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LARGE DEVIATIONS FOR A REACTION–DIFFUSION EQUATION WITH NON-GAUSSIAN PERTURBATIONS

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In this paper we establish a large deviations principle for the non-Gaussian stochastic reaction–diffusion equation (SRDE) $\partial_t v^\varepsilon = \mathscr{L} v^\varepsilon + f(x, v^\varepsilon) + \varepsilon\sigma(x, v^\varepsilon)\ddot{W}_{tx}$ as a random perturbation of the deterministic RDE $\partial_t v^0 = \mathscr{L} v^0 + f(x, v^0)$. Here the space variable takes values on the unit circle S^1 and $\mathscr{L}$ is a strongly-elliptic second-order operator with constant coefficients. The functions f and σ are sufficiently regular so that there is a unique solution to the above SRDE for any continuous initial condition. We also assume that there are positive constants m and M such that $m \leq \sigma(x, y) \leq M$ for all x in S^1 and all y in $\mathbb{R}$. The perturbation $\ddot{W}_{tx}$ is the formal derivative of a Brownian sheet. It is known that if the initial condition is continuous, then the solution will also be continuous, and moreover, if the initial condition is assumed to be Hölder continuous of exponent κ for some $0 < \kappa < \frac{1}{2}$, then the solution will be Hölder continuous of exponent $\kappa/2$ as a function of (t, x). In this paper we establish the large deviations principle for v^ε in the Hölder norm of exponent $\kappa/2$ when the initial condition is Hölder continuous of exponent κ for any $0 < \kappa < \frac{1}{2}$, and when the initial condition is assumed only to be continuous, we establish the large deviations principle for v^ε in the supremum norm. Moreover, we prove that these large deviations principles are uniform with respect to the initial condition.

1. Introduction. In this paper we study the stochastic reaction–diffusion equation (SRDE)

$$\partial_t v^\varepsilon = \mathscr{L} v^\varepsilon + f(x, v^\varepsilon) + \varepsilon\sigma(x, v^\varepsilon)\ddot{W}_{tx}, \tag{1}$$

where $\mathscr{L}$ is a time- and space-invariant second-order elliptic operator, f and σ are functions of sufficient regularity and $\ddot{W}_{tx}$ is the formal derivative of a Brownian sheet. As the parameter ε tends to zero, the solutions v^ε of (1) will tend to solutions of

$$\partial_t v^0 = \mathscr{L} v^0 + f(x, v^0).$$

We seek to establish a large deviations result for v^ε as a random perturbation of v^0, that is, the first term in a logarithmic expansion, as ε tends to zero, of the probabilities $P\{v^\varepsilon \in A\}$, where A is a set in the space in which the solutions to (1) naturally occur. This work thus represents a significant extension of the work of Freidlin (1988), Faris and Jona-Lasinio (1982), Imaykin and Komeč (1988), Zabczyk (1988a) and Zabczyk (1988b), who have

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developed the theory of large deviations for (1) when σ is identically 1. In the case where σ is constant, or even nonconstant but deterministic, the large deviations theory for (1) can be deduced from the contraction principle and standard estimates for the large deviations of Gaussian fields. In our case, where σ in general depends nontrivially upon v^ε, no such approach is possible and truly nonlinear arguments must be used. We also note that in the previously-mentioned works, the large deviations results are in the supremum-norm topology, whereas ours are in the Hölder topologies of exponents κ, for each $0 < \kappa < \frac{1}{4}$.

To place the problem in the proper setting, we consider the following. The space variable we assume to be in $S^1 := \{e^{i\theta}: \theta \in \mathbb{R}\}$; the solutions to (1) will in general not be well-defined functions if the space variables is in $\mathbb{R}^n$ for $n \geq 2$ [see Walsh (1984)], and we enforce periodicity in order to avoid specifying boundary conditions. The differential operator $\mathscr{L}$ we then take to be $\mathscr{L}h := Dh_{xx} - \alpha h$ where D and α are positive constants and where differentiation is with respect to the natural metric on S^1 (see Section 2). As an initial condition we shall take some ζ in $C(S^1)$. The regularity required of f is that there exist constants F and $\bar{f}$ such that for all x in S^1 and all y and z in $\mathbb{R}$,

$$|f(x,y)| \leq F(1 + |y|) \quad \text{and} \quad |f(x,y) - f(x,z)| \leq \bar{f}|y - z|.$$

In order to ensure the existence, uniqueness, and nondegeneracy of the solutions to (1), we also require that $\sigma(x,y)$ be continuous as a function of both arguments and that furthermore there exist positive constants m, M and $\bar{\sigma}$ such that

$$m \leq \sigma(x,y) \leq M \quad \text{and} \quad |\sigma(x,y) - \sigma(x,z)| \leq \bar{\sigma}|y - z|$$

for all x in S^1 and all y and z in $\mathbb{R}$. The random perturbation $\dot{W}_{tx}$ is to be interpreted as the formal derivative of a Brownian sheet W on $\mathbb{R}_+ \times S^1$, W being defined on some underlying (complete) probability triple $(\Omega, \mathscr{F}, P)$. By a Brownian sheet on $\mathbb{R}_+ \times S^1$, we mean a random set function W on the Borel sets of $\mathbb{R}_+ \times S^1$ such that: (i) for A a Borel subset of $\mathbb{R}_+ \times S^1$, $W(A)$ is a zero-mean Gaussian random variable with covariance $v(A)$, v being Lebesgue measure on $(\mathbb{R}_+ \times S^1, \mathscr{B}(\mathbb{R}_+ \times S^1))$; and (ii) for A and B disjoint Borel subsets of $\mathbb{R}_+ \times S^1$, $W(A \cup B) = W(A) + W(B)$ [see Walsh (1984)]. We can make a natural identification of S^1 with the interval $[0, 2\pi]$, and upon doing so, the random field $\overline{W}(t,x) := W([0,t] \times \{e^{i\theta}: 0 \leq \theta \leq x\})$ is a regular Brownian sheet on $\mathbb{R}_+ \times [0, 2\pi]$. Given the Brownian sheet W, stochastic integration against $\dot{W}_{tx}$ follows in the expected way.

With the above in mind, we may rely on Walsh (1984) to see that on the interval $I := [0, T]$, where T is any fixed positive time, there is a unique solution to (1), that is, to the SRDE

$$\begin{aligned} &\partial_t v_\zeta^\varepsilon = \mathscr{L} v_\zeta^\varepsilon + f(x, v_\zeta^\varepsilon) + \varepsilon\sigma(x, v_\zeta^\varepsilon)\dot{W}_{tx}, \qquad (t,x) \in I \times S^1, \\ &v_\zeta^\varepsilon[0] = \zeta, \end{aligned} \tag{2}$$

with ζ in $L^2(S^1)$, when posed in the weak sense—that is, there is a random

field, which is in $L^2(I \times S^1 \times \Omega)$ for which P a.s. we have

$$\int_{S^1} v_\zeta^\varepsilon(t,x)\varphi(x)\,dx = \int_{S^1} \zeta(x)\varphi(x)\,dx + \int_0^t \int_{S^1} v_\zeta^\varepsilon(s,x)\mathscr{L}\varphi(x)\,ds\,dx$$

$$+ \int_0^t \int_{S^1} f\big(x, v_\zeta^\varepsilon(s,x)\big)\varphi(x)\,ds\,dx \tag{3}$$

$$+ \varepsilon \int_0^t \int_{S^1} \varphi(x)\sigma\big(x, v_\zeta^\varepsilon(s,x)\big)W(ds,dx)$$

for every φ in $C^2(S^1)$ and for all t in I.

Our goal is to establish a large deviations principle for the solution to (2). It is well known [see Walsh (1984)] that if the initial condition ζ is continuous, then v_ζ^ε is in $C(I \times S^1)$ and if ζ is Hölder continuous of exponent $0 < 2\kappa < \frac{1}{2}$, then v_ζ^ε is Hölder continuous of exponent κ as a function of (t,x) (to see this last fact we may, for example, combine Proposition 1 in Section 3 and Proposition A.2 in Appendix A). In view of this, we shall prove the large deviations principle for v_ζ^ε with respect to the Hölder norm of exponent $\kappa < \frac{1}{4}$ when ζ is Hölder continuous of exponent 2κ and with respect to the supremum norm on $C(I \times S^1)$ when ζ is assumed only to be continuous.

To fix our notation, we shall let r be the standard metric on S^1—for any x and y in S^1, $r(x,y)$ is the length of the shortest arc of S^1 connecting x to y. Let r' be the metric on $I \times S^1$ given by $r'((t,x),(s,y)) := \sqrt{(t-s)^2 + r^2(x,y)}$ for each (t,x) and (s,y) in $I \times S^1$. For each $0 < \kappa < 1$, define $[\cdot]_\kappa$ to be the Hölder seminorm of exponent κ for real-valued functions on $I \times S^1$, that is,

$$[\varphi]_\kappa := \sup\left\{ \frac{|\varphi(t,x) - \varphi(s,y)|}{\big(r'((t,x),(s,y))\big)^\kappa} : (t,x),(s,y) \in I \times S^1, (t,x) \neq (s,y) \right\}$$

for each mapping $\varphi\colon I \times S^1 \to \mathbb{R}$. Also let $\|\cdot\|_C$ be the supremum norm on real-valued functions of $I \times S^1$. We then define the norms $\|\cdot\|_0 := \|\cdot\|_C$ and for each $0 < \kappa < 1$, $\|\cdot\|_\kappa := \|\cdot\|_C + [\cdot]_\kappa$. For any $0 \le \kappa < 1$, we let C^κ be the vector space of those mappings $\varphi\colon I \times S^1 \to \mathbb{R}$ which are continuous and for which $\|\varphi\|_\kappa$ is finite. For any $0 \le \kappa < 1$, we also denote by ρ_κ the metric on C^κ defined by the norm $\|\cdot\|_\kappa$. For each $0 < \kappa < 1$, we may similarly define the seminorm

$$[\zeta]_{\kappa,S^1} := \sup\left\{ \frac{|\zeta(x) - \zeta(y)|}{\big(r(x,y)\big)^\kappa} : x,y \in S^1, x \neq y \right\}$$

for each mapping $\zeta\colon S^1 \to \mathbb{R}$ and let $\|\cdot\|_{C(S^1)}$ be the supremum norm on real-valued functions on S^1. Then we define the norms $\|\cdot\|_{0,S^1} := \|\cdot\|_{C(S^1)}$ and for $0 < \kappa < 1$, $\|\cdot\|_{\kappa,S^1} := \|\cdot\|_{C(S^1)} + [\cdot]_{\kappa,S^1}$ with the associated vector spaces $C^0(S^1)$ and $C^\kappa(S^1)$ of continuous functions. As a convenience, for any $0 \le \kappa < 1$ and any ζ in $C^\kappa(S^1)$, define C_ζ^κ as the collection of those elements φ of C^κ such that $\varphi[0] = \zeta$, where we have used the obvious notation that if φ maps $I \times S^1$ into $\mathbb{R}$, then for each t in I, $\varphi[t]$ is the mapping from S^1 to $\mathbb{R}$

defined as $(\varphi[t])(x) := \varphi(t, x)$ for all x in S^1. The set C_ζ^κ can be thought of as consisting of those elements of C^κ which begin at ζ. As a further convenience, we set $C_0^\kappa := C_{\mathbf{0}}^\kappa$—those elements φ of C^κ such that $\varphi[0] = \mathbf{0}$. As a final notational convention, let us denote by $\|\cdot\|_{L^2}$ the $L^2(I \times S^1)$ norm on the collection of square-integrable functions on $I \times S^1$, with associated metric ρ_{L^2}.

If we fix a $0 \leq \kappa < \frac{1}{4}$ and a ζ in $C^{2\kappa}(S^1)$, we expect, in analogy with the form of the action functional in Freidlin (1988) and the form of the action functional for ordinary stochastic differential equations [see Freidlin and Wentzell (1984), Theorem 5.3.1 and Varadhan (1984), Section 6], the action functional for v_ζ^ε in C_ζ^κ to be defined as

$$S_\zeta(\varphi) := \begin{cases} \frac{1}{2}\int_{I\times S^1} \left| \frac{\partial_t \varphi - \mathscr{L}\varphi - f(\cdot, \varphi)}{\sigma(\cdot, \varphi)} \right|^2 (t, x)\, dt\, dx, \\ \qquad \text{if } \varphi \in W_2^{1,2}, \varphi[0] = \zeta, \\ \infty, \qquad \text{otherwise,} \end{cases} \tag{4}$$

for all mappings φ from $I \times S^1$ to $\mathbb{R}$, where $W_2^{1,2}$ is the closure of $C^\infty(I \times S^1)$ in the norm

$$\|\varphi\|_{W_2^{1,2}} = \left\{ \int_{I\times S^1} |\varphi|^2 + |\dot{\varphi}_t|^2 + |\dot{\varphi}_x|^2 + |\ddot{\varphi}_{xx}|^2 \, dt\, dx \right\}^{1/2},$$

that is, the Sobolev space of functions on $I \times S^1$ with one square-integrable time derivative and two square-integrable space derivatives. For each $0 \leq \kappa < \frac{1}{4}$, the large deviations principle for v_ζ^ε will consist of the following three assertions:

(A.1) The level sets of S_ζ, that is,

$$\Phi_\zeta(s) := \{\varphi \in C_\zeta^\kappa : S_\zeta(\varphi) \leq s\},$$

are compact sets in C_ζ^κ for each $s \geq 0$.

(A.2) For any positive numbers δ and γ and any φ in C_ζ^κ, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho_\kappa(v_\zeta^\varepsilon, \varphi) < \delta\} > \exp\left(- \frac{S_\zeta(\varphi) + \gamma}{\varepsilon^2} \right)$$

for all $0 < \varepsilon < \varepsilon_0$.

(A.3) For any positive numbers δ, γ and s, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho_\kappa(v_\zeta^\varepsilon, \Phi_\zeta(s)) \geq \delta\} < \exp\left(- \frac{s - \gamma}{\varepsilon^2} \right)$$

for all $0 < \varepsilon < \varepsilon_0$.

In the case where σ is identically 1, v_ζ^ε will be Gaussian and the above large deviations principle in the supremum-norm topology was shown in Freidlin (1988), Faris and Jona-Lasinio (1982), Imaykin and Komeč (1988) and Zabczyk (1988a)—see also Zabczyk (1988b). As we shall see, the proof of the large deviations principle when σ depends nontrivially upon v_ζ^ε involves significant

technical complications in comparison with the $\sigma \equiv$ const. case [see also Freidlin and Wentzell (1984), Section 3.4 and Stroock (1984), Chapter 3].

The body of this paper is divided into four parts. In Section 2 we shall review some necessary facts about stochastic PDE's of the form (2). In Section 3 we shall simplify our task by an application of the contraction principle with an appropriate mapping. This will lead us to study large deviations for a simpler process ψ_ζ^ε. In Section 4 we shall prove a large deviations result for the process ψ_ζ^ε; this is the main work of the paper. Then in Section 5 we shall review the arguments of Sections 3 and 4 to show that the large deviations principle (A.1)–(A.3) in fact holds uniformly over all initial conditions ζ.

2. Stochastic RDE's—some required notation. We now shall quickly review how to represent the solution of (2) using the Green's function of the equation $\partial_t u = \mathscr{L} u$; this notation will be needed in the next section.

