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# THE BEHAVIOR OF SUPERPROCESSES NEAR EXTINCTION

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In this paper we use a martingale problem characterization to study the behavior of finite measure valued superprocesses with a variety of spatial motions. In general the superprocess, when normalized to be a probability, will converge to a point mass at its extinction time. For some spatial motions we prove that there are times near extinction at which the closed support of the process is concentrated near one point.

We obtain a Tanaka formula for the measure of a half space under a one dimensional symmetric stable superprocess of index $\alpha$ and we show this process fails to be a semimartingale if $1 < \alpha \le 2$.

**1. Introduction and statement of results.** The measure valued processes studied here arise as high density limits of branching particle systems [see Watanabe (1968)]. We shall study them using their characterization as solutions to a martingale problem [see Roelly-Coppoletta (1986)]. Let $E$ be a locally compact separable metric space and $M_F(E)$ be the space of finite measures on $E$ with the topology of weak convergence. Let $(A, D(A))$ be the generator of a conservative Feller process with values in $E$. This Feller process governs the spatial motion of the particles in the approximating particle system for a superprocess. Write $(C(M_F(E)), (X_t\colon t \ge 0), \mathscr{F}_t^X)$ for the space of continuous $M_F(E)$ valued paths, the coordinate process and the canonical completed right continuous filtration. We write $m(f)$ for the integral of a measurable function $f$ against a measure $m$. For any $m \in M_F(E)$ there is a unique law $Q^m$ on $\mathscr{F} = \vee_{t \ge 0} \mathscr{F}_t^X$ such that the coordinate process satisfies the martingale problem

$$X_0 = m,$$

$$(1) \qquad X_t(f) = m(f) + \int_0^t X_s(Af)\,ds + M_t(f) \quad \text{for all } f \in D(A),$$

$$M_t(f) \text{ is a continuous } \mathscr{F}_t^X \text{ martingale s.t. } \langle M(f)\rangle_t = \int_0^t X_s(f^2)\,ds.$$

The family $(Q^m\colon m \in M_F(E))$ is a strong Markov family.

Taking $f \equiv 1$ in the martingale problem we see that the total mass process $X_t(1)$ satisfies (on an enlarged probability space)

$$(2) \qquad X_t(1) = m(1) + \int_0^t (X_s(1))^{1/2}\,dB_s$$

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for some Brownian motion ($B_t$: $t \geq 0$). It follows that the extinction time $\xi = \inf\{t \geq 0: X_t(1) = 0\}$ is almost surely finite and that $X_t(1) = 0$ for $t \geq \xi$ [see Ikeda and Watanabe (1981), page 221]. The following result shows that as $t \to \xi$, what mass remains is concentrated near a single point.

THEOREM 1. *For $m \in M_F$ there exists an $E$ valued random variable $F$ such that, with probability* 1,

$$X_t/X_t(1) \to \delta_F \quad \text{as } t \to \xi, \tag{3}$$

*where the convergence is weak convergence of measures. The law of $F$ given the history of the total mass process $\mathscr{H} = \sigma(X_t(1): t \geq 0)$ satisfies*

$$E^m(f(F)|\mathscr{H}) = 1/m(1)\int_E T_\xi f\,dm. \tag{4}$$

Equation (4) implies that the law of $F$ can be constructed as follows. Position a particle in $E$ at random according to the measure $m(\cdot)/m(1)$. Let the particle move according to the underlying spatial motion but independently of the process. Stop the particle at time $\xi$. The final position of the particle will have law $F$.

Although near extinction the mass is concentrated near a single point, the closed support of the measure $S(X_t)$ will typically be spread throughout space. The closed support of a superprocess $X_t$ at a fixed time has been studied in Perkins (1990) and Evans and Perkins (1989). If the spatial motion is a Lévy process on $\mathbb{R}^d$ with Lévy measure $\mu$, then in Evans and Perkins (1989), Theorem 5.1, it is shown that for any $t > 0$,

$$\bigcup_{k=1}^{\infty} S(\mu^{*k} * X_t) \subseteq S(X_t), \qquad P^m\text{-a.s.},$$

where $\mu^{*k}$ is the $k$th fold convolution of $\mu$ with itself. For a symmetric stable superprocess this implies

$$S(X_t) = \varnothing \text{ or } \mathbb{R}^d, \qquad P^m\text{-a.s. } \forall\, t > 0. \tag{5}$$

Similar results for certain Feller processes are obtained. Let $C_0$ be the space of continuous functions on $E$ vanishing at infinity. Consider a Markov jump process with bounded generator $A$ so that

$$Af(x) = \rho\int_E \mu(x, dy)(f(y) - f(x)) \tag{6}$$

with $\rho > 0$ and $\mu$ a probability kernel such that $x \to \int\mu(x, dy)f(y) \in C_0(E)$ for all $f \in C_0(E)$. Then for $t > 0$,

$$\bigcup_{k=1}^{\infty} S\left(\int \cdots \int X_t(dx_1)\mu(x_1, dx_2) \cdots \mu(x_k, \cdot)\right) \subseteq S(X_t), \qquad P^m\text{-a.s.} \tag{7}$$

We shall show that (5) and (7) are far from being sample path properties and that near the time of death there will be exceptional times at which the support is concentrated arbitrarily close to the death point.

We start by examining the case where the spatial motion is a Markov jump process as described above. Note that $Af$ is well defined by (6) for any bounded measurable $f$. A monotone class argument shows that for any bounded measurable $f$ the process $X_t(f)$ is continuous and satisfies the semimartingale decomposition given by the martingale problem (1).

THEOREM 2. *Let $X_t$ be a superprocess over a spatial motion with bounded generator of the form* (6). *For all $\varepsilon > 0$, with probability* 1, *there exist $t_n \uparrow \xi$ such that*

$$S(X_{t_n}) \subseteq B(F, \varepsilon).$$

In Section 2 we give the proofs of Theorems 1 and 2 and some examples. The proofs are almost entirely derived from the martingale problem (1).

The behavior described in Theorem 2 holds true for super Brownian motion. This follows since the radius of the support for super Brownian motion shrinks to zero at death [see Liu and Mueller (1989)]. The restriction in Theorem 2 to superprocesses whose spatial motion has a bounded generator allows us to use the semimartingale decomposition for the measure of a Borel set $X_t(C)$ given by the martingale problem. In general the function $\mathbf{I}_C(x)$ will not be in the domain $D(A)$. Using the decompositions obtained in Section 3 we are able to extend the result in Theorem 2.

COROLLARY 3. *If $X_t$ is a one dimensional symmetric stable superprocess of index $\alpha < \frac{1}{2}$, then the conclusions of Theorem* 2 *hold.*

In Section 3 we look for decompositions for processes $X_t(f)$, where $f$ is not in the domain $D(A)$. We specialize to superprocesses over symmetric stable motions on $\mathbb{R}$ with generators $A^\alpha$, $0 < \alpha \leq 2$. We obtain decompositions for processes $X_t(f)$, where $f$ are certain functions of bounded variation on $\mathbb{R}$. The basis is the following Tanaka-like formula. Let $H_a = \{x\colon x \geq a\}$.

THEOREM 4. *Let $m \in M_F$ and $X_t$ be a symmetric stable superprocess of index $\alpha$ started at $m$. Then*

$$X_t(H_a) = m(H_a) + V_t(a) + M_t(H_a),$$

*where $M_t(H_a)$ is a continuous $L^2$ martingale satisfying $\langle M(H_a)\rangle_t = \int_0^t X_s(H_a)\,ds$ and $V_t(a)$ is jointly continuous on $(0,\infty) \times \mathbb{R}$.*

*If $0 < \alpha < 1$ and $m$ has a bounded density, then $X_t(H_a)$ is a semimartingale and $V_t(a)$ has integrable variation on $[0, T]$ for any $T > 0$.*

*If $1 < \alpha \leq 2$ and $m$ has a bounded density, then $V_t(a)$ has integrable $\phi(\alpha)$ variation on $[0, T]$, where $\phi(\alpha) = 2\alpha/(\alpha + 1)$. If, in addition, the density $u(0, x)$ is uniformly Hölder continuous and satisfies $u(0, a) > 0$, then, with* *probability* 1, $V_t(a)$ *has strictly positive* $\phi(\alpha)$ *variation on* $[0, T]$ *and hence* $X_t(H_a)$ *fails to be a semimartingale.*

COROLLARY 5. *Let f have finite total variation and be constant outside a compact interval. Then there is a decomposition*

$$X_t(f) = m(f) + \int_{\mathbb{R}} V_t(a)\, df(a) + M_t(f), \tag{8}$$

*where* $M_t(f)$ *is a continuous martingale satisfying* $\langle M(f)\rangle_t = \int_0^t X_s(f^2)\, ds$ *and all terms are continuous. If we assume that m has a bounded density, then for* $0 < \alpha < 1$, $X_t(f)$ *is a semimartingale and for* $1 < \alpha \le 2$, $X_t(f)$ *is the sum of a continuous* $L^2$ *martingale and a process of integrable* $\phi(\alpha)$ *variation on* $[0,k]$ *for any* $k > 0$.

*If m also has compact support and* $\alpha = 2$, *then for any function f of locally bounded variation the decomposition* (8) *holds where* $M_t(f)$ *is a local martingale and* $\int V_t(a)\, df(a)$ *has locally finite* $\frac{4}{3}$ *variation.*

**2. Extinction behavior.** The proofs of Theorems 1 and 2 use a time change and renormalization first used by Konno and Shiga. Define

$$C_t = \int_0^t 1/X_s(1)\, ds.$$

In Konno and Shiga (1988), Theorem 2.1, it is shown that, with probability 1, $C_t$ is a homeomorphism between $[0, \xi)$ and $[0, \infty)$. Let $D_t$: $[0, \infty) \to [0, \xi)$ be the continuous strictly increasing inverse to $C_t$. Shiga (1988) uses $D_t$ as a time change together with a renormalization to convert a class of measure valued processes into a class of probability valued processes. The superprocesses studied here do not seem to fall directly into his context. However, the time change will still be useful. By stretching out the interval $[0, \xi)$ into $[0, \infty)$ we can use the behavior at infinity of the time changed process to give information about $X_t$ before death.

