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NECESSARY AND SUFFICIENT CONDITIONS FOR SAMPLE CONTINUITY OF RANDOM FOURIER SERIES AND OF HARMONIC INFINITELY DIVISIBLE PROCESSES[1]

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For very general random Fourier series and infinitely divisible processes on a locally compact Abelian group G, a necessary and sufficient condition for sample continuity is given in terms of the convergence of a certain series. This series expresses a control on the covering numbers of a compact neighborhood of G by certain nonrandom sets naturally associated with the Fourier series (resp. the process). In the nonstationary case, we give a necessary Sudakov-type condition for a probability measure in a Banach space to be a Lévy measure.

1. Introduction. In order to put the results of this paper in proper perspective, we will recall some history of the theory of stochastic processes. For our purposes, a stochastic process is a collection of random variables $(X_t)_{t \in T}$ indexed by a set T. The aim is to find necessary conditions and sufficient conditions for the sample boundedness of the process in terms of simple parameters (usually T is a topological space, and one is also interested in sample continuity; but understanding boundedness is the key to understanding continuity). Historically, a class of processes of special importance has emerged: the case where T is a locally Abelian compact group, or a suitable subset of such a group, and where the process has some "stationarity" property. We will call this case the "stationary case." Its importance stems in part from the fact that it is distinctly easier than the general case, and that consequently it has been the first to be understood. (A second reason for its importance will be described later.) For example, in the Gaussian case, Dudley produced in 1967 a sufficient condition for continuity, the so-called "metric entropy condition." This condition is the finiteness of a certain integral that expresses a control of the growth of the smallest number of balls (for a certain metric associated to the process) needed to cover T, as the radius of these balls goes to 0. This type of condition (that we will call a covering condition) is a common feature of much of the subsequent work. In 1974, Fernique proved that, in the stationary case, Dudley's condition is necessary, thereby providing a complete characterization for continuity of Gaussian processes in an important special case. The complete characterization of sample boundedness and continuity of Gaussian processes was obtained only in 1985 by this author (see [7] which contains more historical information). The main extra difficulty is

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that the entropy condition is ill adapted to the case where the index set T lacks homogeneity, and that a more delicate tool, the so-called majorizing measure, has to be introduced. (This difficulty completely disappears in the stationary case.)

After Fernique's 1974 theorem, there was a desire to understand more than the Gaussian case. This motivated the landmark paper [5] by Marcus and Pisier. In this paper, the authors obtain, in the stationary case, necessary and sufficient conditions for sample continuity of p-stable processes in terms of covering conditions that completely parallel the Dudley–Fernique theorem. An important aspect of this work is that, in contrast with the Gaussian case, sufficient conditions for continuity are very difficult to find in the p-stable case; this is due to the fact that p-stable random variables have a considerably "fatter" tail than Gaussian r.v.'s. A very remarkable fact, which is certainly the main discovery of [5], is that in the stationary case, surprisingly weak conditions suffice for continuity (this is the second remarkable aspect of the stationary case). Actually, the necessary condition of Marcus and Pisier for boundedness of p-stable processes has been extended to the nonstationary case [8], while the understanding of efficient sufficient conditions in that case is certainly not completed. The work of Marcus and Pisier was extended by Marcus [3] to more general processes. An interesting feature of the work of Marcus is that he obtains necessary conditions and sufficient conditions for sample continuity in terms of covering numbers by balls for various metrics. These conditions, which seem to be optimal of their type, never coincide, except in the p-stable case.

The results of [3], [4] and [8] all rely upon the fact that the processes under study can be represented as mixtures of Gaussian processes. Recently, Rosinski [6] has proved the remarkable fact that all infinitely divisible processes can be represented as a mixture of Bernoulli processes (in a sense to be explained below). This gave to this author the feeling that the time was ripe for a full-scale investigation of these processes. As explained above, the natural first case to investigate is the stationary case. This is the object of the present paper. The notion of "harmonic infinitely divisible processes" that we will introduce below seems to be the natural setting for this study. It is more general than the setting of Marcus [3] and, of course, than the p-stable case of [4]. The necessary and sufficient conditions we obtain contain the conditions of Marcus and Pisier [4] and improve those of [3]. These conditions are the convergence of a certain series, which express the rate of growth of covering numbers by certain nonrandom sets. An important new feature is that these sets do not arise naturally as balls for a metric, except in the p-stable case. This is not an accident and we believe that the correct conditions to understand stochastic processes beyond the p-stable case are of this nature. The covering conditions we consider here can be considered as a new way to look at entropy conditions. This new point of view, which was discovered while writing the present paper, can be further extended to reformulate majorizing measure conditions; this reformulation has met surprising success in subsequent work of the author [11, 12]. In particular, the author has succeeded in extending the necessary condition of Theorem 1.1 below to very general processes (thereby also extending the result of [8]). This is, however, very much harder, and the present work exemplifies the characteristics of the stationary case that were explained above: the proof of necessary conditions is much easier than in the nonstationary case and these conditions are also necessary (that is certainly not the case in general).

While many of the ideas of the present paper could be traced back to [5], there are also significant differences. In particular, in the proof of sufficient conditions (that is the hard part of the paper), the very elegant method of [5] (used again in [3]) does not appear to generalize well, and we use the new approach that was invented in [10]. Actually, none of the lemmas of [3] or [4] could be used as they are. Consequently, we have made our paper completely self-contained, and the reader need not know anything about the works [3], [4] and [10]. Indeed, one of the aims of the present paper is to supersede the work of [3] and [5], in the sense that more general results are obtained with comparable, or even smaller, effort. The other aim of the paper is to provide, in a simple case, an introduction to the subsequent work [11, 12].

We now describe our results in a more specific manner. Consider a set T. A positive σ-finite measure ν on $\mathbb{R}^T$ equipped with the cylindrical σ-algebra is called a *Lévy measure* provided

$$\forall\, t \in T, \qquad \int \beta(t)^2 \wedge 1 \, d\nu(\beta) < \infty. \tag{1.1}$$

We will say that a process $(X_t)_{t\in T}$ is a (symmetric; without Gaussian component) infinitely divisible process if there exists a Lévy measure ν (called the Lévy measure of the process) on $\mathbb{R}^T$ such that for each finitely supported family $(\alpha_t)_{t\in T}$ of real numbers, we have

$$E \exp i \sum_{t\in T} \alpha_t X_t = \exp - \int \left(1 - \cos\left(\sum_{t\in T} \alpha_t \beta(t)\right)\right) d\nu(\beta). \tag{1.2}$$

To give examples, consider a Lévy measure η on $\mathbb{R}$ [i.e., $\int t^2 \wedge 1\, d\eta(t) < \infty$]. Consider the measure $\eta \otimes \lambda$ on $\mathbb{R} \times \mathbb{R}^+$ (where λ denotes Lebesgue measure) and the image measure ν of $\eta \otimes \lambda$ under the map $(u, x) \to \beta_{u,x} \in \mathbb{R}^{\mathbb{R}}$, where $\beta_{u,x}(t) = u$ if $x \le t$ and $\beta_{u,x}(t) = 0$ otherwise. Then the process $(X_t)_{t\in\mathbb{R}^+}$ is the Lévy process modeled on η.

A considerably different process is obtained if ν is the image of η under the map $u \to u\beta$, where β is a given element of $\mathbb{R}^{\mathbb{R}}$. In that case, one can write (in distribution) $X_t = \beta(t)Y$, where Y is an infinitely divisible random variable (of Lévy measure η). Observe that, in that case, the process $(X_t)_{t\in\mathbb{R}}$ is sample continuous whenever $t \to \beta(t)$ is continuous; but, of course, the process $(X_t)_{t\in\mathbb{R}}$ is very far from having independent increments. The processes considered in this paper certainly resemble more the second example than the first one.

Since we will deal with locally compact Abelian groups T and characters are definitely complex-valued, we need the notion of a complex-valued infinitely divisible process $(X_t)_{t\in T}$. If we identify $\mathbb{C}$ with $\mathbb{R} \times \mathbb{R}$ this just means that the real-valued process (indexed by two copies of T) that is canonically derived from $(X_t)_{t\in T}$ is infinitely divisible. A concise way to express this is that there is a Lévy measure ν on $\mathbb{C}^T$, that is, a positive measure satisfying

$$\forall\, t \in T, \qquad \int |\beta(t)|^2 \wedge 1\, d\nu(\beta) < \infty, \tag{1.3}$$

such that for each finitely supported family $(\alpha_t)_{t\in T}$ of complex numbers we have

$$E \exp i\mathscr{R}e\Big(\sum_{t\in T} \alpha_t X_t\Big) = \exp - \int \bigg(1 - \cos\Big(\mathscr{R}e\Big(\sum_{t\in T} \alpha_t \beta(t)\Big)\Big)\bigg)\, d\nu(\beta). \tag{1.4}$$

The celebrated Sudakov minoration for Gaussian processes was extended by Marcus and Pisier to p-stable processes, $p > 1$ ([3], Theorem 2.6). Our first result is an extension of their result to the infinitely divisible case. If ν is a Lévy measure on $\mathbb{R}^T$ (or $\mathbb{C}^T$) for $s, t \in T$, $u \in \mathbb{R}$, we set throughout the paper

$$\varphi(s,t,u) = \varphi_\nu(s,t,u) = \int \big(u^2|\beta(s) - \beta(t)|^2\big) \wedge 1\, d\nu(\beta). \tag{1.5}$$

THEOREM 1.1. *There exists a numerical constant C with the following property. Consider a (real or complex) infinitely divisible process indexed by T, with Lévy measure ν. Suppose that we have*

$$P\Big(\sup_{t\in T} |X_t| \ge M\Big) \le 1/5.$$

Then for $u \ge C$, T can be covered by at most e^u sets of type

$$V_t = \Big\{ s \in T;\, \varphi_\nu\Big(s, t, \frac{u}{CM}\Big) \le u \Big\}.$$

It will be shown in Section 6 why this result contains the Sudakov minoration of Marcus and Pisier for p-stable processes, $p > 1$. Unfortunately, this does not contain the case $p = 1$ proved in [9].

We now turn to the case where the index set is a locally compact Abelian group G and we denote its dual group by Γ. For simplicity, we will consider only *complex-valued* infinitely divisible processes $(X_t)_{t\in G}$. We will say that such a process is *harmonic* infinitely divisible if its Lévy measure is supported by $\mathbb{C}\Gamma = \{a\gamma\colon a \in \mathbb{C},\ \gamma \in \Gamma]$. These processes, which need not be stationary, contain the strongly stationary ξ-radial processes of Marcus [3] that themselves contain the strongly stationary p-stable processes of Marcus and Pisier [5]. We will give a necessary condition for sample continuity (of a separable version of) a harmonic infinitely divisible process $(X_t)_{t\in G}$. Under a mild assumption on the Lévy measure ν, this condition is also sufficient.

