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THE ASYMPTOTICS OF STABLE SAUSAGES IN THE PLANE[1]

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In this paper we develop an asymptotic expansion for the ε-neighborhood of the symmetric stable process of order β, $1 < \beta < 2$. Our expansion is in powers of $\varepsilon^{2-\beta}$ with the nth coefficient related to n-fold self-intersections of our stable process.

1. Introduction. In this paper we will develop asymptotic expansions for the area of the ε-neighborhood of X, the symmetric stable process of order β, $1 < \beta < 2$, in $\mathbb{R}^2$. Our asymptotic expansion as $\varepsilon \to 0$ is in powers $(\varepsilon^{2-\beta})^n$ with coefficients γ_n which are random variables related to n-fold self-intersections of X.

The ε-neighborhood of X, known as the stable sausage, is defined as

$$S_\varepsilon(t) = \left\{ y \in \mathbb{R}^2 \,\middle|\, \inf_{0 \le s \le t} |y - X_s| \le \varepsilon \right\}. \tag{1.1}$$

$m(S_\varepsilon(t))$ will denote the area of $S_\varepsilon(t)$, and $c_0(\varepsilon)$ will denote the β-capacity for $B(0, \varepsilon)$, the disc of radius ε centered at 0. We know that

$$c_0(\varepsilon) = \frac{1}{\Gamma(\beta/2)\Gamma(2-\beta/2)} \varepsilon^{2-\beta}. \tag{1.2}$$

THEOREM 1.1. *If $(4k-2)(2-\beta) < 1$, then we can find random variables $\gamma_n(t) \in L^2(d\mathbb{P})$, $n \le k$, such that*

$$m(S_\varepsilon(t)) = \sum_{n=1}^{k} (-1)^{n-1} c_0^n(\varepsilon) \gamma_n(t) + o\big(c_0^k(\varepsilon)\big) \tag{1.3}$$

a.s., and in $L^2(d\mathbb{P})$.

By $\mathbb{P}$ we mean $\mathbb{P}_0$, the probability for our stable process starting at the origin.

Theorem 1.1 will be derived from the next result concerning the area of the stable sausage at a random time. Let ζ be an exponential random variable of mean 1, independent of $X_\cdot$, and let $Q = \mathbb{P} \otimes e^{-t}\,dt$ be the measure for $(X_\cdot, \zeta)$.

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THEOREM 1.2. *If* $(2k-2)(2-\beta)<1$, *then we can find random variables* $\gamma_n \in L^2(dQ)$, $n \le k$, *such that*

$$m(S_\varepsilon(\zeta)) = \sum_{n=1}^{k} (-1)^{n-1} c_0^n(\varepsilon)\gamma_n + o(c_0^k(\varepsilon)) \quad in\ L^2(dQ). \tag{1.4}$$

Theorem 1.1 leads to an asymptotic expansion for $E(m(S_\varepsilon(t)))$, the expected area of the stable sausage, as $\varepsilon \to 0$. By scaling, this will give the asymptotics as $t \to \infty$ for fixed radius. We present the result in this form for comparison with Port (1990).

THEOREM 1.3. *If* $(4k-2)(2-\beta)<1$, *then*

$$\begin{aligned} &E(m(S_1(t)))\\ &\quad = \sum_{n=1}^{k}\left[\frac{-c_0(1)}{2\beta\sin(2\pi/\beta)}\right]^{n-1}\frac{1}{\Gamma(2-(n-1)(2/\beta-1))}t^{1-(n-1)(2/\beta-1)} \qquad (1.5)\\ &\qquad + o(t^{1-(k-1)(2/\beta-1)}), \quad \text{as } t\to\infty. \end{aligned}$$

We now describe briefly the random variables γ_n, known as renormalized intersection local times. Let $f \ge 0$ be a continuous function supported in $B(0,1)$, with $\int f(x)\,d^2x = 1$. Set

$$f_\varepsilon(x) = \frac{1}{\varepsilon^2} f\left(\frac{x}{\varepsilon}\right),$$

so that f_ε is an approximate "delta function." If

$$\alpha_{n,\varepsilon}(t) \doteq \int \cdots \int_{0\le t_1 \le \cdots < t_n < t} \prod_{i=2}^{n} f_\varepsilon(X_{t_i} - X_{t_{i-1}})\,d\bar{t}, \tag{1.6}$$

then $\alpha_{n,\varepsilon}(t)$ can be thought of as an approximation to

$$\int \cdots \int_{0\le t_1 \le \cdots \le t_n < t} \prod_{i=2}^{n} \delta(X_{t_i} - X_{t_{i-1}})\,d\bar{t},$$

hence should measure the amount of n-fold self-intersection. However, as $\varepsilon \to 0$, $\alpha_{n,\varepsilon}(t) \nearrow \infty$ due to the large number of intersectors near the "diagonals" $\{t_i = t_j\}$. To get well-defined random variables, we must renormalize, which in our case means subtracting terms involving lower-order intersections. The precise definition is

$$\begin{aligned} \gamma_{n,\varepsilon}(t) &= \sum_{j=1}^{n} (-h_0(\varepsilon))^{n-j}\binom{n-1}{j-1}\alpha_{j,\varepsilon}(t) \\ &= \int \cdots \int dt_1 \prod_{i=2}^{n}\left(f_\varepsilon(X_{t_i} - X_{t_{i-1}})\,dt_i - h_0(\varepsilon)\delta_{t_{i-1}}(dt_i)\right), \end{aligned} \tag{1.7}$$

where

$$h_0(\varepsilon) = \int f_\varepsilon(x) G_0(x)\, d^2x \tag{1.8}$$

and

$$G_0(x) = \frac{\Gamma((2-\beta)/2)}{\pi 2^\beta \Gamma(\beta/2)} \frac{1}{x^{2-\beta}} \tag{1.9}$$

is the 0-order Green's function for X; hence

$$h_0(\varepsilon) = \frac{1}{\varepsilon^{2-\beta}} \int f(x) G_0(x)\, d^2x. \tag{1.10}$$

Even after our renormalization, we cannot show the convergence of $\gamma_{n,\varepsilon}(t)$, as $\varepsilon \to 0$, for fixed t. Rather, we will show that $\gamma_{n,\varepsilon}(\zeta)$ converges in $L^2(dQ)$, and γ_n of Theorem 1.2 denotes the limit. From γ_n we will define the $\gamma_n(t)$ of Theorem 1.1, and computing $E(\gamma_n(t))$ is our main contribution in Theorem 1.3.

We remark that the preceding renormalization will only work for n satisfying $(2n-1)(2-\beta) < 2$, that is, for larger n, $\gamma_{n,\varepsilon}(\zeta)$ does not converge in $L^2(dQ)$ [see Rosen (1991)]. This provides a theoretical upper bound to the order of asymptotic expansion obtainable in the types of theorems we consider. Our work is based on that of Le Gall (1990), who derived asymptotic expansions to arbitrary order for the area of the Wiener sausage, that is, when X is Brownian motion.

For more detailed results on the topic of intersection local times of stable processes and its applications see Rosen (1991), Le Gall and Rosen (1991) and Le Gall (1987). Dynkin (1988) contains a survey of results on intersection local times.

2. An approximate renormalized intersection local time. Let $B(y, \varepsilon)$ denote the disk centered at y, with radius ε. The 1-capacitory measure for $B(y, \varepsilon)$ with respect to our process is absolutely continuous with respect to Lebesgue measure on $B(y, \varepsilon)$ and its density can be written as $g_{y,\varepsilon}(x) = g_{0,\varepsilon}(x-y) \geq 0$, and has support in $B(y, \varepsilon)$ [Bliedtner and Hansen (1986), page 205 and Getoor (1984), Proposition 2.14]. Let

$$T_{y,\varepsilon} = \inf\{s > 0 | X_s \in B(y, \varepsilon)\}.$$

Then, with $G \doteq G_1$. the 1-potential density,

$$\begin{aligned} \int G(z, \bar{z}) g_{y,\varepsilon}(\bar{z})\, d^2\bar{z} &= E_z(e^{-T_{y,\varepsilon}}) \\ &\doteq E_z(T_{y,\varepsilon} < \zeta) \begin{cases} \leq 1, & \\ = 1, & \text{if } z \in B(y, \varepsilon). \end{cases} \end{aligned} \tag{2.1}$$

This implies that

$$\|L_{y,\varepsilon} f\| \leq \|f\|_{B(y,\varepsilon)}, \tag{2.2}$$

where

$$L_{y,\varepsilon} f(z) \doteq \int G(z,\bar{z}) g_{y,\varepsilon}(\bar{z}) f(\bar{z})\, d^2\bar{z}.$$

We also note that

$$\begin{aligned}\int\left(\int G(z,\bar{z}) g_{y,\varepsilon}(\bar{z})\, d^2\bar{z}\right) d^2y &= \int\left(\int G(z,\bar{z}+y) g_{0,\varepsilon}(\bar{z})\, d^2\bar{z}\right) d^2y \\ &= \int g_{0,\varepsilon}(\bar{z})\, d^2\bar{z} \\ &= c(\varepsilon),\end{aligned} \tag{2.3}$$

where $c(\varepsilon)$ is the 1-capacity for the disk of radius ε. [See Lemma 7.1 for the asymptotics of $c(\varepsilon)$.]

We set

$$\lambda_{n,y,\varepsilon}(t) = \int_{0\le t_1\le \cdots \le t_n \le t} \cdots \int \prod_{i=1}^{n} g_{y,\varepsilon}(X_{t_i})\, dt_i \tag{2.4}$$

and

$$l_{n,\varepsilon}(t) = c^{-n}(\varepsilon) \sum_{j=1}^{n} (-1)^{n-j} \int \lambda_{j,y,\varepsilon}(t)\, d^2y; \tag{2.5}$$

$l_{n,\varepsilon}(t)$ will serve as an approximate renormalized intersection local time. We shall need a systematic notation for products of operators. We denote by $R_{n,m}$ the set of all ordered products $\pi(u;v)$ in the noncommutative variables u, v which contain n factors of u and m factors of v. Equivalently, $R_{n,m}$ consists of all permutations of the $n+m$ factors of $u^n v^m$.

LEMMA 2.1. *For all x, y, n, m we have*

$$E_z\big(\lambda_{n,x,\varepsilon}(\zeta)\lambda_{m,y,\varepsilon}(\zeta)\big) = \sum_{R_{n,m}} \pi(L_{x,\varepsilon}; L_{y,\varepsilon})1(z). \tag{2.6}$$

PROOF. We suppress ε for ease of notation:

$$\begin{aligned}&E_z\big(\lambda_{n,x}(\zeta)\lambda_{m,y}(\zeta)\big) \\ &\quad = E_z\left(\int_{0\le s_1\le \cdots \le s_n<\zeta} \cdots \int \prod_{i=1}^{n} g_x(X_{s_i})\, d\bar{s} \int_{0\le t_1\le \cdots \le t_n<\zeta} \cdots \int \prod_{j=1}^{m} g_y(X_{t_j})\, d\bar{t}\right) \\ &\quad = \sum_{R_{n,m}} E_z\left(\int_{0\le r_1 \cdots \le r_{n+m}<\zeta} \cdots \int \prod_{i=1}^{n+m} g_{\pi_i}(X_{r_i})\, d\bar{r}\right),\end{aligned} \tag{2.7}$$

where π_i is either x or y, depending on whether the ith factor in $\pi(x;y)$ (from the left) is either x or y.