For $t \in [0, \infty)$ define

$$\tilde{Y}_t = X_{D_t},$$
$$Y_t = \tilde{Y}_t / \tilde{Y}_t(1),$$
$$\mathscr{G}_t = \mathscr{F}_{D_t}.$$

Note that $\{Y_t\colon t \ge 0\}$ is a probability valued process. We derive the martingale problem for $Y_t$. For $f \in D(A)$,

$$\begin{aligned}\tilde{Y}_t(f) &= m(f) + \int_0^{D_t} X_s(Af)\, ds + M_{D_t}(f)\\ &= m(f) + \int_0^t \tilde{Y}_s(Af)\tilde{Y}_s(1)\, ds + \tilde{N}_t(f),\end{aligned}$$

where, since $D_t$ is a continuous time change, $\tilde{N}_t(f)$ is a continuous $\mathscr{G}_t$ local martingale satisfying

$$\langle \tilde{N}(f)\rangle_t = \int_0^{D_t} X_s(f^2)\,ds$$

$$= \int_0^t \tilde{Y}_s(f^2)\tilde{Y}_s(1)\,ds.$$

In particular,

$$\tilde{Y}_t(1) = m(1) + \tilde{N}_t(1),$$

$$\langle \tilde{N}(1)\rangle_t = \int_0^t \left(\tilde{Y}_s(1)\right)^2 ds,$$

$$\langle \tilde{N}(f), \tilde{N}(1)\rangle_t = \int_0^t \tilde{Y}_s(f)\tilde{Y}_s(1)\,ds.$$

Applying Itô's formula and noting that $\tilde{Y}_t(1) > 0$ for all $t > 0$ we have

$$Y_t(f) = m(f) + \int_0^t \tilde{Y}_s(Af)\,ds + N_t(f), \tag{9}$$

where $N_t(f)$ is a continuous $\mathscr{G}_t$ local martingale satisfying

$$\langle N(f)\rangle_t = \int_0^t Y_s(f^2) - (Y_s(f))^2\,ds.$$

The martingale problem for $Y_t$ is close to that for the probability valued diffusion known as the Fleming–Viot process [where the drift term in (9) would be replaced by $\int Y_s(Af)\,ds$]. In Konno and Shiga (1988) this "connection" between the martingale problems is used to derive the existence of a continuous density for the Fleming–Viot process in dimension 1 from that for super Brownian motion.

PROOF OF THEOREM 1. First assume $E$ is compact. Take $f \in D(A)$:

$$\left|\int_0^t \tilde{Y}_s(Af)\,ds\right| \le \|Af\|\int_0^t \tilde{Y}_s(1)\,ds$$

$$= \|Af\|\int_0^t X_{D_s}(1)\,ds$$

$$= \|Af\|D_t < \|Af\|\xi.$$

So

$$N_t(f) \ge -m(f) - \|Af\|\xi.$$

For any continuous local martingale $(M_t;\ t \ge 0)$, with probability 1, either $M_t$ converges to a finite limit or $\limsup M_t = -\liminf M_t = \infty$ [see Rogers and Williams (1987), Corollary 4.34.13]. So $N_t(f)$ converges as $t \to \infty$ to a finite limit. Also

$$\left|\int_s^t \tilde{Y}_r(Af)\,dr\right| \le \|Af\|(D_t - D_s) \to 0 \quad \text{as } s, t \to \infty.$$

So $\int_0^t \tilde{Y}_s(Af)\,ds$ converges as $t \to \infty$. Thus $Y_t(f)$ converges a.s. to a finite limit which we call $Y_\infty(f)$.

Since $C(E)$ is separable and $D(A)$ is dense in $C(E)$ we may pick $\{\phi_n\} \subseteq D(A)$ dense in $C(E)$. Off a null set $N$ we have $Y_t(\phi_n) \to Y_\infty(\phi_n)$, $\forall\ n$. Fix $\omega \notin N$. Then by approximation $Y_t(f)$ converges to a finite limit $Y_\infty(f)$ for all $f \in C(E)$. Also $f \to Y_\infty(f)$ is a positive linear functional with $Y_\infty(1) = 1$ and thus arises from a probability which we call $Y_\infty$. For $f \in D(A)$,

$$N_t^2(f) - \int_0^t Y_s(f^2) - (Y_s(f))^2\,ds$$

is a continuous local martingale. Since $N_t(f), Y_s(f^2), Y_s(f)$ all converge to finite limits this local martingale must converge, requiring $Y_\infty(f^2) = (Y_\infty(f))^2$ a.s. So the probability $Y_\infty$ is concentrated on a level set of $f$. But $E$ is a metric space so that $C(E)$ and hence $\{\phi_n\}$ separate points and this forces $Y_\infty = \delta_F$ a.s. for some $F$.

We have been unable to deduce the law of $F$ directly from the martingale problem but it comes immediately from the particle picture. One can approximate the superprocess by a system of branching diffusions [see Dawson, Iscoe and Perkins (1989)]. Let $X_t^n$ be such an approximation where the lifetimes of the particles are of length $1/n$. The initial values $X_0^n = m_n$ are a convergent sequence of finite measures. It follows from the independence of the spatial motion and the branching that

$$E\big(X_t^n(f)|\sigma(X_s^n(1): s \ge 0)\big) = X_t^n(1)\,m^n(1)^{-1}\int T_t f(x)\,dm^n(x).$$

This property is preserved for the limiting superprocess and (4) may be deduced.

When $E$ is only locally compact we can extend the semigroup $T_t$ to $E \cup \{\infty\}$, the one point compactification of $E$, by taking $T_t(\infty, \{\infty\}) = 1$, $T_t(x, \{\infty\}) = 0$ for all $x \in E$, $t > 0$. Working with this new Feller process on $E \cup \{\infty\}$, the above argument gives the existence of a death point $F$ taking values in $E \cup \{\infty\}$ and satisfying (3) and (4). Since $P(\xi < \infty) = 1$, the characterization of the law of $F$ (4) ensures $P(F \in E) = 1$. □

Example. Let $X_t$ be a super Poisson process. Define for $k \in Z_+$,

$$T_s = \inf\big(t \ge 0: X_t(\{0, \dots, k-1\}) = 0\big).$$

In Perkins (1990), Corollary 3.1, it is shown that $T_k \uparrow \xi$ and

$$S_t = \{k, k+1, \dots\} \quad \text{for Lebesgue a.a. } t \text{ in } [T_k, T_{k+1}),\ P^m\text{-a.s.}$$

Theorem 1 shows that only finitely many of the $T_k$'s are distinct. Indeed, with probability 1,

$$0 = T_0 \le T_1 \le \cdots \le T_F \le T_{F+1} = T_{F+2} = \cdots = \xi.$$

PROOF OF THEOREM 2. Take $A \subseteq E$, Borel measurable, and let $f = \mathbf{I}(x \in A)$. We shall use the time changed process $Y_t(f)$. Let $\tilde{B}_t$ be an independent Brownian motion defined if necessary on an extension of the original probability space. Define

$$\bar{B}_t = \int_0^t (Y_s(f)(1 - Y_s(f)))^{-1/2} \mathbf{I}(Y_s(f) \notin \{0,1\})\, dN_s(f) + \int_0^t \mathbf{I}(Y_s(f) \in \{0,1\})\, d\tilde{B}_s,$$

so that $\bar{B}_t$ is a Brownian motion and

$$Y_t(f) = m(f) + \int_0^t \tilde{Y}_s(Af)\, ds + \int_0^t (Y_s(f)(1 - Y_s(f)))^{1/2}\, d\bar{B}_s.$$

If $Y_t(f) = 0$ or 1, then $Y_t$ is supported on $A^c$ or $A$, respectively. So we look for times at which $Y_t(f)(1 - Y_t(f))$ becomes zero. Fix $N \in \mathbf{N}$ and define $Z_t(f) = Y_{N+t}(f)(1 - Y_{N+t}(f))$. By Itô's formula we have

$$\begin{aligned} Z_t(f) &= Z_0(f) + \int_0^t (1 - 2Y_{N+s}(f))(Y_{N+s}(f)(1 - Y_{N+s}(f)))^{1/2}\, d\bar{B}_{N+s} \\ &\quad + \int_0^t (1 - 2Y_{N+s}(f))\tilde{Y}_{N+s}(Af)\, ds - \int_0^t Y_{N+s}(f)(1 - Y_{N+s}(f))\, ds \\ &= Z_0(f) + \int_0^t (\beta_s - Z_s(f))\, ds + \int_0^t (Z_s(f)(1 - 4Z_s(f)))^{1/2}\, dB_s, \end{aligned}$$

where

$$\beta_s = (1 - 2Y_{N+s}(f))\tilde{Y}_s(Af),$$

$$B_t = \int_N^{N+t} \operatorname{sgn}(1 - 2Y_s(f))\, d\bar{B}_s,$$

so that $B_t$ is another Brownian motion. Since the function $(x(1 - 4x))^{1/2}$ satisfies the Yamada–Watanabe criterion [see Rogers and Williams (1987), Theorem 5.40.1], we have a unique solution on the same probability space to the stochastic differential equation

$$X_t = Z_0(f) + \int_0^1 ((1/8) - X_s)\, ds + \int_0^t |X_s(1 - 4X_s)|^{1/2}\, dB_s. \tag{10}$$