We fix a given compact neighborhood of unity K in G. When G is compact, we take $K = G$. We fix a Haar measure $|\cdot|$ of G. When G is compact, this Haar measure is normalized so that $|G| = 1$.

Throughout the paper, we denote by 0 the unit of G and we set

$$
\begin{aligned}
U'_{l,i} &= \{s \in G;\ \varphi(s,0,2^{l+i/2}) \le 2^i\} \\
&= \left\{s \in G;\ \int \left(2^{2l+i}|\beta(s)-\beta(0)|^2\right) \wedge 1\, d\nu(\beta) \le 2^i\right\}
\end{aligned} \tag{1.6}
$$

and we set $U_{l,i} = U'_{l,i} \cap K$.

THEOREM 1.2. *There exists $\varepsilon_0 > 0$ with the following property. Consider a harmonic infinitely divisible process $(X_t)_{t\in G}$. Suppose that for each countable subset $D \subset K$ we have $P(\sup_{t\in D}|X_t| \le M) \ge 1-\varepsilon_0$. Suppose that there exists an integer $l \in \mathbb{Z}$ for which $K \ne U_{l,0}$. Then there exists a smallest integer $l_0 \in \mathbb{Z}$ such that $K \ne U_{l_0,0}$. For $l \ge l_0$, set $i(l) = 0$ if $|U_{l,0}| > 1/e$. If $|U_{l,0}| \le 1/e$ there exists a largest $i(l) \ge 0$ such that $|U_{l,i(l)}| \le e^{-2^{i(l)}}$, and we have $\sum_{l\ge l_0} 2^{-l+i(l)/2} \le CM$, where C depends only on K. Moreover, when G is compact, C is universal.*

This might be the place to observe that $|U|$ is closely related to the covering number of K by translates of U (see Lemma 2.11 below; this is extensively used in [4] and [5]); on the other hand, the sets $U_{l,i(l)}$ do not arise as balls for a natural metric. The definition of $U'_{l,i}$ and convergence of the series $\sum_{l\ge l_0} 2^{-l+i(l)/2}$ are not very intuitive. One of the reasons why it is not apparent that this condition is related to the usual conditions is that a change of variables has been made. It should be mentioned that the condition of Theorem 1.2 can be formulated in many equivalent ways (by other changes of variables, as will be apparent in Section 6). The present formulation is somewhat canonical with respect to the use of the random distances that are basic for its proof. An alternative attractive formulation would be to replace the definition of $U'_{l,i}$ by

$$\{s \in G;\ \varphi(s,0,2^l) \le 2^{-i}\}$$

and the series $\sum_{l\ge l_0} 2^{-l+i(l)/2}$ by $\sum_{l\ge l_0} 2^{-l+i(l)}$.

We now introduce a condition on ν.

$H(B,\delta)$: for all $s,t \in G$, $u \ge 0$, we have

$$
\begin{aligned}
&\int_{u|\beta(s)-\beta(t)|\ge 1} u|\beta(s)-\beta(t)|(\log(eu|\beta(s)-\beta(t)|))^{\delta}\, d\nu(\beta) \\
&\qquad \le B\varphi_\nu(s,t,u).
\end{aligned} \tag{1.7}
$$

Let us first comment on that condition. Obviously, if a family of measures ν satisfies condition $H(B,\delta)$, so does any mixture of this family. In this case where ν is supported by a ray, that is, ν is the image of a measure η on $\mathbb{R}^+$ by the map $x \to \beta_0 x$ from $\mathbb{R}^+$ to $\mathbb{C}^G$, condition $H(B,\delta)$ means that for each $a > 0$,

$$\int_{1/a}^{\infty} at(\log eat)^{\delta}\, d\eta(t) \le B\int_0^{\infty} (a^2t^2 \wedge 1)\, d\eta(t). \tag{1.8}$$

This condition is not intuitive either. But let us note that it is satisfied in particular if $\eta([2t,\infty[) \leq 2^{-\beta}\eta([t,\infty[)$ for some $\beta > 1$. Our argument, however, does not require that much, and (1.8) is simply the formulation of a weaker condition for which the same proof works. For $p > 1$, p-stable processes will satisfy condition $H(B,\delta)$ for some $B > 0$, $\delta > 1$. Condition $H(B,\delta)$ is not satisfied for 1-stable processes. While the case of 1-stable processes has been settled in [8], there seem to be difficulties of a new nature in the extension of that result to the present setting.

THEOREM 1.3. *Suppose that condition $H(B,\delta)$ holds where $\delta > 1$. If $K = U_{l,0}$ for all $l \in \mathbb{Z}$, the process $(X_t)_{t\in K}$ is continuous. Otherwise suppose that there is a smallest integer $l_0 \in \mathbb{Z}$ for which $U_{l_0,0} \neq K_1$. For $l \geq l_0$, set $i(l) = 0$ if $|U_{l,0}| > 1/e$. Suppose that when $|U_{l,0}| < 1/e$ there exists a largest integer $i(l) \geq 0$ such that $|U_{i,i(l)}| \leq e^{-2^{i(l)}}$. Suppose that $\sum_{l\geq l_0} 2^{-i+i(l)/2} < \infty$. Then the process $(X_t)_{t\in G}$ is a.s. continuous. If L is a compact neighborhood of 0 such that $L + L \subset K$, for each $\varepsilon > 0$ there is a number $C(\varepsilon)$, depending only on ε, B, δ, K and L such that*

$$P\left(\sup_{s,t\in L}|X_s - X_t| \geq C(\varepsilon)\sum_{l\geq l_0} 2^{-l+i(l)/2}\right) \leq \varepsilon.$$

In Section 6, we will show how to recover from Theorems 1.2 and 1.3 the results of Marcus and Pisier in the p-stable case $1 < p < 2$; and we will briefly indicate how one could recover many of the results of Marcus [3].

We now turn to random Fourier series. There is no direct connection between the random Fourier series we consider and infinitely divisible processes, although the results and the proofs have many similarities. We consider random Fourier series of the type $X_t = \sum_{\gamma\in\Gamma} f_\gamma\gamma(t)$, where $(f_\gamma)_{\gamma\in\Gamma}$ are independent symmetric random variables. The reader observes that there are no coefficients in front of f_γ; the coefficients are absorbed by f_γ, as our formulation does not require that the functions f_γ have the same distribution.

We set

$$\theta(s,\lambda) = \sum_{\gamma\in\Gamma} E\left(\lambda^2 f_\gamma^2|\gamma(s) - 1|^2 \wedge 1\right),$$

and for $l \in \mathbb{Z}$, $i \geq 0$, we set

$$V_{l,i} = \{s \in K, \theta(s, 2^{l+i/2}) \leq 2^i\}.$$

THEOREM 1.4. *There exists $\epsilon_0 > 0$ with the following property. Assume that $P(\sup_{t\in K}|X_t| \geq M) \leq \varepsilon_0$. Suppose that there is an integer $l \in \mathbb{Z}$ such that $V_{l,0} \neq K$. Then there is a smallest integer l_0 such that $V_{l_0,0} \neq K$. For $l \geq l_0$, we set $i(l) = 0$ if $|V_{l,0}| > 1/e$. If $|V_{l,0}| \leq 1/e$ there is a largest integer $i(l)$ such that $|V_{l,i(l)}| \leq e^{-2^{i(l)}}$, and*

$$\sum_{l\geq l_0} 2^{-l+i(l)/2} \leq CM,$$

where C depends on K only (and C is universal if $K = G$).

REMARK. Since the quantity $\theta(s, \lambda)$ might be hard to compute, the reader might feel that Theorem 1.4 is impractical. However, if one replaces in the definition of $V_{l,i}$ the value of $\theta(s, \lambda)$ by a lower bound, one sees easily that this decreases the value of $i(l)$, and thus that one still gets a necessary condition.

Before we turn to sufficient conditions, we must mention one more (well known [4]) necessary condition that is not captured by Theorem 1.4. For $\gamma \in \Gamma$, we consider the largest number a_γ such that $P(|f_\gamma| \geq a_\gamma) \geq 1/2$.

PROPOSITION 1.5. *There exists* $\varepsilon_0 > 0$ *with the following property. Consider the distance* d *on* G *given by* $d(s,t) = (\sum_{\gamma \in \Gamma} a_\gamma |\gamma(s) - \gamma(t)|^2)^{1/2}$. *Denote by* $N_k(K)$ *the smallest number of balls for* d *of radius* 2^{-k} *needed to cover* K. *Then if* $P(\sup_{t \in K} |X_t| \geq M) \leq \varepsilon_0$, *we have*

$$\sum_k 2^{-k} (\log N_k(K))^{1/2} \leq CM,$$

where C *depends on* K *only.*

THEOREM 1.6. *Assume that for some* $\delta > 1$, $B > 0$, *we have for all* $\gamma \in \Gamma$,

$$u|a_\gamma| \leq 1 \Rightarrow \int_{|f_\gamma| \geq 1/u} u|f_\gamma| \big(\log(eu|f_\gamma|)\big)^\delta \, dP \leq BE\big(u^2|f_\gamma|^2 \wedge 1\big).$$

Then, if l_0 *and the sequence* $(i(l))$ *are defined as in Theorem* 1.4 *and if* $N_k(K)$ *is defined as in Proposition* 1.5, *the series* $X_t = \sum_{\gamma \in \Gamma} f_\gamma \gamma(t)$ *is almost surely continuous whenever* $\sum_{l \geq l_0} 2^{-l + i(l)/2} < \infty$ *and* $\sum_k 2^{-k} (\log N_k(u))^{1/2} < \infty$. *Moreover, for* $\varepsilon > 0$ *we have*

$$P\left(\sup_{t \in K} |X_t| \geq C(\varepsilon) \left(\sum_{l \geq l_0} 2^{-l + i(l)/2} + \sum_k 2^{-k} (\log N_k(K))^{1/2} \right) \right) \leq \varepsilon,$$

where $C(\varepsilon)$ *depends only on* ε, B, δ *and* K (*and only on* B, β *and* ε *if* G *is compact*).

REMARK. If $\theta(s, \lambda)$ is replaced by an upper bound, one sees easily that this increases the value of $i(l)$, and thus that one still obtains a sufficient condition.