Let $\{\phi_k;\ k = 1, 2, \dots\}$ be an orthonormal basis of $L^2(S^1)$ consisting of eigenfunctions of $\mathscr{L}$ and let $\{\lambda_k;\ k = 1, 2, \dots\}$ be the corresponding eigenvectors. For concreteness, we shall take

$$\phi_{2k-1}(x) := \frac{1}{\pi^{1/2}} \Re(x^k) \quad \text{and} \quad \phi_{2k}(x) := \frac{1}{\pi^{1/2}} \Im(x^k),$$

$$x \in S^1,\ k = 1, 2, \dots,$$

with $\phi_0 \equiv 1/(2\pi)^{1/2}$, where $\Re(x)$ and $\Im(x)$ are the real and imaginary parts of any element of S^1. The corresponding eigenvalues are then

$$\lambda_{2k-1} = \lambda_{2k} = Dk^2 + \alpha, \qquad k = 1, 2, \dots,$$

with $\lambda_0 = \alpha$. The Green's function for the equation $\partial_t u = \mathscr{L} u$ is

$$G_t(x, y) := \chi\{t \geq 0\} \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y), \qquad (t, x, y) \in \mathbb{R}_+ \times S^1 \times S^1, \tag{5}$$

where for any set A, χA is the indicator function of the set A. For ζ in $L^2(S^1)$, set $\mathscr{T}_\zeta$ to be

$$\mathscr{T}_\zeta(t, x) := \int_{S^1} G_t(x, y) \zeta(y)\, dy, \qquad (t, x) \in I \times S^1.$$

If we have any f' in $L^2(I \times S^1)$, the unique solution of

$$\partial_t u = \mathscr{L} u + f', \qquad (t, x) \in I \times S^1,$$
$$u[0] = \zeta,$$

can be represented as

$$u(t, x) = \mathscr{T}_\zeta(t, x) + \int_{I \times S^1} G_{t-s}(x, y)\, f'(s, y)\, ds\, dy, \qquad (t, x) \in I \times S^1.$$

If we now take σ' in $L^\infty(\Omega \times I \times S^1)$ such that σ' is P-a.s. continuous as a function of (t, x) in $I \times S^1$ and such that for each (t, x) in $I \times S^1$, $\sigma'(t, x)$ is

measurable with respect to

$$\mathscr{F}_t := \sigma\{W(A): A \in \mathscr{B}([0,t] \times S^1)\}, \qquad t \in I, \tag{6}$$

then we may represent the solution of

$$\partial_t u = \mathscr{L}u + f' + \sigma' \dot{W}_{tx}, \qquad (t,x) \in I \times S^1,$$
$$u[0] = \zeta,$$

as

$$u(t,x) = \mathscr{T}_\zeta(t,x) + \int_{I\times S^1} G_{t-s}(x,y) f'(s,y)\, ds\, dy$$
$$+ \int_{I\times S^1} G_{t-s}(x,y)\sigma'(s,y) W(ds,dy), \qquad (t,x) \in I \times S^1,$$

up to P-a.s. uniqueness. From this we see that the solution of (2) must P-a.s. satisfy the stochastic integral equation

$$v_\zeta^\varepsilon(t,x) = \mathscr{T}_\zeta(t,x) + \int_0^t \int_{S^1} G_{t-s}(x,y) f\big(y, v_\zeta^\varepsilon(s,y)\big)\, ds\, dy$$
$$+ \varepsilon \int_0^t \int_{S^1} G_{t-s}(x,y)\sigma\big(y, v_\zeta^\varepsilon(s,y)\big) W(ds,dy). \tag{7}$$

We note without further elaboration that this integral equation naturally defines a Picard iteration of $\mathscr{F}_t$-predictable elements of $L^2(I \times S^1 \times \Omega)$ and that standard procedures may be used to show the existence and uniqueness of an $\mathscr{F}_t$-predictable solution in $L^2(I \times S^1 \times \Omega)$ [see Walsh (1984), Chapter 3 or Kallianpur (1980), Section 5.1]. Proposition A.1 in the Appendix, in conjunction with well-known results on the continuity of random fields [see Garsia (1972), Lemma 1, Walsh (1984), Theorem 1.1 and Adler (1981), Lemma 3.3.3] allow us to show that if ζ is in $C^{2\kappa}(S^1)$, where $0 \le \kappa < \frac{1}{4}$, then there must exist a version of the $L^2(I \times S^1 \times \Omega)$ solution to (7) which is P-a.s. in C^κ—see Proposition A.2. The reader is referred to Walsh (1984) and Marcus (1974) for general discussions of stochastic parabolic PDE's.

3. A simplification. Instead of directly trying to prove assertions (A.1)–(A.3), we can simplify our task by using the contraction principle to remove the term $f(\cdot, v_\zeta^\varepsilon)$. Recall for reference that if $\{X^\varepsilon\}$ is a collection of random elements of a Polish space $\mathscr{X}$ with metric ρ, then we say that X^ε has a large deviations principle with action functional I: $\mathscr{X} \to [0,\infty]$ if:

(B.1) The level set $\Phi(s) := \{x \in \mathscr{X}: I(x) \le s\}$ is a compact set in $\mathscr{X}$ for each $s \ge 0$.

(B.2) For any positive numbers δ and γ and any x in $\mathscr{X}$, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho(X^\varepsilon, x) < \delta\} > \exp\left(-\frac{I(x) + \gamma}{\varepsilon^2}\right)$$

for all $0 < \varepsilon < \varepsilon_0$.

(B.3) For any positive numbers δ, γ and s, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho(X^\varepsilon, \Phi(s)) \geq \delta\} < \exp\left(-\frac{s-\gamma}{\varepsilon^2}\right)$$

for all $0 < \varepsilon < \varepsilon_0$.

Let us now quickly review some of the essential arguments of Freidlin (1988), fixing a ζ in $L^2(S^1)$. If $\sigma \equiv 1$ and $f \equiv 0$, then v_ζ^ε would be Gaussian and the large deviations principle for v_ζ^ε in $L^2(I \times S^1)$ could be described by the covariance kernel of v_ζ^ε. If f is not identically zero but $\sigma \equiv 1$, then we can use the contraction principle to reduce the problem to $f \equiv 0$. Specifically, for φ in $L^2(I \times S^1)$, define B_ζ as the unique solution of the integral equation

$$\begin{aligned}(B_\zeta\varphi)(t,x) &= \mathscr{T}_\zeta(t,x) + \varphi(t,x) \\ &\quad + \int_{I\times S^1} G_{t-s}(x,y) f\big(y, (B_\zeta\varphi)(s,y)\big)\, ds\, dy, \\ &\qquad\qquad (t,x) \in I \times S^1. \end{aligned} \tag{8}$$

Then we have that

$$v_\zeta^\varepsilon = B_\zeta \psi_\zeta^\varepsilon, \tag{9}$$

where

$$\psi_\zeta^\varepsilon(t,x) = \varepsilon \int_{I\times S^1} G_{t-s}(x,y)\sigma\big(y, v_\zeta^\varepsilon(s,y)\big) W(ds, dy), \qquad (t,x) \in I \times S^1,$$

that is, ψ_ζ^ε is the solution of the SPDE

$$\begin{aligned} &\partial_t \psi_\zeta^\varepsilon = \mathscr{L}\psi_\zeta^\varepsilon + \varepsilon\sigma\big(\cdot, v_\zeta^\varepsilon\big)\ddot{W}_{tx}, \qquad (t,x) \in I \times S^1. \\ &\psi_\zeta^\varepsilon[0] = \mathbf{0}. \end{aligned} \tag{10}$$

If $\sigma \equiv 1$, then $\psi_\zeta^\varepsilon = \varepsilon X$, where X is a Gaussian element of $L_0^2(I \times S^1) := \{\varphi \in L^2(I \times S^1)\colon \varphi[0] = \mathbf{0}\}$; this places us in the framework of Freidlin and Wentzell [(1984), Section 3.4] so that the action functional for ψ_ζ^ε, at least in the ρ_{L^2} topology, is easily found. If it can be shown that the mapping B_ζ is continuous in the ρ_{L^2} topology, the action functional for v_ζ^ε in $L^2(I \times S^1)$ is immediate from the contraction principle [Varadhan (1984)] once we know the large deviations principle for ψ_ζ^ε.

In our current problem where σ depends nontrivially upon v_ζ^ε, we still have the representation (9), where ψ_ζ^ε is the solution to (10), so we may still find the action functional for ψ_ζ^ε instead of v_ζ^ε. We may rewrite the SPDE (10) as

$$\begin{aligned} &\partial_t \psi_\zeta^\varepsilon = \mathscr{L}\psi_\zeta^\varepsilon + \varepsilon\sigma\big(\cdot, B_\zeta\psi_\zeta^\varepsilon\big)\ddot{W}_{tx}, \qquad (t,x) \in I \times S \\ &\psi_\zeta^\varepsilon[0] = \mathbf{0}. \end{aligned} \tag{11}$$

Comparing this and (2), we see that we have moved all the nonlinear effects to the diffusion term $\sigma(\cdot, B_\zeta\psi_\zeta^\varepsilon)$. From (11) one expects that the action functional

for ψ_ζ^ε in C_0^κ for a fixed $0 \le \kappa < \frac{1}{4}$ and ζ in $C^{2\kappa}(S^1)$ should be

$$\tilde{S}_\zeta(\varphi) := \begin{cases} \frac{1}{2}\int_{I\times S^1} \left| \frac{\partial_t\varphi - \mathscr{L}\varphi}{\sigma(\cdot, B_\zeta\varphi)} \right|^2 (t,x)\, dt\, dx, & \text{if } \varphi \in W_2^{1,2}, \\ \infty, & \text{if } \varphi \notin W_2^{1,2}, \end{cases} \tag{12}$$

for all real-valued functions φ on $I \times S^1$. This action functional is considerably simpler and easier to manipulate than (4), since the operator $\varphi \mapsto (\partial_t\varphi - \mathscr{L}\varphi)/\sigma(\cdot, B_\zeta\varphi)$ is merely a scaling of the linear operator $\varphi \mapsto \partial_t\varphi - \mathscr{L}\varphi$. By (9), if B_ζ is continuous in the ρ_κ topology and in addition is one-to-one, then a large deviations principle for ψ_ζ^ε in C_0^κ with action functional $\tilde{S}_\zeta$ immediately gives us that there will be a large deviations principle for v_ζ^ε in C_ζ^κ with action functional $\tilde{S}_\zeta(B_\zeta^{-1})$. That B_ζ is in fact continuous and injective as a mapping from C_0^κ to C^κ for each $0 \le \kappa < \frac{1}{4}$ is contained in the following result, which is essentially an extension of Lemma 4 of Freidlin (1988).

PROPOSITION 1. *For each $0 \le \kappa < \frac{1}{4}$ and each $\tilde{\zeta}$ in $C^{2\kappa}(S^1)$, the mapping $B_{\tilde{\zeta}}$: $C_0^\kappa \to C_{\tilde{\zeta}}^\kappa$ is invertible. Furthermore, for each $0 \le \kappa < \frac{1}{4}$, there is a constant η_κ depending only on κ such that for all φ_1 and φ_2 in C_0^κ and all ζ_1 and ζ_1 in $C^{2\kappa}(S^1)$,*

$$\|B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2\|_\kappa \le \eta_\kappa(\|\zeta_1 - \zeta_2\|_{2\kappa, S^1} + \|\varphi_1 - \varphi_2\|_\kappa).$$

PROOF. The invertibility of $B_{\tilde{\zeta}}$ is obvious as we may rewrite (8) as

$$\varphi(t,x) = \big(B_{\tilde{\zeta}}\varphi\big)(t,x) - \mathscr{T}_{\tilde{\zeta}}(t,x) - \int_{I\times S^1} G_{t-s}(x,y)\, f\big(y, \big(B_{\tilde{\zeta}}\varphi\big)(s,y)\big)\, ds\, dy,$$
$$(t,x) \in I \times S^1,$$

for all φ in C_0^0 and $\tilde{\zeta}$ in $C^0(S^1)$.

To prove the continuity of B, fix a $0 \le \kappa < \frac{1}{4}$, a ζ_1 and ζ_2 in $C^{2\kappa}(S^1)$ and a φ_1 and φ_2 in C_0^κ. Define

$$\mu_1(t) := \sup_{(s,y)\in[0,t]\times S^1} |B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2|$$

and if $\kappa > 0$,

$$\mu_2(t) := \sup_{\substack{(s,y),(r,z)\in[0,t]\times S^1 \\ (s,y)\neq(r,z)}} \left\{ \frac{|(B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2)(s,y) - (B_{\zeta_1}\varphi - B_{\zeta_2}\varphi_2)(r,z)|}{(r'((s,y),(r,z)))^\kappa} \right\}$$

for each $0 \le t \le T$. Using the results of Appendix B [in particular (B.2)], we have that

$$\mu_1(t) \le \|\zeta_1 - \zeta_2\|_{C(S^1)} + \|\varphi_1 - \varphi_2\|_C + \bar{f}\int_0^t \mu_1(s)\, ds \tag{13}$$

for each $0 \le t \le T$, so that by Gronwall's inequality

$$\|B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2\|_C \le e^{\bar{f}T}\big(\|\zeta_1 - \zeta_2\|_{C(S^1)} + \|\varphi_1 - \varphi_2\|_C\big), \tag{14}$$

completing the proof when $\kappa = 0$. When $\kappa > 0$, we can use Proposition A.1 in Appendix A and η'_κ as in Appendix B to see that

$$\begin{aligned}\mu_2(t) &\le \eta'_\kappa\|\zeta_1 - \zeta_2\|_{2\kappa,S^1} + \|\varphi_1 - \varphi_2\|_\kappa \\ &\quad + \bar{f}L_\kappa\left\{\int_0^t\int_{S^1}|(B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2)(s,y)|^2\,ds\,dy\right\}^{1/2}\end{aligned} \tag{15}$$

for each $0 \le t \le T$. Observe that for each (s,y) in $I \times S^1$,

$$\begin{aligned}\big|(B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2)(s,y)\big| &\le \big|(B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2)(s,y) \\ &\qquad -(B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2)(0,y)\big| + |\zeta_1(y) - \zeta_2(y)| \\ &\le \mu_2(s)T^\kappa + \|\zeta_1 - \zeta_2\|_{2\kappa,S^1}\end{aligned}$$

so that we may continue inequality (15) as

$$\begin{aligned}\mu_2(t) &\le \big(\eta'_\kappa + \bar{f}L_\kappa(2\pi T)^{1/2}\big)\|\zeta_1 - \zeta_2\|_{2\kappa,S^1} + \|\varphi_1 - \varphi_2\|_\kappa \\ &\quad + \bar{f}L_\kappa T^\kappa(2\pi)^{1/2}\left\{\int_0^t\mu_2^2(s)\,ds\right\}^{1/2}\end{aligned}$$

for all $0 \le t \le T$, which we square to find that

$$\begin{aligned}\mu_2^2(t) &\le 2\Big\{\big(\eta'_\kappa + \bar{f}L_\kappa(2\pi T)^{1/2}\big)\|\zeta_1 - \zeta_2\|_{2\kappa,S^1} + \|\varphi_1 - \varphi_2\|_\kappa\Big\}^2 \\ &\quad + 2\bar{f}^2L_\kappa^2T^{2\kappa}2\pi\int_0^t\mu_2^2(s)\,ds\end{aligned}$$

for all $0 \le t \le T$. Another application of Gronwall's inequality then is sufficient to imply that

$$\begin{aligned}\big[B_{\zeta_1}\varphi_1 - B_{\zeta_2}\varphi_2\big]_\kappa &\le 2^{1/2}e^{2\pi\bar{f}^2L_\kappa^2T^{2\kappa+1}} \\ &\quad\times\Big(\big(\eta'_\kappa + \bar{f}L_\kappa(2\pi T)^{1/2}\big)\|\zeta_1 - \zeta_2\|_{2\kappa,S^1} + \|\varphi_1 - \varphi_2\|_\kappa\Big).\end{aligned}$$

This and (14) imply the desired result. □

The reader may easily check using (8) that indeed $S_\zeta(\varphi) = \tilde{S}_\zeta(B_\zeta^{-1}\varphi)$ for all ζ in $C^0(S^1)$ and all φ in C_ζ^0. Thus *if for a fixed* $0 \le \kappa < \frac{1}{4}$ *and* ζ *in* $C^{2\kappa}(S^1)$, *the random field* ψ_ζ^ε *has a large deviations principle in* C_0^κ *with action functional* (12), *then* v_ζ^ε *will have the large deviations principle* (A.1)–(A.3).

