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BOOK REVIEW

GEOFFREY GRIMMETT, *Percolation*. Springer, Berlin, 1989.

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The field of rigorous percolation theory has grown so fast during the past 10 years that Kesten's 1982 book on the subject is largely out of date. Even Grimmett's 1989 book, reviewed here, is not completely current, although it remains the best available source of information about percolation theory.

The book is organized in a very logical way, and so a glance at its list of contents gives a quick but useful overview of the subject. It contains an accessible and complete account of most of the important results that had been proved prior to its preparation, especially in those parts of the theory that have been largely worked out. On the other hand, somewhat more detail about scaling theory and critical exponent inequalities might have been desirable. The chapter notes serve as a useful guide to further reading since they contain references to several topics that are not discussed in the text, and notes, added in proof, about almost all the major results that appeared after the book was written. The book makes it clear that the main open problems lie in the area of critical exponents and scaling theory. It should be useful both as a textbook either for a graduate course or for independent study and as a reference work.

The percolation model was introduced in 1957 by Broadbent and Hammersley as a model for the flow of fluid through a porous medium. Many variations are possible, but the simplest case, to be described shortly, is the nearest-neighbour, translation- and rotation-invariant, independent model.

We create a random graph whose set of vertices is $\mathbb{Z}^d$ and (in the nearest-neighbour case) each of whose edges joins a pair of nearest neighbours in $\mathbb{Z}^d$ (i.e., a pair $\{x, y\}$ with $\sum_{i=1}^{d}|x_i - y_i| = 1$). We do this by randomly assigning one of the two statuses *occupied* and *vacant* to each potential edge (nearest-neighbour pair) in $\mathbb{Z}^d$, the assignment being made independently over edges, and the probability that an edge is occupied being the same for all edges. The edge set of the random graph is the random set of occupied edges. This model depends on a single parameter, usually denoted p and called the *edge-density* or simply the *density*, namely, the probability that a given nearest-neighbour edge is occupied.

In describing the model, we have made three assumptions: first, that only nearest neighbours in $\mathbb{Z}^d$ can be joined by edges; second, that the probability that an edge is present is the same for all edges; and, finally, that there is no statistical dependence among the statuses of edges. By confining his attention for the most part to this simple model, Grimmett avoids many of the technical

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details that made Kesten's book rather daunting for the beginner. Variants of the model arise when one relaxes these assumptions. In the notes to Chapter 1, Grimmett gives references to discussions of these and other variants. There are many models that are more or less closely related to percolation, the most important of which is first-passage percolation. Grimmett's book gives short accounts—which are by no means complete, nor intended to be so—of some of these.

The detailed study of the percolation model focuses for the most part on the statistical properties of the connected clusters in the random graph. Principally, one studies the distribution of the random set $C(x)$ of sites in $\mathbb{Z}^d$ that belong to the same connected cluster as a fixed site x (which, by translation invariance, may be taken to be the origin 0 of $\mathbb{Z}^d$), and especially the distribution of the extended real-valued random variable $|C(0)|$, the cardinality of $C(0)$. When, as in the example given here, the model depends on a parameter p, it is of particular interest to understand how this distribution varies with p. The fundamental fact about this model, and the basic reason it is of interest to mathematical physicists, is that it undergoes a phase transition. What this means is that there is an abrupt change in the properties of the model as the parameter passes smoothly through a certain value, called its *critical value*.

To exhibit the phase transition quantitatively, we introduce the so-called *percolation probability*—the probability that, when the edge-density is p, the origin lies in an infinite connected cluster; in symbols,

$$\theta(p) := P_p(|C(0)| = \infty).$$

Hammersley (1963) made the elementary but useful observation that one can construct the models at all possible edge densities simultaneously (i.e., *couple* them) in such a way that if an edge is occupied at density p, then it is also occupied at every higher density. From this it is easy to deduce the intuitively obvious fact that $\theta(p)$ is nondecreasing in p. Evidently, $\theta(0) = 0$ and $\theta(1) = 1$. Therefore, the set $\{p\colon \theta(p) = 0\}$ is an interval containing 0 but not 1. We define $p_c = p_c(d)$ to be the supremum of this interval. The starting point, both historically and logically, for the investigation of the model is the result (due to Broadbent and Hammersley) that $p_c \in (0, 1)$; that is, the model undergoes a phase transition at a nontrivial value of the parameter. (The exact value of p_c is known only when $d = 2$, in which case it equals $\frac{1}{2}$.) We thus have

$$\theta(p)\begin{cases} = 0, & 0 \le p < p_c, \\ > 0, & p_c < p \le 1, \end{cases}$$

so that p_c breaks the parameter space up into two *regimes* or *phases*; the subcritical phase $[0, p_c)$ and the supercritical phase $(p_c, 1]$, on which the behaviour of the model differs both qualitatively and quantitatively. Thus further study of the model falls naturally into three areas, namely, detailed investigations of what happens (i) in the subcritical regime, (ii) in the supercritical regime and (iii) at and near the critical point.