We shall compare the paths of $Z_t(f)$ with those of the solution $X_t$ to (10). We shall show that 0 is a recurrent point for $X_t$ and a comparison theorem will force $Z_t(f)$ to vanish infinitely often. Define

$$T_N = \inf\left(t \ge 0: \tilde{Y}_{N+t}(1) \ge \left(\tfrac{1}{8}\|Af\|\right)\right),$$

which is a $\mathscr{G}_{N+t}$ stopping time. For $s \le T_N$,

$$|\beta_s| = |(1 - 2Y_{N+s}(f))\tilde{Y}_{N+s}(Af)| \le \tfrac{1}{8}.$$

So by comparison theorem for one dimensional diffusions [see Rogers and Williams (1987), Theorem 5.43.1] we have

$$Z_t(f) \le X_t \quad \text{for } t \le T_N,\ P^m\text{-a.s.}$$

(We have applied the comparison theorem up to a stopping time.) $X_t$ takes values in $[0, \frac{1}{4}]$. Indeed, replacing the drift in (10) by $(\frac{1}{8} - X_s)^+$ [or $(\frac{1}{8} - X_s)^-$], we obtain an equation with solution $X_t \equiv \frac{1}{4}$ (or $X_t \equiv 0$). The comparison theorem then ensures $X_t \in [0, \frac{1}{4}]$ a.s. The speed measure $m(dx)$ and scale function $p(x)$ for (10) can be calculated as

$$p(x) = (\tfrac{1}{2})\int_{1/8}^{x} (y(1-4y))^{-1/4}\,dy, \qquad m(dx) = 4(x(1-4x))^{-3/4}.$$

Letting $T = \inf(t \ge 0\colon X_t \in \{0, \frac{1}{4}\})$ we may deduce from the speed and scale [see Karatzas and Shreve (1988), Propositions 5.22 and 5.32] that for $x \in (0, \frac{1}{4})$, $P_x(T < \infty) = 1$ and $P_x(X_T = 0) = (p(\frac{1}{4}) - p(x))/(p(\frac{1}{4}) - p(0))$. A further comparison argument shows that the points $\{0, \frac{1}{4}\}$ are not traps and hence that $X_t$ is a recurrent process and will hit 0 at arbitrarily large times. Thus on the set $\{T_N = \infty\}$, $Z_t(f)$ must hit zero infinitely often as $t \to \infty$. Since $\tilde{Y}_s(1) \to 0$ as $s \to \infty$, $P(T_N = \infty) \uparrow 1$ as $N \to \infty$. So, with probability 1,

(11) there exist $t_n \uparrow \infty$ so that $Y_{t_n}(f) = 0$ or 1.

Given $\varepsilon > 0$, let $(A_m)$ be a countable collection of open balls of radius $\varepsilon/2$ that cover $E$. Fix $\omega$ so that (11) holds simultaneously for all $f_m = \mathbf{I}(x \in A_m)$. Find $m_0(\omega)$ so that $F(\omega) \in A_{m_0(\omega)}$. Since $Y_t \to \delta_F$, then $Y_t(A_{m_0}) \to 1$. So there exist $t_n \uparrow \infty$ so that $Y_{t_n}(\mathbf{I}(x \notin A_{m_0})) = 0$ and

$$S\big(X_{D_{t_n}}\big) = S\big(Y_{t_n}\big) \subseteq B(F, \varepsilon) \quad \text{for all } n. \qquad \square$$

EXAMPLE. We examine a simple nontrivial superprocess. Let $E = \mathbb{Z}$ and the underlying spatial motion be symmetric random walk leaving each state at rate 1. Then if we write $X_t(j)$ for $X_t(\{j\})$, the martingale problem reduces to a system of linked stochastic differential equations. For $j \in \mathbb{Z}$,

$$X_t(j) = X_0(j) + \int_0^t (\tfrac{1}{2})(X_s(j+1) + X_s(j-1)) - X_s(j)\,ds + \int_0^t (X_s(j))^{1/2}\,dB_s^j, \tag{12}$$

where $\{B_t^j\}_j$ are independent Brownian motions. So we consider the superprocess as a diffusion on $\mathbb{R}_+^{\mathbb{Z}}$. As $t \to \xi$ the process $(X_t(j))_j$ traces a continuous path leading to the origin. Theorem 1 implies that this path will approach the origin tangentially to one of the axes. Theorem 2 implies that it will touch this axis at an infinite number of points that accumulate at the origin. Thus $X_t(j)$, $j \ne F$, will simultaneously vanish infinitely often before $\xi$.

**3. Decompositions.** The martingale problem (1) gives a decomposition as a continuous semimartingale for each process $X_t(f)$, where $f$ is in the domain of the generator $A$ of the spatial motion. We consider the problem of finding similar decompositions for more general measurable $f$. Let $b\mathscr{E}$ be the space of bounded measurable functions on $E$. Perkins has shown (private communication) that if the semigroup of the underlying process satisfies a continuity condition, for instance,

$$\begin{gathered}\exists\, C, \beta_1, \beta_2 > 0 \text{ such that for all } 0 \le \delta \le \nu,\ f \in b\mathscr{E}, \\ \|T_{\nu+\delta} f - T_\nu f\|_\infty \le C \|f\|_\infty \delta^{\beta_1}(\nu^{-\beta_2} \vee 1),\end{gathered} \tag{13}$$

then for each $f \in b\mathscr{E}$, with probability 1,

$$t \to X_t(f) \text{ is continuous on } (0, \infty). \tag{14}$$

We examine the case where $X_t$ is a one dimensional symmetric stable superprocess. The following lemma shows that hypothesis (13) is satisfied with $\beta_1 = \beta_2 = 1$. This and other technical lemmas needed for this section are proved in Section 4.

LEMMA 6. *Let $T_t = T_t^\alpha$ be the semigroup of a one dimensional symmetric stable process with index $\alpha$. Then for $\nu, \delta > 0$ with $\delta \le \nu$,*

$$\|T_{\nu+\delta} f - T_\nu f\|_\infty \le (2/\alpha) \|f\|_\infty \delta/\nu. \tag{15}$$

We will need the existence of a density for $X_t$ when $1 < \alpha \le 2$. We state the necessary results as a theorem.

THEOREM 7. *Let $m \in M_F(\mathbb{R})$ have a continuous density $u(x)$. Let $\alpha \in (1, 2]$ and $X_t$ be a one dimensional symmetric $\alpha$-stable superprocess starting at $m$ defined on a probability space $(\Omega, \mathscr{F}, P)$. Then $X_t$ has a density $X(t, x)$ which is continuous on $[0, \infty) \times \mathbb{R}$. There is a space-time white noise $W_{t,x}$ defined on an enlargement of $(\Omega, \mathscr{F}, P)$ such that, for all $f \in C^\infty(\mathbb{R})$ of compact support,*

$$X_t(f) = m(f) + \int_0^t X_s(Af)\, ds + \int_0^t \int_{\mathbb{R}} \sqrt{X(s,x)}\, f(x)\, dW_{s,x}, \quad \forall\, t \ge 0. \tag{16}$$

*For fixed $x \in \mathbb{R}, t \ge 0$,*

$$X(t,x) = T_t u(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y - x) \sqrt{X(s,y)}\, dW_{s,y}. \tag{17}$$

*If $u(x)$ is bounded and uniformly Hölder continuous, then there exist $\gamma > 0$ and $C$ depending only on $m$ and $\alpha$ such that*

$$E\big((X(t,x) - X(s,x))^2\big) \le C(t-s)^\gamma \quad \text{for all } t, s > 0. \tag{18}$$

The existence of a jointly continuous density satisfying (16) is proved for the Brownian case in Reimers (1989) and for all $0 < \alpha \le 2$ in Konno and Shiga (1988), Theorem 1.4. Equation (17) is established during the proof in Konno and Shiga [although they consider more general initial measures and thus work on $[t_0, \infty)$ for $t_0 > 0$, it is easy to extend (17) to $[0, \infty)$ for initial measures that have a continuous density]. The proof uses moment estimates of the type in (18) but since we cannot point to exactly what we need we give a proof of (18) in Section 4.

The proof of Theorem 4 will use the following well known Green's function representation for $X_t(\phi)$, $\phi \in b\mathscr{E}$:

$$X_t(\phi) = m(T_t\phi) + \int_0^t \int_E T_{t-s}\phi(x)\, dZ_{s,x}, \tag{19}$$

where $T_t$ is the semigroup of the underlying motion and $Z_{s,x}$ is an orthogonal martingale measure satisfying

$$\left\langle \int_0^t \int_E f(s,x)\, dZ_{s,x} \right\rangle_t = \int_0^t X_s(f^2(s,\cdot))\, ds \tag{20}$$

for any measurable $f(s,x)$ such that $E(\int_0^t X_s(f^2(s,\cdot))\, ds) < \infty$, $\forall\ t$. For the equivalence of the Green's function representation and the martingale problem (1), see Meleard and Roelly-Coppoletta (1988), Theorem I–7.

Proof of Theorem 4. From the Green's function representation we have

$$\begin{aligned} X_t(H_a) &= m(T_t H_a) + \int_0^t \int_{\mathbb{R}} T_{t-s} H_a\, dZ_{s,x} \\ &= m(H_a) + \left[ \int_0^t \int_{\mathbb{R}} H_a\, dZ_{s,x} \right] \\ &\quad + \left[ m((T_t - I)H_a) + \int_0^t \int_{\mathbb{R}} (T_{t-s} - I) H_a\, dZ_{s,x} \right] \\ &=: m(H_a) + M_t(H_a) + V_t(a). \end{aligned}$$

The second term in the decomposition is a martingale satisfying $\langle M(H_a) \rangle_t = \int_0^t X_s(H_a)\, ds$.

Lemma 8. *Fix $k > 0$, $p \geq 6$ an even integer. Choose $1/k \leq s < t \leq k$, $-k \leq a < b \leq k$ such that $t - s < 1$, $b - a < 1$. Then*

$$E^m\big((M_t(H_b) - M_s(H_a))^p\big) \leq C_{p,k,m}\big((t-s)^{p/2} + (b-a)^{\alpha p/2(1+\alpha)}\big).$$

The lemma is proved in Section 4. Using Kolmogorov's continuity criterion we may and do pick a jointly continuous version of $M_t(H_a)$ on $(0, \infty) \times \mathbb{R}$.

