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REACTION–DIFFUSION EQUATIONS WITH RANDOMLY PERTURBED BOUNDARY CONDITIONS

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We consider semilinear parabolic PDE's with fast oscillating boundary conditions. The central limit theorem and limit theorems for probabilities of large deviations for the solutions of such problems are proved.

1. Introduction. Let n be a positive integer. Consider a system of reaction–diffusion (RD) equations:

$$\begin{aligned} &\frac{\partial u_k(t,x)}{\partial t} = \frac{D_k}{2}\frac{\partial^2 u_k(t,x)}{\partial x^2} + f_k(x; u_1,\dots,u_n), \qquad |x|<1, t>0, \\ &u_k(0,x) = g_k(x), \qquad \left.\frac{\partial u_k(t,x)}{\partial x}\right|_{x=\pm 1} = 0, \qquad k=1,\dots,n. \end{aligned} \tag{1}$$

Here D_k are positive constants. We assume that the nonlinear terms $f_k(x, u_1, \dots, u_n)$ are Lipschitz continuous and that initial functions $g_k(x)$ are continuous. The system (1) defines a semiflow $u_t = (u_1(t,\cdot),\dots,u_n(t,\cdot))$ in the space $C_{[-1,1]}$ of continuous functions on $[-1,1]$ with values in R^n. This semiflow can have a rather rich set of ω-limit behaviors (limit behaviors for $t\to\infty$). In particular, it can have several stationary points or solutions which are periodic in t. In the case of one equation $(n=1)$ under some minor additional assumptions the stable and unstable stationary points exhaust all possible ω-limit sets.

If the system (1) is subjected to small (of intensity $\varepsilon \ll 1$) random perturbations, the solution $u_t^\varepsilon = (u_1^\varepsilon(t,\cdot),\dots,u_n^\varepsilon(t,\cdot))$, if such a solution exists, will be a random process in a proper functional space. As in the finite-dimensional case, the question of describing the deviations of u_t^ε from u_t arises when ε is small. There are several problems connected with these deviations. First, we are interested in the convergence u_t^ε to the nonperturbed semiflow u_t when $\varepsilon \downarrow 0$. This is a law of large numbers type result. The second class of problems concerns normal deviations of u_t^ε from u_t in the corresponding functional space. One can choose the normalizing coefficient $\lambda(\varepsilon)$ in such a way that the normalized deviations

$$w^\varepsilon(t,x) = \frac{u^\varepsilon(t,x) - u(t,x)}{\lambda(\varepsilon)}$$

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converge to a nondegenerate Gaussian field $w(t, x)$ when $\varepsilon \downarrow 0$. This is a result of the central limit theorem type.

The deviations of u_t^ε from u_t of order 1 when $\varepsilon \downarrow 0$ are described by the limit theorems for large deviations. As in the case of finite-dimensional dynamical systems, such deviations define the behavior of the perturbed system on very large time intervals. The limit theorems for the probabilities of the deviations of order 1 define the behavior of the invariant measures of the perturbed system when $\varepsilon \downarrow 0$, as well as the transitions between different stable stationary points of the semiflow u_t and the main term of the time of exit from a neighborhood of a stable point.

If we are interested in the exit problem from a neighborhood of order $\lambda_1(\varepsilon)$, $\lim_{\varepsilon \downarrow 0} \lambda_1(\varepsilon) = 0$, of a stable stationary point of the semiflow u_t and $\lambda_1(\varepsilon)\lambda^{-1}(\varepsilon) \to \infty$ when $\varepsilon \downarrow 0$, where $\lambda^{-1}(\varepsilon)$ is the normalizing coefficient in the central limit theorem, we need the limit theorem for large but not very large deviations [of order $\lambda_1(\varepsilon)$]. The probabilities of such deviations also tend to 0 when $\varepsilon \downarrow 0$, but not so fast as for the deviations of order 1.

Of course, all these limit theorems depend on the kind of perturbations which are considered. There are many more possibilities for introduction of the perturbations in the case of PDEs than in the case of finite-dimensional dynamical systems. There exist several papers where small random perturbations of the equations (1) are considered: In [2], [3], [7] and [8] the large deviations of order 1 are studied; results of central limit theorem type are considered in [6].

Here we consider the perturbations of the boundary conditions. We restrict ourselves, for brevity, to the case of one equation, though most of the results can be proved for the systems.

Consider the following problem:

$$(2)\qquad \begin{aligned} &\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + f(x, u^\varepsilon), \qquad |x| < 1, t > 0,\\ &\left.\frac{\partial u^\varepsilon(t,x)}{\partial x}\right|_{x=\pm 1} = \pm\xi_\pm\left(\frac{t}{\varepsilon}\right), \qquad u^\varepsilon(0,x) = g(x).\end{aligned}$$

We assume that the function f is Lipschitz continuous with continuous derivative in u, and $|f(x,u)| < A + L|u|$ for some constants A and L. The process $(\xi_+(t), \xi_-(t))$ we suppose to be stationary with mean 0 and correlation matrix

$$\begin{pmatrix} K_{++}(\tau) & K_{+-}(\tau) \\ K_{-+}(\tau) & K_{--}(\tau) \end{pmatrix}.$$

We assume that this process satisfies some mixing properties, at least that

$$(3)\qquad \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} \xi_\pm(s)\, ds = 0$$

for any $t \geq 0$, the limit being in probability. This is true if $K_{++}(t), K_{--}(t) \to 0$

when $t \to \infty$. Later on we shall introduce stronger assumptions on the mixing properties.

If (3) is fulfilled, one can consider problem (2) as a small, in the average sense, perturbation of the problem

$$\frac{\partial u(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + f(x,u), \qquad t > 0,\ |x| < 1, \tag{4}$$
$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{|x|=1} = 0, \qquad u(0,x) = g(x).$$

We assume for simplicity that there exists a constant $C < \infty$ such that $P\{|\xi_{\pm}(t)| \le C\} = 1$.

THEOREM 1. *For any $T > 0$, $\delta > 0$,*

$$\lim_{\varepsilon \downarrow 0} P\left\{ \sup_{\substack{0 \le t \le T \\ |x| \le 1}} |u^{\varepsilon}(t,x) - u(t,x)| > \delta \right\} = 0.$$

PROOF. Denote by $p(\tau, x, y)$ the transition density for the diffusion process corresponding to the operator $(D/2)d^2/dx^2$ in the interval $[-1, 1]$ and having reflection at the ends of the interval:

$$p(\tau,x,y) = \frac{1}{\sqrt{2\pi D\tau}}\left[\sum_{k=-\infty}^{\infty} e^{-(y-x-4k)^2/2D\tau} + \sum_{k=-\infty}^{\infty} e^{-(y+x+4k+2)^2/2D\tau}\right]. \tag{5}$$

Introduce the random field

$$w^{\varepsilon}(t,x) = \int_{-1}^{1} p(t,x,y) g(y)\, dy + \sum_{+,-} \int_0^t p(t-s, x, \pm 1)\, d\zeta_{\pm}^{\varepsilon}(s), \tag{6}$$

where $\zeta_{\pm}^{\varepsilon}(t) = \int_0^t \xi_{\pm}(s/\varepsilon)\, ds$. The function $w^{\varepsilon}(t,x)$ is the solution of the problem

$$\frac{\partial w^{\varepsilon}}{\partial t} = \frac{D}{2}\frac{\partial^2 w^{\varepsilon}}{\partial x^2}, \qquad t > 0,\ |x| < 1,$$
$$\left.\frac{\partial w^{\varepsilon}}{\partial x}\right|_{x=\pm 1} = \pm \xi_{\pm}\left(\frac{t}{\varepsilon}\right), \qquad w^{\varepsilon}(0,x) = g(x),$$

and then the function $u^{\varepsilon}(t,x)$ satisfies the following integral equation:

$$u^{\varepsilon}(t,x) = w^{\varepsilon}(t,x) + \int_0^t ds \int_{-1}^{1} dy\, p(t-s,x,y) f(y, u^{\varepsilon}(s,y)). \tag{7}$$