Simple arguments, using the subadditivity of measures in the first case and an appropriate zero–one law in the second, show that if $p < p_c$, then almost surely all connected clusters are finite, whereas when $p > p_c$, with probability 1 there exists at least one infinite cluster in $\mathbb{Z}^d$.

The theory is much more developed in the two-dimensional than the general case (because the geometry of two-dimensional Euclidean space facilitates many proofs). More is known and the proofs have been polished to a high degree so that a very elegant treatment is now possible. Grimmett devotes a chapter to this special case.

The subcritical phase. The most important results about the subcritical phase are listed below. Grimmett gives clear and detailed discussions of these results and their interconnections. As above, let $|C(0)|$ be the size (volume) of the connected cluster containing the origin. Let $\partial S(n) = \{x \in \mathbb{Z}^d \colon \sum_{i=1}^{d} |x_i| = n\}$ and let $R(0) = \max\{n \colon C(0) \cap \partial S(n) \neq \varnothing\}$ be the *radius* of this cluster. Then if $p < p_c$, the distribution of each of the random variables $|C(0)|$ and $R(0)$ is proper (has no atom at ∞) and has an exponentially decaying tail. It follows from this that $E_p|C(0)| < \infty$ whenever $p < p_c$.

Let $\tau_p(0, e_n)$, the connectivity function, be the probability that the point $e_n = (n, \dots, 0)$ in $\mathbb{Z}^d$ belongs to the same connected cluster as the origin. Then if $p < p_c$, $\tau_p(0, e_n)$ decays exponentially in n, its rate of decay being the same as that of $P_p(R(0) \geq n)$.

It should be emphasised that this succinct statement of what is known about the subcritical phase does not accurately reflect the relative depth of the various results, or the tortuous historical path to their discovery. Grimmett's chapter notes do a better job.

In particular, the fact that $E_p|C(0)| < \infty$ for $p < p_c$ is deep and has intimate logical and historical links both with the fact that $p_c(2) = \frac{1}{2}$ and with the exponential decay of the tail of the radius of the cluster containing the origin. It was proved for the case $d = 2$ by Kesten (1980) and in the general case independently by Menshikov (1986) and Aizenman and Barsky (1987).

The supercritical phase. Perhaps the most important feature of the supercritical phase is the uniqueness of the infinite cluster whose existence is guaranteed by the definition of p_c (and the zero–one law mentioned earlier). This fact is relatively easily accessible in two dimensions [Harris (1960)]. The first proof valid in higher dimensions was given by Aizenman, Kesten and Newman (1987). A simplification of this proof [due to Gandolfi, Grimmett and Russo (1988)] appears in Grimmett's book. However, a wonderfully elegant proof of this fact, applicable to a wide variety of models, was later discovered by Burton and Keane (1989)—too late for its details to appear in the body of Grimmett's book. Nevertheless the notes to the relevant chapter contain a clear summary (added in proof).

Another basic fact about the supercritical model is that if $p > p_c$, then in a sufficiently thick slab, there is with probability 1 an infinite connected cluster. More precisely, suppose that $p > p_c$. Then there exists an integer K depending on p so that if $\mathbb{Z}^d$ is replaced in the definition of the model by the set

$$[-K, K]^{d-2} \times \mathbb{Z}^2,$$

then with positive probability, the connected cluster containing the origin is infinite. In fact the same is true if $\mathbb{Z}^d$ is replaced by a set of the form $[-K, K]^{d-2} \times \mathbb{Z}^+ \times \mathbb{Z}$ [where $\mathbb{Z}^+ = \mathbb{Z} \cap [0, \infty)$]. The proof of this result—believed for many years prior to its proof—appears in papers by Barsky, Grimmett and Newman (1990) and Grimmett and Marstrand (1990). In the first of those papers, the main theorem (a summary of whose proof is in Grimmett's book) is the weaker result that the conclusion of the first statement of this paragraph is true if $p > p_c^+$, where p_c^+ is the critical value for percolation in the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}^+$. In the second paper, which appeared after Grimmett's book was published, p_c^+ is shown to coincide with p_c. This means that the hypothesis $p > p_c^+$, which occurs in the statement of several theorems in Grimmett's book, can be replaced by the aesthetically more satisfying hypothesis $p > p_c$.