The proof of Theorems 1.2–1.4 relies upon the fact that both strongly stationary infinitely divisible processes and random Fourier series can be represented as a mixture of random Fourier series $\sum_{\gamma \in M} a_\gamma \varepsilon_\gamma \gamma(t)$ (ε_γ being a Bernoulli sequence) for which necessary and sufficient conditions for continuity are known. In the case of infinitely divisible processes, this representation is made possible by some remarkable recent results of Rosinski [6] that are also essential in proving Theorem 1.1. Suitable inequalities are required to control the random distances involved in these representations. The inequalities we prove are simple, yet they are somewhat different from those used in previous work ([3, 5]). Section 2 is devoted to these and other tools. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorems 1.2 and 1.4. In Section 5 we prove Theorems 1.3 and 1.6. Section 6 is devoted to recovering the p-stable case.

2. Tools. Our result will rely on recent representation results of Rosinski [6] that allow the representation of an infinitely divisible process as a mixture of Bernoulli series. Our first task is to explain the special case of these results that is suitable for our needs. Consider a (real) infinitely divisible process $(X_t)_{t \in T}$ with Lévy measure ν. Since ν is σ-finite, there is a probability measure m on $\mathbb{R}^T$ such that $\nu \ll m$. Consider a Radon–Nikodym derivative $h = d\nu/dm$. For $x \in \mathbb{R}^T$, $u \in \mathbb{R}^+$, we set

$$R(u,x) = 1_{[0,\, h(x)]}(u).$$

Observe the trivial but essential fact that $R(\cdot, x)$ is nonincreasing. Denote Lebesgue measure on $\mathbb{R}^+$ by λ. Obviously, ν is the image of $\lambda \otimes m$ by the map $(u, x) \to R(u, x)x$.

Consider a r.v. $Z \geq 0$ with $P(Z \geq t) = e^{-t}$, and i.i.d. copies Z_i of Z. Set $\Gamma_i = \sum_{j \leq i} Z_j$. Consider a Bernoulli sequence (ε_i) [i.e., (ε_i) is independent and $P(\{\varepsilon_i = 1\}) = 1/2 = P(\{\varepsilon_i = -1\})$] and an i.i.d. sequence Y_i valued in $\mathbb{R}^T$ and distributed like m. We assume that each of the sequences $(\Gamma_i), (\varepsilon_i), (Y_i)$ is independent of the others.

THEOREM 2.1 (Rosinski [6]). *The process*

$$\left(\sum_{i \geq 1} \varepsilon_i R(\Gamma_i, Y_i) Y_i(t) \right)_{t \in T} \tag{2.1}$$

is distributed like $(X_t)_{t \in T}$.

It will be convenient to assume that the basic probability space is a product $\Xi \times \Omega$, and the basic probability Pr is a product $Q \otimes P$. We will assume that $\varepsilon_i(\xi, \omega)$ depends on ξ only, while Γ_i, Y_i depend on ω only.

Conditionally on ω, the process $(X_t)_{t \in T}$ is thus represented as a Bernoulli series; in the strongly stationary case, the series will be a random Fourier series.

Previous work ([3, 5]) makes heavy use (in the more restricted context of p-stable or ξ-radial processes) of the random distance

$$d_\omega(s,t) = \left(\sum_{i \geq 1} R(\Gamma_i, Y_i)^2 |Y_i(s) - Y_i(t)|^2 \right)^{1/2}.$$

A key ingredient in these arguments was sharp bounds for $P(d_\omega(s,t) \leq a)$, $a > 0$. However, in those works, the process (X_t) conditioned on ω was a

Gaussian process, and thus

$$\int \exp i\lambda(X_s - X_t)\, dQ(\xi) = \exp -\frac{\lambda^2}{2} d_\omega^2(s,t).$$

Combined with our knowledge of $E \exp i\lambda(X_s - X_t)$ and an exponential Chebyshev inequality, this yields a bound for $P(d_\omega(s,t) \le a)$, of the right order (see, e.g., [5], Lemma 2.2). This argument does not work any longer, and we will have to proceed directly (which is fortunately easy). The following deserves no proof.

LEMMA 2.2. *Consider a nondecreasing function h on $\mathbb{R}^+$. Then for $\alpha > 0$,*

$$\alpha \sum_{i=1}^{\infty} h(\alpha i) \le \int_0^\infty h(x)\, d\lambda(x) \le \alpha\left(h(0) + \sum_{i=1}^{\infty} h(\alpha i)\right).$$

LEMMA 2.3. *Consider $s, t \in T$. Let*

$$W_i = \left(u^2 R^2(\alpha i, Y_i)|Y_i(s) - Y_i(t)|^2\right) \wedge 1. \tag{2.2}$$

Then

$$\alpha^{-1}\varphi(s,t,u) - 1 \le \sum_{i \ge 1} EW_i \le \alpha^{-1}\varphi(s,t,u).$$

PROOF. For $\beta \in \mathbb{R}^T$ we have, by Lemma 2.2, that

$$\begin{aligned}\alpha \sum_{i=1}^{\infty} & u^2 R^2(\alpha i, \beta)|\beta(s) - \beta(t)|^2 \wedge 1 \\ &\le \int_0^\infty \left(u^2 R^2(x,\beta)|\beta(s) - \beta(t)|^2 \wedge 1\right) d\lambda(x) \\ &\le \alpha\left(1 + \sum_{i=1}^{\infty} u^2 R^2(\alpha i, \beta)|\beta(s) - \beta(t)|^2 \wedge 1\right).\end{aligned}$$

We now apply this for $\beta = Y_1$, and we take expectations. Since the sequence (Y_i) is equidistributed of law m, we get

$$\begin{aligned}\alpha \sum_{i=1}^{\infty} EW_i &\le \int_0^\infty \left(u^2 R^2(x,\beta)|\beta(s) - \beta(t)|^2 \wedge 1\right) d\lambda(x)\, dm(\beta) \\ &\le \alpha\left(1 + \sum_{i=1}^{\infty} EW_i\right).\end{aligned}$$

Since ν is the image of $\lambda \otimes m$ under the map $(x, \beta) \to R(x,\beta)\beta$, the middle quantity is $\varphi(s,t,u)$. □

Several of our inequalities are based on the amazingly simple following lemma.

LEMMA 2.4. *Consider independent random variables* $0 \le W_i \le 1$. *Then*:

(i) $E\exp - \sum W_i \le \exp - \frac{1}{2}\sum EW_i$,
(ii) $E\exp \sum W_i \le \exp 2\sum EW_i$.

PROOF. To prove (i), we observe that $\exp - x \le 1 - x/2$ for $0 \le x \le 1$, so that

$$E\exp - W_i \le E\left(1 - \frac{W_i}{2}\right) = 1 - \frac{1}{2}EW_i \le \exp - \frac{1}{2}EW_i.$$

Now $E\exp - \sum W_i = \prod E\exp - W_i$.

To prove (ii), we proceed similarly, noting now that for $0 \le x \le 1$ we have $\exp x \le 1 + 2x$. □

PROPOSITION 2.5. (i) *Suppose that* $\varphi(t, s, u) \ge 2\alpha$. *Then*

$$P\left(\sum_{i\ge 1} R(\alpha i, Y_i)^2 |Y_i(s) - Y_i(t)|^2 \wedge \frac{1}{u^2} \le \frac{\varphi(s,t,u)}{8\alpha u^2}\right) \le \exp - \frac{\varphi(s,t,u)}{8\alpha}.$$

(ii) *For* $A \ge 4\alpha^{-1}\varphi(s,t,u)$ *we have*

$$P\left(\sum_{i\ge 1} R(\alpha i, Y_i)^2 |Y_i(s) - Y_i(t)|^2 \wedge \frac{1}{u^2} \ge \frac{A}{u^2}\right) \le \exp - \frac{A}{2}.$$

PROOF. Define W_i by (2.2). Thus, by Lemma 2.3, we have

$$\alpha^{-1}\varphi(s,t,u) \ge \sum_{i\ge 1} EW_i \ge \alpha^{-1}\varphi(s,t,u) - 1 \ge \varphi(s,t,u)/2\alpha.$$

(i) This follows from Lemma 2.4(i) and the inequality $P(Z \le A) \le \exp A \cdot E\exp - Z$, used for $Z = \sum W_i$, $A = \varphi(s,t,u)/8\alpha$.

(ii) This follows from Lemma 2.4(ii) and the inequality $P(Z \ge A) \le \exp - A \cdot E\exp Z$, used for $Z = \sum W_i$. □

To prove convergence of a series $\sum \varepsilon_i y_i$, $y_i \in \mathbb{R}^T$, we will, following the method of [4], decompose it in "small" and "large" terms. The small terms will be handled by Proposition 2.5. We now build the tools to control the large terms. These will be based on the following elementary fact.

PROPOSITION 2.6. *Consider an independent sequence* $H_i \ge 0$ *of r.v.'s and* $\delta > 0$. *Suppose that*

$$\sum_{i\ge 1} EH_i 1_{\{H_i \ge 1\}}(\log eH_i)^\delta \le A,$$

where $A > 2$. *Then*

$$P\left(\sum_{i\ge 1} H_i 1_{\{H_i \ge 1\}} \ge 4A\right) \le C(\log A)^{-\delta},$$

where C *is universal.*

PROOF. Observe that, since $\log eu \geq 1$ for $u \geq 1$, we have

$$\sum_{i \geq 1} EH_i 1_{\{H_i \geq 1\}} \leq A.$$

Set $a_1 = A(\log A)^{-\delta}$, $a_2 = A$,

$$H_i^1 = H_i 1_{\{1 \leq H_i < a_1\}}, \qquad H_i^2 = H_i 1_{\{a_1 \leq H_i < a_2\}}, \qquad H_i^3 = H_i 1_{\{H_i \geq a_2\}}.$$

Thus

$$P\Big(\sum_{i \geq 1} H_i 1_{\{H_i \geq 1\}} \geq 4A\Big) \leq \sum_{j \leq 3} P\Big(\sum_{i \geq 1} \big[H_i^j - EH_i^j\big] \geq A\Big)$$
$$=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).$$

For simplicity set $\Delta(t) = t(\log et)^\delta$ for $t \geq 1$, $\Delta(t) = 0$ for $t < 1$. We have

$$\sum_{i \geq 1} P(H_i > a_2)\Delta(a_2) \leq A,$$

which yields (III) $\leq C(\log A)^{-\delta}$. To evaluate (I) and (II), we use

$$P\Big(\sum_{i \geq 1} \big[H_i^j - EH_i^j\big] \geq A\Big) \leq A^{-2} E\Big(\sum_{i \geq 1} \big[H_i^j - EH_i^j\big]\Big)^2$$
$$\leq A^{-2} \sum_{i \geq 1} E\big(H_i^j - EH_i^j\big)^2$$
$$\leq A^{-2} \sum_{i \geq 1} E\big(H_i^j\big)^2$$
$$\leq A^{-2} a_j \sum_{i \geq 1} EH_i^j.$$

For $j = 1$ we use $\sum_{i \geq 1} EH_i^1 \leq \sum_{i \geq 1} EH_i 1_{\{H_i \geq 1\}} \leq A$ to get (I) $\leq C(\log A)^{-\delta}$.
For $j = 2$ we observe that

$$(\log ea_2)^\delta H_i 1_{\{H_i \geq a_2\}} \leq \Delta(H_i)$$

so that

$$\sum_{i \geq 1} EH_i^2 \leq (\log ea_2)^{-\delta} \sum_{i \geq 1} E\Delta(H_i) \leq A(\log ea_1)^{-\delta}$$

and thus (II) $\leq C(\log A)^{-\delta}$. □

PROPOSITION 2.7. *Consider $s, t \in T$ and $u \geq 1$, $0 \leq \alpha < 1$. Assume that condition $H(B, \delta)$ in* (1.5) *holds. Set*

$$H_i = uR(\alpha i, Y_i)|Y_i(s) - Y_i(t)|.$$

Then

$$\sum_{i \geq 1} EH_i 1_{\{H_i \geq 1\}} (\log eH_i)^\delta \leq \alpha^{-1} B\varphi(s, t, u).$$

PROOF. For convenience, set $\Delta(t) = t(\log et)^\delta$ for $t \geq 1$ and $\Delta(t) = 0$ for $t < 1$.