As explained at the beginning of this section, the path $t \to X_t(H_a)$ is continuous for each $a$. For fixed $a_0$ and $k > 0$ the functions $G_\varepsilon(t) = X_t((a_0 - \varepsilon, a_0 + \varepsilon))$ are continuous on $[1/k, k]$ and decrease monotonically to the function $X_t(\{a\})$. But it follows from the existence of a jointly continuous density for $1 < \alpha \leq 2$ and from the characterization of $X_t$ as a bounded multiple of a Hausdorff measure function for $0 < \alpha \leq 1$ [see Perkins (1988)] that

(21) $$X_t(\{a\}) = 0 \quad \text{for all } t > 0,\ a \in \mathbb{R} \text{ almost surely.}$$

By Dini's theorem $G_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ uniformly over $t \in [1/k, k]$. Together these facts imply that $X_t(H_a)$ is jointly continuous on $(0, \infty) \times \mathbb{R}$. The continuity of $X_t(H_a)$ and $M_t(H_a)$ now imply that of $V_t(a)$.

We now consider the variation of the process $t \to V_t(a)$ for fixed $a$. Without loss of generality we consider $a = 0$ and write $H$ for $H_0$. Now

$$m((T_t - I)H) = m((T_t - I)(H)\mathbf{I}(x \geq 0)) - m((I - T_t)(H)\mathbf{I}(x < 0))$$

is the difference of two decreasing processes and so of bounded variation. It remains to check the variation of

(22) $$W_t := \int_0^t \int_{-\infty}^{\infty} (T_{t-s} - I) H \, dZ_{s,x}.$$

Using the isometry for $Z_{s,x}$ [equation (20)] we have

(23) $$E(W_t^2) = \int_0^t m T_s\big(((T_{t-s} - I)H)^2\big)\, ds \leq m(1)t.$$

An upper bound for the expected value of the size of an increment of $W_t$ can be obtained using this isometry. We delay the calculations and state the result as a lemma.

LEMMA 9. *If $m$ has a bounded density, then there is a constant $C$ depending only on $T, \alpha, m$ such that for $0 \leq s \leq t \leq T$, $|t - s| \leq s$,*

$$E\big((W_t - W_s)^2\big) \leq C \begin{cases} (t-s)^{(\alpha+1)/\alpha}, & \text{if } \alpha > 1, \\ (t-s)^2, & \text{if } \alpha < 1. \end{cases}$$

Since we are interested in a continuous version of $W_t$, it is enough to check the variation over one sequence of decreasing nested partitions. Let $\Delta = T/n$ and $s_j = j\Delta$. If $\alpha > 1$, then using (23) and Lemma 9,

$$E\left(\sum_{j=1}^{n} |W_{s_j} - W_{s_{j-1}}|^{\phi(\alpha)}\right) \leq \sum_{j=2}^{n} \Big(E\big((W_{s_j} - W_{s_{j-1}})^2\big)\Big)^{\phi(\alpha)/2} + E\big(|W_{s_1}|^{\phi(\alpha)}\big)$$
$$\leq CT.$$

So $W_t$ and hence $V_t(0)$ has integrable $\phi(\alpha)$ variation on $[0, T]$. Similarly if $\alpha < 1$, then $V_t(0)$ has integrable variation on $[0, T]$.

We now assume that $u(0, x)$ is bounded, uniformly Hölder continuous and satisfies $u(0,0) > 0$. If $1 < \alpha \leq 2$, then $X_t$ has a jointly continuous density $u(t, x)$ and

$$\int_0^t \int_{\mathbb{R}} f(s,x)\, dZ_{s,x} = \int_0^t \int_{\mathbb{R}} f(s,x) \sqrt{u(s,x)}\, dW_{s,x},$$

where $W_{s,x}$ is a space-time white noise (see Theorem 7).

We split an increment of $W_t$ into three parts as follows. Fix $n$ and let $t_j = j/n$:

$$\begin{aligned} W_{t_{j+1}} - W_{t_j} &= \int_0^{t_j}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - T_{t_j-s}\right)H\,dZ_{s,x} \\ &\quad + \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - I\right)(H)\sqrt{u(t_j,x)}\,dW_{s,x} \\ &\quad + \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - I\right)(H)\left(\sqrt{u(s,x)} - \sqrt{u(t_j,x)}\right)dW_{s,x} \\ &=: \zeta_j + \epsilon_j + \eta_j. \end{aligned}$$

We wish to show that $W_t$ has strictly positive $\phi(\alpha)$ variation. We will first show that $|\eta_j|^{\phi(\alpha)}$ is small and does not contribute to the variation. Then noting that $\zeta_j$ is $\mathscr{F}_{t_j}$ measurable, we will show that conditional on $\mathscr{F}_{t_j}$, $\epsilon_j$ has a mean zero normal distribution with variance greater than $C^{n-1}X_{t_j}(B(0,n^{-1/\alpha}))$. Since $X_t$ has a density $u(t,x)$ bounded away from zero at $t = x = 0$, this variance will be of the order of $n^{-(\alpha+1)/\alpha}$ and the increment $|\zeta_j + \epsilon_j|^{\phi(\alpha)}$ will be of the order of $n^{-1}$. In these calculations $C$ will be a constant whose value is unimportant and may change from line to line:

$$\begin{aligned} E\left(|\eta_j|^2\right) &= E\left(\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - I\right)^2(H)\left(\sqrt{u(s,x)} - \sqrt{u(t_j,x)}\right)^2 ds\,dx\right) \\ &\le \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - I\right)^2(H)\left[E\left(u(s,x) - u(t_j,x)\right)^2\right]^{1/2} ds\,dx \\ &\le Cn^{-\gamma}\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - I\right)^2(H)\,ds\,dx, \end{aligned}$$

where $\gamma > 0$ from (18), which uses the Hölder continuity of $u(0,x)$. Using the bound $(T_r - I)H(x) \le Cr|x|^{-\alpha} \wedge 1$ [see (37)], we have

$$\begin{aligned} \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}}\left(T_{t_{j+1}-s} - I\right)^2(H)\,ds\,dx &\le Cn^{-1}\left(\int_0^{n^{-1/\alpha}} dx + \int_{n^{-1/\alpha}}^{\infty} n^{-2}|x|^{-2\alpha}\,dx\right) \\ &\le Cn^{-(\alpha+1)/\alpha}. \end{aligned}$$

So

$$\begin{aligned} E\left[\sum_{j=0}^{[nT]-1}|\eta_j|^{\phi(\alpha)}\right] &\le \sum_{j=0}^{[nT]-1}\left(E\left(\eta_j^2\right)\right)^{\phi(\alpha)/2} \\ &\le C\sum_{j=0}^{[nT]-1}\left(n^{-(\alpha+1)/\alpha}n^{-\gamma}\right)^{\phi(\alpha)/2} \qquad (24) \\ &= Cn^{-\gamma\alpha/(\alpha+1)}. \end{aligned}$$

Conditional on $\mathscr{F}_{t_j}$, $\epsilon_j$ has a normal mean zero distribution with variance

$$X_{t_j}\left(\int_{t_j}^{t_{j+1}}\left(T_{t_{j+1}-s} - I\right)^2 H\,ds\right).$$

Let $Y_t$ be a symmetric $\alpha$-stable process under $P_0$:

$$|(T_r - I)H(x)| = P_0\big(Y_1 \geq |x|/r^{1/\alpha}\big) \geq P_0\big(Y_1 \geq 2^{1/\alpha}\big)\mathbf{I}\big(|x| \leq (2r)^{1/\alpha}\big).$$

So

$$\int_{t_j}^{t_{j+1}} \big(T_{t_{j+1}-s} - I\big)^2 H\, ds \geq \int_{(2n)^{-1}}^{n^{-1}} \big(P_0\big(Y_1 \geq 2^{1/\alpha}\big)\big)^2 \mathbf{I}\big(|x| \leq (n)^{-1/\alpha}\big)\, ds = C_2 n^{-1}\mathbf{I}(|x| \leq n^{-1/\alpha}),$$

where $C_2 = P_0(Y_1 \geq 2^{1/\alpha})^2/2$.

Let $N$ have a normal mean zero variance one distribution under $P_0$:

$$Q\big(|\epsilon_j|^{\phi(\alpha)} \geq \kappa n^{-1}|\mathscr{F}_{t_j}\big) \geq P_0\big(N^2 \geq C_2^{-1}\kappa^{2/\phi(\alpha)} n^{(1-2\phi(\alpha)^{-1})}/X_{t_j}(B(0,n^{-1/\alpha}))\big) \geq \big(\tfrac{1}{5}\big)\mathbf{I}\big[X_{t_j}(B(0,n^{-1/\alpha})) \geq C_2^{-1}\kappa^{2/\phi(\alpha)} n^{-1/\alpha}\big]$$

using $P_0(N^2 \geq 1) \geq \frac{1}{5}$. Since $\zeta_j$ is $\mathscr{F}_{t_j}$-measurable,

$$Q\big(|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa n^{-1}|\mathscr{F}_{t_j}\big) \geq \big(\tfrac{1}{10}\big)\mathbf{I}\big[X_{t_j}(B(0,n^{-1/\alpha})) \geq C_2^{-1}\kappa^{2/\phi(\alpha)} n^{-1/\alpha}\big]. \tag{25}$$

The idea is that since $X_t(dx)$ has a continuous density the event on the right-hand side of (25) should occur frequently (at least for small $\kappa$) and at each of these times there is a $\frac{1}{10}$ chance that $|\epsilon_j + \zeta_j|$ will contribute to the $\phi(\alpha)$ variation.