An induction argument based on the following calculation will then complete the proof of Lemma 2.1:

$$
\begin{aligned}
&E_z\left(\int_{0\le r_1\le \cdots \le r_j<\zeta} \cdots \int \left(\prod_{i=1}^{j} g_{\pi_i}(X_{r_i})\right) F(X_{r_j})\, d\bar r\right)\\
&= E_z\left(\int_{0\le r_1\le \cdots \le r_j<\infty} \cdots \int \left(\prod_{i=1}^{j-1} g_{\pi_i}(X_{r_i})\right) e^{-r_j} g_{\pi_j}(X_{r_j}) F(X_{r_j})\, d\bar r\right)\\
&= E_z\left(\int_{0\le r_1 \cdots \le r_{j-1}<\infty} \cdots \int \left(\prod_{i=1}^{j-1} g_{\pi_i}(X_{r_i})\right) \int_{r_{j-1}}^{\infty} e^{-r_j} g_{\pi_j}(X_{r_j}) F(X_{r_j})\, d\bar r\right)\\
&= E_z\left(\int_{0\le r_1\le \cdots \le r_{j-1}<\infty} \cdots \int \left(\prod_{i=1}^{j-1} g_{\pi_i}(X_{r_i})\right)\right.\\
&\qquad \left. \times e^{-r_{j-1}} \int_0^{\infty} e^{-s} g_{\pi_j}(X_s\circ\theta_{r_{j-1}}) F(X_s\circ\theta_{r_{j-1}})\, ds\, d\bar r\right)\\
&= E_z\left(\int_{0\le r_1 \cdots r_{j-1}<\zeta} \cdots \int \prod_{i=1}^{j-1} g_{\pi_i}(X_{r_i}) L_{\pi_j} F(X_{r_{j-1}})\, dr\right).
\end{aligned}
\tag{2.8}
$$

□

Let $\Theta_\varepsilon = \{|z-x|, |z-y|, |x-y| \ge 4\varepsilon\}$.

LEMMA 2.2. *For all z, n, m we have*

$$
\begin{aligned}
&\int\int E_z(\lambda_{n,x,\varepsilon}(\zeta)\lambda_{m,y,\varepsilon}(\zeta))\, d^2x\, d^2y\\
&= \sum_{R_{n,m}} \int_{\Theta_\varepsilon}\int \pi(L_{x,\varepsilon}; L_{y,\varepsilon})1(z)\, d^2x\, d^2y + O(\varepsilon^2 c(\varepsilon)).
\end{aligned}
\tag{2.9}
$$

PROOF. Consider some $\pi(x;y) \in \mathbb{R}_{n,m}$. If exactly i y's precede the first x in $\pi(x;y)$ (from the left), then, by (2.2),

$$
\begin{aligned}
&\int\int_{|y-z|\le 4\varepsilon} \pi(L_{x,\varepsilon}; L_{y,\varepsilon})1(z)\, d^2x\, d^2y\\
&\le \int_{|y-z|\le 4\varepsilon} d^2y \int d^2x\, L^i_{y,\varepsilon}(L_{x,\varepsilon}1)(z).
\end{aligned}
\tag{2.10}
$$

We apply (2.3) for the d^2x integral, then (2.2) again to bound (2.10) by

$$c(\varepsilon)\int_{|y-z|\le 4\varepsilon} L^i_{y,\varepsilon}1(z)\,d^2y \le c(\varepsilon)\int_{|y-z|\le 4\varepsilon} d^2y = O(c(\varepsilon)\varepsilon^2). \tag{2.11}$$

If no y's precede the first x, we first do the dy integral. The cases $|z-x|\le 4\varepsilon$ or $|x-y|\le 4\varepsilon$ are handled similarly. □

If $\pi\in R_{n,m}$, we denote by $\pi(x\tilde{;}y)$ the polynomial of degree $n+m$ obtained from π as follows: If $\pi_i=\pi_i-1$, then in $\pi(x\tilde{;}y)$ the ith factor will be π_i-1, for example, if

$$\pi(x;y) = xxyxyyyx,$$

then

$$\pi(x\tilde{;}y) = x(x-1)yxy(y-1)(y-1)x.$$

Recall (2.5).

LEMMA 2.3. *For all x, y, n, m, we have*

$$E_z(c^n(\varepsilon)l_{n,\varepsilon}(\zeta)c^m(\varepsilon)l_{m,\varepsilon}(\zeta)) = \sum_{R_{n,m}}\iint_{\Theta_\varepsilon}\pi(L_{x,\varepsilon}\tilde{;}L_{y,\varepsilon})1(z)\,d^2x\,d^2y + O(\varepsilon^2c(\varepsilon)). \tag{2.12}$$

PROOF. This follows easily from the proofs of the preceding lemmas if we note that

$$c^n(\varepsilon)l_{n,\varepsilon}(\zeta) = \int d^2y \int_{0\le t_1\le\cdots\le t_n<\zeta}\cdots\int dt_1\,g_{y,\varepsilon}(X_{t_1})\prod_{i=2}^n\{g_{y,\varepsilon}(X_{t_i})\,dt_i-\delta_{t_{i-1}}(dt_i)\}. \tag{2.13}$$

□

We let $D_{n,m}$ denote that subset of $R_{n,m}$ consisting of products $\pi(x;y)$ such that neither the x nor the y factors are completely separated, that is, such that in $\pi(x\tilde{;}y)$ at least one x factor has been replaced by $x-1$ and at least one y factor has been replaced by $y-1$.

LEMMA 2.4. *For all z, n, m and $\pi\in D_{n,m}$, we have*

$$\iint_{\Theta_\varepsilon}\pi(L_{x,\varepsilon}\tilde{;}L_{y,\varepsilon})1(z)\,d^2x\,d^2y = O(\varepsilon^2c(\varepsilon)). \tag{2.14}$$

PROOF. We can rewrite

$$\pi(L_{x,\varepsilon}; L_{y,\varepsilon})1(z) = \int \cdots \int \prod_{i=1}^{n+m} G(z_{i-1}, z_i) g_{\pi_i}(z_i)\, d\bar{z} \tag{2.15}$$

with $z_0 = z$.

$\pi(L_{x,\varepsilon} \tilde{;} L_{y,\varepsilon})$ differs from (2.15) in that some factors $G(z_{i-1}, z_i) g_{\pi_i}(z_i)\, dz_i$ are replaced by

$$G(z_{i-1}, z_i) g_{\pi_i}(z_i)\, dz_i - \delta_{z_{i-1}}(dz_i). \tag{2.16}$$

Let $\bar{n}$ denote the largest i with $\pi_i = x$ for which we have such a replacement, and let $\bar{m}$ denote the largest i with $\pi_i = y$ for which we have the replacement (2.16). Since $\pi \in D_{m,n}$, such $\bar{m}$, $\bar{n}$ exist, and we can assume $\bar{n} < \bar{m}$. Note that if $\bar{m} = n + m$, then (2.14) is zero. Thus, $\pi_{\bar{n}} = x$, $\pi_{\bar{n}+1} = y$, $\pi_{\bar{n}-1} = x$, $\pi_{\bar{m}} = y$, $\pi_{\bar{m}+1} = x$, $\pi_{\bar{m}-1} = y$ and $\bar{n} + 1 < \bar{m} < n + m$. Therefore, in the expansion of $\pi(L_{x,\varepsilon} \tilde{;} L_{y,\varepsilon})$ we have factors

$$\left[G(z_{\bar{n}-1}, z_{\bar{n}}) g_x(z_{\bar{n}})\, dz_{\bar{n}} - \delta_{z_{\bar{n}-1}}(dz_{\bar{n}})\right]\left[G(z_{\bar{n}}, z_{\bar{n}+1}) g_y(z_{\bar{n}+1})\right], \tag{2.17}$$

$$\left[G(z_{\bar{m}-1}, z_{\bar{m}}) g_y(z_{\bar{m}})\, dz_{\bar{m}} - \delta_{z_{\bar{m}-1}}(dz_{\bar{m}})\right]\left[G(z_{\bar{m}}, z_{\bar{m}+1}) g_x(z_{\bar{m}})\right]. \tag{2.18}$$

Since both $z_{\bar{n}-1}$ and $z_{\bar{n}}$ are in $B(x, \varepsilon)$, by (2.1) the $dz_{\bar{n}}$ integral of

$$\left[G(z_{\bar{n}-1}, z_{\bar{n}}) g_x(z_{\bar{n}})\, dz_{\bar{n}} - \delta_{z_{\bar{n}-1}}(dz_{\bar{n}})\right] G(x, z_{\bar{n}+1})$$

is zero, and therefore we can replace the second factor of (2.17) by

$$\left(G(z_{\bar{n}}, z_{\bar{n}+1}) - G(x, z_{\bar{n}+1})\right) g_y(z_{\bar{n}+1}). \tag{2.19}$$

Similarly, the second factor of (2.18) can be replaced by

$$\left(G(z_{\bar{m}}, z_{\bar{m}+1}) - G(y, z_{\bar{m}+1})\right) g_x(z_{\bar{m}+1}). \tag{2.20}$$

Let

$$\omega_\varepsilon G(z) = \sup_{|a| \le \varepsilon} |G(z + a) - G(z)|, \tag{2.21}$$

then (2.19) is bounded in absolute value by [see (2.24)]

$$c\omega_\varepsilon G(x - y) g_y(z_{\bar{n}+1})$$

and (2.20) by

$$c\omega_\varepsilon G(x - y) g_x(z_{\bar{m}+1}).$$

We now use (2.12), noting that

$$\int g_y(z_{\bar{n}+1})\, dz_{\bar{n}+1} = \int g_x(z_{\bar{m}+1})\, dz_{\bar{m}+1} = c(\varepsilon)$$

to get

$$\left|\pi(L_{x,\varepsilon} \tilde{;} L_{y,\varepsilon}) 1(z)\right| \le c(\omega_\varepsilon G(x - y))^2 c^2(\varepsilon) L_{\pi_1,\varepsilon} 1(z). \tag{2.22}$$

Say $\pi_1 = x$. Then

$$\iint_{\Theta_\varepsilon} \left|\pi(L_{x,\varepsilon}, L_{y,\varepsilon})1(z)\right| d^2x\, d^2y \tag{2.23}$$
$$\leq cc^2(\varepsilon)\int L_{x,\varepsilon}1(z)\left(\int_{|x-y|\geq 4\varepsilon}(\omega_\varepsilon G(x-y))^2\, d^2y\right) d^2x.$$

But we know [Rosen (1991)] that

$$|\omega_\varepsilon G(u)| \leq c\frac{\varepsilon}{u^{1+(2-\beta)}}, |u| \geq 4\varepsilon, \tag{2.24}$$

hence the inner integral of (2.23) is bounded by

$$c\varepsilon^2\int_{|u|\geq 4\varepsilon}\frac{1}{u^{2+2(2-\beta)}}\, d^2u = c\varepsilon^2\frac{1}{\varepsilon^{2(2-\beta)}}; \tag{2.25}$$

hence (2.23) is bounded by [see (2.3)]

$$c\varepsilon^2\int L_{x,\varepsilon}1(z)\, d^2x = cc(\varepsilon)\varepsilon^2. \qquad \square$$

3. Area of the stable sausage. We clearly have that

$$m(S_\varepsilon(t)) = \int 1_{\{T_{\varepsilon,x} < t\}}\, d^2x, \tag{3.1}$$

so that

$$E_z\left(m(S_\varepsilon(\zeta))^2\right) = \iint \mathbb{P}_z(T_{\varepsilon,x} < \zeta, T_{\varepsilon,y} < \zeta)\, d^2x\, d^2y. \tag{3.2}$$

LEMMA 3.1.

$$E_z(m(S_\varepsilon(\zeta)))^2 = \iint_{\Theta_\varepsilon} \mathbb{P}_z(T_{\varepsilon,x} < \zeta, T_{\varepsilon,y} < \zeta)\, d^2x\, d^2y + O(\varepsilon^2 c(\varepsilon)). \tag{3.3}$$

PROOF.

$$\iint_{|x-z|\leq 4\varepsilon} \mathbb{P}_z(T_{\varepsilon,x} < \zeta, T_{\varepsilon,y} < \zeta)\, d^2x\, d^2y \leq \int \mathbb{P}_z(T_{\varepsilon,y} < \zeta)\left(\int_{|x-z|\leq 4\varepsilon} d^2x\right) d^2y$$
$$\leq c\varepsilon^2\int \mathbb{P}_z(T_{\varepsilon,y} < \zeta)\, d^2y$$
$$= cc(\varepsilon)\varepsilon^2$$

by (2.1) and (2.2). The case of $|y-z| \leq 4\varepsilon$ or $|x-y| \leq 4\varepsilon$ is handled similarly. $\square$

We introduce

$$H_{x,\varepsilon} f(z) \doteq P^1_{B(x,\varepsilon)} f(z) = E_z\big(e^{-T_{x,\varepsilon}} f(X_{T_{x,\varepsilon}})\big). \tag{3.4}$$

$H_{x,\varepsilon}$ is known as the first-order hitting operator for $B(x,\varepsilon)$.