We recall that $R(x,y) = 1_{[0,h(y)]}(x)$, where $h = d\nu/dm$. Thus we have

$$\Delta(H_i) = 1_{[0,h(Y_i)]}(\alpha i) 1_{\{u|Y_i(s)-Y_i(t)|\geq 1\}} \Delta(u|Y_i(s) - Y_i(h)|).$$

Thus

$$E\Delta(H_i) = E\big(1_{[0,h(Y_1)]}(\alpha i) 1_{\{u|Y_1(s)-Y_1(t)|\geq 1\}} \Delta(u|Y_1(s) - Y_i(t)|)\big).$$

Since $\sum_i 1_{[0,h(Y_1)]}(\alpha i) \leq \alpha^{-1} h(Y_1)$, we get

$$\sum_i E\Delta(H_i) \leq \alpha^{-1} E\big(h(Y_1) 1_{\{u|Y_1(s)-Y_1(t)|\geq 1\}} \Delta(u|Y_1(s) - Y_1(t)|)\big).$$

Since $h = d\nu/dm$ and Y_1 is distributed like m, we have

$$\sum_i E\Delta(H_i) \leq \alpha^{-1} \int_{\{u|\beta(s)-\beta(t)|\geq 1\}} \Delta(u|\beta(s) - \beta(t)|)\, d\nu(\beta).$$

The conclusion follows from the definition of condition $H(B,\delta)$. □

We have proved inequalities that will handle the case of infinitely divisible processes. We now prove similar (but easier) inequalities that will handle random Fourier series. The following is a consequence of Lemma 2.4 and of the inequalities $P(Z \leq A) \leq \exp A \cdot E \exp - Z$, $P(Z \geq A) \leq \exp - A \cdot E \exp Z$.

PROPOSITION 2.8. *Consider independent r.v. $(h_\gamma)_{\gamma\in\Gamma}$. Then*

(i) $$P\left(\sum_{\gamma\in\Gamma} (h_\gamma^2 \wedge 1) \leq \frac{1}{4} \sum_{\gamma\in\Gamma} E(h_\gamma^2 \wedge 1)\right) \leq \exp\left(-\frac{1}{4} \sum_{\gamma\in\Gamma} E(h_\gamma^2 \wedge 1)\right).$$

(ii) *For $A \geq 4\sum_{\gamma\in\Gamma} E(h_\gamma^2 \wedge 1)$, we have*

$$P\left(\sum_{\gamma\in\Gamma} h_\gamma^2 \wedge 1 \geq A\right) \leq \exp - A/2.$$

The following is an immediate consequence of Proposition 2.6.

PROPOSITION 2.9. *Consider independent r.v. $(h_\gamma)_{\gamma\in\Gamma}$, $h_\gamma \geq 0$ and $\delta > 0$. Then for some universal constant C we have*

$$\sum_\gamma E\left(uh_\gamma 1_{\{uh_\gamma\geq 1\}} (\log euh_\gamma)^\delta\right) \leq 2^k$$

$$\Rightarrow P\left(\sum_\gamma uh_\gamma 1_{\{uh_\gamma \geq 1\}} \geq 2^{k+2}\right) \leq Ck^{-\delta}.$$

Finally, let us recall some general facts.

PROPOSITION 2.10. *Consider vectors* $(x_i)_{i\geq 1}$ *in a Banach space. Suppose that* $P(\|\sum_{i\geq 1}\varepsilon_i x_i\| \leq M) \geq 3/4$. *Then*

$$E\left\|\sum_{i\geq 1}\varepsilon_i x_i\right\| \leq CM.$$

PROOF. This has been known for a long time. It can, for example, be obtained by combining the result of [7] with the inequality

$$P\left(\left|\sum_{i\geq 1}\varepsilon_i x_i\right| \geq \frac{1}{4}\left(\sum x_i^2\right)^{1/2}\right) \geq \frac{1}{4}$$

for (x_i) real ([1], page 31). □

Consider two sets K, A of a group G. Let $K' = K + K = \{x + y; x, y \in K\}$ and $K'' = K' + K$. We denote by $N(K, A)$ the smallest number of translates of A by elements of K that can cover A.

LEMMA 2.11 ([4], page 16)

(i) $$N(K, A) \geq \frac{|K|}{|K' \cap A|},$$

(ii) $$N(K, A - A) \leq \frac{|K''|}{|K' \cap A|}.$$

3. Sudakov minoration. The major difference with previous work is that the process is represented conditionally as a series $\sum \varepsilon_i x_i$, *not* as a Gaussian process. Thus it is not possible to apply Sudakov minoration for Gaussian r.v.'s directly. Instead we will use the version of Sudakov minoration for Bernoulli processes that was recently established by the author. Since there was hope at the time that the structure of Bernoulli processes was soon going to be elucidated, this result was not published. There is no point to giving the proof now, as this result has been included in the book [2] where it is discussed in detail. We set

$$B_1 = \left\{(x_i) \in \mathbb{R}^{\mathbb{N}}; \sum_{i\geq 1}|x_i| \leq 1\right\}, \qquad B_2 = \left\{(x_i) \in \mathbb{R}^{\mathbb{N}}; \sum_{i\geq 1}x_i^2 \leq 1\right\}.$$

THEOREM 3.1 (Sudakov minoration for Bernoulli processes). *There exists a universal constant* C_0 *with the following property. Consider a subset* X *of* $\mathbb{R}^{\mathbb{N}}$. *Set* $r(X) = E\sup_{x\in X}|\sum_{i\geq 1}\varepsilon_i x_i|$. *Given* $\varepsilon > 0$, *consider the set*

$$B = \varepsilon B_2 + C_0 r(X) B_1 = \{x + x'; x \in \varepsilon B_2, x' \in C_0 r(X) B_1\}.$$

Then the smallest number of translates of B *by points of* X *needed to cover* X *is less than or equal to* $\exp(C_0 r(X)^2/\varepsilon^2)$.

While the sets $\varepsilon B_2 + C_0 r(X) B_1$ turn out to be the correct choice, they are not so easy to manipulate. In the present case, to handle them, we will use the following simple fact.

LEMMA 3.2. *Consider* $\varepsilon, a, b > 0$. *If* $x \in \varepsilon B_2 + aB_1$, *then*

$$\left(\sum_{i\geq 1} x_i^2 \wedge b^2\right)^{1/2} \leq \varepsilon + \sqrt{ba}\,.$$

PROOF. By definition $x_i = y_i + z_i$, where $\sum_{i\geq 1} y_i^2 \leq \varepsilon^2$, $\sum_{i\geq 1}|z_i| \leq a$. Now

$$|x_i| \wedge b \leq |y_i| + |z_i| \wedge b.$$

By the triangle inequality,

$$\left(\sum_{i\geq 1} x_i^2 \wedge b^2\right)^{1/2} \leq \varepsilon + \left(\sum_{i\geq 1} z_i^2 \wedge b^2\right)^{1/2} \leq \varepsilon + \sqrt{ba}\,,$$

since $z_i^2 \wedge b^2 \leq z_i b$. □

LEMMA 3.3 (See, e.g., [2]). *Consider symmetric independent r.v.'s* $(X_i)_{i\geq 1}$ *taking values in a Banach space and numbers* $(a_i)_{i\geq 1}$, $|a_i| \leq 1$. *Then*

$$P\left(\left\|\sum_{i\geq 1} a_i X_i\right\| \geq u\right) \leq 2P\left(\left\|\sum_{i\geq 1} X_i\right\| \geq u\right).$$

We now prove Theorem 1.1. We consider a number $A > 0$ to be determined later, and we assume that T cannot be covered by $\leq e^u$ sets $V_t = \{s \in T;\ \varphi_\nu(s, t, u/AM) \leq u\}$. Thus we can find $N > e^u$ and points $(s_i)_{i\leq N}$ such that $\varphi_\nu(s_i, s_j, u/AM) > u$ for $i, j \leq N$, $i \neq j$.

We now fix once and for all α such that if we set

$$H_\alpha = \{\forall\, i, \Gamma_i \leq \alpha i\} \subset \Omega, \quad \text{then } P(H_\alpha) > 9/10.$$

It follows from Proposition 2.5(i) that if $u \geq 2\alpha$ we have

$$P\left(\sum_{k\geq 1} R(\alpha k, Y_k)^2 |Y_k(s_i) - Y_k(s_j)|^2 \wedge \left(\frac{AM}{u}\right)^2 \leq \frac{A^2M^2}{8\alpha u}\right)$$
$$\leq \exp -\frac{u}{8\alpha}.$$

Consider $n \leq N$. Set

$$G = \left\{\forall\, i, j \leq n,\ \sum_{k\geq 1} R(\alpha k, Y_k)^2 |Y_k(s_i) - Y_k(s_j)|^2 \wedge \left(\frac{AM}{u}\right)^2 \geq \frac{A^2M^2}{8\alpha u}\right\}.$$

Then

$$P(G) \geq 1 - n^2 \exp -\frac{u}{8\alpha}.$$

Taking

$$\exp\frac{u}{32\alpha} \le n \le 2\exp\frac{u}{32\alpha}$$

gives $P(G) > 9/10$ if $u \ge C$.