4. The large deviations principle for ψ_ζ^ε. We shall now prove the large deviations principle for ψ_ζ^ε. In our proofs we shall have need, however, of an

even simpler process than ψ_ζ^ε. For any ζ in $C^0(S^1)$ and any ψ in C_0^0, let $\psi_\zeta^{\psi,\varepsilon}$ be the Gaussian process defined by freezing the diffusion term $\sigma(\cdot, B_\zeta\psi_\zeta^\varepsilon)$ at $\sigma(\cdot, B_\zeta\psi)$, that is,

$$\psi_\zeta^{\psi,\varepsilon}(t,x) = \varepsilon\int_{I\times S^1} G_{t-s}(x,y)\sigma\big(y,(B_\zeta\psi)(s,y)\big)W(ds,dy), \tag{16}$$
$$(t,x)\in I\times S^1.$$

Exactly in analogy to Freidlin (1988), we can compute that $(1/\varepsilon)\psi_\zeta^{\psi,\varepsilon}$ has the self-adjoint and compact covariance operator

$$\begin{aligned}\big(A_\zeta^\psi\varphi\big)(t,x) = &\int_{I\times S^1}\int_{I\times S^1} G_{t-s}(x,y)\\ &\times\sigma^2\big(y,(B_\zeta\psi)(s,y)\big)G_{r-s}(z,y)\,ds\,dy\,\varphi(r,z)\,dr\,dz,\\ &(t,x)\in I\times S^1,\end{aligned} \tag{17}$$

for all φ in $L_0^2(I\times S^1)$. We can then use the results of Freidlin [(1988), Section 3.4] to see that $\psi_\zeta^{\psi,\varepsilon}$ satisfies a large deviations principle in $L_0^2(I\times S^1)$ with action functional

$$\tilde S_\zeta^\psi(\varphi) := \begin{cases}\frac{1}{2}\|\big(A_\zeta^\psi\big)^{-1/2}\varphi\|_{L^2}, & \text{if } \varphi\in \operatorname{Range}\big(A_\zeta^\psi\big)^{1/2},\\ \infty, & \text{if } \varphi\notin \operatorname{Range}\big(A_\zeta^\psi\big)^{1/2},\end{cases} \tag{18}$$

for all φ in $L_0^2(I\times S^1)$, where $(A_\zeta^\psi)^{-1/2}\varphi$ is defined as the unique element $\hat\varphi$ of $L_0^2(I\times S^1)$ which is orthogonal to the nullspace of A_ζ^ψ such that $(A_\zeta^\psi)^{1/2}\hat\varphi = \varphi$. Some manipulation of (18) shows us that

$$\tilde S_\zeta^\psi(\varphi) := \begin{cases}\dfrac{1}{2}\displaystyle\int_{I\times S^1}\left|\frac{\partial_t\varphi - \mathscr{L}\varphi}{\sigma(\cdot, B_\zeta\psi)}\right|^2(t,x)\,dt\,dx, & \text{if } \varphi\in W_2^{1,2},\\ \infty, & \text{if } \varphi\notin W_2^{1,2},\end{cases} \tag{19}$$

for all φ in $L_0^2(I\times S^1)$. We state this fact as a lemma.

LEMMA 1. *For each ζ in $C^0(S^1)$ and each ψ in C_0^0, the random field $\psi_\zeta^{\psi,\varepsilon}$ satisfies a large deviations principle in $L_0^2(I\times S^1)$ with action functional $\tilde S_\zeta^\psi$.*

PROOF. The claim is a direct application of Theorems 3.4.2 and Lemma 3.4.1 in Freidlin and Wentzell (1984). For future reference, we shall recall here the proof of the upper bound; for any positive numbers δ, γ and s, we show that there is an $\varepsilon_0 > 0$ such that

$$P\Big\{\rho_{L^2}\big(\psi_\zeta^{\psi,\varepsilon}, \tilde\Phi_\zeta^\psi(s)\big)\ge\delta\Big\} < e^{-(s-\gamma)/\varepsilon^2}, \tag{20}$$

whenever $0 < \varepsilon < \varepsilon_0$, where we have defined $\tilde{\Phi}_\zeta^\psi(s) := \{\varphi \in L_0^2(I \times S^1)$: $\tilde{S}_\zeta^\psi(\varphi) \leq s\}$. Let us first denote by $\{\hat{\lambda}_\zeta^\psi(k)$: $k = 1, 2, \dots\}$ the eigenvalues of the covariance operator A_ζ^ψ as in (17), with $\{\hat{\phi}_\zeta^\psi(k)$; $k = 1, 2, \dots\}$ being the corresponding orthonormal collection of eigenvectors. We assume that the eigenvalues are ordered so that $\{\hat{\lambda}_\zeta^\psi(k)$; $k = 1 = 2, \dots\}$ is a nonincreasing sequence. Letting $\langle \cdot, \cdot \rangle$ be the inner product in $L^2(I \times S^1)$, if we define

$$\psi_\zeta^{N,\psi,\varepsilon} := \sum_{1 \leq k \leq N} \langle \psi_\zeta^{\psi,\varepsilon}, \hat{\phi}_\zeta^\psi(k) \rangle \hat{\phi}_\zeta^\psi(k) \tag{21}$$

for each positive N, then $\psi_\zeta^{\psi,\varepsilon} = \lim_N \psi_\zeta^{N,\psi,\varepsilon}$ in $L_0^2(I \times S^1)$, P-a.s. The coefficients $\{\langle \psi_\zeta^{\psi,\varepsilon}, \hat{\phi}_\zeta^\psi(k) \rangle$; $k = 1, 2, \dots\}$ are independent Gaussian random variables with

$$E\left[\langle \psi_\zeta^{\psi,\varepsilon}, \hat{\phi}_\zeta^\psi(k) \rangle^2\right] = \hat{\lambda}_\zeta^\psi(k)$$

for all $k = 1, 2, \dots$.

As in Theorem 3.4.2 of Freidlin and Wentzell (1984), we should next find an N large enough that $\psi_\zeta^{N,\psi,\varepsilon}$ is appropriately close to $\psi_\zeta^{\psi,\varepsilon}$. Let us now show that in fact this N will *not* depend on the parameters ψ and ζ; this will be critical for the results of Section 5. It is well known [Gohberg and Goldberg (1981), Theorem 3.9.1] that for all $k = 1, 2, \dots$, we have the representations

$$\begin{aligned} \hat{\lambda}_\zeta^\psi(k) &= \min_{E \in \mathscr{E}_{k-1}} \max_{\substack{\|\varphi\|_{L^2} = 1 \\ \varphi \perp E}} \langle A_\zeta^\psi \varphi, \varphi \rangle, \\ \hat{\lambda}_0^0(k) &= \min_{E \in \mathscr{E}_{k-1}} \max_{\substack{\|\varphi\|_{L^2} = 1 \\ \varphi \perp E}} \langle A_0^0 \varphi, \varphi \rangle, \end{aligned} \tag{22}$$

where $\mathscr{E}_{k-1}$ is the collection of all $k-1$-dimensional linear subspaces of $L_0^2(I \times S^1)$ and $\hat{\lambda}_0^0(k)$ is the kth eigenvalue of A_0^0, with A_0^0 being the covariance operator of the form (17) with $\psi = \mathbf{0}$ and $\zeta = \mathbf{0}$. Some basic manipulations of (17) show that for any φ in $L_0^2(I \times S^1)$,

$$\frac{m^2}{M^2} \langle A_0^0 \varphi, \varphi \rangle \leq \langle A_\zeta^\psi \varphi, \varphi \rangle \leq \frac{M^2}{m^2} \langle A_0^0 \varphi, \varphi \rangle.$$

From this and (22) we in particular have the bound

$$\hat{\lambda}_\zeta^\psi(k) \leq \frac{M^2}{m^2} \hat{\lambda}_0^0(k) \tag{23}$$

for all $k = 1, 2, \dots$.

We now continue, keeping (23) in mind. We can take an N' large enough that

$$\frac{2s}{\delta} \frac{M^2}{m^2} \lambda_0^0(N') < \frac{1}{2}. \tag{24}$$

Then we calculate

$$
\begin{aligned}
P\Big\{\rho_{L^2}\big(\psi_\zeta^{\psi,\varepsilon}, \psi_\zeta^{N',\psi,\varepsilon}\big) \geq \delta\Big\} &\leq e^{-s/\varepsilon^2} E\left[\exp\left(\frac{s}{\delta^2}\,\frac{\|\psi_\zeta^{\psi,\varepsilon} - \psi_\zeta^{N',\psi,\varepsilon}\|_{L^2}^2}{\varepsilon^2}\right)\right] \\
&= e^{-s/\varepsilon^2} \prod_{k=N'+1}^{\infty} E\left[\exp\left(\frac{s}{\delta^2}\,\frac{\langle \psi_\zeta^{\psi,\varepsilon}, \hat{\phi}_\zeta^{\psi}(k)\rangle^2}{\varepsilon^2}\right)\right] \\
&= e^{-s/\varepsilon^2} \prod_{k=N'+1}^{\infty} \left(1 - \frac{2s\hat{\lambda}_\zeta^{\psi}(k)}{\delta^2}\right)^{-1/2} \qquad (25) \\
&= e^{-s/\varepsilon^2} \exp\left(\frac{3}{2}\,\frac{s}{\delta^2} \sum_{k=N'+1}^{\infty} \hat{\lambda}_\zeta^{\psi}(k)\right) \\
&\leq e^{-s/\varepsilon^2} \exp\left(\frac{3s}{2\delta^2}\,\text{trace}\, A_\zeta^{\psi}\right).
\end{aligned}
$$

In our calculation we have used that $|\log(1 - z)| \leq 3|z|/2$ whenever $|z| < \frac{1}{2}$. Note also that

$$
\begin{aligned}
\text{trace}\, A_\zeta^{\psi} &= \sum_{k=1}^{\infty} \hat{\lambda}_\zeta^{\psi}(k) = \sum_{k=1}^{\infty} E\big[\langle \psi_\zeta^{\psi,\varepsilon}, \hat{\phi}_\zeta^{\psi}(k)\rangle^2\big]/\varepsilon^2 = E\|\psi_\zeta^{\psi,\varepsilon}\|_{L^2}^2/\varepsilon^2 \\
&\leq M^2 \int_{I\times S^1}\int_{I\times S^1} G_{t-s}^2(x,y)\, ds\, dy\, dt\, dx \leq M^2 T \sum_{k=0}^{\infty} (1/2\lambda_k). \qquad (26)
\end{aligned}
$$

Thus when (24) holds, we have the estimate

$$
P\Big\{\rho_{L^2}\big(\psi_\zeta^{\psi,\varepsilon}, \psi_\zeta^{N',\psi,\varepsilon}\big) \geq \delta\Big\} \leq e^{-s/\varepsilon^2} \exp\left(\frac{3s}{4\delta^2} M^2 T \sum_{k=1}^{\infty} \lambda_k^{-1}\right). \qquad (27)
$$

The proof of (20) is completed by writing

$$
\begin{aligned}
P\Big\{\rho_{L^2}\big(\psi_\zeta^{\psi,\varepsilon}, \tilde{\Phi}_\zeta^{\psi}(s)\big) \geq \delta\Big\} &\leq P\Big\{\rho_{L^2}\big(\psi_\zeta^{\psi,\varepsilon}, \psi_\zeta^{N',\psi,\varepsilon}\big) \geq \delta\Big\} \\
&\quad + P\Big\{\tilde{S}_\zeta^{\psi}\big(\psi_\zeta^{N',\psi,\varepsilon}\big) \geq s\Big\} \qquad (28)
\end{aligned}
$$

with the estimate

$$
\begin{aligned}
&P\Big\{\tilde{S}_\zeta^{\psi}\big(\psi_\zeta^{N'\psi,\varepsilon}\big) \geq s\Big\} \\
&\quad \leq e^{-(s-\gamma/2)/\varepsilon^2} E\Big[\exp\big((1 - \gamma/(2s))\,\tilde{S}_\zeta^{\psi}\big(\psi_\zeta^{N',\psi,\varepsilon}\big)/\varepsilon^2\big)\Big] \qquad (29) \\
&\quad = e^{-(s-\gamma/2)/\varepsilon^2} (\gamma/(2s))^{-N'/2}.
\end{aligned}
$$

Then

$$P\left\{\rho_{L^2}\left(\psi_\zeta^{\psi,\varepsilon}, \tilde{\Phi}_\zeta^\gamma(s)\right) \geq \delta\right\}$$
$$\leq e^{-(s-\gamma/2)/\varepsilon^2}\left[\exp\left(\frac{3s}{4\delta^2}M^2T\sum_{k=1}^{\infty}\lambda_k^{-1}\right) + \left(\frac{\gamma}{(2s)}\right)^{-N'/2}\right]$$

from which we have (20) for $\varepsilon > 0$ small enough. □

Comparing (12) and (19) we see that $\tilde{S}_\zeta(\varphi) = \tilde{S}_\zeta^\varphi(\varphi)$ for all φ in C_0^0 and all ζ in $C^0(S^1)$.

Now let us fix for the rest of this section a specific $0 \leq \kappa < \frac{1}{4}$ and a ζ in $C^{2\kappa}(S^1)$. We begin to prove that ψ_ζ^ε has a large deviations principle in C_0^κ with action functional (12). We first show the compactness result.

PROPOSITION 2. *The level set* $\tilde{\Phi}_\zeta(s) := \{\varphi \in C_0^\kappa\colon \tilde{S}_\zeta(\varphi) \leq s\}$ *is compact in* C_0^κ *for each* $s \geq 0$.

PROOF. Fixing a κ' such that $\kappa < \kappa' < \frac{1}{4}$, we use the fact that $C_0^{\kappa'}$ is compactly embedded in C_0^κ to prove that $\tilde{\Phi}_\zeta(s)$ is relatively compact in C_0^κ by showing that it is bounded in $C_0^{\kappa'}$. For any φ in $\tilde{\Phi}_\zeta(s)$, we must have φ in $W_2^{1,2}$, and thus φ tautologically satisfies the PDE

$$\partial_t\varphi = \mathscr{L}\varphi + (\partial_t\varphi - \mathscr{L}\varphi), \qquad (t,x) \in I \times S^1,$$
$$\varphi[0] = \mathbf{0},$$

and hence φ satisfies

$$\varphi(t,x) = \int_{I\times S^1} G_{t-s}(x,y)(\partial_t\varphi - \mathscr{L}\varphi)(s,y)\,ds\,dy, \qquad (t,x) \in I \times S^1,$$

so by Proposition A.1 in Appendix A,

$$\|\varphi\|_{\kappa'} \leq (1+T^{\kappa'})[\varphi]_{\kappa'} \leq (1+T^{\kappa'})L_{\kappa'}M\left\|\frac{\partial_t\varphi - \mathscr{L}\varphi}{\sigma(\cdot, B_\zeta\varphi)}\right\|_{L^2} \tag{30}$$
$$\leq (1+T^{\kappa'})L_{\kappa'}M(2s)^{1/2},$$

that is,

$$\tilde{\Phi}_\zeta(s) \subset \left\{\varphi \in C_0^{\kappa'}\colon \|\varphi\|_{\kappa'} \leq (1+T^{\kappa'})L_{\kappa'}M(2s)^{1/2}\right\}.$$

We prove that $\tilde{\Phi}_\zeta(s)$ is closed in C_0^κ by using the action functionals of the form (19). We can calculate from (19) that for any ψ_1 and ψ_2 in C_0^κ and any φ in $W_2^{1,2}$ such that $\varphi[0] = \mathbf{0}$,

$$\tilde{S}_\zeta^{\psi_1}(\varphi) \leq \tilde{S}_\zeta^{\psi_2}(\varphi)\left(1 + \frac{2M\bar{\sigma}}{m^2}\|B_\zeta\psi_1 - B_\zeta\psi_2\|_C\right) \tag{31}$$
$$\leq \tilde{S}_\zeta^{\psi_2}(\varphi)(1 + \omega_\kappa\|\psi_1 - \psi_2\|_\kappa),$$

with $\omega_\kappa := 2M\bar{\sigma}\eta_\kappa/m^2$, this last inequality resulting from Proposition 1. Now take $\{\varphi_n\}$ in $\tilde{\Phi}_\zeta(s)$ converging to φ in the ρ_κ topology. Then

$$\tilde{S}_\zeta^\varphi(\varphi_n) \le \tilde{S}_\zeta^{\varphi_n}(\varphi_n)\big(1 + \omega_\kappa\|\varphi - \varphi_n\|_\kappa\big) \le s\big(1 + \omega_\kappa\|\varphi - \varphi_n\|_\kappa\big) \tag{32}$$

for all n. Since $\psi \mapsto \tilde{S}_\zeta^\varphi(\psi)$ is lower semicontinuous in the ρ_{L^2} topology as it is the action functional for $\psi_\zeta^{\psi,\varepsilon}$ in $L_0^2(I \times S^1)$, it must also clearly be lower semicontinuous in the ρ_κ topology, so in light of (32) we must have

$$\tilde{S}_\zeta(\varphi) \le \liminf_n \tilde{S}^\varphi(\varphi_n) \le s \tag{33}$$

implying that φ must indeed be in $\tilde{\Phi}_\zeta(s)$, completing the proof. □

We next prove the lower bound.