The density $u(t,x)$ is jointly continuous and $u(0,0) > 0$ so that given $\varepsilon > 0$ we may find $n_0 > 2/T$, $\kappa_0 > 0$, $0 < t_0 < T - (2/n_0)$, so that for all $n \geq n_0$,

$$Q\big(X_t(B(0,n^{-1/\alpha})) < C_2^{-1}\kappa_0^{2/\phi(\alpha)} n^{-1/\alpha} \quad \text{for some } 0 \leq t \leq t_0\big) \leq \varepsilon.$$

Then for $n \geq n_0$,

$$\begin{aligned}
&Q\left(\sum_{j=0}^{[nT]-1} |\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0 t_0/20\right)\\
&\quad\geq Q\left(\sum_{j=0}^{[nT]-1} \mathbf{I}\big(|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0/n\big) \geq n t_0/20\right)\\
&\quad\geq Q\left(\sum_{j=0}^{[nT]-1} \mathbf{I}\big(X_{t_j}(B(0,n^{-1/\alpha})) \geq C_2^{-1}\kappa_0^{2/\phi(\alpha)} n^{-1/\alpha}\big) \geq n t_0\right)\\
&\qquad - Q\left(\sum_{j=0}^{[nT]-1} \mathbf{I}\big(X_{t_j}(B(0,n^{-1/\alpha})) \geq C_2^{-1}\kappa_0^{2/\phi(\alpha)} n^{-1/\alpha}\big) \geq n t_0,\right.\\
&\qquad\qquad\left.\sum_{j=0}^{[nT]-1} \mathbf{I}\big(|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0/n\big) < n t_0/20\right)\\
&\quad\geq (1-\varepsilon) - P_0\big(B([nt_0], 1/10) \leq n t_0/20\big),
\end{aligned}$$

where $B$ has a binomial distribution under $P_0$. This last estimate follows from Lemma 10, taking

$$A_j = \left\{X_{t_j}\left(B(0, n^{-1/\alpha})\right) \geq C_2^{-1}\kappa_0^{2/\phi(\alpha)}n^{-1/\alpha}\right\},$$
$$B_j = \left\{|\varepsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0/n\right\}.$$

LEMMA 10. *On a filtered probability space* $(\Omega, (\mathscr{F}_j)_{j=1,2,\dots}, P)$, *let* $A_n \in \mathscr{F}_n$, $B_n \in \mathscr{F}_{n+1}$ *for* $n = 1, \dots$ *be events satisfying* $P(B_j|\mathscr{F}_j) \geq q\mathbf{I}(A_j)$ *for some* $q \in [0,1]$. *Then for* $1 \leq n \leq N$, $0 \leq a \leq n$,

$$P\left(\sum_{i=1}^{N}\mathbf{I}_{A_i} \geq n, \sum_{i=1}^{N}\mathbf{I}_{B_i} \leq a\right) \leq P_0(B(n,q) \leq a)$$

*where* $B(n,q)$ *has a binomial distribution under* $P_0$ *with parameters* $n, q$.

So for large $n$, by the law of large numbers,

$$Q\left(\sum_{j=0}^{[nT]-1}|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0 t_0/20\right) \geq 1 - 2\varepsilon.$$

But from (24) for large $n$,

$$Q\left(\sum_{j=0}^{[nT]-1}|\eta_j|^{\phi(\alpha)} \geq \kappa_0 t_0/40\right) \leq \varepsilon.$$

Now Minkowski's inequality and Fatou's lemma give

$$Q\left(\sum_{j=0}^{[nT]-1}|W_{t_{j+1}} - W_{t_j}|^{\phi(\alpha)} \geq \kappa_0 t_0/80 \text{ infinitely often}\right) \geq 1 - 3\varepsilon.$$

Since $\varepsilon$ was arbitrary, it follows that the $\phi(\alpha)$ variation of $W_t$ over $[0,T]$ is strictly positive. □

REMARKS. (i) If we remove the restriction that $m \in M_F(\mathbb{R})$ has a bounded density, the decomposition remains valid and we can guarantee $V_t$ has finite variation when $\alpha \in (0,1)$ [or $\phi(\alpha)$ variation when $\alpha \in (1,2)$] over intervals $[S,T]$ bounded away from the origin. The only change required in the proof is in Lemma 9, where the bound remains true provided $0 < S \leq s < t \leq T$ and where the constant now depends on $S, T, \alpha, m$.

(ii) If $1 < \alpha < 2$, then the instantaneous propagation of the support [see (5)] implies that $V_t$ will have strictly positive $\phi(\alpha)$ variation on $[0,T]$ for any $T > 0$. If $\alpha = 2$ and $m \neq 0$, it follows from the absolute continuity results of Evans and Perkins [(1989)] that there is positive probability that for some $s > 0$ the measure $X_s$ will have a uniformly Hölder continuous bounded density that is strictly positive at some point on the boundary of the half space. Thus for any $X_0 \neq 0$ the process $X_t(H_a)$ fails to be a semimartingale.

(iii) By projecting the superprocess onto a line orthogonal to a given half space, we obtain a decomposition for the measure of a half space under a $d$ dimensional symmetric stable superprocess.

(iv) Sugitani (1989) shows that for super Brownian motion in one dimension the local time process $Y(t,x) = \int_0^t X(s,x)\,ds$ is differentiable in $x$ and that if $m$ is atomless, the derivative $D_x Y(t,x)$ is jointly continuous in $t, x$ almost surely. We can easily identify the drift term $V_t(a)$ in the decomposition of $X_t(H_a)$ as $(\frac{1}{2})\,D_x Y(t,a)$.

Take $m \in M_F(\mathbb{R})$ atomless and of compact support. Define $f_a(x) = ((x-a) \vee 0)^2$. We may find $f_n \in D(A)$ so that $f_n \uparrow f_a(x)$ and $Af_n \to \mathbf{I}(x > a)$ bounded pointwise. We have enough domination [e.g., $E(\sup_{t \le T} X_t(f_a^2)) < \infty$)] to take limits in the martingale problem and obtain

$$(26)\qquad \begin{aligned} X_t(f_a) &= m(f_a) + \int_0^t X_s(\mathbf{I}(x > a))\,ds + M_t(f_a) \\ &= m(f_a) + \int_a^\infty Y(t,x)\,dx + M_t(f_a). \end{aligned}$$

We wish to differentiate (26) twice with respect to $a$ and again we have enough domination. Thus for a fixed $t$,

$$(27)\quad 2X_t((x-a)\vee 0) = 2m((x-a)\vee 0) + Y(t,a) + M_t(2(x-a)\vee 0).$$

Now continuity of both sides in $t$ gives (27) for all $t$. Repeating the argument and using the continuity of $D_x Y(t,x)$ gives

$$X_t(H_a) = m(H_a) + (\tfrac{1}{2})\,D_x Y(t,a) + M_t(H_a).$$

Proof of Corollary 5. Let $f$ be a function of finite total variation which is constant outside a compact interval. By adding a constant we may assume that $\lim_{x \to -\infty} f(x) = 0$. Letting $\bar{f}(x) = \lim_{y \downarrow x} f(y)$ we have that $\bar{f}$ is right continuous and $\bar{f}(x) = f(x)$ except at countably many points. Using (21) we see that $X_t(\bar{f}) = X_t(f)$ and $M_t(\bar{f}) = M_t(f)$ for all $t > 0$ since

$$\langle M(\bar{f}) - M(f)\rangle_t = \int_0^t X_s\big((\bar{f} - f)^2\big)\,ds = 0, \qquad \forall\, t > 0.$$

Since $V_t(a)$ is continuous it will be enough to obtain the decomposition for right continuous $f$. Hence we assume $f$ is right continuous so that $f(x) = \int_{-\infty}^\infty \mathbf{I}(x \ge a)\,df(a)$. This integral is over a compact region by assumption, so using Fubini's theorem and the continuity of $V_t(a)$ we may integrate the decomposition for $X_t(\mathbf{I}(x \ge a))$ to obtain

$$X_t(f) = m(f) + \int V_t(a)\,df(a) + \int M_t(H_a)\,df(a).$$

Also $\int M_t(H_a)\,df(a)$ is a continuous version of

$$\int df(a) \int_0^t \int_{\mathbb{R}} \mathbf{I}(x \ge a)\,dZ_{s,x} = \int_0^t \int_{\mathbb{R}} f(x)\,dZ_{s,x},$$

using a stochastic Fubini theorem [see Walsh (1986), Theorem 2.6]. So $M_t(f) := \int M_t(H_a)\,df(a)$ is a continuous $L^2$ martingale such that $\langle M(f)\rangle_t = \int_0^t X_s(f^2)\,ds$ and the decomposition is complete. For $p \ge 1$,

$$E\left(\left|\int V_t(a)\,df(a) - \int V_s(a)\,df(a)\right|^p\right) \le C_f \int E(|V_t(a) - V_s(a)|^p)\,|df(a)|.$$

So noting that the estimate obtained in Theorem 4 for $E(|V_t(a) - V_s(a)|^2)$ is uniform in $a$ for $s, t \in [0, k]$, the desired results on the variation of $\int V_t(a)\, df(a)$ follow.

Suppose now that $\alpha = 2$ and the support of $X_0$ is contained in $B(0, R)$. The finite speed of the support $S_t$ of super Brownian motion [see Dawson, Iscoe and Perkins (1989), Theorem 1.1] implies that setting $T_r = \inf\{t \geq 0\colon S_t \not\subseteq B(0,r)\}$, then $S_{T_r} \subseteq \overline{B(0,r)}$ for $r \geq R$ and the stopping times $T_r$ satisfy $P(T_t \uparrow \infty) = 1$. For a function $f$ of locally bounded variation define

$$f_r = \begin{cases} f(r), & \text{for } x \geq r, \\ f(x), & \text{for } x \in (-r, r), \\ f(-r), & \text{for } x \leq -r. \end{cases}$$

Then for $r \geq R$,

$$\begin{aligned} X_{t \wedge T_r(f)} &= X_{t \wedge T_r}(f_r) \\ &= m(f_r) + \int V_{t \wedge T_r}(a)\, df_r(a) + M_{t \wedge T_r}(f_r) \\ &= m(f) + \int V_{t \wedge T_r}(a)\, df(a) + M_{t \wedge T_r}(f_r). \end{aligned}$$

Define $M_t(f) = M_t(f_r)$ for $t \leq T_r$. Then $M_t(f)$ is well defined since $M_t := M_t(f_{r_1}) - M_t(f_{r_2})$ is a continuous martingale which satisfies $\langle M \rangle_t = \int_0^t X_s((f_{r_1} - f_{r_2})^2)\, ds = 0$ for $t \leq T_{r_1} \leq T_{r_2}$. Thus $M_t(f)$ is a local martingale with $\langle M(f) \rangle_t = \int_0^t X_s(f_r^2)\, ds = \int_0^t X_s(f^2) ds$ for $t \leq T_r$. Since $\int V_t(a)\, df_r(a)$ has finite $\frac{4}{3}$ variation on compacts, then so does $\int V_t(a)\, df(a)$. □

Before proving Corollary 3 we shall need a preliminary lemma.