For any $0 < \delta < t$, we have

$$\left|\int_0^t p(t-s, x, \pm 1)\, d\zeta_\pm^\varepsilon(s)\right| \le \left|\int_0^{t-\delta} p(t-s, x, \pm 1)\, d\zeta_\pm^\varepsilon(s)\right| + \left|\int_{t-\delta}^t p(t-s, x, \pm 1)\, d\zeta_\pm^\varepsilon(s)\right|. \tag{8}$$

Taking into account that $p(\tau, x, y) < \text{const}/\sqrt{\tau}$ in any bounded interval $0 < \tau < T$, as follows from (5), and with our assumption that $|\dot{\zeta}_\pm^\varepsilon(t)| \le C < \infty$, we conclude that for $\delta \le t \le T$ the second term in the right-hand side of (8) is bounded by

$$\left|\int_{t-\delta}^t p(t-s, x, \pm 1)\, d\zeta_\pm^\varepsilon(s)\right| \le \text{const} \cdot C \int_{t-\delta}^t \frac{ds}{\sqrt{t-s}} = 2C \cdot \text{const}\sqrt{\delta}\,. \tag{9}$$

The first integral in (8) we bound after integrating by parts:

$$\begin{aligned} &\left|\int_0^{t-\delta} p(t-s, x, \pm 1)\, d\zeta_\pm^\varepsilon(s)\right| \\ &\le |\zeta_\pm^\varepsilon(s) p(t-s, x, \pm 1)| \Big|_0^{t-\delta} + \left|\int_0^{t-\delta} ds \frac{\partial p(t-s, x, \pm 1)}{\partial s} \zeta_\pm^\varepsilon(s)\right| \\ &\le \max_{0 \le s \le T} |\zeta_\pm^\varepsilon(s)| \max_{\delta \le s \le T} \left[\left|\frac{\partial p(s, x, \pm 1)}{\partial s}\right| + p(s, x, \pm 1)\right] \\ &\le C_1(\delta) \max_{0 \le s \le T} |\zeta_\pm^\varepsilon(s)|. \end{aligned} \tag{10}$$

Since we assumed that the correlation function of the stationary process $\xi_\pm(s)$ tends to 0 and that $|\xi_\pm(s)| \le C$, the process $\zeta_\pm^\varepsilon(t)$ tends to 0 uniformly in $t \in [0, T]$ in probability when $\varepsilon \downarrow 0$ (see Theorem 7.2.1 in [4]). Thus we conclude from (10) that for any $0 < \delta < T$ and $\delta_1 > 0$,

$$\lim_{\varepsilon \downarrow 0} P\left\{ \max_{\substack{\delta \le t \le T \\ -1 \le x \le 1}} \left|\int_0^{t-\delta} p(t-s, x, \pm 1)\, d\zeta_\pm^\varepsilon(s)\right| > \delta_1 \right\} = 0. \tag{11}$$

From (6), (8), (9) and (11) we derive that for any $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} P\left\{ \max_{\substack{0 \le t \le T \\ |x| \le 1}} \left| w^\varepsilon(t, x) - \int_{-1}^1 g(y) p(t, x, y)\, dy \right| > \delta \right\} = 0. \tag{12}$$

From (7) we obtain the following inequality for the function $v^\varepsilon(t) = \max_{0 \le s \le t, |x| \le 1} |u^\varepsilon(s, x) - u(s, x)|$:

$$v^\varepsilon(t) \le 2K \int_0^t v^\varepsilon(s)\, ds + \max_{\substack{0 \le t \le T \\ |x| \le 1}} \left| w^\varepsilon(t, x) - \int_{-1}^1 g(y) p(t, x, y)\, dy \right|,$$

where K is the Lipschitz constant for the function f. From the last inequality and (12), using the Gronwall lemma, we derive the statement of Theorem 1. □

REMARK. The solution $u^\varepsilon(t,x)$ of problem (2) can be represented as a continuous or even smooth, in a proper norm, image of the solution $W^\varepsilon(t,x)$ of the corresponding linear problem. The proof of Theorem 1 consists of two parts: first, the proof of the convergence of $W^\varepsilon(t,x)$ to the solution of the nonperturbed linear problem as $\varepsilon \downarrow 0$, and second, the use of continuity of the transformation $W^\varepsilon \to u^\varepsilon$. In a sense most of our results are proved in a similar way. The main part of the proof usually consists of proving the limit theorem for the linear problems in a norm strong enough to provide the continuity of the transformation of the solution of the linear problem to the solution of the nonlinear problem.

2. Normal deviations. Now we want to consider the difference $u^\varepsilon(t,x) - u(t,x)$. It tends to 0 when $\varepsilon \downarrow 0$ uniformly in $0 \le t \le T$, $|x| \le 1$, but we can expect that after dividing by $\sqrt{\varepsilon}$ it will be asymptotically Gaussian. We have the following initial boundary problem for $v^\varepsilon(t,x) = \varepsilon^{-1/2}(u^\varepsilon(t,x) - u(t,x))$:

$$\tag{13} \begin{gathered} \frac{\partial v^\varepsilon}{\partial t} = \frac{D}{2}\frac{\partial^2 v^\varepsilon}{\partial x^2} + f_2'(x, \tilde{u}^\varepsilon(t,x))v^\varepsilon, \qquad t > 0,\ |x| < 1, \\ v^\varepsilon(0,x) = 0, \qquad \frac{\partial v^\varepsilon}{\partial x}(t, \pm 1) = \pm\frac{1}{\sqrt{\varepsilon}}\xi_\pm\left(\frac{t}{\varepsilon}\right). \end{gathered}$$

Here $\tilde{u}^\varepsilon(t,x) \to u(t,x)$ when $\varepsilon \downarrow 0$ in the sense of Theorem 1, $f_2'(x,u) = \partial f/\partial u(x,u)$. Denote $\zeta_\pm^\varepsilon(t) = \int_0^t \xi_\pm(s/\varepsilon)\,ds$, $\zeta^\varepsilon(t) = (\zeta_+^\varepsilon(t), \zeta_-^\varepsilon(t))$. It is well known that under some assumptions about mixing properties of the process $\xi(t) = (\xi_+(t), \xi_-(t))$ the process $(1/\sqrt{\varepsilon})\zeta_t^\varepsilon$ converges weakly in the space of continuous functions $\mathbb{C}_{0T}$ to a Gaussian process, namely, the Brownian motion $(W_+(t), W_-(t))$ with $W_\pm(0) = 0$ and covariance matrix $(a_{\lambda\mu})$, where

$$a_{\lambda\mu} = \int_{-\infty}^{\infty} K_{\lambda\mu}(\tau)\,d\tau, \qquad K_{\lambda\mu}(\tau) = E\xi_\lambda(t)\xi_\mu(t+\tau), \qquad \lambda,\mu \in \{+,-\}.$$

Taking all these things into account, we can expect that $v^\varepsilon(t,x)$ converges in one or another sense to the solution $v(t,x)$ of the following linear initial boundary problem:

$$\tag{14} \begin{gathered} \frac{\partial v}{\partial t}(t,x) = \frac{D}{2}\frac{\partial^2 v(t,x)}{\partial x^2} + f_2'(x, u(t,x))v(t,x), \qquad t > 0,\ |x| < 1, \\ \frac{\partial v}{\partial x}(t, \pm 1) = \pm\dot{W}_\pm(t), \qquad v(0,x) = 0. \end{gathered}$$

We will see that this is actually true, but we need to overcome several obstacles.