Set now $x_{i,k} = R(\Gamma_i, Y_i)Y_i(s_k)$. Since $R(\cdot, y)$ decreases, we have shown that $P(W) > 4/5$, where

$$W = \left\{\forall\, k, l \le n,\ \sum_{i\ge 1} (x_{i,k} - x_{i,l})^2 \wedge \left(\frac{AM}{u}\right)^2 \ge \frac{A^2M^2}{8\alpha u}\right\}.$$

On the other hand, since $\Pr(\sup_{t\in\{s_1,\dots,s_n\}}|X_t| \ge M) \le 1/5$, we can find $\omega \in W$ such that, setting $x_k^\omega(\eta) = \sum_{i\ge 1}\varepsilon_i(\eta)x_{i,k}(\omega)$,

$$Q\Big(\sup_{k\le n}|x_k^\omega(\eta)| \le M\Big) > 3/4.$$

By Lemma 2.10 we have $E(\sup_{k\le n}|x_k^\omega|) \le C_i M$. Now take $\varepsilon = AM/2\sqrt{8\alpha u}$, $b = AM/u$, $a < AM/32\alpha$.

Set $B' = \varepsilon B_2 + aB_1$. By Lemma 3.2 and the definition of W, we see that $x_k \notin x_l + B'$ if $k, l \le n$, $k \ne l$, where $x_k = (x_{i,k})_{i\ge 1}$. Set $B = \frac{1}{2}B'$. Then the set $Z = \{x_k;\ k \le n\}$ cannot be covered by n translates of B', since no two of these points can belong to the same translate of B. If we take $A > 64\alpha C_0 C_1$ (universal), we can assume $a > 2C_0C_1M$, so that by Theorem 3.1,

$$\exp\frac{u}{32\alpha} \le n \le \exp\frac{4C_0C_1^2M^2}{\varepsilon^2} = \exp\frac{2^7C_0C_1^2\alpha u}{A^2},$$

that is,

$$\frac{1}{32\alpha} \le \frac{2^7C_0C_1^2\alpha}{A^2}.$$

This is a contradiction if $A^2 > 2^{12}C_0C_1^2\alpha^2$. This concludes the proof. □

4. Proof of Theorems 1.2 and 1.4.

First, we prove Theorem 1.2.

For convenience, we denote by C a constant that depends on K only and that can vary at each occurrence. (The reader will check that when $G = K$ is compact, C is universal.)

Consider a compact neighborhood L of the unit 0 of G such that $L - L \subset K$ (if $K = G$, take $L = G$). Our first task is to prove that $2^{-l_0} \le CM$. This is a consequence of the following elementary lemma.

LEMMA 4.1. *There exist two constants $\varepsilon_0, C > 0$ such that if a random variable X satisfies*:

(i) $\forall\, t \in \mathbb{R}$, $E\exp itX = \exp -\int_0^\infty(1 - \cos tx)\,d\nu(x)$,
(ii) $P(|X| \le M) \ge 1 - \varepsilon_0$,

then $\int_0^\infty (x/CM)^2 \wedge 1\,d\nu(x) \le 1$.

PROOF. Consider $t_0 > 0$ such that $t_0 M \leq \pi$, so that $\cos tX \geq \cos t_0 M$ for $|X| \leq M$, $0 \leq t \leq t_0$. Thus, since $P(|X| \leq M) \geq 1 - \varepsilon_0$, we have

$$E \cos tX \geq -\varepsilon_0 + (1 - \varepsilon_0)\cos t_0 M$$

and hence for $t \leq t_0$ we have

$$\int_0^\infty (1 - \cos tx)\, d\nu(x) \leq B =: -\log(-\varepsilon_0 + (1 - \varepsilon_0)\cos t_0 M).$$

Integrating over t for $0 \leq t \leq t_0$, we get

$$\int_0^\infty \left(1 - \frac{\sin t_0 x}{t_0 x}\right) d\nu(x) \leq B.$$

Observe that for some $\alpha > 0$ we have $1 - \sin u/u \geq \alpha(1 \wedge u^2)$. Now we choose ε_0 small enough and C large enough that $-\log(-\varepsilon_0 + (1 - \varepsilon_0)\cos 1/C) \leq \alpha$, and we take $t_0 = 1/MC$. □

We wish to avoid giving a separate argument to show that $i(l)$ is well defined. Let us agree to set $i(l) = \infty$ when $i(l)$ is not well defined. Observe that $|U_{l,k}|$ is a decreasing function of l, so that $i(l)$ increases. We define l_1 as the smallest integer such that $i(l_1) \geq 1$ and $|L|e^{2^{i(l_1)}} > 1$. If $i(l) < \infty$ for each l, we set $h(l) = i(l)$. Otherwise there is a smallest l_2 such that $i(l_2) = \infty$. We set $h(l) = i(l)$ for $l < l_2$. Since $i(l_2) = \infty$, the inequality $|U_{l_2,i}| \leq e^{-2^i}$ occurs for arbitrarily large values of i, and thus we can find $N > i(l_2 - 1)$ such that $|U_{l_2,N}| \leq e^{-2^N}$. We set $h(l) = N$ for $l \geq l_2$. Thus the sequence $(h(l))$ increases.

It follows from Lemma 2.11 that $N(L, U'_{l,h(l)}) \geq |L|e^{2^{h(l)}}$. This implies that we can find a subset Q_l of L such that card $Q_l \geq |L|e^{2^{h(l)}}$ and $s - t \notin U'_{l,h(l)}$ for $s, t \in Q_l$, $s \neq t$. We recall the crucial fact that $|\beta(s) - \beta(t)| = |\beta(s - t) - 1|$ for $\beta \in \mathbb{C}\Gamma$. Thus we have for $s, t \in Q_l$, $s \neq t$,

$$\int_{\mathbb{C}\Gamma} 2^{2l+h(l)}|\beta(s) - \beta(t)|^2 \wedge 1\, d\nu(\beta) \geq 2^{h(l)}. \tag{4.1}$$

We now appeal to Theorem 2.1 to represent the process (X_t) as a mixture of random Fourier series. Since, by the law of large numbers, we have $\lim_{i\to\infty} \Gamma_i/i = 1$ a.s., there exists a number α such that $P(A) \geq 7/8$, where $A = \{\sup_{i\geq 1} \Gamma_i/i \leq \alpha\}$. Moreover, A is independent of the sequence (Y_i). Since the functions $R(u, y)$ are nonincreasing in u, we have for $\omega \in A$ that

$$d_\omega^2(s,t) \geq \sum_{i\geq 1} (R(\alpha i, Y_i)|Y_i(s) - Y_i(t)|)^2.$$

If we use Proposition 2.5(i), we see that

$$2^{h(l)} \geq 2\alpha \Rightarrow P\left(d_\omega^2(s,t) \leq \frac{2^{h(l)}}{8\alpha 2^{2l+h(l)}}\right) \leq \exp -\frac{2^{h(l)}}{8\alpha}.$$

We thus get

$$(4.2)\qquad 2^{h(l)} \geq 2\alpha \Rightarrow P\big(d_\omega(s,t) \leq 2^{-l-3}\alpha^{-1/2}\big) \leq \exp - \frac{2^{h(l)}}{8\alpha}.$$

For $k \geq 1$ we set

$$B_k = \{l \geq l_1;\, h(l) = k\}.$$

Thus $h(l) = k$ for $l \in B_k$. We set $D = \{k \geq 1,\ B_k \neq \varnothing\}$, and for $k \in D$ we denote by $l(k)$ the smallest element of B_k.

Consider the smallest integer k_0 such that $2^{k_0} \geq 2\alpha$ and

$$\sum_{k \geq k_0} \exp(-2^k/16\alpha) \leq 1/8.$$

We set $D_0 = \{k \in D;\ k \geq k_0\}$. For $k \in D_0$ consider a subset S_k of $Q_{l(k)}$ such that $\operatorname{card} S_k \leq \exp(2^k/32\alpha)$ and

$$(4.3)\qquad \operatorname{card} S_k \geq \min\big(\exp(2^k/32\alpha) - 1;\ |L| \exp 2^k\big).$$

It follows from (4.2) that, for $k \in D_0$, we have

$$P\big(\forall\, s,t \in S_k,\ s \neq t,\ d_\omega(s,t) \geq 2^{-l(k)-3}\alpha^{-1/2}\big) \geq 1 - \exp - \frac{2^k}{16\alpha},$$

since we have assumed $\operatorname{card} S_k \leq \exp(2^k/32\alpha)$. Thus we can find $B \subset A$ such that

$$P(B) \geq P(A) - \sum_{k \geq k_0} e^{-2^k/16\alpha} \geq 3/4$$

and for $\omega \in B$,

$$\forall\, k \in D_0, \forall\, s,t \in S_k,\ s \neq t, \qquad d_\omega(s,t) \geq 2^{-l(k)-3}\alpha^{-1/2}.$$

Since we have $\Pr(\sup_{t \in K}|X_t| \leq M) \geq 3/4$, we can, by Fubini's theorem, find $\omega \in B$ such that

$$Q\Big(\sup_{t \in X}|X_t^\omega| \leq M\Big) \geq 1/2.$$

We now combine Lemma 2.10 with the results of [4] to see that for a constant C depending on K only we have

$$\sum_{k \in D_o} 2^{-l(k)}\alpha^{-1}(\log \operatorname{card} S_k)^{1/2} \leq CM.$$

From (4.3), since $l(k) \geq l_0$ and since $2^{-l_0} \leq CM$, we get that

$$\sum_{k \in D} 2^{-l(k)+k/2} \leq CM.$$

Since again $l(k) \geq l_0$, $2^{-l_0} \leq CM$, it follows that

$$\sum_{k \in D} 2^{-l(k)+k/2} \leq CM.$$

On the other hand, we have

$$\sum_{l\in B_k} 2^{-l+h(l)/2} = \sum_{l\in B_k} 2^{-l+k/2} \le 2^{-l(k)+k/2+1}.$$

Thus we have shown that

$$\sum_{l\ge l_1} 2^{-l+h(l)/2} \le CM.$$

Since obviously

$$\sum_{l_o\le l<l_1} 2^{-l+h(l)/2} \le C2^{-l_0} \le CM,$$

we have shown that $\sum_{l\ge l_0} 2^{-l+h(l)/2} \le CM$. In particular, $h(l)$ is bounded independently of N. Letting $N\to\infty$ shows that each $i(l)$ is finite and that $\sum_{l\ge l_0} 2^{-l+i(l)/2} \le CM$. □

The proof of Theorem 1.4 is very similar; we observe that, since each r.v. (f_γ) is assumed to be symmetric, the process $X_t = \sum_{\gamma\in\Gamma} f_\gamma\gamma(t)$ is distributed like $\sum_{\gamma\in\Gamma}\varepsilon_\gamma f_\gamma\gamma(t)$, where ε_γ is a Bernoulli sequence independent of f_γ; thus we can represent X_t as a mixture of random Fourier series $X_t^\omega = \sum\varepsilon_\gamma f_\gamma(\omega)\gamma(t)$. The only change in the proof from that of Theorem 1.2 is to replace the use of Proposition 2.5(i), by the use of Proposition 2.8(i).