Lemma 11. *Let $X_t$ be a one dimensional symmetric stable superprocess of index $\alpha$ started at $m \in M_F(\mathbb{R})$. Let $f(x) = |x|^{-\beta}$ for $0 \leq \beta < \frac{1}{2}$. Then there exists a continuous version $\tilde{X}_t(f)$ of $X_t(f)$ on $(0, \infty)$.*

Remark. For $1 < \alpha \leq 2$ it follows from the existence of a jointly continuous density for $X_t$ that the process $t \to X_t(|x|^{-\beta})$ is continuous on $(0, \infty)$ for any $\beta < 1$. For $0 < \alpha \leq 1$ it follows from the characterization of $X_t$ as a bounded density times a deterministic measure function [see Perkins (1988)] that $t \to X_t(|x|^{-\beta})$ is continuous on $(0, \infty)$ for any $\beta < \alpha$. The content of the lemma is thus in the range $0 < \alpha < \frac{1}{2}$. The proof, which simply checks the Kolmogorov continuity criterion, is given in Section 4.

Proof of Corollary 3. Fix an open ball $B = (a, b)$ of finite radius in $\mathbb{R}$. From Theorem 4 and the following remark we have the decomposition

$$X_t(B) = X_0(B) + V_t + M_t(B),$$

where $\langle M(B) \rangle_t = \int_0^t X_s(B)\, ds$ and $V_t = V_t(a) - V_t(b)$ has finite variation on

$[S,T]$ for any $0 < S < T < \infty$. Define

$$v_t = \begin{cases} \lim_{h \to 0+} (V_{t+h} - V_t)/h, & \text{if this limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

We will find an upper bound on $|v_s|$. Note that for any $\delta > 0$, $T_\delta \mathbf{I}_B(x)$ is a $C^\infty$ function vanishing at infinity. Let $g(x) = \sup_{\delta > 0} |AT_\delta \mathbf{I}_B(x)|$. Scaling arguments as in the proof of Lemma 6 show there exists $C$ such that $g(x) \le g_0(x) \equiv C(|x-a|^{-\alpha} + |x-b|^{-\alpha})$. For fixed $0 < s < t$,

$$\begin{aligned} &|X_t(T_\delta \mathbf{I}_B) - X_s(T_\delta \mathbf{I}_B) - M_t(T_\delta \mathbf{I}_B) + M_s(T_\delta \mathbf{I}_B)| \\ &\quad = \left| \int_s^t X_r(AT_\delta \mathbf{I}_B)\, dr \right| \\ &\quad \le \int_s^t X_r(g_0)\, dr = \int_0^t \tilde{X}_r(g_0)\, dr, \end{aligned}$$

where $\tilde{X}_r(g_0)$ is the continuous version of $X_r(g_0)$ guaranteed by Lemma 11. Letting $\delta \downarrow 0$,

$$\begin{aligned} |V_t - V_s| &= |X_t(B) - X_s(B) - M_t(B) + M_s(B)| \\ &\le \int_s^t \tilde{X}_r(g_0)\, dr. \end{aligned} \tag{28}$$

$\tilde{X}_r(g_0)$ is locally bounded by Lemma 11 and so $V_t$ is absolutely continuous and $|v_r| \le \tilde{X}_t(g_0)$ for a.a. $r$ $Q^m$-a.s. Now we follow the proof of Theorem 2. We set $Y_t(B) = X_{D_t}(B)/X_{D_t}(1)$ and $Z_t = Y_{N+t}(B)(1 - Y_{N+t}(B))$. Then $Z_t$ satisfies

$$Z_t = Z_0 + \int_0^t \left((1 - 2Y_{N+s}(B))v_{D_{N+s}} - Z_s\right) ds + \int_0^t (Z_s(1 - 4Z_s))^{1/2}\, dB_s.$$

Now the comparison argument of Theorem 2 will work provided we can show

$$P\left(|v_{D_{N+s}}| \le 1/8 \text{ for a.a. } s\right) \to 1 \quad \text{as } N \to \infty. \tag{29}$$

But since $t \to D_t$ is a $C^1$ diffeomorphism, we have $|v_{D_r}| \le \tilde{X}_{D_r}(g_0)$ for a.a. $r$ $Q^m$-a.s. By continuity $\tilde{X}_{D_r}(g_0) \to 0$ as $r \to \infty$ and (29) follows. □

**4. Proof of lemmas.** This section contains the proof of several technical lemmas needed in Sections 2 and 3. Throughout $C$ will denote a constant whose exact value is unimportant and may change from line to line.

Proof of Lemma 6. We use the scaling relation for the one dimensional symmetric stable density $p_t(x) = t^{-1/\alpha} p_1(t^{-1/\alpha}x)$ and the facts that $p_1(x)$ is smooth, unimodal and satisfies $p_1(x) \le C(1 \wedge |x|^{-1-\alpha})$:

$$\begin{aligned} \|T_{\nu+\delta} f - T_\nu f\|_\infty &\le \|f\|_\infty \int_{-\infty}^\infty |p_{\nu+\delta}(x) - p_\nu(x)|\, dx \\ &\le 2\|f\|_\infty \int_0^\infty \int_\nu^{\nu+\delta} |(d/dt) p_t(x)|\, dt\, dx. \end{aligned}$$

From the scaling relation, for $x \geq 0$,

$$\begin{aligned}
&|(d/dt)p_t(x)| \\
&\quad = |(-1/\alpha)\big(t^{-(1+1/\alpha)}p_1(t^{-1/\alpha}x) + t^{-(1+2/\alpha)}x(d/dx)p_1(t^{-1/\alpha}x)\big)| \\
&\quad \leq (1/\alpha t)\big(p_t(x) - x(d/dx)p_t(x)\big)
\end{aligned}$$

since $(d/dx)p_t(x) \leq 0$. Thus

$$\begin{aligned}
\|T_{\nu+\delta}f - T_\nu f\|_\infty &\leq (2/\alpha)\|f\|_\infty \int_\nu^{\nu+\delta}(1/t)\int_0^\infty \big(p_t(x) - x(d/dx)p_t(x)\big)\,dx\,dt \\
&= (2/\alpha)\|f\|_\infty \int_\nu^{\nu+\delta}(1/t) \\
&\qquad \times\left((1/2) + \lim_{R\to\infty}\left(-Rp_t(R) + \int_0^R p_t(x)\,dx\right)\right)dt \\
&= (2/\alpha)\|f\|_\infty \int_\nu^{\nu+\delta}(1/t)\,dt \\
&\leq (2/\alpha)\|f\|_\infty\,\delta/\nu.
\end{aligned}$$

□

PROOF OF EQUATION (18). From (17),

$$\begin{aligned}
X(t,x) - X(s,x) &= (T_t - T_s)u(x) \\
&\quad + \int_0^s\int\big(p_{t-r}(x-y) - p_{s-r}(x-y)\big)\sqrt{X(r,y)}\,dW_{r,y} \\
&\quad + \int_s^t\int p_{t-r}(x-y)\sqrt{X(r,y)}\,dW_{r,y}.
\end{aligned}$$

Find $C, \beta \in (0,1]$ such that $|u(x) - u(y)| \leq C|x-y|^\beta$ for all $x, y \in R$. Then

$$\begin{aligned}
\|(T_t - T_s)u\| &\leq \|(T_{t-s} - I)u\| \\
&\leq CE_0\big(|Y_{t-s}|^\beta\big) \\
&\leq C(t-s)^{\beta/\alpha}.
\end{aligned}$$

The stable density satisfies

$$\begin{aligned}
&p_1(x) \leq C\big(|x|^{-(1+\alpha)} \wedge 1\big), \\
&p_t(x) = t^{-1/\alpha}p_1(t^{-1/\alpha}x).
\end{aligned} \tag{30}$$

So

$$\begin{aligned}
E\left(\left(\int_s^t\int_{-\infty}^\infty p_{t-r}(x-y)\sqrt{X(r,y)}\,dW_{r,y}\right)^2\right) &= \int_s^t m\big(T_r(p_{t-r}^2(x-\cdot))\big)\,dr \\
&\leq \|u\|\int_0^{t-s}\int_\infty^{-\infty} p_r^2(x)\,dx\,dr \\
&\leq C\int_0^{t-s} r^{-1/\alpha}\,dr \\
&= C(t-s)^{(\alpha-1)/\alpha}.
\end{aligned}$$

Similarly, if $(t-s)^{1/2} \le s$,

$$E\left(\left(\int_{s-(t-s)^{1/2}}^{s}\int_{-\infty}^{\infty}(p_{t-r}-p_{s-r})\sqrt{X(r,y)}\,dW_{r,y}\right)^2\right) \le C(t-s)^{(\alpha-1)/2\alpha}$$

and if $s \le (t-s)^{1/2}$,

$$E\left(\left(\int_0^s\int_{-\infty}^{\infty}(p_{t-r}-p_{s-r})\sqrt{X(r,y)}\,dW_{r,y}\right)^2\right) \le C(t-s)^{(\alpha-1)/2\alpha}.$$

Finally $\|p_t - p_s\| \le C(t-s)s^{-(\alpha+1)/\alpha}$ for $0 < s < t$ so if $(t-s)^{1/2} \le s$,

$$\begin{aligned}
&E\left(\left(\int_0^{s-(t-s)^{1/2}}\int_{-\infty}^{\infty}(p_{t-r}(x-y)-p_{s-r}(x-y))\sqrt{X(r,y)}\,dW_{r,y}\right)^2\right)\\
&\quad\le Cm(1)\int_{(t-s)^{1/2}}^{s}(t-s)^2 r^{-2(\alpha+1)/\alpha}\,dr\\
&\quad\le C(t-s)^{(3/2)-(1/\alpha)}.
\end{aligned}$$

□

PROOF OF LEMMA 8.