Most of the remaining results about the supercritical phase address the behaviour of finite clusters. Again, these and other results are explained clearly and at length in Grimmett's book. In some cases there is exponential decay. For example, it is shown that for $p > p_c$, the radius of a finite cluster has an exponentially decaying tail. The so-called *truncated connectivity* $\tau_p^f(0, e_n)$—the probability that 0 and e_n belong to the same finite connected cluster—also decays exponentially as $n \to \infty$. However, the tail of the distribution of the size of a finite cluster decays subexponentially. In fact, if $p > p_c$, then both the lim inf and the lim sup as $n \to \infty$ of

$$\frac{-\log P_p(|C(0)| = n)}{n^{(d-1)/d}}$$

are positive and finite.

At and near the critical point. Most questions about the sub- and supercritical phases have been rigorously settled. This cannot be said of those questions involving the behaviour of the model at or near the critical value of the parameter, where many of the interesting questions remain open. Grimmett's treatment of these topics is somewhat sketchy. To some extent this reflects the paucity of the current state of knowledge, but some of the papers Grimmett refers to in his chapter notes can be consulted with profit.

The exact value of the critical value is known only when $d = 2$, where it equals $\frac{1}{2}$, as Kesten showed in 1980. Bounds are available in other dimensions and asymptotically, $p_c(d) \sim 1/2d$ as $d \to \infty$ [Kesten (1990)]. Grimmett does not go into detail on this topic, but does give some references in the notes to Chapter 1.

The most basic question about the behaviour of the model at its critical value remains open except in special cases. It is this: Is $\theta(p_c) = 0$? This is known to be true when $d = 2$. Harris (1960) showed that $\theta(\frac{1}{2}) = 0$ and this, together with Kesten's result that $p_c(2) = \frac{1}{2}$, confirms that $\theta(p_c) = 0$. In 1984

Aizenman and Newman introduced a condition which they called the triangle condition. It is conjectured to hold when the dimension is more than 6 but not in low dimensions. Aizenman and Barsky (1991) showed that under the triangle condition it is indeed the case that $\theta(p_c) = 0$. Hara and Slade (1990) have verified that the triangle condition holds for sufficiently large d, although the smallest d for which their proof works is not particularly close to 6. Thus it is known that $\theta(p_c) = 0$ when d is 2 or is very large, but not otherwise. The triangle condition is certainly violated in low dimensions so that even if Hara and Slades's result can be improved to show that the triangle condition holds whenever it is conjectured to do so, the problem of showing that $\theta(p_c) = 0$ will remain open in lower dimensions.

Motivated by analogies with other models, statistical physicists conjecture that certain *power laws* hold at p_c and as p approaches p_c. The exponents in such power laws are called *critical exponents* and are believed to be universal in the sense that they are likely to depend only on the dimension and not on the detailed structure of the underlying lattice. Many power laws are believed to hold. So far none has been rigorously established, although partial results are available. An example of a power law conjectured to hold *at* p_c is this: It is believed that $P_{p_c}(|C(0)| = n)$ decays as an inverse power of n. By convention, this power is written $1 + 1/\delta$. A power law believed to hold as p approaches p_c is that $\theta(p)$ is supposed to behave in some suitable sense as $(p - p_c)^\beta$ as $p \downarrow p_c$.

Scaling theory predicts that certain relations hold among the different critical exponents. For example, it is believed that $\beta(\delta + 1) = 2 - \alpha$, where β and δ are the exponents introduced above, and α is another critical exponent whose definition is more complicated. Kesten (1987a, b) has made substantial progress on scaling theory in two dimensions.

As for the values of these critical exponents, it is predicted that when the dimension is sufficiently large (i.e., > 6), they will all be equal to the corresponding values for percolation on a tree. Aizenman and Newman (1984) and Barsky and Aizenman (1991) have shown that when the triangle condition is satisfied, certain power laws hold in a strong sense with critical exponents taking these predicted values. In lower dimension, the values of the critical exponents will be different. No such value has been rigorously determined.

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