Let us mention that Proposition 1.5 follows from the result of [4], since the random Fourier series $\sum_\gamma \varepsilon_\gamma a_\gamma \gamma(t)$ has to converge by standard arguments.

5. Proof of Theorems 1.3 and 1.5. We first prove Theorem 1.3. We first dispose of the case where $U_{l,0} = K$ for all $l\in\mathbb{Z}$. In that case, we have

$$\forall, s\in K, \forall\, l\in\mathbb{Z}, \qquad \int 2^{2l}|\beta(s)-\beta(0)|^2 \wedge 1\, d\nu(\beta) \le 1$$

so that $\beta(s)=\beta(0)$ ν-a.e. Going back to (1.4), one sees that the process $(X_t)_{t\in K}$ is (in distribution) such that $X_t = X_0$ so is certainly continuous.

For convenience, we assume now that Ω is a product $\Omega_1\times\Omega_2$ and $P = P_1\otimes P_2$, and that for $\omega\in\Omega$, $Y_i(\omega)$ depends only on ω_1 and $\Gamma_i(\omega)$ on ω_2. It is clear by definition that the sequence $i(l)$ increases. We define a sequence m_k by induction as follows:

$$m_0 = \max\{l;\, i(l) = i(l_0)\},$$
$$m_{k+1} = \max\{l > m_k;\, \forall\, l', m_k < l' \le l, i(l') = i(l)\}.$$

This means that the function $i(l)$ jumps between m_k and m_k+1, and that these are the only jumps. Note that $i(m_k)\ge k$, and that the sequence $i(m_k)$ increases strictly. By definition of $i(l)$, we have

$$|U_{l,i(l)+1}| \ge e^{-2^{i(l)+1}} \tag{5.1}$$

and for $s\in U_{l,i(l)+1}$ we have

$$\varphi(s,0,2^{l+i(l)/2}) \le \varphi(s,0,2^{l+i(l)/2+1/2}) \le 2^{i(l)+1}.$$

We set $\lambda_k = 2^{m_k + i(m_k)/2}$, $U_k = U_{m_k, i(m_k)+1}$. Thus for $s \in U_k$ we have $\varphi(s, 0, \lambda_k) \le 2^{i(m_k)+1}$. Consider now $\alpha < 1$, which will be fixed until further notice. Consider $s \in U_k$. Set

$$S_{i,s} = R(\alpha i, Y_i)|Y_i(s) - Y_i(0)|^2.$$

We apply Proposition 2.5(ii) with $A = \alpha^{-1}2^{-i(m_k)+3}$. Since $i(m_k) \ge k$, we get

$$P_1\left(\sum_{i\ge 1} S_{i,s}^2 \wedge \frac{1}{\lambda_k^2} \ge \alpha^{-1}2^{-2m_k+3}\right) \le \exp - 2^{k+2}. \tag{5.2}$$

We now use Proposition 2.7 to see that if we set $H_i = \lambda_k S_{i,s}$, we have

$$\sum_{i\ge 1} EH_i 1_{\{H_i \ge 1\}}(\log eH_i)^\delta \le \alpha^{-1}B2^{i(m_k)+1}.$$

We can assume $B \ge 1$. We apply Proposition 2.6; since $\log(\alpha^{-1}B2^{i(m_k)+1}) \ge C^{-1}\log(k+1)$ and since $2^{i(m_k)+1}/\lambda_k = 2^{-m_k+i(m_k)/2+1}$, we have

$$P_1\left(\sum_{i\ge 1} S_{i,s}1_{\{S_{i,s}\ge 1/\lambda_k\}} \ge \alpha^{-1}B2^{-m_k+i(m_k)/2+3}\right) \le \frac{C}{(k+1)^\delta}. \tag{5.3}$$

Given $\omega_1 \in \Omega_1$, we define U_{k,α,ω_1} as the set of points $s \in U_k$ that satisfy

$$\sum_{i\ge 1} S_{i,s}^2 \wedge \frac{1}{\lambda_k^2} \le \alpha^{-1}2^{-2m_k+3}, \tag{5.4}$$

$$\sum_{i\ge 1} S_{i,s}1_{\{S_{i,s}\ge 1/\lambda_k\}} \le \alpha^{-1}CB2^{-m_k+i(m_k)/2+3}. \tag{5.5}$$

It follows from (5.2), (5.3) and Fubini's theorem that

$$P_1(|U_{k,\alpha,\omega_1}| \ge |U_k|/2) \ge 1 - b_k,$$

where $b_k \le C(k+1)^{-\delta}$ (for a new C) is the term of a summable series. Consider the event

$$B'_{p,\alpha} = \{\forall\, k \ge p, |U_{k,\alpha,\omega_1}| \ge |U_k|/2\} \subset \Omega_1.$$

Thus $P_1(B'_{p,\alpha}) \ge 1 - \sum_{k\ge p} b_k$.

By definition of l_0, we have

$$\forall\, s \in K, \qquad \varphi(s, 0, 2^{-l_0}) \le 1.$$

By Lemma 2.3 and Proposition 2.7, we have

$$E\left(\sum_{i\ge 1} 2^{2l_0}S_{i,s}^2 \wedge 1\right) \le \alpha^{-1},$$

$$E\left(\sum_{i\ge 1} 2^{l_0}S_{i,s}1_{\{2^{l_0}S_{i,s}\ge 1\}}\right) \le \alpha^{-1}CB.$$

Consider the set

$$K_{\omega_1} = \left\{ s \in K;\ \sum_{i \geq 1} 2^{2l_0} S_{i,s}^2 \wedge 1 \leq \frac{4p|K|}{|L|}\alpha^{-1}; \right.$$

$$\left. \sum_{i \geq 1} 2^{l_0} S_{i,s} 1_{\{2^{l_0} S_{i,s} \geq 1\}} \leq 4p \frac{|K|}{|L|}\alpha^{-1} CB \right\}.$$

It then follows from Fubini's theorem that if we set

$$B'' = \left\{\omega_1;\ |K_{\omega_1}| \geq |K| - |L|/2\right\},$$

we have $P_1(B'') \geq 1 - 1/p$. We set $B_{p,\alpha} = B'' \cap B'_{p,\alpha}$. We set $H_\alpha = \{\omega_2;\ \forall\ i \geq 1,\ \Gamma_i(\omega_2) \geq \alpha i\}$.

First basic observation. Consider $\omega_1 \in B''$, $\omega_2 \in H_\alpha$, $\omega = (\omega_1, \omega_2)$. Consider $s, t \in G$ such that $s \in t + U_{k,\alpha,\omega_1}$. Then for some constant $C(\alpha)$ depending on α only,

$$(5.6)\quad Q\big(|X_s^\omega - X_t^\omega| \geq C(\alpha)\big(B2^{-m_k + i(m_k)/2+1} + u2^{-m_k+1}\big)\big) \leq 4\exp - u^2.$$

Proof. Theorem 2.1 shows that conditionally on ω, $X_s - X_t$ is distributed like $\sum \varepsilon_i a_i$, where $a_i = R(\Gamma_i, Y_i)(Y_i(s) - Y_i(t))$. Since $\Gamma_i \geq \alpha i$ and $R(\cdot, y)$ decreases, we have

$$|a_i| \leq R(\alpha i, Y_i)|Y_i(s) - Y_i(t)|.$$

Observe now the crucial fact that, since Y_i is the multiple of a character, we have

$$|Y_i(s) - Y_i(t)| = |Y_i(s-t) - Y_i(0)|.$$

Set $J = \{i \geq 1;\ |a_i| \geq 1/\lambda_k\}$. Since $s - t \in U_{k,\alpha,\omega_1}$, we have by (5.4) [resp. (5.5)] that

$$\sum_{i \notin J} |a_i|^2 \leq \alpha^{-1} 2^{-2m_k+3},$$

$$\sum_{i \in J} |a_i| \leq \alpha^{-1} CB 2^{-m_k + i(m_k)/2+3}.$$

Thus we have

$$\left|\sum_{i \geq 1} \varepsilon_i a_i\right| \geq u + \alpha^{-1} CB2^{-m_k+i(m_k)/2+3} \Rightarrow \left|\sum_{i \notin J} \varepsilon_i a_i\right| \geq u.$$

We now apply the standard sub-Gaussian inequality to both the real and imaginary parts of a_i; we thus see that this latter event has a probability less than or equal to $4\exp(-u^2(\sum_{i \notin J}|a_i|^2)^{-1})$. This completes the proof. □

Second basic observation. While the proof of Theorem 1.3 relies, in the end, on a standard chaining argument, there is a difficulty starting the chaining. The next few lines address this difficulty. The same argument as above shows that for $\omega_1 \in B''$, $s - t \in K_{\omega_1}$, we have

$$Q\big(|X_s^\omega - X_t^\omega| \geq 2^{-l_0} C(\alpha, p)(1 + u)\big) \leq \exp - u^2,$$

where $C(\alpha, p)$ depends on α, p only. We claim that $L \subset K_{\omega_1} - K_{\omega_1}$. Indeed, take $s \in L$. Since $L - L \subset K$, we have $L \subset s + K$. Since $|K \setminus K_{\omega_1}| \leq |L|/2$, we have $|(s + K) \setminus (s + K_{\omega_1})| < |L|/2$. Thus we have $|L \cap (s + K_{\omega_1})| > |L|/2$. Since $|K \setminus K_{\omega_1}| < |L|/2$, we have $|L \cap K_{\omega_1}| > |L|/2$. Thus $K_{\omega_1} \cap s + K_{\omega_1} \neq \varnothing$, that is, $s \in K_{\omega_1} - K_{\omega_1}$. It follows that for $s - t \in L$, we have

$$Q\big(|X_s^\omega - X_t^\omega| \geq 2^{-l_0} C(\alpha, p)(1 + u)\big) \leq 2 \exp - u^2. \tag{5.7}$$

Note that $U_{k, \alpha, \omega} = -U_{k, \alpha, \omega}$, so that $V_k = U_{k, \alpha, \omega} - U_{k, \alpha, \omega} = U_{k, \alpha, \omega} + U_{k, \alpha, \omega}$. It follows from (5.6) that whenever $s - t \in V_k$ we have

$$Q\big(|X_s^\omega - X_t^\omega| \geq C(\alpha)\big(B 2^{-m_k + i(m_k)/2} + u 2^{-m_k} 2^{-m_k}\big)\big) \leq 8 \exp - u^2. \tag{5.8}$$

For $k \geq p$, we have $|U_{k, \alpha, \omega_1}| \geq |U_k|/2 \geq \frac{1}{2} e^{-2^{i(m_k)+1}}$. From (the proof of) Lemma 2.11 we can find a set $R_k \subset L$ such that $L \subset R_k + V_k$, and card $R_k \leq 2|K''| e^{2^{i(m_k)+1}}$.