$$\begin{aligned}
&E\big((M_t(H_b)-M_s(H_a))^p\big)\\
&\quad\le 2^{p-1}E\big((M_t(H_b)-M_t(H_a))^p+(M_t(H_a)-M_s(H_a))^p\big)\\
&\quad\le C_pE\left(\left(\int_s^t X_r(H_a)\,dr\right)^{p/2}+\left(\int_0^t X_r([a,b))\,dr\right)^{p/2}\right)\\
&\quad\le C_p\big((t-s)^{p/2}+(b-a)^{\alpha p/2(1+\alpha)}\big)\sup_{r\in[0,k]}E\big(X_r^{p/2}(1)\big)\\
&\qquad+C_p\int_{t\wedge(b-a)^{\alpha/(1+\alpha)}}^{t}E\big(X_r^{p/2}([a,b))\big)\,dr,
\end{aligned}\tag{31}$$

where we used Burkholder's inequality in the second step. An exact formula for the expectations in the right-hand side of (31) is known [see Dynkin (1988)] but the following upper bound can be derived from the particle picture as in Perkins (1988), Proposition 2.6(a)(i). For $\phi$: $\mathbb{R}^d \to [0,\infty]$ Borel measurable, $p \in N$,

$$E^m(X_t^p(\phi)) \le p^p\big(G(\phi,t)+E^m(X_t(\phi))\big)^{p-1}E^m(X_t(\phi)),\tag{32}$$

where $G(\phi,t) = \int_0^t \sup_y T_s\phi(y)\,ds$. Thus $E^m(X_r^{p/2}(1)) \le C_{p,m,k}$ $\forall$ $r \in [0,k]$. Also we calculate using (30) that

$$G\big(\mathbf{I}_{[a,b)}(x),t\big) \le C_k\begin{cases}(b-a), & \text{if } \alpha > 1,\\ (b-a)\log(1/(b-a)), & \text{if } \alpha = 1,\\ (b-a)^{\alpha}, & \text{if } \alpha < 1\end{cases}\tag{33}$$

and that $E^m(X_t([a,b))) \le C_{k,m}(1 \wedge (b-a)t^{-1/\alpha})$. Substituting (33) and (32) into (31) gives the correct bound. □

PROOF OF LEMMA 9. Without loss of generality we take $T \le 1$. From (22),

$$
\begin{aligned}
&E\left[(W_t - W_s)^2\right] \\
(34)\quad &= E\left[\left(\int_s^t \int_{-\infty}^{\infty} (T_{t-r} - I) H \, dZ_{r,x} + \int_0^s \int_{-\infty}^{\infty} (T_{t-r} - T_{s-r}) H \, dZ_{r,x}\right)^2\right] \\
&= \int_s^t mT_r\left(((T_{t-r} - I)H)^2\right) dr + \int_0^s mT_r\left(((T_{t-r} - T_{s-r})H)^2\right) dr.
\end{aligned}
$$

For fixed $x \ge 0$,

$$
\begin{aligned}
(T_{r+\delta} - T_r)H(x) &= P_0(Y_{r+\delta} \ge -x) - P_0(Y_r \ge -x) \\
&= P_0\left(Y_1 \in \left[x/(r+\delta)^{1/\alpha}, x/r^{1/\alpha}\right]\right) \\
&\le |(x/r^{1/\alpha}) - (x/(r+\delta)^{1/\alpha})| p_1\left(x/(r+\delta)^{1/\alpha}\right).
\end{aligned}
$$

$C$ will be a constant depending only on $T$, $\alpha$, $m$ whose value may change from line to line. Using the bound on the stable density (30) we have for $r \ge \delta$,

$$
(35)\qquad |(T_{r+\delta} - T_r)H(x)| \le C(\delta|x|^{-\alpha} \wedge \delta|x|r^{-(\alpha+1)/\alpha}),
$$

for $r \le \delta$,

$$
(36)\qquad |(T_{r+\delta} - T_r)H(x)| \le P_0\left(Y_1 \in \left[|x|/\delta^{1/\alpha}, \infty\right)\right) \le C\delta|x|^{-\alpha} \wedge 1
$$

and for $r > 0$,

$$
(37)\qquad |(T_r - I)H(x)| \le P_0\left(Y_1 \in [|x|/r^{1/\alpha}, \infty)\right) \le Cr|x|^{-\alpha} \wedge 1.
$$

Find a constant $K$ so that the densities of the measures $mT_r$ are bounded by $K$ for all $r \ge 0$. Recall that the lemma requires $t - s \le s$. From (35), for $0 \le r \le s - (t-s)$,

$$
\begin{aligned}
&mT_r\left(((T_{t-r} - T_{s-r})H)^2\right) \\
&\quad \le C(t-s)^2\left(K(s-r)^2\int_0^{(s-r)^{1/\alpha}} dx + K\int_{(s-r)^{1/\alpha}}^1 |x|^{-2\alpha}\, dx + mT_r(|x| \ge 1)\right) \\
&\quad \le C(t-s)^2 \begin{cases} 1 + (s-r)^{(1/\alpha)-2}, & \text{if } \alpha \ne \frac{1}{2}, \\ 1 + \log^+\left((s-r)^{-1}\right), & \text{if } \alpha = \frac{1}{2}. \end{cases}
\end{aligned}
$$

From (36), for $s - (t-s) \le r \le s$,

$$
\begin{aligned}
&mT_r\left(((T_{t-r} - T_{s-r})H)^2\right) \\
&\quad \le 2CK\int_0^{(t-s)^{1/\alpha}} dx + 2CK(t-s)^2\int_{(t-s)^{1/\alpha}}^1 |x|^{-2\alpha}\, dx \\
&\qquad + 2C(t-s)^2 mT_r(|x| \ge 1) \\
&\quad \le C\begin{cases} (t-s)^{(2\wedge(1/\alpha))}, & \text{if } \alpha \ne \frac{1}{2}, \\ (t-s)^2\left(1 + \log^+\left((t-s)^{-1}\right)\right), & \text{if } \alpha = \frac{1}{2}. \end{cases}
\end{aligned}
$$

So

$$\int_0^s mT_r\Big(\big((T_{t-r} - T_{s-r})H\big)^2\Big)\,dr$$

$$\leq C(t-s)^2 \int_0^{s-(t-s)} \begin{Bmatrix} 1 + (s-r)^{(1/\alpha)-2} \\ 1 + \log^+\big((s-r)^{-1}\big) \end{Bmatrix} dr$$

$$+ C\int_{s-(t-s)}^{s} \begin{Bmatrix} (t-s)^{(2\wedge(1/\alpha))} \\ (t-s)^2\Big(1 + \log^+\big((t-s)^{-1}\big)\Big) \end{Bmatrix} dr$$

$$\leq C\begin{cases} (t-s)^2, & \text{if } \alpha < 1, \\ (t-s)^{(\alpha+1)/\alpha}, & \text{if } \alpha > 1. \end{cases}$$

Similar arguments give an upper bound of no larger order for the first term in (34). □

Proof of Lemma 10. Define $\tau_0 = 0$, $\tau_j(\omega) = \inf(m > \tau_{j-1}\colon \omega \in A_m)$ for $j = 1, \dots$ ($+\infty$ if this set is empty). Then

$$P\left(\sum_{j=1}^{N} \mathbf{I}(A_j) \geq n, \sum_{j=1}^{N} \mathbf{I}(B_j) \leq a\right) \leq P\left(\tau_n < \infty, \sum_{j=1}^{n} \mathbf{I}(B_{\tau_j}) \leq a\right).$$

We shall show by induction on $n$ that

$$P\left(\tau_n < \infty, \sum_{j=1}^{n} \mathbf{I}(B_{\tau_j}) \leq a\right) \leq P_0(B(n,q) \leq a) \quad \text{for } a = 0, 1, \dots, n. \tag{38}$$

I claim that

$$P\big(B_{\tau_j} \cap (\tau_j < \infty) | \mathscr{F}_{\tau_j}\big) \geq q\mathbf{I}(\tau_j < \infty). \tag{39}$$

To see this pick $C \in \mathscr{F}_{\tau_j}$ and note that $(\tau_j = n) \subseteq A_n$:

$$P\big(B_{\tau_j} \cap (\tau_j < \infty) \cap C\big) = \sum_{n=1}^{\infty} P\big(B_n \cap (\tau_j = n) \cap C\big)$$

$$\geq q \sum_{n=1}^{\infty} P\big(A_n \cap (\tau_j = n) \cap C\big)$$

$$= qP\big(C \cap (\tau_j < \infty)\big).$$

The case $n = 1$ follows immediately from (39). Assume (38) is true for $n =$

$1, \ldots, k-1$. Note that $B_{\tau_{k-1}} \in \mathscr{F}_{\tau_k}$ and $\{\tau_k < \infty\} \subseteq \{\tau_{k-1} < \infty\}$. Then

$$\begin{aligned}
&E\left(P\left(\tau_k < \infty, \sum_{j=1}^{k} \mathbf{I}(B_{\tau_j}) \leq a | \mathscr{F}_{\tau_k}\right)\right) \\
&\quad = E\left(\mathbf{I}\left(\sum_{j=1}^{k-1} \mathbf{I}(B_{\tau_j}) = a\right) P\left(B^c_{\tau_k} \cap (\tau_k < \infty) | \mathscr{F}_{\tau_k}\right)\right) \\
&\quad\quad + P\left(\sum_{j=1}^{k-1} \mathbf{I}(B_{\tau_j}) \leq a - 1, \tau_k < \infty\right) \\
&\quad \leq (1-q) P\left(\sum_{j=1}^{k-1} \mathbf{I}(B_{\tau_j}) = a, \tau_k < \infty\right) \\
&\quad\quad + P\left(\sum_{j=1}^{k-1} \mathbf{I}(B_{\tau_j}) \leq a - 1, \tau_k < \infty\right) \quad \text{using (39)} \\
&\quad \leq (1-q) P\left(\sum_{j=1}^{k-1} \mathbf{I}(B_{\tau_j}) \leq a, \tau_{k-1} < \infty\right) \\
&\quad\quad + qP\left(\sum_{j=1}^{k-1} \mathbf{I}(B_{\tau_j}) \leq a - 1, \tau_{k-1} < \infty\right) \\
&\quad \leq (1-q) P_0(B(k-1, q) \leq a) + qP_0(B(k-1,q) \leq a-1) \\
&\quad = P_0(B(k,q) \leq a). \qquad \square
\end{aligned}$$