PROPOSITION 3. *For any positive numbers δ and γ and any φ in C_0^κ, there is an $\varepsilon_0 > 0$ such that*

$$P\big\{\rho_\kappa(\psi_\zeta^\varepsilon, \varphi) < \delta\big\} > \exp\left(-\frac{\tilde{S}_\zeta(\varphi) + \gamma}{\varepsilon^2}\right) \tag{34}$$

for all $0 < \varepsilon < \varepsilon_0$.

PROOF. We shall use a Girsanov change of measure. First, we may assume that φ is in $W_2^{1,2}$; if not, then $\tilde{S}_\zeta(\varphi) = \infty$ and the result is vacuously true. For the purposes of Section 5, let us fix $s_0 \ge \tilde{S}_\zeta(\varphi)$. Define $Z_\zeta^{\varphi,\varepsilon} := \psi_\zeta^\varepsilon - \varphi$ and observe that $Z_\zeta^{\varphi,\varepsilon}$ satisfies the stochastic PDE

$$\begin{aligned} &\partial_t Z_\zeta^{\varphi,\varepsilon} = \mathscr{L}Z_\zeta^{\varphi,\varepsilon} + \varepsilon\sigma\big(\cdot, B_\zeta(Z_\zeta^{\varphi,\varepsilon} + \varphi)\big)\ddot{W}_{tx} - (\partial_t\varphi - \mathscr{L}\varphi), \\ &\qquad\qquad (t,x) \in I \times S^1, \\ &Z_\zeta^{\varphi,\varepsilon}[0] = \mathbf{0}. \end{aligned}$$

If we now define a new noise process W^ε such that

$$W(A) = W^\varepsilon(A) + \frac{1}{\varepsilon}\int_A \left(\frac{\partial_t\varphi - \mathscr{L}\varphi}{\sigma\big(\cdot, B_\zeta(Z_\zeta^{\varphi,\varepsilon} + \varphi)\big)}\right)(s,y)\,ds\,dy$$

for all sets A in $\mathscr{B}(I \times S^1)$, then $Z_\zeta^{\varphi,\varepsilon}$ also satisfies the stochastic PDE

$$\begin{aligned} &\partial_t Z_\zeta^{\varphi,\varepsilon} = \mathscr{L}Z_\zeta^{\varphi,\varepsilon} + \varepsilon\sigma\big(\cdot, B_\zeta(Z_\zeta^{\varphi,\varepsilon} + \varphi)\big)\ddot{W}_{tx}^\varepsilon, \quad (t,x) \in I \times S^1, \\ &Z_\zeta^{\varphi,\varepsilon}[0] = \mathbf{0}. \end{aligned} \tag{35}$$

Define a new measure P^ε on $(\Omega, \mathscr{F})$ by

$$\frac{dP^\varepsilon}{dP} = \exp\left[\frac{1}{\varepsilon}I^\varepsilon + \frac{1}{\varepsilon^2}S^{Z_\zeta^{\varphi,\varepsilon}+\varphi}(\varphi)\right],$$

where for convenience we have set

$$I^\varepsilon := \int_{I\times S^1}\left(\frac{\partial_t\varphi - \mathscr{L}\varphi}{\sigma\big(\cdot, B_\zeta(\psi_\zeta^\varepsilon + \varphi)\big)}\right)(s,y)W^\varepsilon(ds,dy).$$

We can easily show by Girsanov's theorem that P^ε is a probability measure and that W^ε is a P^ε-Brownian sheet. Recalling the calculation (31), let us define $\delta' = \delta'(s_0,\delta,\gamma) > 0$ by $\sigma' := \min\{\delta, \gamma/(2\omega_\kappa s_0)\}$. Then for all $\varepsilon > 0$,

$$\begin{aligned} P\big\{\rho_\kappa(\psi_\zeta^\varepsilon,\varphi) < \delta\big\} &\geq E^\varepsilon\left[\chi\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}\frac{dP}{dP^\varepsilon}\right] \\ &\geq E^\varepsilon\left[\chi\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}\exp\left[\frac{-I^\varepsilon}{\varepsilon}\right]\right]\exp\left(-\frac{\tilde{S}_\zeta(\varphi)+\gamma/2}{\varepsilon^2}\right). \end{aligned}$$

Jensen's inequality gives us that

$$\frac{E^\varepsilon\big[\chi\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}\exp(-I^\varepsilon/\varepsilon)\big]}{P^\varepsilon\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}} \geq \exp\left\{\frac{E^\varepsilon\big[\chi\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}(-I^\varepsilon/\varepsilon)\big]}{P^\varepsilon\big\{\|Z_\|^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}}\right\}$$

and also that

$$\begin{aligned} \frac{E^\varepsilon\big[\chi\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}|I^\varepsilon|\big]}{P^\varepsilon\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}} &\leq \left(\frac{E^\varepsilon\big[\chi\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}(I^\varepsilon)^2\big]}{P^\varepsilon\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}}\right)^{1/2} \\ &\leq \left(\frac{E^\varepsilon\big[(I^\varepsilon)^2\big]}{P^\varepsilon\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}}\right)^{1/2}. \end{aligned}$$

The fact that W^ε is a P^ε-Brownian sheet allows us to write the bound

$$E^\varepsilon\big[(I^\varepsilon)^2\big] = E^\varepsilon\big[2\tilde{S}_\zeta^{Z_\zeta^{\varphi,\varepsilon}+\varphi}(\varphi)\big] \leq 2M^2\tilde{S}_\zeta(\varphi)/m^2.$$

Collecting these inequalities together, we have that

$$\begin{aligned} P\big\{\rho_\kappa(\psi_\zeta^\varepsilon,\varphi) < \delta\big\} \geq P^\varepsilon\big\{\|Z_\zeta^\varepsilon\|_\kappa < \delta'\big\}&\exp\left(-\frac{1}{\varepsilon}\frac{M}{m}\left(\frac{2s_0}{P^\varepsilon\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\big\}}\right)^{1/2}\right) \\ &\times\exp\left(-\frac{\tilde{S}_\zeta(\varphi)+\gamma/2}{\varepsilon^2}\right) \end{aligned} \tag{36}$$

for all $\varepsilon > 0$. It is easy to see that the result is true if we can show that $\lim_{\varepsilon\to 0} P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\} = 1$. Notice that similarly to (26), $E^\varepsilon[\|Z_\zeta^{\varphi,\varepsilon}\|_{L^2}^2] \leq \varepsilon^2(M^2T/2)\Sigma_{k=0}^\infty \lambda_k^{-1}$. From this and Proposition A.2 and (A.8) in Appendix A, it is not difficult to argue that indeed we have $\lim_{\varepsilon\to 0} P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa < \delta'\} = 1$. Taking any κ' such that $\kappa < \kappa' < \frac{1}{2}$, and any constant $L > 0$, by Proposition

A.2 and (A.8) in Appendix A, there must exist an $\varepsilon_1 = \varepsilon_1(\kappa', L) > 0$ such that

$$
\begin{aligned}
P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa \geq \delta'\} &\leq P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_{\kappa'} \geq \beta_L^{\kappa'}\} + P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_{\kappa'} \leq \beta_L^{\kappa'}, \|Z_\zeta^{\varphi,\varepsilon}\|_\kappa \geq \delta'\} \\
&\leq e^{L/\varepsilon^2} + P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_{L^2} \geq \delta''\} \\
&\leq e^{-L/\varepsilon^2} + \varepsilon^2(\delta'')^{-2}(M^2T/2)\sum_{k=0}^{\infty}\lambda_k^{-1}
\end{aligned} \tag{37}
$$

for all $0 < \varepsilon < \varepsilon_1$, where $\delta'' = \delta''(\kappa', \beta_L^{\kappa'}, \delta') > 0$ may be defined as

$$\delta'' := \inf\{\|\varphi\|_{L^2}: \varphi \in C_0^{\kappa'}, \|\varphi\|_{\kappa'} \leq \beta_L^{\kappa'}, \|\varphi\|_\kappa \geq \delta'\};$$

clearly δ'' must be positive (see Appendix C). This computation is sufficient to extract the desired result from (36). □

Finally let us prove the upper bound.

PROPOSITION 4. *For any positive numbers s, δ and γ, there is an $\varepsilon_0 > 0$ such that*

$$P\left\{\rho_\kappa\left(\psi_\zeta^\varepsilon, \tilde{\Phi}_\zeta(s)\right) \geq \delta\right\} < e^{-(s-\gamma)/\varepsilon^2}, \tag{38}$$

whenever $0 < \varepsilon < \varepsilon_0$.

PROOF. The main idea of the proof is to approximate ψ_ζ^ε by a collection of processes of the form $\psi_\zeta^{\psi,\varepsilon}$ as in (16) and to approximate each such $\psi_\zeta^{\psi,\varepsilon}$ by an expansion of the form $\psi_\zeta^{N,\psi,\varepsilon}$ as in (21), and to carry out calculations parallel to (27)–(29).

Note that we may assume that $\gamma \leq s$. Let us take an $s_0 \geq s$ (we do this in anticipation of Section 5) and a number κ' such that $\kappa < \kappa' < \frac{1}{4}$. By Proposition A.2 and (A.8) in Appendix A, if we define the C_0^κ-compact set $K_{s_0} := \{\varphi \in C_0^{\kappa'}: \|\varphi\|_{\kappa'} \leq \beta_{s_0}^{\kappa'}\}$, then there is an $\varepsilon_1 = \varepsilon_1(\kappa', s_0) > 0$ such that

$$P\{\psi_\zeta^\varepsilon \notin K_{s_0}\} < e^{-s_0/\varepsilon^2} \tag{39}$$

for all $0 < \varepsilon < \varepsilon_1$. For convenience, we shall also define

$$\tilde{\Phi}_\zeta^{(\delta)}(s) := \left\{\varphi \in C_0^\kappa: \rho_\kappa\left(\varphi, \tilde{\Phi}_\zeta(s)\right) \geq \delta\right\}.$$

Consider now the set $K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta)}(s)$. This set is ρ_κ-compact, so for any $\delta' > 0$ we can cover it by a finite collection of ρ_κ-spheres of radius δ'. Let us fix for the moment a φ in $K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta)}(s)$. If ψ_ζ^ε is close to φ, then as we shall see in equations (A.9) and (A.10) of Appendix A, the Gaussian process $\psi_\zeta^{\varphi,\varepsilon}$ is a good approximation to ψ_ζ^ε. We may then approximate $\psi_\zeta^{\varphi,\varepsilon}$ by $\psi_\zeta^{N,\varphi,\varepsilon}$, where N is sufficiently large and combining these two approximations, we approximate ψ_ζ^ε by $\psi_\zeta^{N,\varphi,\varepsilon}$ when ψ_ζ^ε is close enough to φ and N is large enough. But in this

case, where ψ_ζ^ε is close to φ and $\psi_\zeta^{N,\varphi,\varepsilon}$ is close to ψ_ζ^ε, then $\psi_\zeta^{N,\varphi,\varepsilon}$ must also be close to φ. To be more explicit, we shall prove:

LEMMA 2. *There is a* $\delta' = \delta'(s_0, \delta, \gamma) > 0$, *a positive integer* $N' = N'(s_0, \delta, \gamma)$ *and an* $\varepsilon_2 = \varepsilon_2(s_0, \delta, \gamma) > 0$ *such that*

$$P\Big\{\rho_\kappa\big(\psi_{\bar\zeta}^\varepsilon, \bar\varphi\big) < \delta', \rho_\kappa\big(\psi_{\bar\zeta}^{N',\bar\varphi,\varepsilon}, \bar\varphi\big) \geq \min\{\gamma/(4\omega_\kappa s_0), \delta/2\}\Big\} < e^{-s_0/\varepsilon^2} \tag{40}$$

for all $\bar\varphi$ *in* C_0^κ, *all* $\bar\zeta$ *in* $C^{2\kappa}(S^1)$ *and all* $0 < \varepsilon < \varepsilon_2$.

Here ω_κ is as in (31). The constant $\gamma/(2\omega_\kappa s_0)$ enters into the estimate since we would like to use (31) to estimate $\tilde S_\zeta(\psi_\zeta^{N',\varphi,\varepsilon})$ by $\tilde S_\zeta^\varphi(\psi_\zeta^{N',\varphi,\varepsilon})$ and the distance $\delta/2$ stems from the fact that necessarily $\tilde S_\zeta(\psi_\zeta^{N',\varphi,\varepsilon}) \geq s$ when $\rho_\kappa(\psi_\zeta^{N',\varphi,\varepsilon}, \varphi) < \delta/2$; recall that we assumed that $\rho_\kappa(\varphi, \tilde\Phi_\zeta(s)) \geq \delta$. The relevant calculation is that

$$\begin{aligned}
P\Big\{\rho_\kappa\big(\psi_\zeta^\varepsilon, \varphi\big) < \delta'\Big\} &< e^{-s_0/\varepsilon^2} + P\Big\{\rho_\kappa\big(\psi_\zeta^{N',\varphi,\varepsilon}, \varphi\big) < \min\{\gamma/(4\omega_\kappa s_0), \delta/2\}\Big\} \\
&< e^{-s_0/\varepsilon^2} + P\Big\{\rho_\kappa\big(\psi_\zeta^{N',\varphi,\varepsilon}, \varphi\big) < \gamma/(4\omega_\kappa s_0), \\
&\qquad\qquad \tilde S_\zeta\big(\psi_\zeta^{N',\varphi,\varepsilon}\big) \geq s\Big\} \\
&< e^{-s_0/\varepsilon^2} + P\Big\{\tilde S_\zeta^\varphi\big(\psi_\zeta^{N',\varphi,\varepsilon}\big)\big(1 + \gamma/(4s_0)\big) \geq s\Big\} \\
&< e^{-s_0/\varepsilon^2} + \big(\gamma/(4s_0)\big)^{-N'/2} \\
&\quad \times \exp\big(-(s/\varepsilon^2)(1 - \gamma/(4s_0))/(1 + \gamma/(4s_0))\big)
\end{aligned} \tag{41}$$

for $0 < \varepsilon < \varepsilon_2$. In the third inequality here we used (31) and the fourth calculation is similar to (29). Since $s(1 - \gamma/(4s_0))/(1 + \gamma/(4s_0)) \geq s - \gamma/2$, we have arrived at the fact that

$$P\Big\{\rho_\kappa\big(\psi_\zeta^\varepsilon, \varphi\big) < \delta'\Big\} < e^{-s_0/\varepsilon^2} + \big(\gamma/(4s_0)\big)^{-N'/2} e^{-(s-\gamma/2)/\varepsilon^2} \tag{42}$$

for all $0 < \varepsilon < \varepsilon_2$, where δ', N' and ε_2 are as in Lemma 2. Note that δ', N' and ε_2 depend solely on s_0, δ and γ and *not* on φ in $K_{s_0} \sim \tilde\Phi_\zeta^{(\delta)}(s)$. Thus inequality (42) holds for *all* φ in $K_{s_0} \sim \tilde\Phi_\zeta^{(\delta)}(s)$ and all $0 < \varepsilon < \varepsilon_2$. But given this estimate, we can simply cover the set $K_{s_0} \sim \tilde\Phi_\zeta^{(\delta)}(s)$ with ρ_κ-spheres of radius δ' and extract a finite subcovering, with centers $\{\varphi_i;\ i = 1, 2, \ldots, L\}$. Then for $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$,

$$\begin{aligned}
P\Big\{\rho_\kappa\big(\psi_\zeta^\varepsilon, \tilde\Phi_\zeta(s)\big) \geq \delta\Big\} &\leq P\big\{\psi_\zeta^\varepsilon \notin K_{s_0}\big\} + P\Big\{\psi_\zeta^\varepsilon \in K_{s_0} \sim \tilde\Phi_\zeta^{(\delta)}(s)\Big\} \\
&< e^{-s_0/\varepsilon^2} + \sum_{1 \leq i \leq L} P\Big\{\rho_\kappa\big(\psi_\zeta^\varepsilon, \varphi_i\big) < \delta'\Big\} \\
&< e^{-s_0/\varepsilon^2} + L\Big(e^{-s_0/\varepsilon^2} + \big(\gamma/(4s_0)\big)^{-N'/2} e^{-(s-\gamma/2)/\varepsilon^2}\Big) \\
&< (L+1)e^{-s_0/\varepsilon^2} + L\big(\gamma/(4s_0)\big)^{-N'/2} e^{-(s-\gamma/2)/\varepsilon^2},
\end{aligned} \tag{43}$$

so we easily have (38) for $\varepsilon > 0$ small enough.