LEMMA 3.2. *For any n, and for x, y such that $|z-x| \geq 4\varepsilon$, $|z-y| \geq 4\varepsilon$, $|x-y| \geq 4\varepsilon$, we have*

$$\begin{aligned} \mathbb{P}_z(T_{x,\varepsilon} < \zeta, T_{y,\varepsilon} < \zeta) = & \sum_{i,j\leq n} (-1)^{i+j} \sum_{N_{i,j}} \pi(H_{x,\varepsilon}; H_{y,\varepsilon}) 1(z) \\ & + E_z\Big((H_{y,\varepsilon} H_{x,\varepsilon})^{n-1} H_{y,\varepsilon} 1(X_{T_{x,\varepsilon}}); T_{y,\varepsilon} < T_{x,\varepsilon} < z\Big) \\ & + E_z\Big((H_{x,\varepsilon} H_{y,\varepsilon})^{n-1} H_{x,\varepsilon} 1(X_{T_{y,\varepsilon}}); T_{x,\varepsilon} < T_{y,\varepsilon} < \zeta\Big), \end{aligned} \tag{3.5}$$

where $N_{i,j} \subseteq R_{i,j}$ is the set of ordered products $\pi(x;y)$ such that in $\pi(x;y)$ no contiguous letters are the same.

REMARKS. (i) The last two terms in (3.5) are error terms which are controlled in Lemma 3.3.

(ii) $N_{i,j}$ is precisely the set of products π satisfying

$$\pi(x \tilde{;} y) = \pi(x; y).$$

We also note that $N_{i,j} = \varnothing$ unless $j = i-1$, i or $i+1$.

PROOF OF LEMMA 3.2. We again suppress ε. For any u, v, we define inductively

$$\begin{aligned} A^1_{u,v} &= T_u, \\ A^2_{u,v} &= A^1_{u,v} + T_v \circ \theta_{A^1_{u,v}}, \\ A^3_{u,v} &= A^2_{u,v} + T_u \circ \theta_{A^2_{u,v}}, \\ A^{2k}_{u,v} &= A^{2k-1}_{u,v} + T_v \circ \theta_{A^{2k-1}_{u,v}}, \\ A^{2k+1}_{u,v} &= A^{2k}_{u,v} + T_u \circ \theta_{A^{2k}_{u,v}}. \end{aligned}$$

We first show that

$$\begin{aligned} P_z(T_x < T_y < \zeta) = & \sum_{i=1}^n P_z\big(A^{2i}_{x,y} < \zeta\big) \\ & - \sum_{i=1}^n P_z\big(A^{2i+1}_{y,x} < \zeta\big) + P_z\big(T_x < A^1_{y,x}; A^{2n+1}_{y,x} < \zeta\big). \end{aligned} \tag{3.6}$$

The verification of (3.6) proceeds inductively:

$$P_z(T_x < T_y < \zeta) = P_z\big(A^1_{x,y} < A^2_{x,y} < \zeta\big) - P_z\big(T_y < A^1_{x,y} < A^2_{x,y} < \zeta\big). \tag{3.7}$$

By definition,

$$\{A^1_{x,y} < A^2_{x,y} < \zeta\} = \{A^2_{x,y} < \zeta\} \tag{3.8}$$

and

$$\begin{aligned} P_z\big(T_y < A^1_{x,y} < A^2_{x,y} < \zeta\big) &= P_z\big(A^1_{y,x} < A^2_{y,x} < A^2_{y,x} < \zeta\big) \\ &\quad - P_z\big(T_x < A^1_{y,x} < A^2_{y,x} < A^3_{y,x} < \zeta\big) \\ &= P_z\big(A^3_{y,x} < \zeta\big) - P_z\big(T_x < A^1_{y,x};\, A^3_{y,x} < \zeta\big) \end{aligned} \tag{3.9}$$

and so on.

We next show, for example,

$$P_z\big(A^{2i+1}_{y,x} < \zeta\big) = H_{y,\varepsilon}(H_{x,\varepsilon}H_{y,\varepsilon})^i 1(z). \tag{3.10}$$

This is done inductively. It is true for $i = 0$ by (3.4). The induction step is (we take, e.g., j even)

$$\begin{aligned} E_z\Big(A^j_{y,x} < \zeta;\, F\big(X_{A^j_{y,x}}\big)\Big) &= E_z\Big(\exp\big(-A^j_{y,x}\big);\, F\big(X_{A^j_{y,x}}\big)\Big) \\ &= E_z\Big(\exp\Big[-\Big(A^{j-1}_{y,x} + T_x \circ \theta_{A^{j-1}_{y,x}}\Big)\Big] F\Big(X_{A^{j-1}_{x,y} + T_x \circ \theta^{j-1}_{A_{y,x}}}\Big)\Big) \\ &= E_z\Big(\exp\big(-A^{j-1}_{y,x}\big)\big\{\exp(-T_x) F(X_{T_x})\big\} \circ \theta_{A^{j-1}_{y,x}}\Big) \\ &= E_z\Big(\exp\big(-A^{j-1}_{y,x}\big) E_{X_{A^{j-1}_{y,x}}}\big(\exp(-T_x) F(X_{T_x})\big)\Big) \\ &= E_z\Big(A^{j-1}_{y,x} < \zeta;\, H_{x,\varepsilon} F\big(X_{A^{j-1}_{y,x}}\big)\Big), \end{aligned} \tag{3.11}$$

which yields Lemma 3.2. □

We now control the error terms in (3.5).

LEMMA 3.3. *If* $(2k-1)(2-\beta) < 2$, *then*

$$\iint_{\Theta_\varepsilon} E_z\Big(\big(H_{y,\varepsilon}H_{x,\varepsilon}\big)^{k-1} H_{y,\varepsilon}1(X_{T_x});\, T_y < T_x < \zeta\Big)\, dx\, dy = O\big(\varepsilon^{\bar\alpha} c^{2k}(\varepsilon)\big), \tag{3.12}$$

where $\bar\alpha = \inf(2 - (2k-1)(2-\beta), \frac{1}{2}(2-\beta)) > 0$.

PROOF. The integrand in (3.12) is the error term in (3.5), hence less than

$$\big(H_{y,\varepsilon}H_{x,\varepsilon}\big)^k H_{y,\varepsilon}1(z), \tag{3.13}$$

by the method of (3.7), (3.8) and the proof of Lemma 3.2.

Now, by (2.1), if $\bar{y} \in B_{y,\varepsilon}$,

$$
\begin{aligned}
H_{x,\varepsilon} f(\bar{y}) &\le \|f\| \int G(\bar{y}, \bar{x}) g_x(\bar{x})\, d\bar{x} \\
&\le cG(x-y) \|f\| \int g_x(x)\, d\bar{x} \qquad (3.14)\\
&= cc(\varepsilon) G(x-y) \|f\|;
\end{aligned}
$$

hence, beginning on the right, (3.13) is bounded by

$$cc(\varepsilon)^{2k+1} G(z-y) G^{2k}(y-x); \qquad (3.15)$$

hence (3.12) is bounded by

$$
\begin{aligned}
&cc(\varepsilon)^{2k+1} \int_{|z-y| \ge 4\varepsilon} G(z-y) \left(\int_{|x-y| \ge 4\varepsilon} G^{2k}(x-y)\, d^2x \right) d^2y \\
&\le cc(\varepsilon)^{2k+1} \int_{|u| \ge 4\varepsilon} G^{2k}(u)\, d^2u \\
&= cc(\varepsilon)^{2k+1} \left(\int_{4\varepsilon \le |u| \le 1} \frac{1}{u^{2k(2-\beta)}}\, d^2u + O(1) \right) \qquad (3.16)\\
&= cc(\varepsilon)^{2k+1} \left(\frac{\varepsilon^2}{\varepsilon^{2k(2-\beta)}} + O\left(\log\left(\frac{1}{\varepsilon} \right) \right) \right) \\
&= c\left(c(\varepsilon)\varepsilon^2 + c(\varepsilon)^{2k+1} \log\left(\frac{1}{\varepsilon} \right) \right),
\end{aligned}
$$

where $\log(1/\varepsilon)$ appears only if $2k(2-\beta) = 2$. □

The following lemma summarizes the results of this section.

LEMMA 3.4. *If* $(2k-1)(2-\beta) < 2$, *then*

$$
\begin{aligned}
E_z\left(m(S_\varepsilon(\zeta))^2 \right) = &\sum_{i,j \le k} (-1)^{i+j} \iint_{\Theta_\varepsilon} \sum_{N_{i,j}} \pi(H_{x,\varepsilon}; H_{y,\varepsilon}) 1(z)\, dx\, dy \qquad (3.17)\\
&+ O(\varepsilon^{\bar{\alpha}} c^{2k}(\varepsilon)).
\end{aligned}
$$

4. The cross terms. We let $S_{i,j} \subseteq R_{i,j}$ denote the set of ordered products $\pi(x; y)$ such that at least one of the letters x, y has the property that it never appears twice in a row. We can refine this further by setting $S_{i,j}^1$ ($S_{i,j}^2$) to be that subset of $S_{i,j}$ in which x (respectively y) never appears twice in a row in $\pi(x; y)$. We note that

$$S_{i,j}^1 \cap S_{i,j}^2 = N_{i,j}. \qquad (4.1)$$

Recall the definition of $\lambda_{n,y,\varepsilon}$ from (2.4).