LEMMA 12. *Fix $m \in M_F(E)$, $t > 0$, $1 \leq p < \infty$ and let $X_t$ be a superprocess over a spatial motion with semigroup $(T_t\colon\ t \geq 0)$. If $f \in L^p(mT_t)$, then $E(X_t^p|f|)) \leq q(t) mT_t(|f|^p)$, where $q(t)$ is a polynomial with coefficients depending on $m(1)$ and $p$ which is of order $[p] = \sup(n \in N\colon n \leq p)$.*

PROOF.

$$\begin{aligned}
E(X_t^p(|f|)) &\leq E\left(X_t^{p-1}(1) X_t(|f|^p)\right) \\
&\leq E\left(\left(1 + X_t^{[p]}(1)\right) X_t(|f|^p)\right) \\
&= mT_t(|f|^p) + E\left(X_t^{[p]}(1) X_t(|f|^p)\right).
\end{aligned}$$

An exact formula to evaluate such product moments is known [see Dynkin (1988)]. This formula will collapse to give the desired result. Alternatively, arguing from the particle picture as in Perkins (1988), Proposition 2.6(a)(i) gives an upper bound for positive measurable $f$:

$$E\left(X_t^n(1) X_t(f)\right) \leq n!\left(t + m(1)\right)^n mT_t(f). \qquad \square$$

PROOF OF LEMMA 11. By the remark following the statement of the lemma we give a proof only when $\alpha \leq \frac{1}{2}$. Set $f_n = f \wedge n$. From the Green's function representation and Burkholder's inequality we have

$$E\big((X_t(f_n) - X_s(f_n))^p\big) \le C_p|m(T_t - T_s) f_n|^p + C_p E\bigg(\bigg(\int_s^t X_r\big((T_{t-r} f_n)^2\big)\,dr\bigg)^{p/2}\bigg) + C_p E\bigg(\bigg(\int_0^s X_r\big((T_{t-r} - T_{s-r})^2 f_n\big)\,dr\bigg)^{p/2}\bigg). \tag{40}$$

By Lemma 12 the left-hand side of (40) will converge to $E((X_t(f) - X_s(f))^p)$ as $n \to \infty$ provided $p\beta < 1$. Fix $0 < 1/k \le s \le t < k < \infty$. Set $\Delta = t - s$. We assume $\Delta \le 1/2k$ and $k \ge 1$. Pick $\varepsilon \in (0, \frac{1}{2})$ so that $(2 + \varepsilon)\beta < 1$. Set $p = 2 + \varepsilon$, $\theta = 1 - \varepsilon/4$.

We shall show that the third term in (40) is bounded by a constant $\times \Delta^{1+(\epsilon/8)}$. The first and second terms have a similar bound and are easier. Thus the Kolmogorov continuity criterion will be satisfied and there will exist a continuous version on $[1/k, k]$ for any $k$, and hence on $(0, \infty)$.

We break the third term of (40) itself into three parts. Using Lemma 12 we have

$$\begin{aligned} &E\bigg(\bigg(\int_0^{\Delta^\theta} X_{s-r}\big((T_{r+\Delta} - T_r)^2 f_n\big)\,dr\bigg)^{p/2}\bigg) \\ &\quad\le \Delta^{\theta((p/2)-1)} \int_0^{\Delta^\theta} q(s-r)\, m T_{s-r}\big((T_{r+\Delta} + T_r)^p f\big)\,dr \\ &\quad\le 2^p \Delta^{\theta p/2} m(1) \sup_{r\in[0,k]} q(r) \sup_{r\in[1/k,k]} \|T_r(f^p)\|_\infty \\ &\quad\le C\Delta^{1+(\epsilon/8)}. \end{aligned}$$

Using Lemma 6 we have

$$\begin{aligned} &E\bigg(\bigg(\int_{s-(1/2k)}^{s} X_{s-r}\big((T_{r+\Delta} - T_r)^2 f_n\big)\,dr\bigg)^{p/2}\bigg) \\ &\quad\le E\bigg(\bigg(\int_{s-(1/2k)}^{s} X_{s-r}(1)(2/\alpha)^2 \|T_{1/4k} f\|_\infty^2 \Delta^2 (4k)^2\,dr\bigg)^{p/2}\bigg) \\ &\quad\le C\Delta^2. \end{aligned}$$

The remaining part is

$$\begin{aligned} &E\bigg(\bigg(\int_{\Delta^\theta}^{s-(1/2k)} X_{s-r}\big((T_{r+\Delta} - T_r)^2 f_n\big)\,dr\bigg)^{p/2}\bigg) \\ &\quad\le C\int_{\Delta^\theta}^{s-(1/2k)} q(s-r)\, m T_{s-r}\big((T_{r+\Delta} - T_r)^p f_n\big)\,dr. \end{aligned} \tag{41}$$

To get an estimate on $|T_{r+\Delta} f_n(y) - T_r f_n(y)|$ we argue as in Lemma 6. For $y \geq 0$,

$$
\begin{aligned}
&|T_{r+\delta} f_n(y) - T_r f_n(y)| \\
&\quad \leq \int_{-\infty}^{\infty} |p_{r+\Delta}(x) - p_r(x)||x-y|^{-\beta}\, dx \qquad (42)\\
&\quad \leq (2/\alpha)\int_0^{\infty}\int_r^{r+\Delta}(1/t)(p_t(x) - x(d/dx)p_t(x))|x-y|^{-\beta}\, dt\, dx.
\end{aligned}
$$

Using the bounds on the stable density (30) we have for $0 \leq y \leq 2t^{1/\alpha}$, $t \leq k$,

$$
\begin{aligned}
&\int_0^{\infty} p_t(x)|x-y|^{-\beta}\, dx \\
&\quad \leq \int_0^{4t^{1/\alpha}} Ct^{-1/\alpha}|x-y|^{-\beta}\, dx + \int_{4t^{1/\alpha}}^{\infty} Ctx^{-(1+\alpha)}2^{-\beta}t^{-\beta/\alpha}\, dx \qquad (43)\\
&\quad \leq Ct^{-\beta/\alpha}.
\end{aligned}
$$

For $y \geq 2t^{1/\alpha}$, $t \leq k$,

$$
\begin{aligned}
&\int_0^{\infty} p_t(x)|x-y|^{-\beta}\, dx \\
&\quad \leq \int_0^{1/\alpha} Ct^{-1/\alpha}2^{\beta}y^{-\beta}\, dx + \int_{t^{1/\alpha}}^{y/2} Ctx^{-(1+\alpha)}2^{\beta}y^{-\beta}\, dx \qquad (44)\\
&\qquad + \int_{y/2}^{y} Ct2^{1+\alpha}y^{-(1+\alpha)}|x-y|^{-\beta}\, dx + \int_y^{\infty} Ctx^{-(1+\alpha)}(x-y)^{-\beta}\, dx \\
&\quad \leq Cy^{-\beta}.
\end{aligned}
$$

There is a series expansion for the one dimensional stable density for $\alpha < 1$ [see Zolotarev (1986), page 90]. This may be differentiated term by term and used to prove the bound $(d/dx)p_1(x) \leq C_1(1 \wedge |x|^{-(2+\alpha)})$. Hence by scaling we have $|x(d/dx)p_t(x)| \leq C_1(t^{-1/\alpha} \wedge tx^{-(1+\alpha)})$ and the same calculations as in (43) and (44) give the same bounds for $\int_0^{\infty} x(d/dx)p_t(x)|x-y|^{-\beta}\, dx$. Substituting into (42) we obtain for $y \geq 0$, $\Delta \leq r \leq k$,

$$
\begin{aligned}
|T_{r+\Delta} f_n(y) - T_r f_n(y)| &\leq C\int_r^{r+\Delta} dt \begin{cases} t^{-1-(\beta/\alpha)}, & \text{for } y \leq 2t^{1/\alpha},\\ t^{-1}y^{-\beta}, & \text{for } y \geq 2t^{1/\alpha}\end{cases} \\
&\leq C\Delta \begin{cases} r^{-1-(\beta/\alpha)}, & \text{for } y \leq 2r^{1/\alpha},\\ r^{-1}y^{-\beta}, & \text{for } y \geq 2r^{1/\alpha}.\end{cases}
\end{aligned}
$$

Noting that $p_s(x) \le C_k(1 \wedge |x|^{-(1+\alpha)})$ for $s \in [1/2k, k]$ [see (30)] we have for $\Delta \le r \le s - (1/2k)$,

$$\begin{aligned}
&\|T_{s-r}\big((T_{r+\Delta} - T_r)^p(f_n)\big)\|_\infty \\
&= 2\Big\|\int_0^\infty p_{s-r}(\cdot - y)(T_{r+\Delta} - T_r)^p(f_n)(y)\,dy\Big\|_\infty \\
&\le C\Delta^p\bigg(\int_0^{2r^{1/\alpha}} r^{-(1+(\beta/\alpha))p}\,dy \\
&\qquad + \int_{2r^{1/\alpha}}^k y^{-\beta p} r^{-p}\,dy + \sup_z \int_k^\infty r^{-p}\big(1 \wedge |z-y|^{-(1+\alpha)}\big)\,dy\bigg) \\
&\le C\Delta^p r^{-p}.
\end{aligned}$$

Finally substituting into (41) we obtain

$$\begin{aligned}
&E\left(\left(\int_{\Delta^\theta}^{s-(1/2k)} X_{s-r}\big((T_{r+\Delta} - T_r)^2 f_n\big)\,dr\right)^{p/2}\right) \\
&\le C\int_{\Delta^\theta}^{s-(1/2k)} m(1) \sup_{u\in[0,k]} q(u)\,\Delta^p r^{-p}\,dr \\
&\le C\Delta^{1+(\varepsilon/8)}. \qquad \square
\end{aligned}$$

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