First, we should introduce a generalized solution of the problem (14). Let $p(t, x, y)$ be the function introduced by (5). Denote

$$z^0(t,x) = \sum_{+,-} \int_0^t p(t-s, x, \pm 1)\, dW_\pm(s), \qquad |x| < 1, t \geq 0. \tag{15}$$

A function $v(t, x)$ is called the generalized solution of the problem (14) if it satisfies the linear equation

$$v(t,x) = z^0(t,x) + \int_0^t ds \int_{-1}^1 dy\, p(t-s,x,y)\, f_2'(y, u(s,y)) v(s,y). \tag{16}$$

One can see from (15) that $z^0(t, x)$ is continuous in $t \geq 0$, $|x| < 1$, with probability 1. Consider the function

$$h(t,x) = \int_0^t z^0(s,x)\, ds.$$

We deduce from (15) by integrating by parts that

$$\begin{aligned} h(t,x) &= \sum_{+,-} \int_0^t dt_1 \int_0^{t_1} p_1'(t_1 - s, x, \pm 1) W_\pm(s)\, ds \\ &= \sum_{+,-} \int_0^t W_\pm(s)\, ds \int_s^t p_1'(t_1 - s, x, \pm 1)\, dt_1 \\ &= \sum_{+,-} \int_0^t W_\pm(s)\, ds\, p(t-s, x, \pm 1), \end{aligned}$$

where

$$p_1'(t, x, \pm 1) = \frac{\partial p(t, x, \pm 1)}{\partial t}.$$

For any $0 < \delta < t$ we can write

$$\begin{aligned} h(t,x) = &\sum_{+,-} \int_0^{t-\delta} W_\pm(s) p(t-s, x, \pm 1)\, ds \\ &+ \sum_{+,-} \int_{t-\delta}^t W_\pm(s) p(t-s,x,\pm 1)\, ds. \end{aligned} \tag{17}$$

One can see from (5) that $p(\tau, x, y) < \text{const}/\sqrt{\tau}$, $0 < \tau \leq T < \infty$. Using the last inequality, we can bound the second term in (17):

$$\begin{aligned} \left| \sum_{+,-} \int_{t-\delta}^t W_\pm(s) p(t-s, x, \pm 1)\, ds \right| &\leq C_1 \sum_{+,-} \max_{0 \leq s \leq t} |W_\pm(s)| \int_0^\delta \frac{ds}{\sqrt{s}} \\ &\leq C_2 \sqrt{\delta} \sum_{+,-} \max_{0 \leq s \leq t} |W_\pm(s)|. \end{aligned}$$

We have from this that for almost any trajectory $(W_+(s), W_-(s))$, $0 \leq s \leq T$, the second term in (17) can be made less than any $\lambda > 0$ if δ is small enough. For any fixed $\delta > 0$ the first term in (17) is continuous in x. This, together with the fact that $\lim_{t \downarrow 0} h(t,x) = 0$ uniformly in $|x| < 1$, implies the existence of $\lim_{x\to 1} h(t,x)$ and $\lim_{x\to -1} h(t,x)$ uniformly in $t \in [0,T]$. It is not difficult to check using the explicit formula for $p(t,x,y)$, that for any $\gamma > 0$,

$$\lim_{x\to \pm 1} (1-x^2)^{\gamma} z^0(t,x) = 0 \quad \text{with probability 1.}$$

Let $a(x)$ be a continuous function in $[-1,1]$, $a(\pm 1) = 0$, $a(x) > 0$ for $|x| < 1$, $\int_{-1}^{1} a^{-1}(x)\,dx < \infty$, $\lim_{x\to\pm 1} a(x)(1-x^2)^{\gamma} = 0$ for some $\gamma < 0$. The typical example of such a function is $a(x) = (1-x^2)^{\alpha}$, $0 < \alpha < 1$.

Denote by $C_a = C_a([0,T] \times (-1,1))$ the space of all continuous functions $u(t,x)$, $t \in [0,T]$, $|x| < 1$, such that $\lim_{x\to\pm 1} a(x)u(t,x) = 0$ uniformly in $t \in [0,T]$. The norm in this space is defined as follows:

$$\|u\|_a = \sup_{\substack{0\le t\le T\\ |x|<1}} a(x)|u(t,x)|.$$

We denote by $\hat{C}_a$ the subspace of C_a, consisting of $u(t,x)$ such that $h(t,x) = \int_0^t u(s,x)\,ds$ has uniform limit in $t \in [0,T]$ when $x \to 1$ and when $x \to -1$, provided with the norm

$$|||\, u \,||| = \|u\|_a + \sup_{\substack{1\le t\le T\\ |x|<1}} \left| \int_0^t u(s,x)\,ds \right|.$$

As was explained above, the function $Z^0(t,x)$ defined by (15) belongs to the space $\hat{C}_a$ with probability 1.

LEMMA 1. *There exists a generalized solution $v(t,x)$ of the problem* (14) *belonging to $\hat{C}_a$. Such a solution is unique.*

Taking into account that $Z^0 \in \hat{C}_a$, one can prove Lemma 1 by successive approximations, using bounds given in Lemmas 2 and 3 in a little more general situation. The proof is omitted.

So the field $v(t,x)$, which as we expect is the limit field for the normalized difference $v^{\varepsilon}(t,x)$, does not belong to the space $C_{[0,T]\times[-1,1]}$. It belongs only to $\hat{C}_a$, and we shall prove that v^{ε} converges weakly to v in the space $\hat{C}_a$.

The solution $u^{\varepsilon}(t,x)$ of the problem (2) can be obtained from $\zeta^{\varepsilon}(t) = (\zeta_+^{\varepsilon}(t), \zeta_-^{\varepsilon}(t))$, $\zeta_{\pm}^{\varepsilon}(t) = \int_0^t \xi_{\pm}(s/\varepsilon)\,ds$, as the product of two mappings: the linear mapping $\zeta \to w$ defined by the problem

$$\frac{\partial w(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 w(t,x)}{\partial x^2}, \qquad w(0,x) = g(x), \qquad \frac{\partial w}{\partial x}(t,\pm 1) = \pm\dot{\zeta}_{\pm}(t),$$

and the mapping $w \to u$ given by (7). The mappings can be expressed in the integral form

$$(18)\quad w(t,x) = \int_{-1}^{1} p(t,x,y)g(y)\,dy + \sum_{+,-} \int_0^t p(t-s,x,\pm 1)\,d\zeta_{\pm}(s),$$

where $p(t,x,y)$ is defined by (5), and

$$(19)\qquad u(t,x) = w(t,x) + \int_0^t ds \int_{-1}^{1} dy\, p(t-s,x,y)\, f(y,u(s,y)).$$

LEMMA 2. *The mapping* $w \to u - w$: $C_a \to C = C([0,T] \times [-1,1])$ *is Lipschitz continuous and Fréchet differentiable as are the mappings* $w \to u$: $C_a \to C_a$ *and* $w \to u$: $C \to C$.

PROOF. Let there be given two functions $w_1(t,x), w_2(t,x)$, $\|w_i\|_a < \infty$. Define $u_i^{(0)} = w_i$, $u_i^{(n+1)}(t,x) = w_i(t,x) + \int_0^t ds \int_{-1}^1 dy\, p(t-s,x,y) f(y, u_i^{(n)}(s,y))$. We have

$$u_i^{(1)}(t,x) - u_i^{(0)}(t,x) = \int_0^t ds \int_{-1}^1 p(t-s,x,y) f(y, w_i(s,y))\, dy,$$

$$u_i^{(n+1)}(t,x) - u_i^{(n)}(t,x)$$
$$= \int_0^t ds \int_{-1}^1 dy\, p(t-s,x,y)\left[f(y, u_i^{(n)}(s,y)) - f(y, u_i^{(n-1)}(s,y))\right], \quad n > 0,$$

$$\sup_{\substack{0 \le t \le T \\ |x| \le 1}} |u_i^{(1)}(t,x) - u_i^{(0)}(t,x)|$$
$$\le \sup_{\substack{0 \le t \le T \\ |x| < 1}} \int_0^t ds \int_{-1}^1 p(t-s,x,y)\left[A + L|w_i(s,y)|\right] dy$$
$$\le AT + L\|w_i\|_a \int_{-1}^1 a^{-1}(y)\, dy \int_0^T d\tau \sup_{x,y} p(\tau,x,y) = A_{T,i} < \infty.$$

We used here that $|f(x,u)| < A + L|u|$, $\int_{-1}^1 a^{-1}(y)\, dy < \infty$, and the bound $\sup_{x,y} p(\tau,x,y) \sim 2/\sqrt{2\pi\tau}$ as $\tau \downarrow 0$.