For $K \geq p + 1$, we consider a map φ_k: $R_{k+1} \to R_k$ such that $t - \varphi_k(t) \in V_k$. Consider $N \geq p$. For $t \in R_N$, we define by decreasing induction $t(k) \in R_k$ by $t(N) = t$, $t(k) = \varphi_k(t(k + 1))$. Thus

$$X_t^\omega - X_{t(p)}^\omega = \sum_{k=p}^{N-1} X_{t(k+1)}^\omega - X_{t(k)}^\omega.$$

Observe that there are at most card R_{k+1} variables of the type $X_{t(k+1)}^\omega - X_{t(k)}^\omega$ and that $t(k + 1) - t(k) \in V_k$. Thus from (5.8) we see that

$$\begin{aligned} &Q\Big(\sup_{t \in R_N} |X_{t(k+1)}^\omega - X_{t(k)}^\omega| \geq C(\alpha)\big(2^{-m_k + i(m_k)/2} + u_k 2^{-m_k}\big)\Big) \\ &\qquad \leq 8 \operatorname{card}(R_{k+1}) \exp - u_k^2. \end{aligned} \tag{5.9}$$

We now take $u_k = \sqrt{\log \operatorname{card}(R_{k+1})} + k + 1$. We have

$$\sum_{k \geq p} 2^{-m_k} u_k \leq \sum_{k \geq p} 2^{-m_k}\big(k + C + 2^{i(m_{k+1})/2 + 1}\big).$$

We now observe that by definition of m_{k+1} we have $i(m_{k+1}) = i(m_k + 1)$, so that

$$2^{-m_k} 2^{i(m_{k+1})/2} \leq 2\big(2^{-(m_k+1)} 2^{i(m_k+1)/2}\big)$$

and thus

$$\sum_{k \geq p} 2^{-m_k} 2^{i(m_{k+1})/2} \leq 2 \sum_{l \geq l_0} 2^{-l + i(l)/2}.$$

Finally, it follows from (5.9) that, since $m_k \geq k + l_0$, we have

$$Q\left(\sup_{t \in R_N} |X_t^\omega - X_{t(p)}^\omega| \geq C(\alpha)\left(\sum_{l \geq l_0} 2^{-l+i(l)/2}\right)\right) \leq 8e^{-p^2}. \tag{5.10}$$

On the other hand, from (5.7) we have

$$Q\left(\sup_{t \in R_p} |X_t^\omega - X_0^\omega| > 2^{-l_0} C(p,\alpha)(1+u)\right) \leq 8e^{-u^2} \operatorname{card} R_p. \tag{5.11}$$

If we take $u = p + \sqrt{\log \operatorname{card} R_p}$ in (5.11) and combine with (5.10), we get

$$Q\left(\sup_{t \in R_N} |X_t^\omega - X_0^\omega| \geq C(p,\alpha)\left(\sum_{l \geq l_0} 2^{-l+i(l)/2}\right)\right) \leq 16e^{-p^2}$$

where $C(p,\alpha)$ depends only on p, α, K,B and δ. As N is arbitrary, we get

$$Q\left(\sup_{t \in L} |X_t^\omega - X_0^\omega| \geq C(p,\alpha)\left(\sum_{l \geq l_0} 2^{-l+i(l)/2}\right)\right) \leq 16e^{-p^2}. \tag{5.12}$$

We can now conclude the proof of Theorem 1.3. Given $\varepsilon > 0$, we pick p such that $p^{-1} + \sum_{k \geq p} b_k \leq \varepsilon$, $16e^{-p^2} \leq \varepsilon$, and we pick α such that $P_2(H_\alpha) \geq 1 - \varepsilon$. Then (5.12) holds for $\omega = (\omega_1, \omega_2)$, $\omega_1 \in B_{p,\alpha}$, $\omega_2 \in H_\alpha$. By Fubini's theorem, we have

$$\Pr\left(\sup_{s,t \in L} |X_t - X_s| \geq C(p,\alpha)\left(\sum_{l \geq l_0} 2^{-l+i(l)/2}\right)\right) \leq 3\varepsilon.$$

The claim concerning a.s. continuity follows from Fubini's theorem, and the fact that P-a.s., the random Fourier series X_t^ω is a.s. bounded (as we have shown), so that it is a.s. continuous by Theorem 1.1 of [4]. □

The proof of Theorem 1.6 is rather similar, but does require an extra idea. It is easy to see from (1.5) that for all $u > 0$, for some constant C depending on δ only,

$$\begin{aligned}\int_{u|f_\gamma| \geq 1} u|f_\gamma|\left(\log eu|f_\gamma|\right)^\delta dP &\leq CB\left(E\left(u^2|f_\gamma|^2 \wedge 1\right) + ua_\gamma\left(\log^+ ua_\gamma\right)\right)^\delta \\ &\leq CB\left(E\left(u^2|f_\gamma|^2 \wedge 1\right) + u^2a_\gamma^2\right).\end{aligned}$$

In order to use Proposition 2.8 to prove a substitute of (5.3), we desire that

$$\sum_\gamma E\left(\lambda_k|f_\gamma|^2|\gamma(s) - 1|^2 \wedge 1\right) \leq 2^{i(m_k)+1}; \ \lambda_k^2 \sum_\gamma a_\gamma^2|\gamma(s) - 1|^2 \leq 2^{i(m_k)}.$$

This is done by using the sets

$$V_k = V_{m_k, i(m_k)+1} \cap \{s \in K;\, d(s,0) \leq 2^{-m_k}\}$$

instead of U_k. It is easily seen that

$$\sum_k 2^{-m_k}(\log 1/|V_k|)^{1/2} \le C\left(\sum_{l\ge l_0} 2^{-l+i(l)/2} + \sum_k 2^{-k}(\log N_k(K))^{1/2}\right),$$

and this is exactly what we need to complete the proof as above. □

6. The p-stable and ξ-radial cases. Consider a positive measure θ on $\mathbb{R}^+$ and a probability measure m on $\mathbb{R}^T$. Consider the case where ν is the image of $\theta \otimes m$ by the map $(x, \beta) \to x\beta$. Then, following Marcus [3], we say that the process is *ξ-radial*. Set $\Phi(u) = \int_0^\infty x^2u^2 \wedge 1\, d\theta(x)$. Then

$$\varphi_\nu(s,t,u) = \int_{\mathbb{C}^T} \Phi(|\beta(s) - \beta(t)|u)\, dm(\beta). \tag{6.1}$$

Of particular importance is the case when θ has density t^{-p-1} with respect to Lebesgue measure λ. In this case, the process is called *p-stable*. It is easily seen that for some constant $c(p)$ we have $\Phi(u) = c(p)u^p$. Thus if we define the distance d_p on T by

$$d_p^p(s,t) = \int_{\mathbb{C}^T} c(p)^{-1}|\beta(s) - \beta(t)|^p\, dm(\beta),$$

we see that $\varphi(s,t,u) = d_p^p(s,t)u^p$.

For $p > 1$ the sets V_t of Theorem 1.1 are thus of the type

$$\{s \in T; d_p(s,t) \le CMu^{-1+1/p}\}.$$

Setting $q = 1 - 1/p$, $v = CMu^{-1/q}$, Theorem 1.1 means that T can be covered by at most $\exp(CM/v)^q$ balls (for d) of radius v, which is the formulation of [4], Theorem 26.

We now turn to the case where $T = G$ is a locally compact group. We still assume $p > 1$. We denote by $B(\varepsilon)$ the ball for d_p centered at 0 of radius ε. For simplicity of notation, we set $N_k(A) = N(A, B(2^{-k}))$. We now relate Theorems 1.2 and 1.4 with the usual entropy conditions.

PROPOSITION 6.1. *Suppose $p \ge 1$. Then for some constant C depending on p, k only we have*

$$\sum_{l\ge l_0} 2^{-l+i(l)/2} \le C\left(\sum_k 2^{-k}(\log N_k(K))^{1/q}\right), \tag{6.2}$$

$$\sum_k 2^{-k}(\log N_k(K))^{1/q} \le C\left(\sum_{l\ge l_0} 2^{-l+i(l)/2}\right). \tag{6.3}$$

The reader will note that $\log N_k(K) = 0$ when $K \subset B(2^{-k})$. We observe that $U_{l,i}^0 = B(2^{-l+i(1/p-1/2)})$. By the definition of l_0, we have

$$K \not\subset B(2^{-l_0}), \qquad K \subset B(2^{-l_0+1}).$$

This implies in particular that $N_{l_0-1}(K) = 1$, and (easily) that $N_{l_0+1}(K) > 1$.

For convenience we denote by C a constant that depends only on p and K and that may vary at each occurrence.