LEMMA 4.1. *For any n and for any x, y, z such that $|x - y| \geq 4\varepsilon$, $|x - z| \geq 4\varepsilon$ we have*

$$E_z(T_{x,\varepsilon} < \zeta; \lambda_{n,y,\varepsilon}(\zeta)) = \sum_{i=1}^{n+1} (-1)^{i-1} \sum_{S^1_{i,n}} \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z). \tag{4.2}$$

PROOF. We again suppress ε's and define the random measure on $\mathbb{R}^n_+$:

$$\bigwedge_{n,y}(B) \doteq \int \cdots \int_{B \cap D_n} \prod_{i=1}^{n} g_y(X_{t_i})\, dt \tag{4.3}$$

with $D_n = \{0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \zeta\}$. We use the notation

$$\bigwedge_{n,y}(B; F) \doteq \bigwedge_{n,y}(B \cap F). \tag{4.4}$$

Setting $t_0 = 0$, $t_{n+1} = \zeta$, we have

$$1_{\{T_x < \zeta\}} \lambda_{n,y}(\zeta) = \sum_{j=0}^{n} \bigwedge_{n,y}(t_j < T_x < t_{j+1}). \tag{4.5}$$

As in Section 3 we have

$$\begin{aligned} \bigwedge_{n,y}(t_j < T_x < t_{j+1}) &= \bigwedge_{n,y}(t_j < t_j + T_x \circ \theta_{t_j} < t_{j+1}) \\ &\quad - \sum_{i=0}^{j-1} \bigwedge_{n,y}(t_i < T_x < t_{i+1}; t_j < t_j + T_x \circ \theta_{t_j} < t_{j+1}), \end{aligned} \tag{4.6}$$

and proceeding in this manner we find that

$$\begin{aligned} &1_{\{T_x < \zeta\}} \lambda_{n,y}(\zeta) \\ &\quad = \sum_{i=1}^{n+1} (-1)^{i-1} \sum_{\substack{A \subseteq \{0,1,\ldots,n\} \\ |A| = i}} \bigwedge_{n,y} \Big(\bigcap_{k \in A} \{t_k + T_x \circ \theta_{t_k} < t_{k+1}\} \Big). \end{aligned} \tag{4.7}$$

We show by induction that

$$E_z\Bigg(\bigwedge_{n,y} \Big(\bigcap_{i \in A} \{t_i + T_x \circ \theta_{t_i} < t_{i+1}\} \Big) \Bigg) = \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z), \tag{4.8}$$

where $\pi(x, y) \in S^1_{i,n}$ is determined by the fact that its x factors occur precisely between those ith and $(i+1)$th y factors for which $i \in A$.

Assume first that $0 \in A$, $n \notin A$. Set $A_0 = A - \{0\}$. Set

$$B(A) = \bigcap_{i \in A} \{t_i + T_x \circ \theta_{t_i} < t_{i+1}\}. \tag{4.9}$$

We have

$$
\begin{aligned}
E_z\Big(\bigwedge_{n,y}(B(A))\Big) &= E_z\Big(\bigwedge_{n,y}(T_x < t_1; B(A_0))\Big)\\
&= E_z\Bigg(\int\cdots\int_{T_x<t_1<\cdots<t_n<\zeta} 1_{B(A_0)}\prod_{j=1}^n g_y(X_{t_j})\,dt_j\Bigg)\\
&= E_z\Bigg(\int\cdots\int_{T_x<t_1<\cdots<t_n<\infty} e^{-t_n}1_{B(A_0)}\prod_{j=1}^n g_y(X_{t_j})\,dt_j\Bigg)\\
&= E_z\Big(e^{-T_x}\Big\{\bigwedge_{n,j}(B(A_0))\Big\}\circ\theta_{T_x}\Big)\\
&= E_z\Big(e^{-T_x}E_{X_{T_x}}\Big(\bigwedge_{n,y}(B(A_0))\Big)\Big)\\
&= E_z\big(e^{-T_x}\pi_0(H_{x,\varepsilon}; L_{y,\varepsilon})1(X_{T_x})\big)\\
&= H_{x,\varepsilon}\pi_0(H_{x,\varepsilon}; L_{y,\varepsilon})1(z),
\end{aligned}
\tag{4.10}
$$

where π_0 is obtained from π by dropping the first factor.

If $n \in A_0$, we proceed similarly, except that in the fourth line we have

$$\exp\big[-(t_n + T_x\circ\theta_{t_n})\big]1_{B(A_0-\{n\})}$$

instead of

$$e^{-t_n}1_{B(A_0)}.$$

If on the other hand $0 \notin A$, then if also $n \notin A$, we have

$$
\begin{aligned}
&E_z\Big(\bigwedge_{n,y}(B(A))\Big)\\
&\quad= E_z\Bigg(\int\cdots\int_{0<t_1<\cdots<t_n<\infty} e^{-t_n}1_{B(A)}\prod_{j=1}^n g_y(X_{t_j})\,dt_j\Bigg)\\
&\quad= E_z\Bigg(\int_0^\infty e^{-t_1}\Bigg\{\int\cdots\int_{t_1\le t_2<\cdots<t_n<\infty} e^{-(t_n-t_1)}1_{B(A)}\prod_{j=2}^n g_y(X_{t_j})\,dt_j\Bigg\}g_y(X_{t_1})\,dt_1\Bigg)\\
&\quad= E_z\Big(\int_0^\infty e^{-t_1}\Big\{\bigwedge_{n-1,y}(B(A-1))\Big\}\circ\theta_{t_1}g_y(X_{t_1})\,dt_1\Big)\\
&\quad= E_z\Big(\int_0^\infty e^{-t_1}E_{X_{t_1}}\Big(\bigwedge_{n-1,y}(B(A-1))\Big)g_y(X_{t_1})\,dt_1\Big)\\
&\quad= L_{y,\varepsilon}\pi_0(H_{x,\varepsilon}; L_{y,\varepsilon})1(z),
\end{aligned}
\tag{4.11}
$$

where $A - 1$ denotes the set of integers obtained by subtracting 1 from each integer in A.

The case $n \in A$ is handled as before. □

LEMMA 4.2. *If* $(2k-1)(2-\beta) < 2$, *then for any* $n \le k$ *we have*

$$E_z\left(\left(\int \lambda_{n,y,\varepsilon}(\zeta)\,d^2y\right) m(S_\varepsilon(\zeta))\right) = \iint_{\Theta_\varepsilon} \sum_{i=1}^{(n+1)\wedge k} (-1)^{i-1} \sum_{S^1_{i,n}} \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z)\,dx\,dy + O(\varepsilon^{\bar{\alpha}} c^{2k}(\varepsilon)). \tag{4.12}$$

PROOF. As before we have

$$E_z\left(\left(\int \lambda_{n,y,\varepsilon}(\zeta)\,dy\right) m(S_\varepsilon(\zeta))\right) = \iint E_z(\lambda_{n,y,\varepsilon}(\zeta); T_x < \zeta)\,d^2x\,d^2y. \tag{4.13}$$

We have

$$\begin{aligned} &\iint_{|x-y|\le 4\varepsilon} E_z(\lambda_{n,y,\varepsilon}(\zeta); T_x < \zeta)\,d^2x\,d^2y \\ &\quad \le \int E_z(\lambda_{n,y,\varepsilon}(\zeta)) \left(\int_{|x-y|\le 4\varepsilon} d^2x\right) d^2y \\ &\quad \le c\varepsilon^2 \int E_z(\lambda_{n,y,\varepsilon}(\zeta))\,d^2y \\ &\quad \le c\varepsilon^2 c(\varepsilon), \end{aligned} \tag{4.14}$$

as in the proof of Lemma 2.2. The case $|x-z| \le 4\varepsilon$ follows similarly, while for $|y-z| \le 4\varepsilon$ we can use the previous lemma and need only control:

$$\iint_{\Theta_\varepsilon} \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z)\,d^2x\,d^2y, \tag{4.15}$$

for $\pi \in S^1_{i,n}$, $1 \le i \le n+1$.

Using (2.2) and (3.14), our last integral is bounded by

$$\begin{aligned} c \iint_{|y-z|\le 4\varepsilon} c(\varepsilon) G(x-y)\,dx\,dy &\le c c(\varepsilon) \int_{|y-z|\le 4\varepsilon} dy \\ &\le c c(\varepsilon) \varepsilon^2. \end{aligned} \tag{4.16}$$

Finally, we show that if $\pi \in S^1_{k+1,k}$, then

$$\iint_{\Theta_\varepsilon} \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z)\, d^2x\, d^2y = O(\varepsilon^{\bar{\alpha}} c^{2k}(\varepsilon)), \tag{4.17}$$

but $\pi \in S^1_{k+1,k}$ is only possible if $\pi(x;y) = xyxy \cdots xyx$, that is, $\pi \in S^2_{k+1,k}$. The analysis used in the proof of Lemma 3.3 then establishes (4.17). □

As in Section 2, we can now state the following.

LEMMA 4.3. *If* $(2k-1)(2-\beta)$, *then, for any* $n \le k$,

$$\begin{aligned} &E_z(c^n(\varepsilon) l_{n,\varepsilon}(\zeta) m(S_\varepsilon(\zeta))) \\ &= \iint_{\Theta_\varepsilon} \sum_{i=1}^{(n+1)\wedge k} (-1)^{i-1} \sum_{S^1_{i,n}} \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z)\, dx\, dy + O(\varepsilon^{\bar{\alpha}} c^{2k}(\varepsilon)). \end{aligned} \tag{4.18}$$

5. Analysis of stable sausages.

LEMMA 5.1. *If* $(2k-1)(2-\beta) < 2$, *then*

$$\begin{aligned} &E_z\left(\left\{ m(S_\varepsilon(\zeta)) + \sum_{n=1}^{k} (-c(\varepsilon))^n l_{n,\varepsilon}(\zeta) \right\}^2\right) \\ &= \sum_{i,j\le k} (-1)^{i+j} \\ &\quad \times \iint_{\Theta_\varepsilon} \left\{ 2 \sum_{S^1_{i,j} - N_{i,j}} [\pi(L_{x,\varepsilon}; L_{y,\varepsilon}) - \pi(H_{x,\varepsilon}; L_{y,\varepsilon})] 1(z)\, dx\, dy \right. \\ &\qquad + \sum_{N_{i,j}} [\pi(L_{x,\varepsilon}; L_{y,\varepsilon}) - \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) \\ &\qquad \left. - \pi(L_{x,\varepsilon}; H_{y,\varepsilon}) + \pi(H_{x,\varepsilon}; H_{y,\varepsilon})] 1(z) \right\} dx\, dy + O(\varepsilon^{\bar{\alpha}} c^{2k}(\varepsilon)). \end{aligned} \tag{5.1}$$

REMARK. We will control the terms on the right-hand side in Lemmas 5.2 and 5.3.

PROOF OF LEMMA 5.1. We first expand the square in (5.1), then use Lemmas 4.3, 3.4, 2.3 and 2.4 to find that

$$E_z\left(\left\{m(S_\varepsilon(\zeta)) + \sum_{n=1}^{k} (-c(\varepsilon))^n l_{n,\varepsilon}(\zeta)\right\}^2\right)$$

$$= E_z\left(m\left(S_\varepsilon(\zeta)^2\right)\right) + 2\sum_{n=1}^{k} (-1)^n E_z(c^n(\varepsilon) l_{n,\varepsilon}(\zeta) m(S_\varepsilon(\zeta)))$$

$$+ \sum_{i,j\le k} (-1)^{i+j} E_z\left(c^i(\varepsilon) l_{i,\varepsilon}(\zeta) c^j(\varepsilon) l_{j,\varepsilon}(\zeta)\right)$$

$$(5.2)\qquad = \sum_{i,j\le k} (-1)^{i+j} \int_{\Theta_\varepsilon} \sum_{N_{i,j}} \pi(H_{x,\varepsilon}; H_{y,\varepsilon}) 1(z)\, dx\, dy$$

$$+ 2\sum_{i,j\le k} (-1)^{i-1+j} \int_{\Theta_\varepsilon} \sum_{S^1_{i,j}} \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) 1(z)\, dx\, dy$$

$$+ \sum_{i,j\le k} (-1)^{i+j} \int_{\Theta_\varepsilon} \sum_{S_{i,j}} \pi(L_{x,\varepsilon}; L_{y,\varepsilon}) 1(z)\, dx\, dy + O(\varepsilon^{\bar\alpha} c^{2k}(\varepsilon)),$$

and reorganizing gives Lemma 5.1. □

LEMMA 5.2. *If* $\pi \in N_{i,j}$, *then*

$$(5.3)\qquad \iint_{\Theta_\varepsilon} \{\pi(L_{x,\varepsilon}; L_{y,\varepsilon}) - \pi(H_{x,\varepsilon}; L_{y,\varepsilon}) - \pi(L_{x,\varepsilon}; H_{y,\varepsilon}) + \pi(H_{x,\varepsilon}; H_{y,\varepsilon})\} 1(z)\, dx\, dy = O(\varepsilon c^2(\varepsilon)).$$

REMARK. Note the appearance of $\varepsilon c^2(\varepsilon)$, as opposed to $\varepsilon^2 c(\varepsilon)$.