In a similar way one can check that

$$\sup_{\substack{0 \le t \le T \\ |x| \le 1}} |u_1^{(1)}(t,x) - u_1^{(0)}(t,x) - u_2^{(1)}(t,x) - u_2^{(0)}(t,x)|$$
$$\le L\|w_1 - w_2\|_a \int_{-1}^1 a^{-1}(y)\, dy \int_0^T d\tau \sup_{x,y} p(\tau,x,y) \le L_1\|w_2 - w_1\|_a,$$

$$\sup_{\substack{0 \le t_1 \le T \\ |x| \le 1}} |u_i^{(n+1)}(t_1,x) - u_i^{(n)}(t_1,x)| \le \int_0^t ds\, L \sup_{\substack{0 \le t_1 \le T \\ |x| \le 1}} |u_i^{(n)}(t_1,x) - u_i^{(n-1)}(t_1,x)|$$
$$\le A_{T,i} \frac{(Lt)^n}{n!},$$

$$\sup_{\substack{0 \le t_1 \le T \\ |x| \le 1}} |u_1^{(n+1)}(t_1,x) - u_1^{(n)}(t_1,x) - u_2^{(n+1)}(t_1,x) + u_2^{(n)}(t_1,x)|$$
$$\le \int_0^t ds\, L \sup_{\substack{0 \le t_1 \le T \\ |x| \le 1}} |u_1^{(n)}(t_1,x) - u_1^{(n-1)}(t_1,x) - u_2^{(n)}(t_1,x) + u_2^{(n-1)}(t_1,x)|$$
$$\le L_1\|w_1 - w_2\|_a \frac{(Lt)^n}{n!}.$$

Now, taking into account that $u_i = w_i + \sum_{n=0}^{\infty}(u_i^{(n+1)} - u_i^{(n)})$, we obtain the first statement:

$$\|u_i - w_i\| \le A_{T,i}e^{LT}, \qquad \|u_1 - w_1 - u_2 + w_2\| \le L_1\|w_1 - w_2\|_a e^{LT}.$$

Here $\|b(t,x)\| = \max_{0\le t\le T,|x|\le 1}|b(t,x)|$. From the last bound and the continuity of the derivative $\partial f(x,u)/\partial u$, Fréchet differentiability of the mapping $w \to u - w$: $C_a \to C_{[0,T]\times[-1,1]}$ follows. The Fréchet differential of this mapping is the linear transformation $\delta w \to \delta u - \delta w$ defined by the integral equation

$$\begin{aligned}(20)\qquad &\delta u(t,x) - \delta w(t,x)\\ &\quad = \int_0^t ds\int_{-1}^1 dy\, p(t-s,x,y)\, f_u'(y,u(s,y))\,\delta u(s,y).\end{aligned}$$

The last statement of the lemma is a corollary of the Lipschitz continuity and the differentiability of the mapping $w \to u - w$: $C_a \to C_{[0,T]\times[-1,1]}$. □

Let $\mathbb{C} = \mathbb{C}_{0T}$ be the space of all pairs $\zeta(t) = (\zeta_+(t), \zeta_-(t))$ of continuous functions on $[0,T]$, $\zeta_\pm(0) = 0$. Denote by $C^\gamma = C_{0T}^\gamma$ the subspace of $\mathbb{C}$ consisting of Hölder continuous functions with the norm

$$\|\zeta\|_\gamma = \max_{+,-}\ \sup_{0\le t_1<t_2\le T}\frac{|\zeta_\pm(t_1) - \zeta_\pm(t_2)|}{(t_2-t_1)^\gamma}, \qquad 0<\gamma<1.$$

LEMMA 3. *Let $a(x) = (1-x^2)^\alpha$, $\alpha > 1-2\gamma$, $0<\gamma<1/2$. Then the mapping $\zeta \to w$ defined by* (18) *is continuous as a mapping from C^γ to C_a.*

PROOF. We have for $\zeta_1, \zeta_2 \in C^\gamma$,

$$\begin{aligned}
&w_1(t,x) - w_2(t,x)\\
&\quad = \sum_{+,-}\int_0^t p(t-s,x,\pm 1)\, d[\zeta_{1,\pm}(s) - \zeta_{2,\pm}(s)]\\
&\quad = \sum_{+,-}\int_0^t p(t-s,x,\pm 1)\, d[\zeta_{1,\pm}(s) - \zeta_{2,\pm}(s) - \zeta_{1,\pm}(t) + \zeta_{2,\pm}(t)]\\
&\quad = \sum_{+,-}\Big\{[\zeta_{1,\pm}(t) - \zeta_{2,\pm}(t) - \zeta_{1,\pm}(0) + \zeta_{2,\pm}(0)]\,p(t,x,\pm 1)\\
&\qquad\quad + \int_0^t[\zeta_{1,\pm}(s) - \zeta_{2,\pm}(s) - \zeta_{1,\pm}(t) + \zeta_{2,\pm}(t)]\,p_1'(t-s,x,\pm 1)\,ds\Big\},
\end{aligned}$$

where $p_1'(\tau,x,y) = (\partial p/\partial\tau)(\tau,x,y)$. We get from this equality that

$$\begin{aligned}
&a(x)|w_1(t,x) - w_2(t,x)|\\
(21)\qquad &\quad \le \|\zeta_1 - \zeta_2\|_\gamma \sum_{+,-}\Big\{t^\gamma p(t,x,\pm 1)a(x)\\
&\qquad\qquad + \int_0^t (t-s)^\gamma p_1'(t-s,x,\pm 1)a(x)\,ds\Big\}.
\end{aligned}$$

Since $a(x)p(t, x, \pm 1) \le c_1 t^{(\alpha-1)/2}$ and $a(x)p_1'(t, x, \pm 1) \le c_2 t^{(\alpha-3)/2}$ for $|x| < 1$, where c_i are constants, we conclude from (21) that

$$\|w_1 - w_2\|_a \le \|\zeta_1 - \zeta_2\|_\gamma C_3 \sup_{0\le t\le T} \left\{ t^{\gamma+(\alpha-1)/2} + \int_0^t \tau^{\gamma+(\alpha-3)/2}\, d\tau \right\}$$
$$\le C_4 \|\zeta_1 - \zeta_2\|_\gamma. \quad \square$$

LEMMA 4. *The mapping* $\zeta \to \int_0^t w(s,x)\,ds$ *is continuous as a mapping* $\mathbb{C}_{0T} \to C_{[0,T]\times[-1,1]}$.

PROOF. It is simple:

$$\int_0^t w(s,x)\,ds = \int_0^t ds \int_{-1}^1 dy\, p(s,x,y) g(y)\, dy + \sum_{+,-} \int_0^t du \int_0^u p(u-s, x, \pm 1)\, d\zeta_\pm(s). \tag{22}$$

The last integral in (22) is equal to

$$\int_0^t d\zeta_\pm(s) \int_s^t p(u-s, x, \pm 1)\, du = \int_0^t d\zeta_\pm(s) \int_0^{t-s} p(\tau, x, \pm 1)\, d\tau$$
$$= \int_0^t p(t-s, x, \pm 1) \zeta_\pm(s)\, ds.$$

Taking this into account, we have from (22):

$$\sup_{\substack{0\le t\le T\\ |x|<1}} \left| \int_0^t w_1(s,x)\,ds - \int_0^t w_2(s,x)\,ds \right|$$
$$\le \|\zeta_1 - \zeta_2\| \sup_{|x|\le 1} \sum_{+,-} \int_0^T p(\tau, x, \pm 1)\, d\tau = \text{const} \cdot \|\zeta_1 - \zeta_2\|,$$

where $\|\cdot\|$ means the uniform norm in $\mathbb{C}_{0T}$. $\square$

Consider two separable functional Banach spaces B_1 and B_2 provided with Borel σ-fields. Let $f\colon B_1 \to B_2$ be a measurable mapping which is Fréchet differentiable at a point $x_1 \in B_1$. The following statement is a simple corollary of the definitions of the weak convergence and the Fréchet differentiability.