Now we must prove the lemma.

PROOF OF LEMMA 2. For ease of exposition, we shall define $\tilde{\delta} := \min\{\gamma/(4\omega_\kappa s_0), \delta/2\}$. Referring to Proposition A.2 in Appendix A and (A.9) and (A.10), we see that we can find a $\delta' = \delta'(s_0, \tilde{\delta}) > 0$ such that $\delta' \le \tilde{\delta}/3$ and an $\varepsilon_3 = \varepsilon_3(s_0, \tilde{\delta}) > 0$ such that

$$P\Big\{\rho_\kappa\big(\psi^{\varepsilon}_{\bar{\zeta}}, \bar{\varphi}\big) < \delta', \rho_\kappa\big(\psi^{\varepsilon}_{\bar{\zeta}}, \psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}\big) \ge \tilde{\delta}/\varepsilon\Big\} < e^{-2s_0/\varepsilon^2} \tag{44}$$

for all $\bar{\zeta}$ in $C^{2\kappa}(S^1)$, all $\bar{\varphi}$ in C_0^κ and all $0 < \varepsilon < \varepsilon_3$. This estimate is of fundamental importance in that it allows us to approximate the non-Gaussian process $\psi^{\varepsilon}_{\bar{\zeta}}$ by the Gaussian process $\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}$. To complete the proof of the lemma, we now need to carry out an estimate similar to (27) showing that $\psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}$ is close enough to $\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}$ when N is large enough. Fix any κ' such that $\kappa < \kappa' < \frac{1}{4}$. We can use Proposition A.2 and (A.8) in Appendix A to find an $\varepsilon_4 = \varepsilon_4(\kappa', s_0) > 0$ such that

$$P\Big\{\|\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}\|_{\kappa'} \ge \beta^{\kappa'}_{2s_0}\Big\} < e^{-2s_0/\varepsilon^2}$$

for all $\bar{\zeta}$ in $C^{2\kappa}(S^1)$, all $\bar{\varphi}$ in C_0^κ and all $0 < \varepsilon < \varepsilon_4$. We can also partition the sample space Ω as

$$\begin{aligned}
&P\Big\{\rho_\kappa\big(\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}, \psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}\big) \ge \tilde{\delta}/3\Big\} \\
&\quad\le P\Big\{\|\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}\|_{\kappa'} \ge \beta^{\kappa'}_{2s_0}\Big\} + P\Big\{S^{\bar{\varphi}}_{\bar{\zeta}}\big(\psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}\big) \ge 2s_0\Big\} \\
&\qquad + P\Big\{\|\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}\|_{\kappa'} \le \beta^{\kappa'}_{2s_0}, \|\psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}\|_{\kappa'} \le (1 + T^{\kappa'})L_{\kappa'}M(4s_0)^{1/2}, \\
&\qquad\qquad \rho_\kappa\big(\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}, \psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}\big) \ge \tilde{\delta}/3\Big\}
\end{aligned}$$

for each $\bar{\zeta}$ in $C^{2\kappa}(S^1)$, each $\bar{\varphi}$ in C_0^κ, each $\varepsilon > 0$ and each positive integer N, where we have used the fact that $\tilde{\Phi}^{\bar{\varphi}}_{\bar{\zeta}y}(2s_0) \subset \{\varphi \in C_0^{\kappa'}\colon \|\varphi\|_{\kappa'} \le (1 + T^{\kappa'})L_{\kappa'}M(4s_0)^{1/2}\}$, the proof of which is similar to calculation (30). Set $\tilde{\delta}' = \tilde{\delta}'(\kappa', s_0, \tilde{\delta}) > 0$ to be

$$\begin{aligned}
\tilde{\delta}' := \inf\Big\{&\rho_{L^2}(\psi_1, \psi_2)\colon \|\psi_1\|_{\kappa'} \le \beta^{\kappa'}_{2s_0}, \\
&\|\psi_2\|_{\kappa'} \le (1 + T^{\kappa'})L_{\kappa'}M(4s_0)^{1/2}, \rho_\kappa(\psi_1, \psi_2) \ge \tilde{\delta}/3\Big\}.
\end{aligned}$$

Clearly $\tilde{\delta}'$ is positive (see Appendix C); then using a calculation similar to (29),

$$\begin{aligned}
P\Big\{\rho_\kappa\big(\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}, \psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}\big) \ge \tilde{\delta}/3\Big\} &< e^{-2s_0/\varepsilon^2} + (1/4)^{-N/2}e^{-3s_0/(2\varepsilon^2)} \\
&\quad + P\Big\{\rho_{L^2}\big(\psi^{\bar{\varphi},\varepsilon}_{\bar{\zeta}}, \psi^{N,\bar{\varphi},\varepsilon}_{\bar{\zeta}}\big) \ge \tilde{\delta}'\Big\}
\end{aligned} \tag{45}$$

for all $\bar{\zeta}$ in $C^{2\kappa}(S^1)$, all $\bar{\varphi}$ in C_0^κ, each positive integer N and all $0 < \varepsilon < \varepsilon_4$. Using (23), it is evident that there is an integer $N' = N'(s_0, \tilde{\delta}')$ large enough that

$$\sup_{\substack{\bar{\varphi}\in C_0^\kappa \\ \bar{\zeta}\in C^{2\kappa}(S^1)}} \frac{4s_0}{(\tilde{\delta}')^2}\lambda^{\bar{\varphi}}_{\bar{\zeta}}(N') \le \frac{4s_0}{(\tilde{\delta}')^2}\lambda^0_0(N') < \frac{1}{2}.$$

Note that since $\delta' \le \tilde{\delta}/3$, if $\rho_\kappa(\psi_{\bar{\zeta}}^{\varepsilon}, \bar{\varphi}) < \delta'$ and $\rho_\kappa(\psi_{\bar{\zeta}}^{N',\bar{\varphi},\varepsilon}, \bar{\varphi}) \ge \tilde{\delta}$, then by the triangle inequality, either $\rho_\kappa(\psi_{\bar{\zeta}}^{\varepsilon}, \psi_{\bar{\zeta}}^{\bar{\varphi},\varepsilon}) \ge \tilde{\delta}/3$ or $\rho_\kappa(\psi_{\bar{\zeta}}^{N',\bar{\varphi},\varepsilon}, \psi_{\bar{\zeta}}^{\bar{\psi},\varepsilon}) \ge \tilde{\delta}/3$. Combining (44) and (45) and using a calculation analogous to (27), then

$$
\begin{aligned}
&P\left\{\rho_\kappa\left(\psi_{\bar{\zeta}}^{\varepsilon}, \bar{\varphi}\right) < \delta', \rho_\kappa\left(\psi_{\bar{\zeta}}^{N',\bar{\varphi},\varepsilon}, \bar{\varphi}\right) \ge \tilde{\delta}\right\} \\
&\quad \le P\left\{\rho_\kappa\left(\psi_{\bar{\zeta}}^{\varepsilon}, \bar{\varphi}\right) < \delta', \rho_\kappa\left(\psi_{\bar{\zeta}}^{\varepsilon}, \psi_{\bar{\zeta}}^{\bar{\varphi},\varepsilon}\right) \ge \frac{\tilde{\delta}}{3}\right\} + P\left\{\rho_\kappa\left(\psi_{\bar{\zeta}}^{\bar{\varphi},\varepsilon}, \psi_{\bar{\zeta}}^{N',\bar{\varphi},\varepsilon}\right) \ge \frac{\tilde{\delta}}{3}\right\} \\
&\quad < e^{-3s_0/(2\varepsilon^2)}\left(2 + \left(\frac{1}{4}\right)^{-N'/2} + \exp\left(\frac{9s_0M^2T}{8(\tilde{\delta}')^2}\sum_{k=1}^{\infty}\lambda_k^{-1}\right)\right)
\end{aligned}
$$

for all $\bar{\zeta}$ in $C^{2\kappa}(S^1)$, all $\bar{\varphi}$ in C_0^κ and all $0 < \varepsilon < \min\{\varepsilon_3, \varepsilon_4\}$. Note that δ', N', $\tilde{\delta}'$, ε_3 and ε_4 depend solely on s_0, δ and γ, and on the arbitrarily chosen κ'; this gives us the lemma.

This also completes the proof of Proposition 4. □

Propositions 2–4 give us a large deviations principle for $\{\psi_\zeta^\varepsilon\}$ in C_0^κ, and, as we noted at the end of Section 3, this is sufficient to imply the large deviations principle (A.1)–(A.3). We state this as a theorem.

THEOREM 1. *For each* $0 \le \kappa < \frac{1}{4}$ *and each fixed* ζ *in* $C^{2\kappa}(S^1)$, *claims* (A.1)–(A.3) *are satisfied; that is,* v_ζ^ε *satisfies a large deviations principle in* C_ζ^κ *with action functional* S_ζ *as given by* (4).

Having given a proof of (A.1)–(A.3), we mention now that there are some other more or less equivalent approaches. There are essentially two problems to overcome. The first is to prove a large deviations principle for ψ_ζ^ε, the nearly Gaussian process [see estimate (44)], and then use the regularity of the mapping B_ζ to transfer the result back to v_ζ^ε. The other problem is to show some form of exponential tightness so that one can strengthen the large deviations principle from $L^2(I \times S^1)$, which is the natural space in which to analyze large deviations for v_ζ^ε or ψ_ζ^ε, to the finer topology of C^κ. We have here first showed the large deviations result for ψ_ζ^ε, using the exponential tightness of (A.8) to transfer from $L_0^2(I \times S^1)$ to C_0^κ, and then used Proposition 1 to transfer this result concerning ψ_ζ^ε to one concerning v_ζ^ε. The two problems could also be solved in the other order, or in fact they could be solved simultaneously, using results such as Exercise 2.1.20 of Deuschel and Stroock (1989). For a better understanding of the abstract basis for this last approach, see the paper by Ioffe (1991).

5. The uniform large deviations principle. The main result of this paper is Theorem 1. Let us now review our arguments to show that for each fixed $0 \le \kappa < \frac{1}{4}$, the large deviations principle for v_ζ^ε in C^κ holds uniformly with respect to all initial conditions ζ in $C^{2\kappa}(S^1)$. For each fixed $0 \le \kappa < \frac{1}{4}$, we

wish to prove that:

(C.1) For each $s \geq 0$ and each compact set K in $C^{2\kappa}(S^1)$, the set

$$\Phi_K(s) := \bigcup_{\zeta \in K} \Phi_\zeta(s)$$

is a compact set in C^κ, with $\Phi_\zeta(s)$ as in claim (A.1).

(C.2) For any positive numbers δ, γ and s_0, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho_\kappa(v_\zeta^\varepsilon, \varphi) < \delta\} > \exp\left(-\frac{S_\zeta(\varphi) + \gamma}{\varepsilon^2}\right)$$

for all $0 < \varepsilon < \varepsilon_0$, ζ in $C^{2\kappa}(S^1)$ and φ in $\Phi_\zeta(s_0)$.

(C.3) For any positive numbers δ, γ and s_0, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho_\kappa(v_\zeta^\varepsilon, \Phi_\zeta(s)) \geq \delta\} < \exp\left(-\frac{s - \gamma}{\varepsilon^2}\right)$$

for all $0 < \varepsilon < \varepsilon_0$, ζ in $C^{2\kappa}(s^1)$ and $0 \leq s \leq S_0$.

This stronger version of (A.1)–(A.3) is necessary, for example, in proving the large deviations principle for the invariant measure of v^ε [see Sowers (1991)]. In general, if $\{X_y^\varepsilon\}$ is a collection of random elements of some Polish space $\mathscr{X}$ with metric ρ, where $\varepsilon > 0$ and y is a parameter with values in some topological space $\mathscr{Y}$, then we say that X_y^ε satisfies a large deviations principle with action functionals I_y: $\mathscr{X} \to [0, \infty]$ *uniformly* over a class $\mathscr{A}$ of subsets of $\mathscr{Y}$ if:

(D.1) For each $s \geq 0$ and each subset K of $\tilde{A} := \bigcup_{A \in \mathscr{A}} A$ which is compact in the topology inherited by $\tilde{A}$, the set

$$\Phi_K(s) := \bigcup_{y \in K} \Phi_y(s)$$

is a compact set in $\mathscr{X}$, where $\Phi_y(s) := \{x \in \mathscr{X}: I_y(x) \leq s\}$ for each y in $\tilde{A}$.

(D.2) For any positive numbers δ, γ and s_0, and any set A in $\mathscr{A}$, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho(X_y^\varepsilon, x) < \delta\} > \exp\left(-\frac{I_y(x) + \gamma}{\varepsilon^2}\right)$$

for all $0 < \varepsilon < \varepsilon_0$, y in A and x in $\Phi_y(s_0)$.

(D.3) For any positive numbers δ, γ and s_0, and any set A in $\mathscr{A}$, there is an $\varepsilon_0 > 0$ such that

$$P\{\rho(X_y^\varepsilon, \Phi_y(s)) \geq \delta\} < \exp\left(-\frac{s - \gamma}{\varepsilon^2}\right)$$

for all $0 < \varepsilon < \varepsilon_0$, y in A and $0 \leq s \leq s_0$;

see Freidlin and Wentzell [(1984), page 92]. The large deviations principle (C.1)–(C.3) is thus uniform with respect to all sets of initial conditions.

We shall prove (C.1)–(C.3) in a manner analogous to our proof of Theorem 1. We shall first extend the contraction principle to cover uniformity with respect to a parameter; this will require some easily-verified regularity of the mappings $\{B_\zeta; \zeta \in C^{2\kappa}(S^1)\}$ and $\{\tilde{S}_\zeta; \zeta \in C^{2\kappa}(S^1)\}$ for each $0 \le \kappa < \frac{1}{4}$. The uniform large deviations principle (C.1)–(C.3) for any $0 \le \kappa < \frac{1}{4}$ will then follow if we can show that the large deviations principle for ψ_ζ^ε in C_0^κ is uniform for all ζ in $C^{2\kappa}(S^1)$. It will be a simple matter to check that Propositions 2–4 do indeed hold uniformly with respect to the parameter ζ.

To begin, we have the following version of the contraction principle.