PROOF OF LEMMA 5.2. Recall that when $\pi \in N_{i,j}$ the x and y factors in $\pi(x, y)$ alternate.

We first write

$$\{\pi(L_{x,\varepsilon}; L_{y,\varepsilon}) - \pi(H_{x,\varepsilon}; L_{y,\varepsilon})\} 1(z)$$

as a sum of terms, each of which contains one factor $L_{x,\varepsilon} - H_{x,\varepsilon}$ followed by $L_{y,\varepsilon}$, or 1. Since $(L_{x,\varepsilon} - H_{x,\varepsilon})1 = 0$ by (2.1), we can extract a factor

$c(\varepsilon)\omega_\varepsilon G(x-y)$ as in the proof of Lemma 2.4. As described there, we can then bound the integral by

$$(5.4)\qquad \begin{aligned} c^2(\varepsilon) \iint_{\substack{|z-x|\geq 4\varepsilon \\ |x-y|\geq 4\varepsilon}} G(z-x)\omega_\varepsilon G(x-y)\,dx\,dy \\ \leq \varepsilon c^2(\varepsilon)\left(\int_{4\varepsilon\leq|u|\leq 1} 1/u^{1+(2-\beta)}\,d^2u + O(1)\right) + O(\varepsilon c^2(\varepsilon)), \end{aligned}$$

since $(2-\beta)<1$.

Similarly, we write

$$\{\pi(L_{x,\varepsilon};H_{y,\varepsilon}) - \pi(H_{x,\varepsilon};H_{y,\varepsilon})\}1(z)$$

as a sum of terms, each of which has a factor $L_{x,\varepsilon}-H_{x,\varepsilon}$. However, if such a factor is followed by $H_{y,\varepsilon}$, then we cannot immediately extract a $c(\varepsilon)\omega_\varepsilon G(x-y)$. But since, as noted, $\{L_{x,\varepsilon}-H_{x,\varepsilon}\}1=0$, we can replace $H_{y,\varepsilon}(\bar{x},\cdot)$ by

$$(5.5)\qquad H_{y,\varepsilon}(\bar{x},\cdot) - H_{y,\varepsilon}(x,\cdot).$$

Now, by (2.1),

$$(5.6)\qquad \{H_{y,\varepsilon}(\bar{x},\cdot) - H_{y,\varepsilon}(x,\cdot)\}1 = \int\{G(\bar{x},\bar{y}) - G(x,\bar{y})\}g_{y,\varepsilon}(\bar{y})\,d\bar{y},$$

so that, if (5.5) is followed by $H_{x,\varepsilon}(\bar{y},\cdot)$ we can replace this in turn by

$$H_{x,\varepsilon}(\bar{y},\cdot) - H_{x,\varepsilon}(y,\cdot),$$

introducing an error term containing the factor (5.6), which is bounded by $c(\varepsilon)\omega_\varepsilon G(x-y)$. We continue in this manner until we reach a factor of the form $L_{x,\varepsilon}$ or 1. This completes the proof of Lemma 5.2. □

LEMMA 5.3. *If* $\pi\in S^1_{i,j}-N_{i,j}$, *then*

$$(5.7)\qquad \iint_{\Theta_\varepsilon}\{\pi(L_{x,\varepsilon};L_{y,\varepsilon}) - \pi(H_{x,\varepsilon};L_{y,\varepsilon})1(z)\,dx\,dy\} = O(\varepsilon c^2(\varepsilon)).$$

PROOF. We can write the integrand as a sum of terms each of which contains a factor $L_{x,\varepsilon}-H_{x,\varepsilon}$ followed by $L_{y,\varepsilon}$, so that Lemma 5.3 follows from the first half of the proof of Lemma 5.2. □

Summarizing, we have the following lemma.

LEMMA 5.4. *If* $(2k-2)(2-\beta)<1$, *then*

$$(5.8)\qquad \left\| m(S_\varepsilon(\zeta)) + \sum_{n=1}^{k}(-c(\varepsilon))^n l_{n,\varepsilon}(\zeta)\right\|_2 = O(\varepsilon^{\tilde{\alpha}/2}c^k(\varepsilon)),$$

where

$$\tilde{\alpha} = \inf\left(1 - (2k - 2)(2 - \beta), \tfrac{1}{2}(2 - \beta)\right).$$

REMARK. $\tilde{\alpha} < \alpha$, since $2 - \beta < 1$.

6. Renormalized intersection local times. As in the introduction, let $f(x) \geq 0$ be a continuous function with support in $B(0,1)$, and $\int f(x)\,d^2x = 1$. We set $f_\varepsilon(x) = (1/\varepsilon^2) f(x/\varepsilon)$ and

$$h(\varepsilon) = \int f_\varepsilon(x) G(x)\, d^2x. \tag{6.1}$$

We now define

$$\begin{aligned} \Gamma_{n,\varepsilon}(t) &= \sum_{j=1}^{n} (-h(\varepsilon))^{n-j} \binom{n-1}{j-1} \alpha_{j,\varepsilon}(t) \\ &= \int_{0 \leq t_1 \leq \cdots \leq t_n < t} \cdots \int dt_1 \prod_{i=2}^{n} \left(f_\varepsilon(X_{t_i} - X_{t_{i-1}})\, dt_i - h(\varepsilon) \delta_{t_{i-1}}(dt_i) \right), \end{aligned} \tag{6.2}$$

with $\alpha_{j,\varepsilon}(t)$ defined in (1.6).

LEMMA 6.1. *If* $(2k - 1)(2 - \beta) < 2$, *then*

$$\| l_{k,\varepsilon}(\zeta) - \Gamma_{k,\varepsilon}(\zeta) \|_2 \leq c\varepsilon^{\alpha/2}, \tag{6.3}$$

where $\alpha = 2 - (2k - 1)(2 - \beta) > 0$.

Applying this to any $j \leq k$, we have

$$\begin{aligned} \| l_{j,\varepsilon}(\zeta) - \Gamma_{j,\varepsilon}(\zeta) \|_2 &\leq c\varepsilon^{(1/2)(2 - (2j-1)(2-\beta))} \\ &= c\varepsilon^{(k-j)(2-\beta) + \alpha/2}. \end{aligned} \tag{6.4}$$

Combined with Lemma 5.4 we have the following corollary immediately.

COROLLARY 6.2. *If* $(2k - 2)(2 - \beta) < 1$, *then*

$$\left\| m(S_\varepsilon(\zeta)) + \sum_{n=1}^{k} (-c(\varepsilon))^n \Gamma_{n,\varepsilon}(\zeta) \right\|_2 \leq cc^k(\varepsilon) \varepsilon^{\tilde{\alpha}/2}. \tag{6.5}$$

In addition, we prove in Rosen (1991) that if $(2k - 1)(2 - \beta) < 2$, then, for $j \leq k$, $\Gamma_{j,\varepsilon}(\zeta)$ converges in $L^2(dQ_z)$ and the limits Γ_j satisfy

$$\| \Gamma_{j,\varepsilon}(\zeta) - \Gamma_j \|_2 \leq c\varepsilon^{\alpha/2 + (k-j)(2-\beta)}, \tag{6.6}$$

so that we get the following corollary.

COROLLARY 6.3. *If* $(2k-2)(2-\beta) < 1$, *then*

$$\left\| m(S_\varepsilon(\zeta)) + \sum_{n=1}^{k} (-c(\varepsilon))^n \Gamma_n \right\|_2 \le c c^k(\varepsilon)\varepsilon^{\tilde{\alpha}/2}. \tag{6.7}$$

PROOF OF LEMMA 6.1. We will recast our lemma in a form which will allow the reader to see how it follows from the proof of Proposition 2 of Rosen (1991):

$$\begin{aligned} &\|l_{k,\varepsilon}(\zeta) - \Gamma_{k,\varepsilon}(\zeta)\|_2^2 \\ &\quad = E\big(l_{k,\varepsilon}^2(\zeta)\big) - 2E\big(l_{k,\varepsilon}(\zeta)\Gamma_{k,\varepsilon}(\zeta)\big) + E\big(\Gamma_{k,\varepsilon}^2(\zeta)\big), \end{aligned} \tag{6.8}$$

and we first study each term separately.

By Lemma 2.3,

$$E\big(l_{k,\varepsilon}^2(\zeta)\big) = \sum_{R_{k,k}} c^{-2k}(\varepsilon) \iint_{\Theta_\varepsilon} \pi\big(L_{x,\varepsilon} \,\bar{;}\, L_{y,\varepsilon}\big)1(z)\, d^2x\, d^2y + O(\varepsilon^\alpha). \tag{6.9}$$

Let

$$J(\pi) = \frac{1}{c^{2k}(\varepsilon)} \iint_{\Theta_\varepsilon} \pi\big(L_{x,\varepsilon} \,\bar{;}\, L_{y,\varepsilon}\big)1(z)\, d^2x\, d^2y. \tag{6.10}$$

We will say that a sequence $S = \{i, i+1, \dots, i+l, \bar{i}\}$, $\bar{i} = i + l + 1$, is elementary for π if

$$\pi_j \ne \pi_{i-1} \quad \text{and} \quad \pi_j \ne \pi_{\bar{i}}, \tag{6.11}$$

for all j, $i \le j < \bar{i}$. This implies that $\pi_j = \pi_i$, $i \le j < \bar{i}$.

With such an elementary sequence we associate a function $G_S(z)$ of the $2k$ variables

$$Z_i^j, \qquad j = 1, 2;\ i = 1, \dots, k,$$

defined by

$$G_S(Z) = G\big(\mathscr{I}_{i+1}^j\big) \cdots G\big(\mathscr{I}_{i+l}^j\big) \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} G\big(Z_{i+m}^j - Z_{\bar{i}}^{\bar{j}}\big), \tag{6.12}$$

where

$$\mathscr{I}_{i+m}^j = Z_{i+m}^j - Z_{i+m-1}^j.$$

Using the fact that [see (2.1)]

$$\int G(z, \bar{z}) g_{y,\varepsilon}(\bar{z})\, d\bar{z} = 1, \quad \text{if } z \in B(y, \varepsilon),$$

we find that, with $h_{y,\varepsilon}(\cdot) = (1/c(\varepsilon))g_{y,\varepsilon}(\cdot)$,

$$(6.13)\qquad J(\pi) = \iint_{\Theta_\varepsilon} dx\,dy \int \prod_{i=1}^{k} h_{x,\varepsilon}(Z_i^1) h_{y,\varepsilon}(Z_i^2) \prod_{\varepsilon(\pi)} G_S(Z)\,dZ,$$

where $\varepsilon(\pi)$ is the set of elementary sequences relative to π.