LEMMA 5. *Let* X^ε, $\varepsilon > 0$, *be a random element of the space* B_1 *and assume that* X^ε *is asymptotically* $(x_1, \varepsilon K(\cdot,\cdot))$*-Gaussian.* [*This means that* $(x^\varepsilon - x_1)/\sqrt{\varepsilon}$, $\varepsilon \downarrow 0$, *converges weakly in* B_1 *to the Gaussian distribution with mean* 0 *and correlation function* $K(\cdot,\cdot)$.] *Then* $f(X^\varepsilon)$ *is asymptotically a* $(f(x_1), \varepsilon f_1'(x_1) f_2'(x_1) K(\cdot,\cdot))$*-Gaussian random element of* B_2, *where* $f_i'(x_1)$ *is the Fréchet differential* (*linear operator*) *of the mapping f at the point* x_1, *and the subscript i in* $f_i'(x_1)$ *means that this operator is applied to* $K(\cdot,\cdot)$ *as a function of the i-th argument,* $i = 1, 2$.

We have the linear continuous mapping $\zeta^\varepsilon \to w^\varepsilon: C^\gamma \to C_a$ and the Fréchet differentiable mapping $w^\varepsilon \to u^\varepsilon: C_a \to C_a$. Thus the composition of these two mappings of $\zeta^\varepsilon \to u^\varepsilon$ is a Fréchet differentiable mapping from C^γ to C_a. According to Lemma 5, to prove weak convergence of $(1/\sqrt{\varepsilon})(u^\varepsilon - u)$ to a Gaussian field, we should first of all prove that $\zeta^\varepsilon = (\zeta^\varepsilon_+, \zeta^\varepsilon_-)$ is asymptotically Gaussian. Recall that

$$\zeta^\varepsilon_\pm(t) = \int_0^t \xi_\pm(s/\varepsilon)\, ds,$$

where $\xi_\pm(s)$ are stationary processes, $E\xi_\pm(s) \equiv 0$, and $P\{|\xi_\pm| < C\} = 1$ for some nonrandom $C < \infty$. Some results are known about asymptotic normality of such processes ζ^ε.

Let $\alpha^*(\tau)$ be the strong mixing coefficient for the process $(\xi_+(s), \xi_-(s)) = \xi(s)$:

$$\alpha^*(\tau) = \sup|E\xi\eta - E\xi E\eta|,$$

where the supremum is calculated over all random variables ξ, η such that $|\xi| \le 1$, $|\eta| \le 1$, and ξ is measurable with respect to the σ-field $\mathscr{F}_{\le t}$ generated by the process ξ_s for $s \le t$, and η is measurable with respect to the σ-field $\mathscr{F}_{\ge t+\tau}$ generated by ξ_s for $s \ge t + \tau$.

The weak convergence of $(1/\sqrt{\varepsilon})\zeta^\varepsilon_t$ in the space of continuous functions on $[0, T]$ was proved under the condition that the strong mixing coefficient $\alpha^*(\tau)$ decreases fast enough. But we need a stronger statement: $\varepsilon^{-1/2}\zeta^\varepsilon_t$ converges weakly in the space C^γ with Hölder norm. It turns out that this slightly stronger statement can be proven by a slight modification of the proof of the standard result.

THEOREM 2. *Let $\int_0^\infty \tau^{k-1}\alpha^*(\tau)\, d\tau < \infty$ for some $k > 1$. Then for any $\gamma \in (0, (k-1)/2k)$, $T > 0$, the family of processes $(1/\sqrt{\varepsilon})\zeta^\varepsilon_t$ converges weakly in the space C^γ_{0T} to the two-dimensional Brownian motion with covariance matrix $(a_{\lambda\mu})$, where*

$$a_{\lambda\mu} = \int_{-\infty}^{\infty} K_{\lambda\mu}(\tau)\, d\tau, \qquad K_{\lambda\mu}(\tau) = E\xi_\lambda(s)\xi_\mu(s+\tau), \qquad \lambda, \mu \in (+, -).$$

PROOF. It is known that under the conditions of Theorem 2, $(1/\sqrt{\varepsilon})\zeta^\varepsilon_t$ converges weakly in the space of continuous functions, and the limit process is the Brownian motion with covariance matrix $(a_{\lambda\mu})$ (see [1] and [5]). Thus the finite-dimensional distributions of $(1/\sqrt{\varepsilon})\zeta^\varepsilon_t$ converge to the corresponding distributions of the limit process, and it remains to prove the C^γ_{0T}-tightness of the family of the processes $\eta^\varepsilon_t = (1/\sqrt{\varepsilon})\zeta^\varepsilon_t$. The tightness of the family η^ε_t in the space $\mathbb{C}_{0T}$ in [1] and [5] and in other papers relies on the Kolmogorov-type inequalities:

$$E|\eta^\varepsilon_{t_1} - \eta^\varepsilon_{t_2}|^{2k} \le C|t_2 - t_1|^k,$$

where $C < \infty$ if $\int_0^\infty \tau^{k-1}\alpha^*(\tau)\, d\tau < \infty$. But these inequalities automatically imply not only the $\mathbb{C}_{0T}$-tightness but the C^γ_{0T}-tightness also for $\gamma < (k-1)/2k$. □

THEOREM 3. *Suppose that the process* $\xi(t) = (\xi_+(t), \xi_-(t))$ *has a mixing coefficient* $\alpha^*(\tau)$ *such that* $\int_0^\infty \tau^{k-1}\alpha^*(\tau)\,d\tau < \infty$ *for some* $k > 1$. *Let* $\alpha \in (1/k, 1)$ *and* $a(x) = (1 - x^2)^\alpha$. *Then* $v^\varepsilon(t,x) = (1/\sqrt{\varepsilon})(u^\varepsilon(t,x) - u(t,x))$ *converges when* $\varepsilon \downarrow 0$ *weakly in the space* $\hat{C}_a$ *to the Gaussian random field* $v(t,x)$, $0 \le t \le T$, $|x| < 1$, *which is the generalized solution of problem* (14).

PROOF. Since $\alpha > 1/k$ we have $(1-\alpha)/2 < (k-1)/2k$. Take $\gamma \in ((1-\alpha)/2, (k-1)/2k)$. It follows from Theorem 2 that the process $\eta_t^\varepsilon = (1/\sqrt{\varepsilon})\zeta_t^\varepsilon$ converges weakly in the space C_{0T}^γ to the Brownian motion with covariance matrix $(a_{\lambda\mu})$.

The mapping $\zeta^\varepsilon \to u^\varepsilon$ is the composition of two mappings, $\zeta^\varepsilon \to w^\varepsilon\colon C^\gamma \to \hat{C}_a$ and $w^\varepsilon \to u^\varepsilon\colon \hat{C}_a \to \hat{C}_a$. The first mapping is linear and, according to Lemma 3, is continuous since $1/2 > \gamma > (1-\alpha)/2$. Thus the first mapping is Fréchet differentiable. The second mapping is Fréchet differentiable according to Lemma 2. The differential of the second mapping is the linear transformation $\delta w \to \delta u$ defined by (20). The statement of Theorem 3 follows from Lemma 5 and equalities (16), (18), and (20). □

REMARK. Using Lemma 5, we can describe the limit Gaussian field as a mean zero Gaussian field with correlation function K_{vv}, which is defined as follows: Let

$$K_{z^0z^0}(t_1,x_2,t_2,x_2) = \sum_{\lambda,\mu} a_{\lambda\mu}\int_0^{t_1\wedge t_2} p(t_1-s,x,\lambda)p(t_2-s,x,\mu)\,ds,$$

where $p(t_1-s,x,+) = p(t_1-s,x,1)$ and $p(t_1-s,x,-) = p(t_1-s,x,-1)$; $K_{z^0v}(t_1,x_1,t_2,x_2)$ is the solution of the integral equation [for every (t_2,x_2)]

$$K_{z^0v}(t_1,x_1,t_2,x_2) = K_{z^0z^0}(t_1,x_1,t_2,x_2) + \int_0^{t_1} ds\int_{-1}^1 dy\, p(t_1-s,x_1,y) \times f_u'(s,u(s,y))K_{z^0v}(s,y,t_2,x_2).$$

Then K_{vv} is the solution of the equation [for every (t_1,x_1)]

$$K_{vv}(t_1,x_1,t_2,x_2) = K_{z^0v}(t_1,x_1,t_2,x_2) + \int_0^{t_2} ds\int_{-1}^1 dy\, p(t_2-s,x_2,y) \times f_u'(s,u(s,y))K_{vv}(t_1,x_1,s,y).$$

3. Deviations of order ε^κ, $0 < \kappa < 1/2$. We proved in Section 1 that $u^\varepsilon(t,x) \to u(t,x)$ when $\varepsilon \downarrow 0$ uniformly in $0 \le t \le T$, $|x| \le 1$. The deviations u^ε from u of order $\varepsilon^{1/2}$, $\varepsilon \downarrow 0$, are described in Section 2. It follows from Theorem 3 that the probabilities of deviations of order ε^κ for $\kappa < 1/2$ tend to 0. In this section we consider the asymptotics of the probabilities of the deviations of order ε^κ, $0 < \kappa < 1/2$. Our goal is to obtain rough limit theorems for the probabilities of such deviations. This means that we calculate the logarithmic asymptotics of these probabilities. Such asymptotics are interesting, for example, when we study the exit problem from a neighborhood of size of order ε^{κ}, $\kappa \in (0, 1/2)$, of a stable equilibrium point of (4).