PROPOSITION 5. *Let X_y^ε have the uniform large deviations principle* (D.1)–(D.3). *Let $\{G_y;\ y \in \tilde{A}\}$ be a collection of mappings from $\mathscr{X}$ to another Polish space $\mathscr{X}'$ with metric ρ'. Assume that*:

(i) *For each $s \ge 0$, the set $\Phi_{\tilde{A}}(s)$ is relatively compact in $\mathscr{X}$ for each $s \ge 0$ and $\lim_{y \to y^*, y \in \tilde{A}} I_y(x) = I_{y^*}(x)$ for each y^* in $\tilde{A}$ and each x in $\mathscr{X}$, this limit being uniform as x varies over $\Phi_{\tilde{A}}(s)$.*

(ii) *The family $\{G_y; y \in \tilde{A}\}$ is equicontinuous and $\lim_{y \to y^*, y \in \tilde{A}} G_y(x) = G_{y^*}(x)$ for each y^* in $\tilde{A}$ and each x in $\mathscr{X}$.*

Then $G_y(X_y^\varepsilon)$ has a large deviations principle which is uniform over $\mathscr{A}$ and with action functionals

$$I'_y(x') := \inf\{I_y(x): x \in \mathscr{X} \text{ such that } G_y(x) = x'\}, \qquad x' \in \mathscr{X}', y \in \tilde{A}. \tag{46}$$

PROOF. Note first that if y is in $\tilde{A}$ and x' is in $\mathscr{X}'$ such that $I'_y(x') < \infty$, then from the continuity of G_y and the lower semicontinuity of I_y, we are assured that there is an x in $\mathscr{X}$ such that $G_y(x) = x'$ and $I_y(x) = I'_y(x')$.

Take now a number $s \ge 0$ and a subset K of $\tilde{A}$ which is compact in the topology inherited by $\tilde{A}$. We show that $\Phi'_K(s) := \bigcup_{y \in K} \Phi'_y(s)$ is compact in $\mathscr{X}'$ where $\Phi'_y(s) := \{x' \in \mathscr{X}'\colon I'_y(x') \le s\}$ for each y in $\tilde{A}$. We shall show that $\Phi'_K(s)$ has the Bolzano–Weierstrass property. Take $\{x'_n\}$ in $\Phi'_K(s)$. By the above remarks, there is a sequence $\{y_n\}$ in K and a sequence $\{x_n\}$ in $\Phi_K(s)$ such that for each n, $G_{y_n}(x_n) = x'_n$ and $I_{y_n}(x_n) = I'_{y_n}(x'_n)$. By the compactness result for the uniform large deviations principle for X_y^ε and the compactness of K, we can extract convergent subsequences $\{x_{n'}\}$ of $\{x_n\}$ and $\{y_{n'}\}$ of $\{y_n\}$ with limit points x^* in $\Phi_K(s)$ and y^* in K. By hypothesis (ii), we can show that $\lim_{n'} x'_{n'} = \lim_{n'} G_{y_{n'}}(x_{n'}) = G_{y^*}(x^*)$ and by using hypothesis (i),

$$I_{y^*}(x^*) \le \liminf_{n'} I_{y^*}(x_{n'}) \le \liminf_{n'} I_{y_{n'}}(x_{n'}) \le s.$$

We thus see that $x'_{n'}$ converges to $G_{y^*}(x^*)$ and $G_{y^*}(x^*) \in \Phi'_{y^*}(s) \subset \Phi'_K(s)$, so $\Phi'_K(s)$ has the Bolzano–Weierstrass property and hence is compact in $\mathscr{X}'$.

To prove the lower and upper bounds on the probabilistic behavior of $\{G_y(X_y^\varepsilon)\}$, let us first prove:

LEMMA 3. *For any $\delta_1 > 0$, there is a $\delta_2 > 0$ such that $\rho'(G_y(x), G_y(z)) < \delta_1$ for all y in $\tilde{A}$, all x in $\Phi_{\tilde{A}}(s)$ and all z in $\mathscr{X}$ such that $\rho(z, x) < \delta_2$.*

This is a statement of uniform equicontinuity of the family $\{G_y; y \in \tilde{A}\}$.

PROOF. To prove the lemma, observe that by the equicontinuity of $\{G_y; y \in \tilde{A}\}$, for each x in $\mathscr{X}$ there is a $\delta_x > 0$ such that if z is in $\mathscr{X}$ with $\rho(z, x) < \delta_x$, then $\rho(G_y(z), G_y(x)) < \delta_1/2$ for all y in $\tilde{A}$. Let us cover the closure of $\Phi_{\tilde{A}}(s)$ by open spheres of the form $\{z \in \mathscr{X}\colon \rho(z, x) < \delta_x/2\}$ and extract a finite subcover with centers $\{x_i;\ x = 1, 2, \ldots, N\}$; recall by hypothesis (i) that $\Phi_{\tilde{A}}(s)$ is relatively compact. We can then define $\delta_2 := \min\{\delta_{x_i}/2;\ i = 1, 2, \ldots, N\}$. Indeed, let x be in $\Phi_{\tilde{A}}(s)$ and z be in $\mathscr{X}$ such that $\rho(z, x) < \delta_2$. We then take any x_i such that $\rho(x, x_i) < \delta_{x_i}/2 < \delta_{x_i}$ and hence we will also have by the triangle inequality that $\rho(z, x_i) < \delta_2 + \delta_{x_i}/2 < \delta_{x_i}$. Thus for all y in $\tilde{A}$, $\rho'(G_y(z), G_y(x)) \le \rho'(G_y(z), G_y(x_i)) + \rho'(G_y(x_i), G_y(x)) < \delta_1$. □

From this lemma we may easily complete the proof of Proposition 5. Take positive numbers δ, γ and s_0 and any set A in $\mathscr{A}$. Take now δ_2 as in the lemma such that $\rho'(G_y(z), G_y(x)) < \delta$ for all y in $\tilde{A}$, all x in $\Phi_{\tilde{A}}(s_0)$ and all z in $\mathscr{X}$ with $\rho(z, x) < \delta_2$. First, by using (D.2), we may find an ε_0 such that we can write

$$\begin{aligned} P\Big\{\rho'\big(G_y(X_y^\varepsilon), x'\big) < \delta\Big\} &= P\Big\{\rho'\big(G_y(X_y^\varepsilon), G_y(x_{(x',y)})\big) < \delta\Big\} \\ &\ge P\Big\{\rho\big(X_y^\varepsilon, x_{(x',y)}\big) < \delta_2\Big\} \\ &> \exp\left(-\frac{I_y(x_{(x',y)}) + \gamma}{\varepsilon^2}\right) = \exp\left(-\frac{I_y'(x') + \gamma}{\varepsilon^2}\right) \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$ and all x' in $\Phi_y'(s_0)$, with y being any element of A. Here for each x' in $\Phi_y'(s_0)$ with y in A, we denote by $x_{(x',y)}$ any element of $\Phi_y(s_0) \subset \Phi_A(s_0)$ such that $G_y(x_{(x',y)}) = x'$ and $I_y(x_{(x',y)}) = I_y'(x')$. Alternatively, we may use (D.3) to find an $\varepsilon_0 > 0$ such that we have

$$\begin{aligned} P\Big\{\rho'\big(G_y(X_y^\varepsilon), \Phi_y'(s)\big) \ge \delta\Big\} &= P\Big\{\rho'\big(G_y(X_y^\varepsilon), G_y\big(\Phi_y(s)\big)\big) \ge \delta\Big\} \\ &\le P\Big\{\rho\big(X_y^\varepsilon, \Phi_y(s)\big) \ge \delta_2\Big\} < e^{-(s-\gamma)/\varepsilon^2} \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$, all y in A and all $0 \le s \le s_0$. Here we have used the easily-proved fact that for $s \ge 0$ and y in $\tilde{A}$, $\Phi_y'(s) = G_y(\Phi_y(s))$. This completes the proof of the proposition. □

In our case, we naturally identify X_y^ε with ψ_ζ^ε and G_y with B_ζ. To carry out our arguments, let us fix a $0 \le \kappa < \frac{1}{4}$. The space $\mathscr{X}$ we take to be C_0^κ with the associated metric ρ_κ. The parameter y we take to be ζ, with the topological

space $\mathscr{Y}$ being $C^{2\kappa}(S^1)$ with the topology generated by the norm $\|\cdot\|_{2\kappa, S^1}$. We take $\mathscr{A} := \{C^{2\kappa}(S^1)\}$, that is, the class $\mathscr{A}$ has only one set in it, that set being the entire space $C^{2\kappa}(S^1)$. Of course $\tilde{S}_\zeta$ plays the role of I_y and $\mathscr{X}'$ is C^κ with the natural metric ρ_k. We have already noted that in this situation, the action functional I'_y given by (46) will be equal to S_ζ as in (4).

We first note that hypothesis (ii) of Proposition 5 is true by Proposition 1. The equicontinuity of $\{B_\zeta; \zeta \in C^{2\kappa}(S^1)\}$ follows from Proposition 1 when we take $\zeta_1 = \zeta_2$ and the continuity of $\zeta \mapsto B_\zeta(\varphi)$ for any φ in C_0^κ follows upon setting $\varphi_1 = \varphi_2$ in Proposition 1. Considering now the first hypothesis of Proposition 5, we easily see from the calculation (30) that $\tilde{\Phi}_{C^{2\kappa}(S^1)}(s)$ is contained in $\{\varphi \in C_0^{\kappa'}: \|\varphi\|_{\kappa'} \le (1 + T^{\kappa'})L_{\kappa'}M(2s)^{1/2}\}$ for any κ' with $\kappa < \kappa' < \frac{1}{4}$, so indeed $\tilde{\Phi}_{C^{2\kappa}(S^1)}(s)$ is relatively compact in C_0^κ. In a way analogous to (31), we can calculate from (12) that for any φ in $W_1^{1,2}$ such that $\varphi[0] = \mathbf{0}$ and any ζ_1 and ζ_2 in $C^{2\kappa}(S^1)$,

$$\tilde{S}_{\zeta_1}(\varphi) \le \tilde{S}_{\zeta_2}(\varphi)\big(1 + \omega_\kappa \|\zeta_1 - \zeta\|_{2\kappa, S^1}\big), \tag{47}$$

where ω_κ is as defined in (31). This easily gives the required continuity of the mappings $\zeta \mapsto S_\zeta(\varphi)$.

It now remains only to verify that the large deviations principle for ψ_ζ^ε in C_0^κ is uniform over all ζ in $C^{2\kappa}(S^1)$. We have arranged our proofs in Section 4 so that only minor changes are required.

PROPOSITION 6. *The random fields $\{\psi_\zeta^\varepsilon\}$ satisfy a large deviations principle in C_0^κ with action functionals $\tilde{S}_\zeta$, which is uniform over all ζ in $C^{2\kappa}(S^1)$.*

PROOF. We shall indicate how to modify the proofs of Section 4.

Fix a compact subset K of $C^{2\kappa}(S^1)$ and an $s \ge 0$. As we calculated before in verifying hypothesis (i) of Proposition 5, the set $\tilde{\Phi}_{C^{2\kappa}(S^1)}(s)$ is contained in $\{\varphi \in C_0^{\kappa'}: \|\varphi\|_{\kappa'} \le (1 + T^{\kappa'})L_{\kappa'}M(2s)^{1/2}\}$ for each κ' such that $\kappa < \kappa' < \frac{1}{4}$, so indeed $\tilde{\Phi}_K(s)$ must be relatively compact. To prove that it is also closed, take $\{\varphi_n\}$ in $\tilde{\Phi}_K(s)$ converging in C_0^κ to some φ. Then for each n, there is a ζ_n in K such that φ_n is in $\tilde{\Phi}_{\zeta_n}(s)$. The compactness of K ensures that $\{\zeta_n\}$ has a $C^{2\kappa}(S^1)$-limit point ζ^* in K along some subsequence $\{\zeta'_n\}$ of $\{\zeta_n\}$. Thus analogously to (32)–(33), we have from (47) that

$$\tilde{S}_{\zeta^*}(\varphi) \le \liminf_{n'} \tilde{S}_{\zeta^*}(\varphi_{n'}) \le \liminf_{n'} \tilde{S}_{\zeta_{n'}}(\varphi_{n'})\big(1 + \omega_\kappa \|\zeta^* - \zeta_{n'}\|_{2\kappa, S^1}\big) \le s$$

by using the ρ_κ-lower semicontinuity of $\varphi \mapsto \tilde{S}_{\zeta^*}(\varphi)$ as ensured by Proposition 2. Thus $\tilde{\Phi}_K(s)$ is also closed.

Now take positive numbers s_0, δ and γ and proceed as in the proof of Proposition 3. Note that the estimate (36) will hold for all ζ in $C^{2\kappa}(S^1)$ and all φ in $\tilde{\Phi}_\zeta(s_0)$. We may also check that the calculation (37) is uniform over all ζ in $C^{2\kappa}(S^1)$ and all φ in $W_2^{1,2}$ with $\varphi[0] = \mathbf{0}$:

$$\lim_\varepsilon \sup_{\substack{\zeta \in C^{2\kappa}(S^1) \\ \varphi \in W_2^{1,2}:\, \varphi[0] = \mathbf{0}}} P^\varepsilon\big\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa \ge \delta''\big\} = 0.$$

This fact in conjunction with (36) easily shows that there is an $\varepsilon_0 > 0$ such that (34) holds for all ζ in $C^{2\kappa}(S^1)$, all φ in $\tilde{\Phi}_\zeta(s_0)$ and all $0 < \varepsilon < \varepsilon_0$.

Finally, let us take positive numbers s_0, δ and γ and review the proof of Proposition 4. The essential idea of the proof was to use (39) to show that we may disregard the set $\sim K_{s_0}$ and then to cover the compact set $K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta)}(s)$ by a finite number of δ'-spheres and to use (42) on each sphere. By Proposition A.2 and (A.8) in Appendix A, (39) holds uniformly as ζ varies over $C^{2\kappa}(S^1)$ and we were careful to include uniformity in $\bar{\zeta}$ and $\bar{\varphi}$ in our statement of Lemma 2. Reviewing the arguments of (41), we find that the bound (41) holds in fact for all ζ in $C^{2\kappa}(S^1)$, all φ in $K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta/2)}(s)$, all $0 \le s \le s_0$ and all $0 < \varepsilon < \varepsilon_2$, that is, it also holds when $\rho_\kappa(\varphi, \tilde{\Phi}_\zeta(s)) \ge \delta/2$, where s is any number between zero and s_0. Thus (42) holds for all ζ in $C^{2\kappa}(S^1)$, all φ in $K_{S_0} \sim \tilde{\Phi}_\zeta^{(\delta/2)}(s)$, all $0 \le s \le s_0$ and all $0 < \varepsilon < \varepsilon_2$. The uniformity of (43) will hence result if we can show that L, the number of δ'-spheres used to cover $K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta)}(s)$, is bounded as ζ varies in $C^{2\kappa}(S^1)$ and s varies from zero to s_0. Let us cover the entire set K_{s_0} by ρ_κ-spheres of radius $\delta'' := \min\{\delta', \delta/2\}$ and extract a finite subcover $\{\varphi_i;\ i = 1, 2, \ldots, L'\}$. Now if φ is in $K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta)}(s)$ for some ζ in $C^{2\kappa}(S^1)$ and some $0 \le s \le s_0$, then there is a φ_i such that $\rho_\kappa(\varphi, \varphi_i) < \delta'' < \delta/2$ and by the triangle inequality, then $\rho_\kappa(\varphi_i, \tilde{\Phi}_\zeta(s)) \ge \delta/2$. Thus

$$K_{s_0} \sim \tilde{\Phi}_\zeta^{(\delta)}(s) \subset \bigcup_{i:\, \rho_\kappa(\varphi_i, \tilde{\Phi}_\zeta(s)) \ge \delta/2} \left\{\varphi \in C_0^\kappa:\, \rho_\kappa(\varphi, \varphi_i) < \tilde{\delta''}\right\}$$

$$\subset \bigcup_{i:\, \rho_\kappa(\varphi_i, \tilde{\Phi}_\zeta(s)) \ge \delta/2} \left\{\varphi \in C_0^\kappa:\, \rho_\kappa(\varphi, \varphi_i) < \delta'\right\}.$$

Then we use (38) and (42) to write

$$P\left\{\rho_\kappa\left(\psi_\zeta^\varepsilon, \tilde{\Phi}_\zeta(s)\right) \ge \delta\right\} \le e^{-s_0/\varepsilon^2} + \sum_{i:\, \rho_\kappa(\varphi_i \tilde{\Phi}_\zeta(s)) \ge \delta/2} P\left\{\rho_\kappa\left(\psi_\zeta^\varepsilon, \varphi_i\right) < \delta'\right\}$$

(48)

$$\le e^{-s_0/\varepsilon^2} + L'\left(e^{-s_0/\varepsilon^2} + \left(\gamma/(4s_0)\right)^{-N'/2} e^{-(s-\gamma/2)/\varepsilon^2}\right)$$

for all ζ in $C^{2\kappa}(S^1)$, all $0 \le s \le s_0$ and all $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Our arguments show that L', N', ε_1 and ε_2 depend *solely* on s_0, δ, γ and the arbitrarily chosen constant κ', and *not* on ζ in $C^{2\kappa}(S^1)$ nor on s. This allows us to use (48) to find an $\varepsilon_0 > 0$ such that (38) holds for all ζ in $C^{2\kappa}(S^1)$, all $0 \le s \le s_0$ and all $0 < \varepsilon < \varepsilon_0$. □

This completes the proof of the uniform large deviations principle for v_ζ^ε. We state the result as a theorem.

