (1989a),

$$\int A\, dv = M^{-1} \sum_{n=2}^{\infty} a(n)\beta(n)(n-1), \tag{4.2}$$

$$\int A^2\, dv = M^{-1} \sum_{n=2}^{\infty} a^2(n)\beta(n)(n-1), \tag{4.3}$$

$$\begin{aligned}\sum_u \int [A(\eta^u) - A(\eta)]^2 dv &= M^{-1} \sum_{n=2}^{\infty} a^2(n) \sum_{l+r=n} \beta(l)\beta(r)\\ &\quad + 2M^{-1} \sum_{m,n\geq 1} [a(m+n) - a(n)]^2 \beta(m)\beta(n)(n-1).\end{aligned} \tag{4.4}$$

It is not hard to check that

$$\Delta_x(A) = a(\infty) - a(|x| + 1)$$

for $x \neq 0$, so that

$$V(A) = \left\{a(\infty) + 2\sum_{n=2}^{\infty}[a(\infty) - a(n)]\right\}^2. \tag{4.5}$$

Now take a $\gamma > 0$ such that

$$\sum_{n=1}^{\infty} n^{2\gamma+1}\beta(n) = \infty \tag{4.6}$$

and let

$$a_N(n) = (n \wedge N)^{\gamma}.$$

(If there is no such γ, then there is nothing to prove.) Let A_N be the corresponding function on the configuration space. In the following lemmas, we estimate the terms appearing on the right of (4.2), (4.3), (4.4) and (4.5), when A is replaced by A_N.

LEMMA 4.7. *As* $N \to \infty$,

$$V(A_N) \sim \left\{\frac{\gamma}{\gamma+1}N^{\gamma+1}\right\}^2.$$

PROOF. This follows immediately from a Riemann sum approximation to an integral

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left\{\frac{n}{N}\right\}^{\gamma} = \int_0^1 x^{\gamma}\,dx = \frac{1}{\gamma+1}. \qquad \square$$

LEMMA 4.8. *As* $N \to \infty$,

$$\left\{\int A_N\,dv\right\}^2 = o\left(\int A_N^2\,dv\right).$$

PROOF. For any positive integer L, apply the Schwarz inequality to obtain

$$\sum_{n=2}^{\infty} a(n)\beta(n)(n-1) \le \sum_{n=2}^{L-1} a(n)\beta(n)(n-1) + \left(\sum_{n=2}^{\infty} a^2(n)\beta(n)(n-1)\sum_{n=L}^{\infty}(n-1)\beta(n)\right)^{1/2}.$$

Apply this to a_N, square both sides and use (4.6) to conclude that

$$\limsup_{N\to\infty}\frac{\{\int A_N\,dv\}^2}{\int A_N^2\,dv} \le \sum_{n=L}^{\infty}(n-1)\beta(n).$$

Now let $L \to \infty$. $\square$

LEMMA 4.9. *There is a constant C (independent of γ and N) so that*

$$\sum_u \int [A_N(\eta^u) - A_N(\eta)]^2 \, dv \le C \sum_{n=2}^{\infty} a_N^2(n)\beta(n).$$

PROOF. Since $\beta(n)$ is logconvex,

$$\beta(r)\beta(2l) \le \beta(l)\beta(l+r)$$

for $l \le r$. Therefore,

$$\sum_{l+r=n} \beta(l)\beta(r) \le 2\beta(n) \sum_{l=1}^{n-1} \frac{\beta^2(l)}{\beta(2l)}. \tag{4.10}$$

Using this and assumption (1.11) gives the inequality needed to bound the first term on the right of (4.4). For the second term on the right of (4.4), consider separately the summands corresponding to (i) $m + n \ge N$, (ii) $m \le n$ and $m + n < N$ and (iii) $m > n$ and $m + n < N$:

(i) There is a constant C so that

$$\begin{aligned} &\sum_{m+n \ge N > n} [N^{\gamma} - n^{\gamma}]^2 \beta(m)\beta(n)(n-1) \\ &\quad \le N^{2\gamma} \sum_{n=1}^{N-1} \beta(n)(n-1)B(N-n) \\ &\quad \le CN^{2\gamma} \sum_{n=1}^{N-1} B(n)B(N-n) \le 2CN^{2\gamma}B(N) \sum_{n=1}^{N-1} \frac{B^2(n)}{B(2n)}. \end{aligned} \tag{4.11}$$

The first inequality in (4.11) comes from the definition of $B(N-n)$, the second from assumption (1.10) and the third from the argument that led to (4.10), applied to $B(n)$ [which is also logconvex by (3.10)] instead of $\beta(n)$. By (1.10) again,

$$\frac{B(n) - B(n+1)}{B(n)} = \frac{\beta(n)}{B(n)} \le \frac{C}{n},$$

so that

$$\frac{B(n+1)}{B(n)} \ge 1 - \frac{C}{n}. \tag{4.12}$$

Iterating this gives

$$\frac{B(2n)}{B(n)} \ge \left(1 - \frac{C}{n}\right)^n \to e^{-C}.$$

Using this in (4.11) and using the fact that $B(n)$ is summable (since β has a finite mean) completes the proof of the required bound in case (i).

(ii) There is a constant C so that

$$\sum_{m\le n,\, m+n<N} \beta(m)\beta(n)(n-1)\left[(m+n)^{\gamma}-n^{\gamma}\right]^2 \le C \sum_{m\le n,\, m+n<N} \beta(m)\beta(n)n^{2\gamma}m \le CM\sum_{n=1}^{N} n^{2\gamma}\beta(n),$$

which gives the required bound in this case.

(iii) There is a constant C so that

$$\sum_{m> n,\, m+n<N} \beta(m)\beta(n)(n-1)\left[(m+n)^{\gamma}-n^{\gamma}\right]^2 \le C \sum_{m> n,\, m+n<N} \beta(m)\beta(n)m^{2\gamma}n \le CM\sum_{m=1}^{N} m^{2\gamma}\beta(m),$$

which gives the required bound in the final case.

The remainder of the proof of the second part of Theorem 1.5 is similar to the proof of the second part of Theorem 2.10. By (4.1) applied to the functions A_N and the estimates in Lemmas 4.7, 4.8 and 4.9, there is a constant C independent of N so that

$$\sum_{n=2}^{\infty} a_N^2(n)\beta(n)(n-1) \le CN^{(2\gamma+2)/q}\left\{\sum_{n=2}^{\infty} a_N^2(n)\beta(n)\right\}^{1/p}.$$

Apply Hölder's inequality to the expression in brackets on the right side and divide to conclude that

$$\left\{\sum_{n=2}^{\infty}(n\wedge N)^{2\gamma+1}\beta(n)\right\}^{1-(2\gamma/p(2\gamma+1))} \le CN^{(2\gamma+2)/q}.$$

This implies that

$$\sum_{n=N/2}^{N} n^{\alpha}\beta(n) \le CN^{(2\gamma(\alpha+2-q)+2-q+\alpha q)/(2\gamma+q)},$$

where the value of C has been changed. The limit as $\gamma\to\infty$ of the exponent on the right is $\alpha+2-q$, so that if $\alpha<q-2$, γ can be chosen so that the exponent is negative. To finish the proof, replace N by 2^m and sum on m. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CALIFORNIA 90024-1555