First, we recall some definitions and formulate a simple general lemma.

Let ζ^{ε}, $\varepsilon > 0$, be a family of random elements in a Banach space B_1 with norm $\|\cdot\|_1$ and distance $\rho_1(f, g) = \|f - g\|_1$, $f, g \in B_1$. Denote $\eta^{\varepsilon} = \varepsilon^{-\kappa}\zeta^{\varepsilon}$ and assume that $\|\eta^{\varepsilon}\|_1 \to 0$ when $\varepsilon \downarrow 0$ in probability.

A functional $S^{\eta}(\varphi)$ defined for all $\varphi \in B_1$ is called the normalized action functional [4] for the family η^{ε} in B_1 when $\varepsilon \downarrow 0$, if a positive function $\lambda(\varepsilon)$, $\varepsilon > 0$, $\lim_{\varepsilon \downarrow 0}\lambda(\varepsilon) = \infty$, exists such that the following conditions are fulfilled.

1. For any $\delta, \gamma > 0$ and $\varphi \in B_1$ there exists $\varepsilon_0 > 0$ such that
$$P\{\rho_1(\eta^{\varepsilon}, \varphi) < \delta\} \geq \exp\{-\lambda(\varepsilon)(S^{\eta}(\varphi) + \gamma)\}, \quad \text{for } \varepsilon < \varepsilon_0.$$
2. For any $s, \gamma, \delta > 0$ there exists $\varepsilon_0 > 0$ such that
$$P\{\rho_1(\eta^{\varepsilon}, \Phi_s) > \delta\} \leq \exp\{-\lambda(\varepsilon)(s - \gamma)\}, \quad \text{for } \varepsilon < \varepsilon_0,$$
where $\Phi_s = \{\varphi \in B_1\colon S^{\eta}(\varphi) \leq s\}$.
3. The functional $S^{\eta}(\varphi)$ is semicontinuous from below: If $\lim_{n\to\infty} \rho_1(\varphi_n, \varphi) = 0$, then $S^{\eta}(\varphi) \leq \liminf_{n\to\infty} S^{\eta}(\varphi_n)$; the set Φ_s is compact in B_1.

The product $\lambda(\varepsilon)S^{\eta}(\varphi)$ is called the action functional.

Consider a continuous mapping $f\colon B_1 \to B_2$, where B_2 is another Banach space with the norm $\|\cdot\|_2$ and distance $\rho_2(f, g) = \|f - g\|_2$ for $f, g \in B_2$. Assume that the mapping f is Fréchet differentiable at the point 0, and let the linear continuous operator $f'\colon B_1 \to B_2$ be the Fréchet differential.

Assume for brevity that for any $\psi \in B_2$ there exists at most one element $\varphi \in B_1$ such that $f'\varphi = \psi$, $\varphi = f'\varphi$.

LEMMA 6. *Let $y^{\varepsilon} = (1/\varepsilon^{\kappa})f(\zeta^{\varepsilon})$. The action functional for the family y^{ε}, $\varepsilon > 0$, in B_2 when $\varepsilon \downarrow 0$ is equal to $\lambda(\varepsilon)S^{y}(\varphi)$, where*
$$S^{y}(\varphi) = \begin{cases} S^{\eta}(f'^{-1}\varphi), & \text{if } f'^{-1}\varphi \text{ exists}, \\ +\infty, & \text{if } f'^{-1}\varphi \text{ does not exist}, \end{cases}$$
where $\lambda(\varepsilon)S^{\eta}$ is the action functional for the family $\eta^{\varepsilon} = \varepsilon^{-\kappa}\zeta^{\varepsilon}$ in the space B_1.

This lemma follows easily from the continuity of the operator f'. It is close to Theorem 3.3.1 from [4] and the proof is omitted.

As was shown in the previous section, the mapping
$$\int_0^t \xi(s/\varepsilon)\, ds = \zeta_t^{\varepsilon} \to u^{\varepsilon}(t, x)$$
is continuous and even Fréchet differentiable as a mapping from C^{γ} to $\hat{C}_a$, where $a(x) = (1 - x^2)^{\alpha}$, $\alpha > 1 - 2\gamma$, $0 < \gamma < 1/2$. Recall that C^{γ} is the Banach space of functions $\varphi_s = (\varphi_+(s), \varphi_-(s))$, $\varphi_0 = 0$, $0 \leq s \leq T$, with

the Hölder norm $\|\cdot\|_\gamma$. We denote $\rho_\gamma(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_\gamma$, $\|\varphi\|_0 = \max_{+,-} \max_{0 \le t_1 < t_2 \le T} |\varphi_\pm(t_1) - \varphi_\pm(t_2)|$; the norm $\|\cdot\|_0$, of course, is equivalent to the uniform norm in C_{0T}. The Fréchet differential is defined by (18) and (20).

To calculate the action functional for the family $\varepsilon^{-\kappa}(u^\varepsilon - u)$ in the space $\hat{C}_a$, we should, according to Lemma 6, calculate the action for the family $\eta^\varepsilon = \varepsilon^{-\kappa}\zeta^\varepsilon$, $0 < \kappa < 1/2$, in the space C^γ. There are some results on the large deviations for the family η^ε, $\varepsilon \downarrow 0$, but in the space C_{0T} (see [4] Chapter 7, and some other references therein).

Introduce the functional

$$S_{0T}(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \sum_{\lambda,\mu} a^{\lambda,\mu}\dot{\varphi}_\lambda(s)\dot{\varphi}_\mu(s)\,ds, & \text{for } \varphi_+, \varphi_- \text{ absolutely continuous,} \\ +\infty, & \text{for the rest of } C_{0T}. \end{cases}$$

Here $(a^{\lambda\mu}) = (a_{\lambda\mu})^{-1}$, $a_{\lambda\mu} = \int_{-\infty}^{\infty} K_{\lambda\mu}(\tau)\,d\tau$, $K_{\lambda\mu}(\tau) = E\xi_\lambda(t)\xi_\mu(t+\tau)$, $\lambda, \mu \in \{+,-\}$. We assume for brevity that the matrix $a_{\lambda\mu}$ is not degenerate. Under some assumptions about the process $\xi(t)$, one can show that $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ is the action functional for the family $\eta^\varepsilon = \varepsilon^{-\kappa}\zeta^\varepsilon$ in the space C_{0T}. Let $a > 0$ be such that

$$a^{-1}(z_+^2 + z_-^2) \le \sum_{\lambda,\mu} a^{\lambda,\mu} z_\lambda z_\mu \le a(z_+^2 + z_-^2).$$

If φ is absolutely continuous we have

$$|\varphi(t_1) - \varphi(t_2)| = \left|\int_{t_1}^{t_2} \dot{\varphi}_s\,ds\right| \le \sqrt{|t_2 - t_1| \int_{t_1}^{t_2} |\dot{\varphi}_s|^2\,ds}$$

$$(23) \qquad \le \sqrt{|t_2 - t_1|}\,\sqrt{2S_{0T}(\varphi)a}, \qquad t_1, t_2 \in [0,T],$$

$$\|\varphi\|_{1/2} \le \sqrt{2aS_{0T}(\varphi)}, \qquad \|\varphi\|_0 \le \sqrt{2aS_{0T}(\varphi)T}.$$

It is easy to check that for $0 < \gamma < \gamma_0$,

$$(24) \qquad \|\varphi\|_\gamma \le \|\varphi\|_0^{1-(\gamma/\gamma_0)} \|\varphi\|_{\gamma_0}^{\gamma/\gamma_0}.$$

From (23) and (24) we deduce that

$$(25) \qquad \|\varphi\|_\gamma \le \sqrt{2aS_{0T}(\varphi)}\,T^{(1/2)-\gamma} \quad \text{for } \gamma \in (0, 1/2).$$

We say that the family η^ε is $\varepsilon^{-\mu}$-exponentially bounded in the space C^γ if for any $C > 0$ there exists K such that

$$P\{\|\eta^\varepsilon\|_\gamma \ge K\} \le \exp\left\{-\frac{c}{\varepsilon^\mu}\right\}$$

for ε small enough.