THEOREM 2. *For each* $0 \le \kappa < \frac{1}{4}$, *claims* (C.1)–(C.3) *are satisfied, that is,* v_ζ^ε *satisfies a large deviations principle in* C^κ *with action functionals* S_ζ *as in* (4), *uniformly over all initial conditions* ζ *in* $C^{2\kappa}(S^1)$.

6. Conclusion. We now have the basic large deviations result for the dynamical system (2). With this result, we can investigate many other asymptotic problems concerning v^ε [see Freidlin (1988), Freidlin and Wentzell (1984), Sections 4.2 and 4.4 and Ventsel' and Freidlin (1970)]. For instance, we can consider a domain D containing a stable point of the dynamical system $\partial_t v^0 = \mathscr{L} v^0 + f(x, v^0)$ and look for the most likely place on ∂D at which the trajectories v^ε will exit D as ε tends to zero; we can also find the asymptotic behavior of the first exit time of v^ε from D. Yet another direction is to investigate the asymptotic behavior of the invariant measure of v^ε when v^ε is considered as a function-valued Markov process—see Sowers (1991). Problems such as these have been answered for stochastic differential equations (SDE's) —see Chapter 4 of Freidlin and Wentzell (1984)—but due to the infinite-dimensionality of $C(S^1)$, many of the arguments for SDE's do not immediately apply to v_ξ^ε.

APPENDIX A

We consider here some continuity properties of solutions to stochastic PDE's of the form

$$\begin{aligned} &\partial_t \Xi = \mathscr{L}\Xi + \sigma' \dot{\dot{W}}_{tx}, \qquad (t,x) \in I \times S^1, \\ &\Xi[0] = \mathbf{0}, \end{aligned} \tag{A.1}$$

or equivalently solutions of the integral equation

$$\Xi(t,x) = \int_{I\times S^1} G_{t-s}(x,y)\sigma'(s,y)W(ds,dy), \qquad (t,x) \in I \times S^1, \tag{A.2}$$

where σ' is in $L^\infty(\Omega \times I \times S)$ such that σ' is P-a.s. continuous as a function of (t,x) in $I \times S^1$, and such that $\sigma'(t,x)$ is measurable with respect to $\mathscr{F}_t$ as given in (6) for each t in I. We shall first demonstrate a deterministic result concerning the Green's function G of (5), and then use this to estimate the continuity of Ξ solving (A.2).

Our first result is an enhanced version of Lemma 3.9 of Walsh (1984).

PROPOSITION A.1. *For $0 < \kappa < \frac{1}{4}$, there is a positive number L_κ such that for all (t,x) and (s,y) in $\mathbb{R}_+ \times S^1$,*

$$\left\{\int_{\mathbb{R}_+\times S^1} |G_{t-r}(x,z) - G_{s-r}(y,z)|^2 \, dr\, dz\right\}^{1/2} \leq L_\kappa r'((t,x),(s,y))^\kappa. \tag{A.3}$$

PROOF. We shall first prove the proper variation in the x-direction and then the proper variation in the t-direction.

Take $t > 0$ and x and y in S^1 [notice that if $t = s = 0$ in (A.3), then the left-hand side of (A.3) is zero]. Then

$$\int_{\mathbb{R}_+\times S^1} |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 \, dr\, dz = \sum_{k=0}^{\infty} \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} |\phi_k(x) - \phi_k(y)|^2.$$

Clearly for $k = 0, 1, \ldots,$

$$|\phi_k(x) - \phi_k(y)| \le \frac{2}{\pi^{1/2}}$$

and by the mean value theorem also

$$|\phi_{2k-1}(x) - \phi_{2k-1}(y)| \le \frac{kr(x,y)}{\pi^{1/2}} \le \frac{\lambda_{2k-1}^{1/2} r(x,y)}{(\pi D)^{1/2}},$$

$$|\phi_{2k}(x) - \phi_{2k}(y)| \le \frac{kr(x,y)}{\pi^{1/2}} \le \frac{\lambda_{2k}^{1/2} r(x,y)}{(\pi D)^{1/2}}.$$

We then calculate

$$\begin{aligned} &\int_{\mathbb{R}_+ \times S^1} |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 \, dr\, dz \\ &\quad \le \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} |\phi_k(x) - \phi_k(y)|^{2\kappa} |\phi_k(x) - \phi_k(y)|^{2-2\kappa} \\ &\quad = L_{1,\kappa} r(x,y)^{2\kappa}, \end{aligned} \tag{A.4}$$

where $L_{1,\kappa} := (2^{1-2\kappa}/(\pi D^{\kappa}))\sum_{k=1}^{\infty}(1/\lambda_k^{1-\kappa})$ which is finite since $2(1-\kappa) > 1$.

Now take $0 \le s < t$ and x in S^1. Then

$$\begin{aligned} &\int_{\mathbb{R}_+ \times S^1} |G_{t-r}(x,z) - G_{s-r}(x,z)|^2 \, dr\, dz \\ &\quad = \sum_{k=0}^{\infty} \int_{\mathbb{R}_+} \left(e^{-\lambda_k(t-r)}\chi\{r \le t\} - e^{-\lambda_k(s-r)}\chi\{r \le s\}\right)^2 \phi_k^2(x) \, dr \\ &\quad \le \frac{1}{\pi} \sum_{k=0}^{\infty} \int_{\mathbb{R}_+} \left(e^{-\lambda_k(t-r)}\chi\{r \le t\} - e^{-\lambda_k(s-r)}\chi\{r \le s\}\right)^2 dr \end{aligned}$$

having used the fact that for all $k = 0, 1, \ldots,$ $\max_{x \in S^1} |\phi_k(x)|^2 = 1/\pi$. Our calculation continues as

$$\begin{aligned} &\int_{\mathbb{R}_+} \left(e^{-\lambda_k(t-r)}\chi\{r \le t\} - e^{-\lambda_k(s-r)}\chi\{r \le s\}\right)^2 dr \\ &\quad = \int_0^s \left(e^{-\lambda_k(t-s+r)} - e^{-\lambda_k r}\right)^2 dr + \int_0^{t-s} e^{-2\lambda_k r} \, dr \\ &\quad \le \left(e^{-\lambda_k(t-s)} - 1\right)^2 \frac{1 - e^{-2\lambda_k s}}{2\lambda_k} + \int_0^{t-s} e^{-2\lambda_k r} \, dr \\ &\quad \le \frac{1 - e^{-\lambda_k(t-s)}}{2\lambda_k} + \int_0^{t-s} e^{-\lambda_k r} \, dr \\ &\quad = \frac{3}{2} \int_0^{t-s} e^{-\lambda_k r} \, dr. \end{aligned}$$

Set $\kappa' := 1/(1-2\kappa)$; then $1 < \kappa' < 2$ and using Jensen's inequality,

$$\int_{\mathbb{R}_+} \left(e^{-\lambda_k(t-r)}\chi\{r \le t\} - e^{-\lambda_k(s-r)}\chi\{r \le s\}\right)^2 dr \le \frac{3}{2}\left\{\left(\int_0^{t-s} e^{-\lambda_k r}\, dr\right)^{\kappa'}\right\}^{1/\kappa'}$$

$$\le |t-s|^{2\kappa}\frac{3}{2}\left(\frac{1}{\lambda_k \kappa'}\right)^{1/\kappa'}$$

and hence

$$\text{(A.5)} \qquad \int_{R_+\times S^1} |G_{t-r}(x,z) - G_{s-r}(x,z)|^2\, dr\, dz \le L_{2,\kappa}|t-s|^{2\kappa},$$

where

$$L_{2,\kappa} := \frac{3}{2\pi}\frac{1}{(\kappa')^{\kappa'}}\sum_{k=0}^{\infty}\frac{1}{(\lambda_k)^{1/\kappa'}}$$

which is finite since $2/\kappa' > 1$. We can then take $L_\kappa := L_{1,\kappa}^{1/2} + L_{2,\kappa}^{1/2}$ and complete the proof with (A.4) and (A.5). □

We use this proposition and well-known results on the continuity of random fields [see Garsia (1972), Lemma 1, Walsh (1984), Theorem 1.1 and Adler (1981), Lemma 3.3.3] to establish the following estimate of the Hölder norm of any solution Ξ of (A.2).

PROPOSITION A.2. *For each $0 < \kappa < \frac{1}{4}$, there are positive constants K_κ^0 and K_κ^1 depending only on κ such that*

$$P\{\|\Xi\|_\kappa \ge L\} \le \exp\left(-\left(L/\|\sigma'\|_{L^\infty(\Omega\times I\times S^1)}\right)^2 K_\kappa^1\right)$$

for all $L > 0$ such that $L/\|\sigma'\|_{L^\infty(\Omega\times I\times S^1)} \ge K_\kappa^0$.

PROOF. Fixing a $0 \le \kappa < \frac{1}{4}$, define $\kappa' := (\kappa + \frac{1}{4})/2$ as the midpoint between κ and $\frac{1}{4}$. Since $\Xi[0] = \mathbf{0}$, Ξ will be in C_0^κ if and only if $[\Xi]_\kappa$ is finite, in which case we will have $\|\Xi\|_\kappa \le (1 + T^\kappa)[\Xi]_\kappa$, so we can restrict our interest to $[\Xi]_\kappa$. For convenience, let us define $\sigma'_\infty := \|\sigma'\|_{L^\infty(\Omega_B\times I\times S^1)}$ and normalize Ξ by setting $\tilde{\Xi} := \Xi/\sigma'_\infty$. Then

$$\tilde{\Xi}(t,x) = \int_{I\times S^1} G_{t-s}(x,y)\frac{\sigma'(s,y)}{\sigma'_\infty}W(ds,dy).$$

We shall proceed in a manner similar to Walsh [(1984), Corollary 1.3]. Define $p(u) := L_{\kappa'}2^{\kappa'/2}u^{\kappa'}$ for $u \ge 0$; then by Proposition A.1, we have that

$$\text{(A.6)} \qquad \left\{E|\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)|^2\right\}^{1/2} \le p\left(r'((t,x),(s,y))/2^{1/2}\right)$$

for all (t,x) and (s,y) in $I \times S^1$. Also define the function $\Psi(x) := \exp(x^2/4)$

for all x in $\mathbb{R}$ and denote

$$B := \int_{I\times S^1}\int_{I\times S^1} \Psi\left(\frac{\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)}{p(r'((t,x),(s,y))/2^{1/2})}\right) dt\,dx\,ds\,dy.$$

Consider now $E[B]$ and use Tonelli's theorem to interchange the expectation integral and the Lebesgue integrals; thus our interest turns to

$$E\left[\Psi\left(\frac{\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)}{p(r'((t,x),(s,y))/2^{1/2})}\right)\right]$$

for a fixed (t,x) and (s,y) in $I\times S^1$. If $\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)$ were Gaussian, this expectation would be less than or equal to $2^{1/2}$ in view of (A.6); since $\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)$ is not in general Gaussian, we must adopt another approach. We shall define

$$f(r,z) := \frac{G_{t-r}(x,z) - G_{s-r}(y,z)}{p(r'((t,x),(s,y))/2^{1/2})}\,\frac{\sigma'(r,z)}{\sigma'_\infty}, \qquad (r,z)\in I\times S^1,$$

and let $\tilde{M}_u$ be the right-continuous $\mathcal{F}_t$-martingale

$$\tilde{M}_u := \int_0^u\int_{S^1} f(r,z)W(dr,dz)$$

with associated quadratic variation

$$\langle\tilde{M}\rangle_u := \int_0^u\int_{S^1}(f(r,z))^2\,dr\,dz, \qquad u\geq 0.$$

By standard results, there then is a Brownian motion B, perhaps on an augmented probability triple, such that $\tilde{M}_u = B_{\langle\tilde{M}\rangle_u}$ for all $u\geq 0$. We now can use Proposition A.1 to see that $\sup_{u\geq 0}\langle\tilde{M}\rangle_u \leq 1$ P-a.s., so as a result of well-known properties of Brownian motion [Karatzas and Shreve [(1987), Problem 8.2],

$$\begin{aligned} E\left[\Psi\left(\frac{\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)}{p(r'((t,x),(s,y))/2^{1/2})}\right)\right] &\leq E\left[\Psi\left(\frac{\tilde{M}_\infty^2}{4}\right)\right] \\ &\leq E\left[\exp\left\{\frac{1}{4}\left(\max_{0\leq u\leq 1}|B_u|\right)^2\right\}\right] \\ &= 2^{1/2}. \end{aligned}$$

We consequently know that $E[B]\leq (2\pi T)^2 2^{1/2}$; the same bound, in fact, as if $\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)$ were Gaussian.