We next consider the term

$$(6.14)\qquad \begin{aligned} &E\left(\Gamma_{k,\varepsilon}^2(\zeta)\right) \\ &= E\left(\left[\int\cdots\int_{0\le t_1\le\cdots\le t_k<\zeta} dt_1 \prod_{i=2}^{k} \left(f_\varepsilon(X_{t_i} - X_{t_{i-1}})\,dt_i - h(\varepsilon)\delta_{t_{i-1}}(dt_i)\right)\right]^2\right) \\ &= E\left(\int\cdots\int_{0\le t_1^j\le t_2^j\le\cdots\le t_k^j<\zeta} \prod_{j=1}^{2} dt_1^j \prod_{i=2}^{k} \left(f_\varepsilon(X_{t_i^j} - X_{t_{i-1}^j})\,dt_i^j - h(\varepsilon)\delta_{t_{i-1}^j}(dt_i^j)\right)\right) \\ &= \sum \tilde{I}(D), \end{aligned}$$

where we identify π with the ordering $0, t_i^j$ as before,

$$\tilde{I}(D) = E\left[\int\cdots\int_D \prod_{j=1}^{2} dt_1^j \prod_{i=2}^{k} \left(f_\varepsilon(X_{t_i^j} - X_{t_{i-1}^j})\,dt_i^j - h(\varepsilon)\delta_{t_{i-1}^j}(t_i^j)\right)\right]$$

and D runs over the set of orderings of the $2k$ points t_i^j, $j = 1, 2$, $i = 1, \ldots, k$, such that $0 \le t_1^j \le t_2^j \le \cdots \le t_k^j < \zeta$, $j = 1, 2$.

We now associate a $\pi \in R_{k,k}$ to each ordering D. Simply set $\pi_i = x$ or y depending on whether the ith element in D is a $t_\cdot^1$ or a $t_\cdot^2$. With this π we easily check that

$$(6.15)\qquad \begin{aligned} \tilde{I}(D) &= \int\cdots\int \left(\prod_{\substack{i=2\\ j=1,2}}^{k} f_\varepsilon(\mathscr{P}_i^j)\right) \prod_{\varepsilon(\pi)} G_S(Z)\,dZ \\ &= \iint dx\,dy \int h_{x,\varepsilon}(Z_1^1) h_{y,\varepsilon}(Z_1^2) \prod_{\substack{i=2\\ j=1,2}}^{k} f_\varepsilon(\mathscr{P}_i^j) \prod_{\varepsilon(\pi)} G_S(Z)\,dZ \\ &= I(\pi) + O(\varepsilon^\alpha), \end{aligned}$$

where

$$(6.16)\qquad I(\pi) = \iint_{\Theta_\varepsilon} dx\,dy \int h_{x,\varepsilon}(Z_1^1) h_{y,\varepsilon}(Z_1^2) \prod_{i=2}^{k} f_\varepsilon(\mathscr{P}_i^2) \prod_{\varepsilon(\pi)} G_S(Z)\,dZ,$$

and the last line of (6.15) follows as in the proof of Lemma 2.2, using

$$\sup_a \int f_\varepsilon(z-a)G(z)\,d^2z \le \frac{c}{c(\varepsilon)}.$$

A similar analysis shows that

$$E\left[l_{k,\varepsilon}(\zeta)\Gamma_{k,\varepsilon}(\zeta)\right] = \sum_{R_{k,k}} K(\pi) + O(\varepsilon^\alpha), \tag{6.17}$$

where

$$\begin{aligned} K(\pi) = \iint_{\Theta_\varepsilon} dx\,dy \int &\left(h_{x,\varepsilon}(Z_1^1) \prod_{i=2}^{k} f_\varepsilon(\mathscr{I}_i^1) \right) \\ &\times \left(\prod_{i=1}^{k} h_{y,\varepsilon}(Z_i^2) \right) \prod_{\varepsilon(\pi)} G_S(Z)\,dZ. \end{aligned} \tag{6.18}$$

Thus, we consider

$$\begin{aligned} &J(\pi) - 2K(\pi) + I(\pi) \\ &\quad = \iint_{\Theta_\varepsilon} dx\,dy \int F_{\varepsilon,x}(Z_\cdot^1) F_{\varepsilon,y}(Z_\cdot^2) \prod_{\varepsilon(\pi)} G_s(Z)\,dZ, \end{aligned} \tag{6.19}$$

where

$$F_{\varepsilon,v}(Z_\cdot^j) = \left(\prod_{i=1}^{k} h_{v,\varepsilon}(Z_i^j) \right) - \left(h_{v,\varepsilon}(Z_1^j) \prod_{i=2}^{k} f_\varepsilon(\mathscr{I}_i^j) \right),$$

and it suffices to show (6.19) is $O(\varepsilon^\alpha)$.

We expand

$$\begin{aligned} F_{\varepsilon,v}(Z_\cdot^j) &= \sum_{m=2}^{k} (-1)^{k-m} h_{v,\varepsilon}(Z_1^j) \\ &\quad \times \prod_{i=2}^{m-1} h_{v,\varepsilon}(Z_i^j)\left(h_{v,\varepsilon}(Z_m^j) - f_\varepsilon(\mathscr{I}_m^j)\right) \prod_{i=m+1}^{k} f_\varepsilon(\mathscr{I}_i^j) \end{aligned} \tag{6.20}$$

and write (6.19) as a sum of many terms, each of which has a factor of the form

$$h_{x,\varepsilon}(Z_\cdot^1) - f_\varepsilon(\mathscr{I}_\cdot^1) \tag{6.21}$$

and also a factor of the form

$$h_{y,\varepsilon}(Z_\cdot^2) - f_\varepsilon(\mathscr{I}_\cdot^2); \tag{6.22}$$

the proof now essentially follows the lines of the proof of Proposition 2 of Rosen (1991) if we recall that $\int f_\varepsilon = \int h_{x,\varepsilon} = 1$. □

7. Theorem 1.2.

LEMMA 7.1.

$$kc_0(\varepsilon) \le c(\varepsilon) \le c_0(\varepsilon), \quad \textit{for some } k > 0, 0 < \varepsilon \le 1. \tag{7.1}$$

PROOF. From the resolvent equation,

$$\begin{aligned} 0 \le G_0(x) - G(x) &= G_0 * G(x) \\ &= \frac{1}{(2\pi)^2} \int \frac{e^{ipx}}{p^\beta(1+p^\beta)}\, d^2p \\ &\doteq V(x) \end{aligned} \tag{7.2}$$

is bounded and continuous. Since

$$\frac{1}{c(\varepsilon)} = \inf_\mu \int\int G(x-y)\, d\mu(x)\, d\mu(y),$$

$$\frac{1}{c_0(\varepsilon)} = \inf_\mu \int\int G_0(x-y)\, d\mu(x)\, d\mu(y),$$

where the inf is over all probability measures μ supported on $B(0,\varepsilon)$, (7.2) shows that

$$d(\varepsilon) \doteq \frac{1}{c_0(\varepsilon)} - \frac{1}{c(\varepsilon)} \le \sup \int\int V(x-y)\, d\mu(x)\, d\mu(y) = O(1). \tag{7.3}$$

Thus,

$$c(\varepsilon) = c_0(\varepsilon) \frac{1}{1 - c_0(\varepsilon) d(\varepsilon)}, \tag{7.4}$$

which completes the proof of Lemma 7.1. □

We shall need a more detailed estimate for $d(\varepsilon)$. We have

$$|V(x) - V(0)| \le cx^\delta \int \frac{p^\delta}{p^\beta(1+p^\beta)}\, d^2p, \tag{7.5}$$

for any $\delta \le 1$. Thus, for $|x| \le 1$,

$$|V(x) - V(0)| \le \begin{cases} c|x|, & \text{if } \beta > \frac{3}{2}, \\ c|x|^{2\beta - 2 - \bar\delta}, & \text{if } \beta \le \frac{3}{2} \end{cases}, \tag{7.6}$$

for any $\bar\delta > 0$.

Now, for any $j \geq 2$,

$$\begin{aligned} 2\beta - 2 - \bar{\delta} &= 2 - 2(2-\beta) - \bar{\delta} \\ &= \tfrac{1}{2}(2 - (2j-1)(2-\beta)) + 1 - \bar{\delta} + \left(j - \tfrac{5}{2}\right)(2-\beta) \\ &> \tfrac{1}{2}(2 - (2j-1)(2-\beta)), \end{aligned}$$

for $\bar{\delta} > 0$ sufficiently small.

Since, obviously, we also have

$$1 > \tfrac{1}{2}(2 - (2j-1)(2-\beta)),$$

we have that, if $\alpha = 2 - (2k-1)(2-\beta) > 0$,

$$|V(x) - V(0)| \leq c|x|^{\alpha/2 + (k-j)(2-\beta)}. \tag{7.7}$$

Returning to the proof of Lemma 7.1, we have

$$d(\varepsilon) = V(0) + O(\varepsilon^{\alpha/2 + (k-2)(2-\beta)}), \tag{7.8}$$

and, comparing $h(\varepsilon) = \int f_\varepsilon(x) G(x)$ and $h_0(\varepsilon) = \int f_\varepsilon(x) G_0(x)$, we also have

$$h_0(\varepsilon) = h(\varepsilon) + V(0) + O(\varepsilon^{\alpha/2 + (k-2)(2-\beta)}). \tag{7.9}$$

PROOF OF THEOREM 1.2. We have, for $|x| < 1$,

$$\begin{aligned} \left(\frac{1}{1+x}\right)^n &= \sum_{i=n}^{\infty} (-1)^{i-n} \binom{i-1}{n-1} x^{i-n} \\ &= \sum_{i=n}^{k} (-1)^{i-n} \binom{i-1}{n-1} x^{i-n} + O(x^{k+1-n}), \end{aligned} \tag{7.10}$$

hence from (6.6), (7.4), (7.8) and (7.9) we get, in $L^2(dQ)$,

$$\begin{aligned} &\sum_{n=1}^{k} (-c(\varepsilon))^n \Gamma_{n,\varepsilon}(\zeta) \\ &= \sum_{n=1}^{k} (-c_0(\varepsilon))^n \left(\frac{1}{1 - c_0(\varepsilon) d(\varepsilon)}\right)^n \Gamma_{n,\varepsilon}(\zeta) \\ &= \sum_{n=1}^{k} (-c_0(\varepsilon))^n \left(\sum_{i=n}^{k} (c_0(\varepsilon) d(\varepsilon))^{i-n} \binom{i-1}{n-1}\right) \Gamma_{n,\varepsilon}(\zeta) + O(c_0^{k+1}(\varepsilon)) \\ &= \sum_{i=1}^{k} (-c_0(\varepsilon))^i \left(\sum_{n=1}^{i} \binom{i-1}{n-1} (-d(\varepsilon))^{i-n} \Gamma_{n,\varepsilon}(\zeta)\right) + O(c_0^{k+1}(\varepsilon)) \\ &= \sum_{i=1}^{k} (-c_0(\varepsilon))^i \left(\sum_{n=1}^{i} \binom{i-1}{n-1} (h(\varepsilon) - h_0(\varepsilon))^{i-n} \Gamma_{n,\varepsilon}(\zeta)\right) + O(c_0^{k+1}(\varepsilon)) \\ &= \sum_{i=1}^{k} (-c_0(\varepsilon))^i \gamma_{i,\varepsilon}(\zeta) + O(c_0^{k+1}(\varepsilon)), \end{aligned} \tag{7.11}$$

as follows on comparing (1.7) and (6.2).