LEMMA 7. *Let $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ be the action functional for the family $\eta^\varepsilon = \varepsilon^{-\kappa}\zeta^\varepsilon$, $0 < \kappa < 1/2$, in the space C_{0T} when $\varepsilon \downarrow 0$. Assume that the family η^ε is*

$\varepsilon^{2\kappa-1}$-exponentially bounded in the space C^{γ_0}, $0 < \gamma_0 < 1/2$. Then the restriction of the functional $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ on the space C^{γ}, $0 < \gamma < \gamma_0$, will be the action functional for the family η^ε in C^γ when $\varepsilon \downarrow 0$.

PROOF. For any $\delta, K > 0$ and $\gamma \in (0, \gamma_0)$, denote

$$\delta' = \delta^{\gamma_0/(\gamma_0-\gamma)}K^{-\gamma/(\gamma_0-\gamma)}.$$

We deduce from (24) that

$$\begin{aligned} P\{\|\eta^\varepsilon - \varphi\|_\gamma < \delta\} &\geq P\{\|\eta^\varepsilon - \varphi\|_0 < \delta', \|\eta^\varepsilon - \varphi\|_{\gamma_0} < K\} \\ &\geq P\{\|\eta^\varepsilon - \varphi\|_0 < \delta'\} - P\{\|\eta^\varepsilon - \varphi\|_{\gamma_0} \geq K\}. \end{aligned}$$

From (25) and the last inequality we have

$$\begin{aligned} P\{\|\eta^\varepsilon - \varphi\|_\gamma < \delta\} &\geq P\{\|\eta^\varepsilon - \varphi\|_0 < \delta'\} \\ &\quad - P\Big\{\|\eta^\varepsilon\|_{\gamma_0} \geq K - \sqrt{2aS_{0T}(\varphi)}\, T^{(1-2\gamma)/2}\Big\}. \end{aligned} \tag{26}$$

The first term on the right-hand side of (26) can be bounded from below by

$$\exp\{-\varepsilon^{2\kappa-1}(S_{0T}(\varphi) + \lambda)\}.$$

This is true for any $\lambda > 0$ if $\varepsilon > 0$ is small enough, since $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ is the action for the family η^ε in the norm $\|\cdot\|_0$. The second term in (26) can be bounded from above by $\exp\{-\varepsilon^{2\kappa-1} \cdot 10(S_{0T}(\varphi) + \lambda)\}$ if K is large enough since the family η^ε is $\varepsilon^{2\kappa-1}$-exponentially bounded in the $\|\cdot\|_{\gamma_0}$-norm. From this bound we conclude that for any $\lambda > 0$,

$$P\{\|\eta^\varepsilon - \varphi\|_\gamma < \delta\} > \exp\{-\varepsilon^{2\kappa-1}(S_{0T}(\varphi) + \lambda)\}$$

if ε is small enough.

To prove the upper bound, included in the definition of the action functional, note that if $\rho_0(\eta^\varepsilon, \phi_s) < \delta'$ and $\|\eta^\varepsilon\|_{\gamma_0} < K - \sqrt{2as}\, T^{(1/2)-\gamma}$, then a point $\varphi \in \phi_s$ exists such that

$$\|\eta^\varepsilon - \varphi\|_0 < \delta', \qquad \|\eta^\varepsilon - \varphi\|_{\gamma_0} < K, \qquad \|\eta^\varepsilon - \varphi\|_\gamma < \delta.$$

The last inequality results from the first two and from (24).

We have from this:

$$\begin{aligned} P\{\rho_\gamma(\eta^\varepsilon, \phi_s) \geq \delta\} &\leq P\{\rho_0(\eta^\varepsilon, \phi_s) \geq \delta'\} \\ &\quad + P\Big\{\|\eta^\varepsilon\|_{\gamma_0} \geq K - \sqrt{2as}\, T^{(1/2)-\gamma_0}\Big\}. \end{aligned} \tag{27}$$

The first term in the right-hand side of (27) can be bounded from above due to the fact that $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ is the action functional for η^ε in the $\|\cdot\|_0$-norm. The second term can be made less than $\exp\{-2s\varepsilon^{2\kappa-1}\}$ because of the $\varepsilon^{2\kappa-1}$-exponential boundedness of η^ε in the $\|\cdot\|_{\gamma_0}$-norm by choosing K large enough. We have finally that for any $\delta, \lambda, s > 0$ there exists $\varepsilon_0 > 0$ such that

$$P\{\rho_\gamma(\eta^\varepsilon, \phi_s) > \delta\} \leq \exp\{\varepsilon^{2\kappa-1}(s - \lambda)\}$$

if $\varepsilon \in (0, \varepsilon_0)$.

To finish the proof of this lemma, we should check that $\Phi_s = \{\varphi \in C^\gamma: S_{0T}(\varphi) \le s\}$ is compact in C^γ and that the functional $S_{0T}(\varphi)$ is semicontinuous from below in C^γ. The last statement follows immediately from the semicontinuity of $S_{0T}(\varphi)$ in the $\|\cdot\|_0$-norm. Compactness of the set Φ_s in C^γ follows from the fact that the set $\{\varphi \in C^\gamma: \|\varphi\|_{\gamma_0} \le c\}$ is compact in C^γ, $0 < \gamma < \gamma_0$, for any $c < \infty$ and from semicontinuity of $S_{0T}(\varphi)$. □

THEOREM 4. *Assume that the functional $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ is the action functional for the family $\eta^\varepsilon = \varepsilon^{-\kappa}\zeta^\varepsilon = \varepsilon^{-\kappa}\int_0^t \xi(s/\varepsilon)\,ds$ in the space C_{0T} when $\varepsilon \downarrow 0$, and that the family η^ε is $\varepsilon^{2\kappa-1}$-exponentially bounded in C^{γ_0} for some $\gamma_0 \in (0, 1/2)$. Then the family of random fields*

$$v_\kappa^\varepsilon(t,x) = \varepsilon^{-\kappa}(u^\varepsilon(t,x) - u(t,x)), \qquad \varepsilon \downarrow 0,$$

has the action functional $\varepsilon^{2\kappa-1} \cdot S^\kappa(g)$ in the space $\hat{C}_a$ for $a(x) = (1-x^2)^\alpha$, $1 > \alpha > 1 - 2\gamma_0$, where

$$S^\kappa(g) = \begin{cases} S_{0T}\left(\dfrac{\partial g}{\partial x}(s,1), -\dfrac{\partial g}{\partial x}(s,-1)\right), & \text{if } \dfrac{\partial g}{\partial t} = \dfrac{D}{2}\dfrac{\partial^2 g}{\partial x^2} \\ \qquad + f_u'(x, u(t,x))g(t,x) \text{ for } 0 < t \le T,\ |x| < 1, & \\ \qquad \text{and } \dfrac{\partial g}{\partial x}(s, \pm 1) \text{ are absolutely continuous}, & \\ +\infty, & \text{for the rest of } \hat{C}_a. \end{cases}$$

PROOF. According to Lemma 7, the action functional for the family η^ε in the space C^γ, $0 < \gamma < \gamma_0$, will be the restriction of $\varepsilon^{2\kappa-1}S_{0T}(\varphi)$ to the space C^γ. As was shown in Section 2, the mapping $\zeta^\varepsilon \to u^\varepsilon(t,x)$ is continuous and Fréchet differentiable as a mapping from C^γ to $\hat{C}_a$ if $a = (1-x^2)^\alpha$, $\alpha > 1 - 2\gamma$. The Fréchet differential is defined by (18) and (20). The statement of Theorem 4 now follows from Lemma 6. □

LEMMA 8. *Assume that there exist C_1, C_2 such that*

$$E\exp\left\{z\varepsilon^{-1/2}\int_t^{t+\tau}\xi_\pm\left(\frac{s}{\varepsilon}\right)ds\right\} < \exp\left\{\frac{C_1 z^2 \tau}{2}\right\} \tag{28}$$

for $|z| \le C_2/\sqrt{\varepsilon}$, $\tau > \varepsilon$. Then the family $\eta_t^\varepsilon = \varepsilon^{-\kappa}\int_0^t \xi_{s/\varepsilon}\,ds$, $\varepsilon \downarrow 0$, is $\varepsilon^{2\kappa-1}$-exponentially bounded in the norm C^γ for any $\gamma \in (0, 1/2)$.