We next apply Garsia [(1972), Lemma 1], appropriately adapted to functions on $I\times S^1$. The continuity of the realizations of $\tilde{\Xi}$ implies that P-a.s. for all

(t, x) and (s, y) in $I \times S^1$,

$$|\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)| \le 8\int_0^{r'((t,x),(s,y))} \Psi^{-1}\left(\frac{B}{u^4}\right) dp(u).$$

The last integral can be bounded as

$$\begin{aligned}
&\int_0^{r'((t,x),(s,y))} \Psi^{-1}\left(\frac{B}{u^4}\right) dp(u) \\
&\quad \le 2^{3/2}\int_0^{r'((t,x),(s,y))} \left((\ln_+ B)^{1/2} + \left(\ln_+ u^{-4}\right)^{1/2}\right) dp(u) \\
&\quad = 2^{3/2}(\ln_+ B)^{1/2} p(r'((t,x),(s,y))) \\
&\qquad + 2^{3/2}\int_0^{r'((t,x),(s,y))} \left(\ln_+ u^{-4}\right)^{1/2} dp(u),
\end{aligned}$$

where $\ln_+ z := \max\{\ln z, 0\}$ for $z \ge 0$. Observe now that by standard methods,

$$\tilde{\eta}_1 := \sup_{u>0} \frac{\left(\ln_+ u^{-4}\right)^{1/2}}{u^{\kappa/2-1/8}}$$

is finite. Some straightforward manipulation then reveals that

$$|\tilde{\Xi}(t,x) - \tilde{\Xi}(s,y)| \le \tilde{\eta}_2\left((\ln_+ B)^{1/2} + \tilde{\eta}_3\right) r'((t,x),(s,y))^{\kappa}$$

P-a.s. for all (t, x) and (s, y) in $I \times S^1$, where $\tilde{\eta}_2 := 2^{(\kappa'+3)/2} L_{\kappa'} d^{\kappa'-\kappa}$ and $\tilde{\eta}_3 := \tilde{\eta}_1 d^{\kappa-\kappa'}\kappa'/\kappa$, with $d := \{\pi^2 + T^2\}^{1/2}$. Thus, P-a.s.,

$$[\Xi]_\kappa = \sigma_\infty'[\tilde{\Xi}]_\kappa \le \sigma_\infty'\tilde{\eta}_2\left((\ln_+ B)^{1/2} + \tilde{\eta}_3\right).$$

Using Chebyshev's exponential inequality, then

$$\begin{aligned}
P\{\|\Xi\|_\kappa \ge L\} &\le P\left\{[\Xi]_\kappa \ge \frac{L}{1+T^\kappa}\right\} \le P\left\{(\ln_+ B)^{1/2} \ge \frac{(L/\sigma_\infty')}{\tilde{\eta}_2(1+T^\kappa)} - \tilde{\eta}_3\right\} \\
&\le \left\{(2\pi T)^2 2^{1/2} + 1\right\} \exp\left(-\left(\frac{(L/\sigma_\infty')}{\tilde{\eta}_2(1+T^\kappa)} - \tilde{\eta}_3\right)^2\right),
\end{aligned} \tag{A.7}$$

where we have used the calculation

$$E[\exp(\ln_+ B)] \le E[\exp(\ln B) + \chi\{B < 1\}] \le (2\pi T)^2 2^{1/2} + 1.$$

Defining

$$K_\kappa^0 := \max\left\{2\tilde{\eta}_2\tilde{\eta}_3(1+T^\kappa), 8^{1/2}\tilde{\eta}_2(1+T^\kappa)\left(\ln\left\{(2\pi T)^2 2^{1/2} + 1\right\}\right)^{1/2}\right\}$$

and $K_\kappa^1 := (8\tilde{\eta}_2^2(1+T^\kappa))^{-1}$ gives us the stated result, for from (A.7), then if

$$L/\sigma'_\infty \geq K^0_\kappa,$$

$$P\{\|\Xi\|_\kappa \geq L\} \leq \exp\left\{ -\left(\frac{L/\sigma'_\infty}{\tilde{\eta}_2(1+T^\kappa)} \right)^2 \left(\left(1 - \frac{\tilde{\eta}_2\tilde{\eta}_3(1+T^\kappa)}{(L/\sigma'_\infty)} \right)^2 - \frac{\tilde{\eta}_2^2(1+T^\kappa)^2 \ln\{(2\pi T)^2 2^{1/2}+1\}}{(L/\sigma'_\infty)^2} \right) \right\}$$

$$\leq \exp\left(-\frac{(L/\sigma'_\infty)^2}{\tilde{\eta}_2^2(1+T^\kappa)^2} \left(\left(1-\frac{1}{2}\right)^2 - \frac{1}{8} \right) \right)$$

$$\leq \exp\left(-\frac{1}{8\tilde{\eta}_2^2(1+T^\kappa)^2} \left(\frac{L}{\sigma'_\infty} \right)^2 \right),$$

which is the desired estimate. □

Note that the result of Garsia (1972) in fact only gives us Hölder continuity up to a (t, x)-set of Lebesgue measure zero. By defining the solution of (A.1) in the weak sense, however, as in (3), we can alter the solution on this null set so as to have a Hölder-continuous solution of the SPDE (A.1).

We can now apply Proposition A.2 to the processes (11), (16) and (35). Let us fix a $0 < \kappa < \frac{1}{4}$ and for each $L > 0$, define $\beta_L^\kappa := M(L/K_\kappa^1)^{1/2}$ and $\varepsilon_L^\kappa := \beta_L^\kappa/(MK_\kappa^0)$. Then whenever $0 < \varepsilon < \varepsilon_L^\kappa$,

$$\begin{aligned} &P^\varepsilon\{\|Z_\zeta^{\varphi,\varepsilon}\|_\kappa \geq \beta_L^\kappa\} < e^{-L/\varepsilon^2}, \qquad P\{\|\psi_\zeta^\varepsilon\|_\kappa \geq \beta_L^\kappa\} < e^{-L/\varepsilon^2}, \\ &P\{\|\psi_\zeta^{\varphi,\varepsilon}\|_\kappa \geq \beta_L^\kappa\} < e^{-L/\varepsilon^2}. \end{aligned} \tag{A.8}$$

We can also easily find, for any fixed $0 \leq \kappa < \frac{1}{4}$, the $\delta' > 0$ and $\varepsilon_3 > 0$ such that (44) holds for all $\bar{\zeta}$ in $C^{2\kappa}(S^1)$ and all $\bar{\varphi}$ in C_0^κ when $0 < \varepsilon < \varepsilon_3$. Indeed, if $0 < \kappa < \frac{1}{4}$, define

$$\delta' := \min\{\tilde{\delta}/3, \tilde{\delta}/(3\bar{\sigma}\eta_0)(K_\kappa^1/(2s_0))^{1/2}\} \quad \text{and} \quad \varepsilon_3 := \tilde{\delta}/(3\bar{\sigma}\eta_0 K_\kappa^0 \delta'),$$

where η_0 is as given in Proposition 1. By the definition of stochastic integration it is not hard to check that

$$P\{\rho_\kappa(\psi_\zeta^\varepsilon, \varphi) < \delta', \rho_\kappa(\psi_\zeta^\varepsilon, \psi_\zeta^{\varphi,\varepsilon}) \geq \tilde{\delta}/3\} = P\{\rho_\kappa(\psi_\zeta^\varepsilon, \varphi) < \delta', \|\bar{\Xi}^\varepsilon\|_\kappa \geq \tilde{\delta}/3\},$$

where

$$\bar{\Xi}^\varepsilon(t,x) := \varepsilon \int_{I\times S^1} G_{t-s}(x,y)\big(\sigma(y,(B_\zeta\psi_\zeta^\varepsilon)(s,y)) - \sigma(y, B_\zeta\varphi)(s,y)\big)\chi\{\|\psi_\zeta^\varepsilon - \varphi\|_0^t < \delta'\} W(ds,dy), \quad (t,x) \in I \times S^1,$$

with $\|\psi_\zeta^\varepsilon - \varphi\|_0^t := \sup_{(s,y)\in[0,t]\times S^1} |\psi_\zeta^\varepsilon(s,y) - \varphi(s,y)|$ for each $0 \leq t \leq T$. But

by Proposition A.2 and using an obvious refinement of (13)–(14), we have that

$$P\left\{\rho_\kappa(\psi_\zeta^\varepsilon,\varphi)<\delta',\rho_\kappa(\psi_\zeta^\varepsilon,\psi_\zeta^{\varphi,\varepsilon})\geq\frac{\tilde\delta}{3}\right\}\leq P\left\{\|\bar{\bar\Xi}^\varepsilon\|_\kappa\geq\frac{\tilde\delta}{3}\right\}$$

$$\leq\exp\left(-K_\kappa^1\frac{(\tilde\delta/3)^2}{\varepsilon^2\bar\sigma^2\eta_0^2}\right)<e^{-2s_0/\varepsilon^2} \tag{A.9}$$

when $0<\varepsilon<\varepsilon_3$. If $\kappa=0$, we can simply note that $\|\psi_\zeta^\varepsilon-\psi_\zeta^{\varphi,\varepsilon}\|_C\leq\|\psi_\zeta^\varepsilon-\psi_\zeta^{\varphi,\varepsilon}\|_{1/8}$ and set

$$\delta':=\min\left\{\tilde\delta/3,\tilde\delta/(3\bar\sigma\eta_0)\left(K_{1/8}^1/(2s_0)\right)^{1/2}\right\}\quad\text{and}\quad\varepsilon_3:=\tilde\delta/\left(3\bar\sigma\eta_0K_{1/8}^0\delta'\right)$$

and proceed as in (A.9) to get that

$$P\left\{\rho_0(\psi_\zeta^\varepsilon,\varphi)<\delta',\rho_0(\psi_\zeta^\varepsilon,\psi_\zeta^{\varphi,\varepsilon})\geq\tilde\delta/3\right\}\leq P\left\{\|\bar{\bar\Xi}^\varepsilon\|_{1/8}\geq\tilde\delta/3\right\}<e^{-2s_0/\varepsilon^2} \tag{A.10}$$

for $0<\varepsilon<\varepsilon_3$.

APPENDIX B

In this appendix we estimate the continuity of $\mathscr{T}_\zeta$, where ζ is in $C^{2\kappa}(S^1)$ for some $0\leq\kappa<\frac14$. Our goal is to show that for each $0\leq\kappa<\frac14$, there is a constant $\eta_\kappa'>0$ such that $\|\mathscr{T}_\zeta\|_\kappa\leq\eta_\kappa'\|\zeta\|_{2\kappa,S^1}$ for all ζ in $C^{2\kappa}(S^1)$, that is, that $\mathscr{T}$ is a bounded linear operator from $C^{2\kappa}(S^1)$ to C^κ. These estimates were necessary in (13) and (15) in Proposition 1.

Let us fix a $0\leq\kappa<\frac14$ and a ζ in $C^{2\kappa}(S^1)$. Let us first extend $\mathscr{T}_\zeta$ to $I\times\mathbb{R}$ by defining $u(t,x):=\mathscr{T}_\zeta(t,e^{ix})$ for all (t,x) in $I\times\mathbb{R}$. Then u satisfies the PDE

$$\partial_t u=Du_{xx}-\alpha u,\qquad(t,x)\in I\times\mathbb{R},$$
$$u(0,x)=\zeta(e^{ix}),$$

where differentiation in $\mathbb{R}$ is understood in the normal sense. We can now use the heat kernel in writing

$$u(t,x)=e^{-\alpha t}\int_{\mathbb{R}}\frac{e^{-(y-x)^2/(4Dt)}}{(4\pi Dt)^{1/2}}\zeta(e^{iy})\,dy,\qquad(t,x)\in I\times\mathbb{R}. \tag{B.1}$$

For any (t,x) in $I\times\mathbb{R}$,

$$|\mathscr{T}_\zeta(t,e^{ix})|=|u(t,x)|\leq\|\zeta\|_{C(S^1)}, \tag{B.2}$$

so we may take $\eta_0'=1$. If $\kappa>0$, for any x and y in $\mathbb{R}$ and any t in I,

$$|u(t,x)-u(s,y)|\leq\int_{\mathbb{R}}\frac{e^{-z^2/(4Dt)}}{(4\pi Dt)^{1/2}}|\zeta(e^{i(x+z)})-\zeta(e^{i(y+z)})|\,dz \tag{B.3}$$

$$\leq\|\zeta\|_{2\kappa,S^1}\left(r(e^{ix},e^{iy})\right)^{2\kappa}$$

after changing the variable of integration in (B.1). On the other hand, for any

$0 \le s \le t \le T$ and any x in $\mathbb{R}$, the semigroup property gives us that

$$\begin{aligned}
&|u(t,x) - u(s,x)| \\
&\quad \le |e^{-\alpha(t-s)} - 1| \int_{\mathbb{R}} \frac{e^{-(y-x)^2/(4D(t-s))}}{(4\pi D(t-s))^{1/2}} |u(s,y)|\, dy \\
&\qquad + \int_{\mathbb{R}} \frac{e^{-z^2/2}}{(2\pi)^{1/2}} |u\big(s, x + (2D(t-s))^{1/2} z\big) - u(s,x)|\, dz \\
&\quad \le \alpha |t-s|\, \|\zeta\|_{C(S^1)} + (2D)^{\kappa} \int_{\mathbb{R}} \frac{e^{-z^2/2}}{(2\pi)^{1/2}} z^{2\kappa}\, dz |t-s|^{\kappa} \|\zeta\|_{2\kappa, S^1}
\end{aligned}$$

by using (B.2) and (B.3) and another rearrangement of (B.1). By the triangle inequality, then

$$\begin{aligned}
|\mathscr{T}_\zeta(t,x) - \mathscr{T}_\zeta(s,y)| \le &\left(\pi^{\kappa} + \alpha T^{1-\kappa} + (2D)^{\kappa} \int_{\mathbb{R}} \frac{e^{-z^2/2}}{(2\pi)^{1/2}} z^{2\kappa}\, dz \right) \\
&\times \|\zeta\|_{2\kappa, S^1} \big(r'((t,x),(s,y)) \big)^{\kappa}
\end{aligned}$$

for all (t,x) and (s,y) in $I \times S^1$. This and (B.2) complete the proof when $\kappa > 0$.

APPENDIX C

In this final appendix, we carry out some simple calculations to demonstrate that if $0 \le \kappa \le \kappa' < \frac{1}{4}$, then for any $B > 0$, the ρ_{L^2} topology is equivalent to the ρ_κ topology on

$$\text{Sphere}^{\kappa'}(B) := \{\varphi \in C_0^{\kappa'} : \|\varphi\|_{\kappa'} \le B\}.$$

We used this in defining δ'' in the proof of Proposition 3 and $\tilde{\delta}'$ in the proof of Lemma 2. In particular, we used the fact that on $\text{Sphere}^{\kappa'}(B)$, the Hölder norm of exponent κ is continuous at $\mathbf{0}$ in the ρ_{L^2} topology.

Of course the ρ_{L^2} topology is always contained in the ρ_κ topology; only the other inclusion is interesting. To prove the other direction, let us fix a φ_1 and φ_2 in $\text{Sphere}^{\kappa'}(B)$ such that $\|\varphi\|_{L^2}^{1/2} < \min\{\pi, T\}$, where $\varphi := \varphi_1 - \varphi_2$. Take any (t^*, x^*) in $I \times S^1$ such that $|\varphi(t^*, x^*)| = \|\varphi\|_C$ and define the set

$$A := \big\{(t,x) \in I \times S^1 : |t - t^*| \le \|\varphi\|_{L^2}^{1/2}, r(x, x^*) \le \|\varphi\|_{L^2}^{1/2}\big\}.$$

Then letting v be Lebesgue measure on $I \times S^1$, we have

$$\begin{aligned}
\|\varphi\|_C (v(A))^{1/2} &= \left\{ \int_A |\varphi(t^*, x^*)|^2\, dt\, dx \right\}^{1/2} \\
&\le \|\varphi\|_{L^2} + (2\|\varphi\|_{L^2})^{\kappa'/2} (2B) (v(A))^{1/2}.
\end{aligned}$$

It is easy to argue that $v(A) \geq 2\|\varphi\|_{L^2}$, so that we have the bound

$$\|\varphi_1 - \varphi_2\|_C \leq 2^{-1/2}\|\varphi_1 - \varphi_2\|_{L^2}^{1/2} + 2^{\kappa'/2+1}B\|\varphi_1 - \varphi_2\|_{L^2}^{\kappa'/2}.$$

Hence any ρ_{L^2}-sphere in $\text{Sphere}^{\kappa'}(B)$ of radius $\delta < \min\{\pi, T\}$ is contained in a ρ_0-sphere of radius $2^{-1/2}\delta^{1/2} + 2^{\kappa'/2+1}B\delta^{\kappa'/2}$, so at least we know that the supremum-norm topology of $\text{Sphere}^{\kappa'}(B)$ is contained in the ρ_{L^2} topology of $\text{Sphere}^{\kappa'}(B)$. This completes the proof when $\kappa = 0$. When $\kappa > 0$, we may recall standard interpolation results to see that

$$\|\varphi_1 - \varphi_2\|_\kappa \leq (1 + T^\kappa)[\varphi_1 - \varphi_2]_\kappa \leq (1 + T^\kappa)(2B)^{\kappa/\kappa'}\left(2\|\varphi_1 - \varphi_2\|_C\right)^{1-\kappa/\kappa'}.$$

Thus for $\kappa > 0$, the Hölder-κ topology of $\text{Sphere}^{\kappa'}(B)$ is contained in the supremum-norm topology of $\text{Sphere}^{\kappa'}(B)$, which, as we have seen, is in turn contained in the ρ_{L^2} topology of $\text{Sphere}^{\kappa'}(B)$, concluding our analysis when $\kappa > 0$.

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