Finally, from (6.6) and the preceding, for $2 \le i \le k$,

$$\gamma_{i,\varepsilon}(\zeta) - \sum_{n=1}^{i} \binom{i-1}{n-1} (-V(0))^{i-n} \Gamma_n$$

$$(7.12) \qquad = \sum_{n=1}^{i} \binom{i-1}{n-1} \left[(h(\varepsilon) - h_0(\varepsilon))^{i-n} \Gamma_{n,\varepsilon}(\zeta) - (-V(0))^{i-n} \Gamma_n \right]$$

$$= O(\varepsilon^{(k-i)(2-\beta)+\alpha/2}),$$

which, together with (7.11) proves Theorem 1.2 [recall $\gamma_{1,\varepsilon}(\zeta) = \Gamma_{1,\varepsilon}(\zeta) = \zeta$], with

$$\gamma_i \doteq \sum_{n=1}^{i} \binom{i-1}{n-1} (-V(0))^{i-n} \Gamma_n. \qquad \square$$

8. Asymptotics for nonrandom times.

PROOF OF THEOREM 1.1. The random variables γ_n, $n \le k$, which appear in the statement of Theorem 1.2 are constructed as $L^2(d\mathbb{P} \otimes e^{-t}\,dt)$ limits, hence only defined a.e. We choose a representative of the L^2 equivalence class of γ_n and denote it by $\tilde{\gamma}_n(t, w)$. By Fubini's theorem, we have

$$(8.1) \qquad \tilde{\gamma}_n(t) \doteq \tilde{\gamma}_n(t, w) \in L^2(d\mathbb{P}), \quad \text{for a.e. } t,$$

and we can restate our previous results as

$$(8.2) \qquad \int_0^\infty e^{-t} \left\| m(S_\varepsilon(t)) + \sum_{n=1}^{k} (-c_0(\varepsilon))^n \tilde{\gamma}_n(t) \right\|_2^2 dt \le c c_0^{2(k+\delta)}(\varepsilon),$$

for some $\delta \ge 0$.

We can assume that $k \ge 2$ and that k is the largest integer such that $(2k-2)(2-\beta) < 1$. Choosing $\delta \ge 0$ small, we have

$$(8.3) \qquad 2(k+\delta)(2-\beta) > 1,$$

$$(8.4) \qquad (k+\delta)(2-\beta) < 1.$$

Taking $\varepsilon_i = 1/i^2$, this implies that

$$(8.5) \qquad \begin{aligned} &\int_0^\infty e^{-t} \sum_{i=1}^{\infty} c_0^{-(k+\delta)}(\varepsilon_i) \left\| m\big(S_{\varepsilon_i}(t)\big) + \sum_{n=1}^{k} \big(-c_0(\varepsilon_i)\big)^n \tilde{\gamma}_n(t) \right\|_2^2 dt \\ &\quad \le c \sum_{i=1}^{\infty} c_0^{k+\delta}(\varepsilon_i) = c \sum_{i=1}^{\infty} \left(\frac{1}{i^2} \right)^{(k+\delta)(2-\beta)} < \infty. \end{aligned}$$

Therefore, for a.e. t, we have

$$(8.6) \qquad \lim_{i \to \infty} c_0^{-(k+\delta)/2}(\varepsilon_i) \left[m\big(S_{\varepsilon_i}(t)\big) + \sum_{n=1}^{[k/2]} \big(-c_0(\varepsilon_i)\big)^n \tilde{\gamma}_n(t) \right] = 0$$

a.s. and in $L^2(d\mathbb{P})$. Note that we have simply dropped the terms for $n > [k/2]$, since $\delta > 0$ is small.

By (8.4),

$$\lim_{i\to\infty} c_0^{-(k+\delta)/2}(\varepsilon_{i+1}) - c_0^{-(k+\delta)/2}(\varepsilon_i) = 0, \tag{8.7}$$

and, therefore, for t as above,

$$\lim_{i\to\infty} \sup_{\varepsilon_{i+1}\le\varepsilon\le\varepsilon_i} \Bigg| \sum_{n=1}^{[k/2]} (-c_0(\varepsilon))^{n-((k+\delta)/2)} \tilde{\gamma}_n(t) - \sum_{n=1}^{[k/2]} (-c_0(\varepsilon_i))^{n-((k+\delta)/2)} \tilde{\gamma}_n(t) \Bigg| = 0 \tag{8.8}$$

a.s. and in $L^2(d\mathbb{P})$.

Equation (8.6) now shows that

$$\lim_{i\to\infty} c_0^{-(k+\delta)/2}(\varepsilon_i) m(S_{\varepsilon_i}(t)) - c_0^{-(k+\delta)/2}(\varepsilon_{i+1}) m(S_{\varepsilon_{i+1}}(t)) = 0; \tag{8.9}$$

therefore, using monotonicity, we have that, for any $\varepsilon_{i+1} \le \varepsilon \le \varepsilon_i$,

$$\begin{aligned} c_0^{-(k+\delta)/2}(\varepsilon_i)&\left[m(S_{\varepsilon_{i+1}}(t)) - m(S_{\varepsilon_i}(t))\right] \\ &\le c_0^{-(k+\delta)/2}(\varepsilon) m(S_\varepsilon(t)) - c_0^{-(k+\delta)/2}(\varepsilon_i) m(S_{\varepsilon_i}(t)) \\ &\le \left[c_0^{-(k+\delta)/2}(\varepsilon_{i+1}) - c_0^{-(k+\delta)/2}(\varepsilon_i)\right] m(S_{\varepsilon_i}(t)), \end{aligned}$$

hence, for a.e. t,

$$\lim_{\varepsilon\to 0} c_0^{-(k+\delta)/2}(\varepsilon)\left[m(S_\varepsilon(t)) + \sum_{n=1}^{[k/2]} (-c_0(\varepsilon))^n \tilde{\gamma}_n(t)\right] = 0 \tag{8.10}$$

a.s. and in $L^2(d\mathbb{P})$, for all $0 < \varepsilon \le 1$.

We need (8.10) for all t, but we cannot show this directly. Rather, we redefine the random variables $\tilde{\gamma}_n(t)$.

Take some $s > 0$, such that (8.10) holds for $t = s$, and define

$$\gamma_n(rs, \omega) \doteq r^{2/\beta + n(1-2/\beta)} \tilde{\gamma}_n(s, \omega_r), \tag{8.11}$$

where

$$\omega_r(u) = r^{-1/\beta}\omega(ru). \tag{8.12}$$

The scaling property of the stable process $X_\cdot$ says precisely that $\omega \to \omega_\Gamma$ is a measure-preserving transformation of $(\Omega, d\mathbb{P})$.

In particular, we see that $\gamma_n(rs) \doteq \gamma_n(rs, \omega) \in L^2(d\mathbb{P})$ and

$$\lim_{\varepsilon\to 0} c_0^{-(k+\delta)/2}\left(\frac{\varepsilon}{r^{1/\beta}}\right)\left[m\left(S_{\frac{\varepsilon}{r^{1/\beta}}}(s)\right)(\omega_r) + \sum_{n=1}^{[k/2]} \left(-c_0\left(\frac{\varepsilon}{r^{1/\beta}}\right)\right)^n \tilde{\gamma}_n(s, \omega_r)\right] \tag{8.13}$$

converges a.s. and in $L^2(d\mathbb{P})$. But

$$c_0\left(\frac{\varepsilon}{r^{1/\beta}}\right) = \left(\frac{1}{r^{1/\beta}}\right)^{2-\beta} c_0(\varepsilon) = r^{1-2/\beta}c_0(\varepsilon) \tag{8.14}$$

and

$$\begin{aligned} m\left(S_{\frac{\varepsilon}{r^{1/\beta}}}(s)\right)(\omega_r) &= m\left(\left\{y \inf_{0\le t\le rs} d\left(y, r^{-1/\beta}X_t\right) \le \varepsilon/r^{1/\beta}\right\}\right) \\ &= m\left(r^{-1/\beta}\left\{z \inf_{0\le t\le rs} d(z, X_t) \le \varepsilon\right\}\right) \\ &= r^{-2/\beta}m(S_\varepsilon(rs))(\omega), \end{aligned} \tag{8.15}$$

so that (8.11) and (8.13) give

$$\lim_{\varepsilon\to 0} c_0^{-(k+\delta)/2}(\varepsilon)\left[m(S_\varepsilon(rs)) + \sum_{n=1}^{[k/2]} (-c_0(\varepsilon))^n \gamma_n(rs)\right] = 0 \tag{8.16}$$

a.s. and in $L^2(d\mathbb{P})$, for all $r > 0$. This completes the proof of Theorem 1.1. □

NOTE. If we could show that

$$\left\| m(S_\varepsilon(\zeta)) + \sum_{n=1}^{k} (-c_0(\varepsilon))^n \gamma_n \right\|_{2m} = O(c^{k+\delta}(\varepsilon)),$$

then the preceding should give us terms up to

$$\left[\frac{2m-1}{2m}k\right]$$

instead of $[k/2]$.

9. Asymptotics of the expected value.

PROOF OF THEOREM 1.3. If $(4k-2)(2-\beta) < 1$, then by Theorem 1.1,

$$E(m(S_\varepsilon(1))) = \sum_{n=1}^{k} (-1)^{n-1} c_0^n(\varepsilon) E(\gamma_n(1)) + o(c^k(\varepsilon)), \tag{9.1}$$

as $\varepsilon \to 0$.

Using (8.14) and (8.15), this is equivalent to

$$\begin{aligned} E(m(S_1(t))) &= E(t^{2/\beta}m(S_{t^{-1/\beta}}(1))) \\ &= \sum_{n=1}^{k} (-1)^{n-1} c_0^n E(\gamma_n(1)) t^{1-(n-1)(2/\beta-1)} + o(t^{1-(k-1)(2/\beta-1)}), \end{aligned} \tag{9.2}$$

as $t \to \infty$, where $c_0 = c_0(1)$.

We first intend to prove that

$$(9.3)\quad \lim_{\varepsilon\to 0} E(\gamma_{n,\varepsilon}(1)) = \left[\frac{1}{2\beta\sin(2\pi/\beta)}\right]^{n-1} \frac{1}{\Gamma(2-(n-1)(2/\beta-1))}$$

whenever $(2n-1)(2-\beta) < 2$; later we will show that the preceding limit equals $E(\gamma_n(1))$, which would complete the proof of Theorem 1.3.

Let $p_s(x)$ denote the density of X_s. We have

$$E(\gamma_{n,\varepsilon}(1)) = E\bigg(\int_0^1 dt_1 \int_{t_1}^1 \{f_\varepsilon(X_{t_2}-X_{t_1})\,dt_2 - \langle f_\varepsilon, G_0\rangle \delta_{t_1}(dt_2)\}$$

$$\times \int_{t_2}^1 \cdots \int_{t_{n-1}}^1 \{f_\varepsilon(X_{t_n}-X_{t_{n-1}})\,dt_n - \langle f_\varepsilon, G_0\rangle \delta_{t_{n-1}}(dt_{t_n})\}\bigg)$$

$$(9.4)\quad = \int_0^1 dt_1 \int_{t_1}^1 \{\langle f_\varepsilon, p_{t_2-t_1}\rangle\,dt_2 - \langle f_\varepsilon, G_0\rangle \delta_{t_1}(dt_2)\}$$

$$\times \int_{t_2}^1 \cdots \int_{t_{n-1}}^1 \{\langle f_\varepsilon, p_{t_n-t_{n-1}}\rangle\,dt_n - \langle f_\varepsilon, G_0\rangle \delta_{t_{n-1}}(dt_n)\}$$

$$= \int_0^1 I_\varepsilon^{n-1} 1(t)\,dt,$$

where

$$(9.5)\quad I_\varepsilon g(t) = \int_t^1 \langle f_\varepsilon, p_{s-t}\rangle g(s)\,ds - \langle f_\varepsilon, G_0\rangle g(t).$$

In particular,

$$I_\varepsilon 1(t) = \int_t^1 \langle f_\varepsilon, p_{s-t}\rangle\,ds - \langle f_\varepsilon, G_0\rangle$$

$$(9.6)\quad = \int_0^{1-t} \langle f_\varepsilon, p_s\rangle\,ds - \int_0^\infty \langle f_\varepsilon, p_s\rangle\,ds$$

$$= -\left\langle f_\varepsilon, \int_{1-t}^\infty p_s\,ds\right\rangle.$$

We have that, for any $0 \le \delta \le 1$,

$$|p_s(0) - p_s(x)| \le c\int |1-e^{ipx}| e^{-sp^\beta}\,d^2p$$

$$(9.7)\quad \le cx^\delta \int p^\delta e^{-sp^\beta}\,d^2p$$

$$\le cx^\delta s^{-(2+\delta)/\beta}.$$

Hence, with $p_s(0) = p_1(0)s^{-2/\beta}$,

$$\begin{aligned} I_\varepsilon 1(t) &= -\int_{1-t}^{\infty} p_s(0)\,ds + O(\varepsilon^\delta)(1-t)^{1-(2+\delta)/\beta} \\ &= \frac{p_1(0)}{1-2/\beta}(1-t)^{1-2/\beta} + O(\varepsilon^\delta)(1-t)^{1-(2+\delta)/\beta}. \end{aligned} \tag{9.8}$$

We now show through an appropriate choice of δ that the second term in (9.8) does not contribute to (9.4) in the limit as $\varepsilon \to 0$.