PROOF OF (6.2). For $l \geq l_0$, we denote by $k(l)$ the integer such that

$$k(l) - 1 < l - i(l)(1/p - 1/2) \leq k(l). \tag{6.4}$$

Thus we have $K \cap B(2^{-k(l)}) \subset U_{l,i(l)}$ and thus

$$|K \cap B(2^{-k(l)})| \leq e^{-2^{i(l)}}. \tag{6.5}$$

Consider now a compact neighborhood L of 0 such that $L' = L - L \subset K$. From (6.5) we have that $|L' \cap B(2^{-k(l)})| \leq e^{-2^{i(l)}}$, so by Lemma 2.11(i) we have $N_{k(l)}(K) \geq N_{k(l)}(L) \geq |L| e^{-2^{i(l)}}$ so that

$$2^{i(l)} \leq \log N_{k(l)}(L) + \log^+(1/|L|). \tag{6.6}$$

Now

$$\begin{aligned} 2^{-l+i(l)/2} &= 2^{-l} 2^{i(l)(1/p-1/2)} 2^{i(l)/q} \\ &\leq 2^{-k(l)+1}\Big(\big(\log N_{k(l)}(K)\big)^{1/q} + \big(\log^+(1/|L|)\big)^{1/q}\Big). \end{aligned}$$

Set $A_- = \{l \geq l_0;\ k(l) \leq l_0 - 1\}$. From (6.6) and $N_{l_0-1}(K) = 1$, it follows that $i(l) \leq C$ for $l \in A_-$, so that from (6.4), $l \leq l_0 + C$ and

$$\sum_{l \in A_-} 2^{l-i(l)/2} \leq C 2^{l_0}. \tag{6.7}$$

For $k \geq l_0$ denote by A_k the set

$$A_k = \{l;\ k(l) = k\}.$$

From (6.6) we see that $i(l)$ is bounded on A_k. By (6.4) we see that A_k is bounded. Whenever A_k is not empty, denote by $l(k)$ its largest element. For $l \in A_k$ we have

$$l - i(l)(1/p - 1/2) \geq k - 1 \geq l(k) - i(l(k))(1/p - 1/2) - 1$$

so that

$$i(l)(1/p - 1/2) \leq i(l(k))(1/p - 1/2) + l - l(k) + 1,$$

and thus

$$\begin{aligned} -l + i(l)/2 \leq{} & -l(k) + i(l(k))/2 + (1/p - 1/2)^{-1} \\ & + (l - l(k))\left(\frac{1/q}{1/2 - 1/p}\right). \end{aligned}$$

It follows that

$$\sum_{l \in A_k} 2^{-l+i(l)/2} \leq C 2^{-l(k)+i(l(k))/2}. \tag{6.8}$$

Now, by (6.6), we have

$$\sum 2^{-l(k)+i(l(k))/2} \le C + 2\sum_{k\ge 1} 2^{-k}(\log N_k(L))^{1/q}, \tag{6.9}$$

where the summation on the left is over the values of k for which $A_k \neq \varnothing$. To complete the proof, we combine (6.7), (6.8) and (6.9) and we observe that since $N_{l_0+1}(K) > 1$, we have $2^{-l_0} \le C(\sum_k 2^{-k}(\log N_k(K))^{1/q})$. □

PROOF OF (6.3). Consider m such that $2^{-m}(\log 2)^{1/q-1/2} \le 1$. For $k \ge l_0$, we have $N_k(K) \ge 2$, so for $k \ge l_0 + m$, we can define $l(k)$ by

$$2^{-l(k)} \le 2^{-k-1}(\log N_k(K))^{1/q-1/2} < 2^{-l(k)+1} \tag{6.10}$$

and we have $l(k) \ge l_0$. We define $j(k)$ by

$$2^{j(k)+1} \le \log N_k(K) < 2^{j(k)+2}. \tag{6.11}$$

Our first task is to show that for some j_0 depending on K only, we have

$$j(k) \ge j_0 \Rightarrow i(l(k)) \ge j(k). \tag{6.12}$$

From (6.10) and (6.11), we have

$$2^{-l(k)+j(k)(1/p-1/2)} \le 2^{-k-1}$$

and thus

$$U_{l(k),\,j(k)} \subset K \cap B(2^{-k-1}).$$

Since $B(2^{-k-1}) - B(2^{-k-1}) \subset B(2^{-k})$, we have

$$N(K, B(2^{-k-1}) - B(2^{-k-1})) \ge N(K, B(2^{-k})) = N_k(K).$$

From Lemma 2.1(ii) we get

$$|K \cap B(2^{-k-1})| \le |K' \cap B(2^{-k-1})| \le \frac{|K''|}{N_k(K)} \le |K''|e^{-2^{j(k)+1}}.$$

For $j(k) \ge j_0$, where j_0 depends on K only, this is less than or equal to $e^{-2^{j(k)}}$, and this proves (6.12) by the definition of $i(l)$.

We have from (6.12), when $j(k) \ge j_0$,

$$\begin{aligned} 2^{-k}(\log N_k(K))^{1/q} &\le 2^{-k}(\log N_k(K))^{1/q-1/2}(\log N_k(K))^{1/2} \\ &\le 2^{-l(k)+j(k)/2+3} \\ &\le 2^{-l(k)+i(l(k))/2+3}. \end{aligned} \tag{6.13}$$

Denote $A = \{k \ge l_0 + m\colon j(k) < j_0\}$: Clearly,

$$\sum_{k\in A} 2^{-k}(\log N_k(K))^{1/q} \le C2^{-l_0}. \tag{6.14}$$

Also

$$\sum_{k<l_0+m} 2^{-k}(\log N_k(K))^{1/q} \le C2^{-l_0-m}(\log N_{l_0+m}(K))^{1/q}. \tag{6.15}$$

For $l \geq l_0$ we denote

$$B_l = \{k \geq l_0 + m;\, k \notin A,\, l(k) = l\}.$$

When B_l is not empty, denote by $k(l)$ its smallest element. An argument similar to that used to prove (6.8), but easier, shows that

$$\sum_{k \in B_l} 2^{-k}(\log N_k(K))^{1/q} \leq C 2^{-k(l)}(\log N_{k(l)}(K))^{1/q}. \tag{6.16}$$

From (6.13), we have

$$2^{-k(l)}(\log N_{k(l)}(K))^{1/q} \leq 2^{-l+i(l)/2+3}.$$

Together with (6.14), (6.15) and (6.16), this completes the proof. □

Unfortunately, Theorem 1.3 does not cover the case of 1-stable processes, that is, $\Phi(x) = x$. This case is however completely understood ([5, 8]). Setting $LLt = \log(\max(e, \log t))$, when m is supported by Γ, the necessary and sufficient condition for sample continuity of $(X_t)_{t \in G}$ is the convergence of the series $\sum_k 2^{-k} LLN_k(K)$. It is still of interest to check that even in that case, the condition $\sum_l 2^{-l+i(l)/2} < \infty$ is the correct condition.

PROPOSITION 6.2. *The series $\sum_l 2^{-l+i(l)/2}$ converges if and only if the series $\sum_k 2^{-k} LLN_k(K)$ converges.*

PROOF. To reduce the technicalities, let us assume that G is compact, $K = G$. Thus $U_{l,i} = B(2^{-l+i/2})$. We set $N_k = N_k(G)$.

We assume first the convergence of $\sum_k 2^{-k} LLN_k$. Define B_k as the set of integers l for which $k - 1 \leq l - i(l)/2 \leq k$.

By the definition of $i(l)$, we have

$$|B(2^{-l+i(l)/2})| \leq e^{-2^{i(l)}}.$$

By Lemma 2.11(i) this implies that

$$N_{l-i(l)/2} \geq |B(2^{-l+i(l)/2})|^{-1} \geq e^{2^{i(l)}}$$

so that

$$2^{i(l)} \leq \log N_{l-i(l)/2}.$$

For $l \in B_k$, we have $2^{i(l)} \leq \log N_k$ so that $i(l) \leq \log_2 \log N_k$. Since $l \leq k + i(l)/2$ for $l \in B_k$, we have

$$\sum_{l \in B_k} 2^{-l+i(l)/2} \leq (k + \log_2 \log N_k) 2^{-k+1}$$

and this implies the convergence of $\sum_l 2^{-l+i(l)/2}$.

We now turn to the other direction, which is the most interesting part of this proof. Let k_0 be the smallest for which $LLN_{k_0} \geq 1$. For $k \geq k_0$ we can define $a(k) \geq 0$ by

$$2^{a(k)} \leq LLN_k \leq 2^{a(k)+1}.$$

Define by induction $k_j = \inf\{k > k_{j-1};\ a(k) > a(k_{j-1})\}$. Clearly,

$$\sum_{k \geq k_0} 2^{-k} LLN_k \leq C \sum_{j \geq 0} 2^{-k_j + a(k_j)}. \tag{6.17}$$

Now

$$|B(2^{-k})| \leq 1/N_k \leq e^{-e^{2^{a(k)}}}.$$

Since $B(2^{-k}) = U_{l,2(l-k)}$, the definition of $i(l)$ shows that

$$2^{2(l-k)} \leq e^{2^{a(k)}} \Rightarrow i(l) \geq 2(l-k). \tag{6.18}$$

Consider the interval

$$I_j = [k_j + 2^{a(k_j)-3}, k_j + 2^{a(k_j)-2}[.$$

Since $k_{j+1} > k_j$ and $a(k_{j+1}) > a(k_j)$, the sets I_j are disjoint. For $l \in I_j$ we have

$$2^{2(l-k_j)} \leq 2^{2^{a(k_j)-2}} \leq e^{2^{a(k_j)}}$$

and thus, by (6.18), we have $i(l) \geq 2(l-k_j)$, so that $2^{-l+i(l)/2} \geq 2^{-k_j}$. Since card $I_j = 2^{a(k_j)-3}$, we have

$$\sum_{l \geq 1} 2^{-l+i(l)/2} \geq \sum_{j \geq 0} 2^{-k_j + a(k_j) - 3}.$$

Together with (6.17), this completes the proof. □

It would be too space-consuming to discuss the full generality of Theorem 1.2 of Marcus [3] and its many conditions, so we will instead briefly indicate how one could derive his Corollary 1.3 from our results. The Orlicz norm $\|X\|_T$ of a random variable X is given by

$$\|X\|_T = \inf\left\{a > 0;\ ET\left(\frac{X}{a}\right) \leq 1\right\}.$$

If T satisfies $T(ab) \leq CT(a)T(b)$, we thus have

$$EY(X) \leq a \Rightarrow ET\left(XT^{-1}\left(\frac{1}{Ca}\right)\right) \leq \frac{1}{a} ET(X) \leq 1$$

so that $\|X\|_T \leq 1/T^{-1}(1/Ca)$. Also, if $\|X\|_T \leq 1$, then $ET(X) \leq CT(1/a)$.

For what follows it is actually enough that $T \sim \Phi$ at ∞; but for simplicity let us assume that $T = \Phi$. We can compare the sets $U_{i,l}$ with sets of type $\{x \in U;\ d_T(x,0) \leq \varepsilon\}$, where

$$d_T(x,0) = \inf\left\{a > 0;\ \int T\left(\frac{|\gamma(x) - 1|}{a}\right) dm(\gamma) \leq 1\right\}.$$

We observe that, however, unless $T(x) = x^p$ for some p, there is a loss of information. (This explains why there is a small distance between the conditions of Marcus and the optimal ones.)

If T satisfies $T(ab) \geq ca^p T(b)$, then $T(aX) = b$ implies $EX^p \leq b/cT(a)$, and $U_{i,l}$ can be compared with a ball for the distance d_p given by (2.1). Arguments as in Proposition 2.2 then allow one to derive Corollary 1.3, part I, of [3]. Part II of that corollary is of a similar nature.

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