If $(2k-1)(2-\beta) < 2$, then a fortiori $(k-2)(2-\beta) < 1$, while $(2k-1)(2/\beta - 1) < 2/\beta$, that is, $(2k-2)(2/\beta - 1) < 1$; hence we can choose $0 < \delta < 1$ such that

$$\delta > (k-2)(2-\beta), \tag{9.9}$$

$$\delta/\beta + k(2/\beta - 1) < 1. \tag{9.10}$$

Now consider, for general $\gamma > -1$,

$$\begin{aligned} &\int_a^1 \langle f_\varepsilon, p_{s-a}\rangle (1-s)^\gamma\,ds \\ &\quad = \int_a^1 \langle f_\varepsilon, p_{s-a}\rangle \big[(1-s)^\gamma - (1-a)^\gamma\big]\,ds \\ &\qquad + (1-a)^\gamma \int_a^1 \langle f_\varepsilon, p_{s-a}\rangle\,ds \\ &\quad = \int_0^{1-a} \langle f_\varepsilon, p_s\rangle \big[(1-a-s)^\gamma - (1-a)^\gamma\big]\,ds + O(h(\varepsilon))(1-a)^\gamma, \end{aligned} \tag{9.11}$$

where $h(\varepsilon)$ is defined in (6.1); see also (7.9).

Since we have $p_s(x) \le p_s(0) = cs^{-2/\beta}$, (9.11) is bounded by

$$\begin{aligned} &c\int_0^{1-a} s^{-2/\beta}|(1-a-s)^\gamma - (1-a)^\gamma|\,ds + O(h(\varepsilon))(1-a)^\gamma \\ &\quad = c(1-a)^{\gamma+(1-2/\beta)}\int_0^1 s^{-2/\beta}|(1-s)^\gamma - 1|\,ds + O(h(\varepsilon))(1-a)^\gamma \\ &\quad \le c\big((1-a)^{\gamma+(1-2/\beta)} + h(\varepsilon)(1-a)^\gamma\big) \end{aligned} \tag{9.12}$$

since

$$\int_0^1 s^{-2/\beta}|(1-s)^\gamma - 1|\,ds < \infty, \tag{9.13}$$

for $\gamma > -1$.

Checking the definition (9.5), we have that if

$$|g(s)| \le c(1-s)^\gamma, \qquad \gamma > -1, \tag{9.14}$$

then

$$|I_\varepsilon g(t)| \le c\big((1-t)^{\gamma+(1-2/\beta)} + h(\varepsilon)(1-t)^\gamma\big). \tag{9.15}$$

The error term $g(s)$ in (9.8) satisfied

$$|g(s)| \leq c\varepsilon^{\delta}(1-s)^{1-2/\beta-\delta/\beta};$$

hence an induction argument using (9.15) and (9.10) shows that

$$|I_{\varepsilon}^{n-2}g(t)| \leq c\varepsilon^{\delta}\sum_{j=1}^{n-2}(1-t)^{(j+1)(1-2/\beta)-\delta/\beta}h^{n-2}(\varepsilon) \tag{9.16}$$

and, therefore, with (9.10) we have

$$\left|\int_0^1 I_{\varepsilon}^{h-2}g(t)\,dt\right| \leq c\varepsilon^{\delta}h^{n-2}(\varepsilon) \to 0, \tag{9.17}$$

by (9.9) and (7.9).

Thus we can drop the error term in (9.8) and calculate

$$\int_0^1 I_{\varepsilon}^{n-2}\left(\frac{p_1(0)(1-s)^{1-2/\beta}}{1-2/\beta}\right)(t)\,dt.$$

To this end we compute, for $\gamma > -1$, using (9.7),

$$\begin{aligned} &I_{\varepsilon}((1-s)^{\gamma})(t)\\ &= \int_t^1 \langle f_{\varepsilon}, p_{s-t}\rangle(1-s)^{\gamma}\,ds - \langle f_{\varepsilon}, G_0\rangle(1-t)^{\gamma}\\ &= \int_t^1 \langle f_{\varepsilon}, p_{s-t}\rangle\left[(1-s)^{\gamma}-(1-t)^{\gamma}\right]ds - (1-t)^{\gamma}\left\langle f_{\varepsilon}, \int_{1-t}^{\infty} p_s\,ds\right\rangle\\ &= \int_t^1 p_{s-t}(0)\left[(1-s)^{\gamma}-(1-t)^{\gamma}\right]ds - (1-t)^{\gamma}\int_{1-t}^{\infty}p_s(0)\,ds\\ &\quad + O(\varepsilon^{\delta})\left(\int_t^1 (s-t)^{-(2+\delta)/\beta}|(1-s)^{\gamma}-(1-t)^{\gamma}|\,ds\right.\\ &\qquad\qquad \left. + (1-t)^{\gamma+(1-2/\beta)-\delta/\beta}\right). \end{aligned} \tag{9.18}$$

We first compute

$$\begin{aligned} &\int_t^1 p_{s-t}(0)\left[(1-s)^{\gamma}-(1-t)^{\gamma}\right]ds - (1-t)^{\gamma}\int_{1-t}^{\infty}p_s(0)\,ds\\ &= p_1(0)\int_0^{1-t}s^{-2/\beta}\left[(1-t-s)^{\gamma}-(1-t)^{\gamma}\right]ds\\ &\quad + p_1(0)\frac{(1-t)^{(1-2/\beta)+\gamma}}{1-2/\beta}\\ &= p_1(0)(1-t)^{\gamma+(1-2/\beta)}\left[\int_0^1 s^{-2/\beta}\left[(1-s)^{\gamma}-1\right]ds + \frac{1}{1-2/\beta}\right]. \end{aligned} \tag{9.19}$$

Now, for $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$, we have

$$\begin{aligned}\int_0^1 t^{p-1}\left[(1-t)^{q-1}-1\right]dt &= B(p,q)-\frac{1}{p}\\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}-\frac{1}{p},\end{aligned}\tag{9.20}$$

while both sides of (9.20) have analytic continuations to $\mathrm{Re}(p) > -1$, $p \neq 0$.
Hence the integral in (9.19) equals

$$p_1(0)(1-t)^{\gamma+(1-2/\beta)}\frac{\Gamma(1-2/\beta)\Gamma(1+\gamma)}{\Gamma(1+(1-2/\beta)+\gamma)}.\tag{9.21}$$

Now, by induction assume that we have shown that, in j successive applications of I_ε to 1, we ignore the error term in (9.18). Then, via (9.8) and (9.21), γ successively takes the values $i(1-2/\beta)$, $i = 1, 2, \dots, j$ and the $(j+1)$st error term (9.18) will then be

$$O(\varepsilon^\delta)(1-t)^{(j+1)(1-2/\beta)-\delta/\beta}.$$

Arguing exactly as when $j = 0$, we see that we can ignore this error term in the $\varepsilon \to 0$ limit of (9.4). Hence, with (9.8), (9.20) and (9.21) we have, using $\Gamma(2-2/\beta) = (1-2/\beta)\Gamma(1-2/\beta)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0} E(\gamma_{n,\varepsilon}(1))\\ &\quad= \left(p_1(0)\Gamma\left(1-\frac{2}{\beta}\right)\right)^{n-1}\frac{1}{\Gamma(1+(n-1)(1-2/\beta))}\\ &\qquad\times\int_0^1(1-t)^{(n-1)(1-2/\beta)}\,dt\\ &\quad= \left(p_1(0)\Gamma\left(1-\frac{2}{\beta}\right)\right)^{n-1}\frac{1}{\Gamma(2+(n-1)(1-2/\beta))}.\end{aligned}\tag{9.22}$$

But, for the symmetric stable process of order β,

$$p_1(0) = \frac{1}{2\pi\beta}\Gamma\left(\frac{2}{\beta}\right)\tag{9.23}$$

and

$$\Gamma\left(\frac{2}{\beta}\right)\Gamma\left(1-\frac{2}{\beta}\right) = \frac{1}{\sin(2\pi/\beta)},\tag{9.24}$$

which leads to (9.3). It now remains to show that

$$E(\gamma_n(1)) = \lim_{i\to\infty} E(\gamma_{n,\bar\varepsilon_i}(1)), \tag{9.25}$$

for some subsequence $\bar\varepsilon_i \to 0$.

By (8.11),

$$E(\gamma_n(s)) = s^{-1+(n-1)(2/\beta-1)} E(\tilde\gamma_n(s)). \tag{9.26}$$

As in Section 8, we can assume s is chosen so that, for some subsequence $\varepsilon_l \to \varepsilon$,

$$\begin{aligned}
E(\gamma_n(s)) &= \lim_{l\to\infty} E(\gamma_{n,\varepsilon_l}(s)) \\
&= \lim_{l\to\infty} \sum_{j=1}^{n} (-h_0(\varepsilon_l))^{n-j} \binom{n-1}{j-1} \\
&\quad\times \int\cdots\int_{0\le t_1\le \cdots \le t_j\le s} E\left(\prod_{i=2}^{j} f_{\varepsilon_l}(X_{t_i} - X_{t_{i-1}})\right) d\bar t \\
&= \lim_{l\to\infty} \sum_{j=1}^{n} (-h_0(\varepsilon_l))^{n-j} \binom{n-1}{j-1} s^j \\
&\quad\times \int\cdots\int_{0\le t_i\le \cdots \le t_j\le 1} E\left(\prod_{i=2}^{j} f_{\varepsilon_l}(X_{st_i} - X_{st_{i-1}})\right) d\bar t \\
&= \lim_{l\to\infty} \sum_{j=1}^{n} (-h_0(\varepsilon_l))^{n-j} \binom{n-1}{j-1} s^j \\
&\quad\times \int\cdots\int_{0\le t_1\le \cdots \le t_j\le 1} E\left(\prod_{i=2}^{j} f_{\varepsilon_l}(s^{1/\beta}(X_{t_i} - X_{t_{i-1}}))\right) d\bar t \\
&= s^{1-(n-1)(2/\beta-1)} \lim_{l\to\infty} \sum_{j=1}^{n} \left(-h_0\left(\frac{\varepsilon_l}{s^{1/\beta}}\right)\right)^{n-j} \binom{n-1}{j-1} \\
&\quad\times \int\cdots\int_{0\le t_1 \cdots \le t_j\le 1} E\left(\prod_{i=2}^{j} f_{\varepsilon l}(s^{1/\beta}(X_{t_i} - X_{t_{i-1}}))\right) d\bar t.
\end{aligned} \tag{9.27}$$

Thus, comparing (9.26) and (9.27), we have (9.25) with $\bar\varepsilon_l = (\varepsilon_l)/s^{1/\beta}$. □

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