PROOF. We need to prove that for any $C > 0$ one can find K such that

$$P\{\|\eta^\varepsilon\|_\gamma \ge K\} \le \exp\{-C\varepsilon^{2\kappa-1}\}.$$

Denote

$$A_n^\varepsilon = \max_{0 \le k/2n < (k+1)/2n \le T}\left|\eta^\varepsilon\left(\frac{k+1}{2^n}\right) - \eta^\varepsilon\left(\frac{k}{2^n}\right)\right|.$$

For any $t_1, t_2 \in [0, T]$ we have

$$|\eta^\varepsilon_{t_1} - \eta^\varepsilon_{t_2}| \le 2 \sum_{n:\, 2^{-n} < |t_1 - t_2|} A^\varepsilon_n. \tag{29}$$

In order that $\|\eta^\varepsilon\|_\gamma < K$, it is sufficient that $A^\varepsilon_n \le M 2^{-n\gamma_0}$ for all n, where $M = K(1 - 2^{-\gamma})/2$. It follows from (29) that

$$\begin{aligned} P\{\|\eta^\varepsilon\|_\gamma > K\} &\le 2 \sum_n P\{A^\varepsilon_n > M 2^{-n\gamma}\} \\ &\le 2\Bigg(\sum_{n:\, 2^{-n(1-\gamma)} \le N\varepsilon^\kappa} P\{A^\varepsilon_n > M 2^{-n\gamma}\} \\ &\qquad + \sum_{n:\, 2^{-n(1-\gamma)} > N\varepsilon^\kappa} P\{A^\varepsilon_n > M 2^{-n\gamma}\} \Bigg). \end{aligned} \tag{30}$$

The constant N will be chosen later. Taking into account that $|\xi_\pm(t)| < C$, we have

$$A^\varepsilon_n \le \max_{0 \le t \le T} \left| \varepsilon^{-\kappa} \int_t^{t + (2^{-n})} \xi\left(\frac{s}{\varepsilon}\right) ds \right| \le \frac{\varepsilon^{-\kappa} C}{2^n} \le C N 2^{-\gamma n}$$

for n: $2^{-n(1-\gamma)} \le N\varepsilon^\kappa$. Therefore, all terms of the first sum on the right-hand side of (30) are equal to 0 if M is large enough.

To bound the second sum in (30), we use the exponential Chebyshev inequality. Denote

$$z = \frac{M 2^{-n\gamma}}{K 2^{-n} \varepsilon^{(1/2) - \kappa}}.$$

We have

$$\begin{aligned} &P\{\varepsilon^{-\kappa} |\zeta^\varepsilon_{(k+1)/2^n} - \zeta^\varepsilon_{k/2^n}| > M 2^{-n\gamma}\} \\ &\quad = P\left\{ z \varepsilon^{-1/2} |\zeta^\varepsilon_{(k+1)/2^n} - \zeta^\varepsilon_{k/2^n}| > \frac{M^2 2^{-2n\gamma}}{K 2^{-n} \varepsilon^{1-2\kappa}} \right\} \\ &\quad \le E \exp\left(z \varepsilon^{-1} |\zeta^\varepsilon_{(k+1)/2^n} - \zeta^\varepsilon_{k/2^n}| - \frac{M^2 2^{n(1-2\gamma)}}{K \varepsilon^{1-2\kappa}} \right) \\ &\quad \le \exp\left(- \frac{M^2 2^{n(1-2\gamma)}}{K \varepsilon^{1-2\kappa}} + \frac{C_1 M^2 2^{n(1-2\gamma)}}{2K^2 \varepsilon^{1-2\kappa}} \right). \end{aligned} \tag{31}$$

We used condition (28) of the lemma in the last inequality. We can do this if $z \le C_2/\sqrt{\varepsilon}$, $2^{-n} \ge \varepsilon$. Both these inequalities are true for n such that $2^{-n(1-\gamma)} > N\varepsilon^\kappa$ if N is large enough and ε is small enough,

$$z = \frac{M 2^{n(1-2\gamma)}}{K \varepsilon^{(1/2) - \kappa}} \le \frac{M}{KN\sqrt{\varepsilon}} < \frac{C_2}{\sqrt{\varepsilon}} \quad \text{for } N \text{ large enough.}$$

If $2^{-n(1-\gamma)} \ge N\varepsilon^\kappa$, then $2^{-n} > N^{1/(1-\gamma)} \varepsilon^{\kappa/(1-\gamma)} > \varepsilon N^{1/(1-\gamma)} \varepsilon^{\kappa/(1-\gamma)} > \varepsilon$ for ε small enough, since $\kappa/(1-\gamma) < 1$ for $\kappa, \gamma \in (0, 1/2)$.

From (31) we have for any C,

$$P\{\varepsilon^{\kappa}|\zeta^{\varepsilon}_{(k+1)/2^n} - \zeta^{\varepsilon}_{k/2^n}| > M2^{-n\gamma}\} \le \exp\left(-\frac{M^2 2^{n(1-2\gamma)}}{K\varepsilon^{1-2\kappa}}\right) < \exp(-C\varepsilon^{2\kappa-1})$$

if K is large enough.

From these bounds and from (30) the statement of the lemma follows. □

We will consider some examples in the last section.

4. Deviations of order 1. In this section we study deviations $u^{\varepsilon}(t,x)$ from $u(t,x)$ of order 1 when $\varepsilon \downarrow 0$ in the uniform topology. We denote by $\mathbb{C}^1 = \mathbb{C}^1[0,T]$ the space of all continuously differentiable functions $\varphi(t) = (\varphi_+(t), \varphi_-(t))$, $0 \le t \le T$, $\varphi(0) = 0$ with the norm

$$\|\varphi\|_1 = 2\max_{+,-}\max_{0\le t\le T}|\dot\varphi_{\pm}(t)|;$$

$\|\cdot\|$ means uniform norm in the space $\mathbb{C}_{0T}$ or in $C_{[0,T]\times[-1,1]}$. The other notations were introduced in Sections 1 and 2.

LEMMA 9. *The mapping $\zeta \to w$, defined by formula (6), from $\{\zeta \in \mathbb{C}^1$: $\|\zeta\|_1 \le C\}$ to the space $C_{[0,T]\times[-1,1]}$ is continuous with respect to the $\|\cdot\|$-norms in these spaces.*

PROOF. We have for $\zeta_1, \zeta_2 \in \mathbb{C}^1$, $\|\zeta_i\|_1 < C$,

$$w_1(t,x) - w_2(t,x) = \sum_{+,-}\int_0^t p(t-s,x,\pm 1)\, d(\zeta_{1\pm}(s) - \zeta_{2\pm}(s)),$$

$$\begin{aligned} |w_1 - w_2| &\le \sum_{+,-}\left\{\left|\int_0^{t-\delta} p(t-s,x,\pm 1)\, d(\zeta_{1\pm}(s)-\zeta_{2\pm}(s))\right|\right.\\ &\qquad \left.+\left|\int_{t-\delta}^t p(t-s,x,\pm 1)\, d(\zeta_{1\pm}(s)-\zeta_{2\pm}(s))\right|\right\}\\ &\le 2C\int_{t-\delta}^t ds\sum_{+,-}p(t-s,x,\pm 1) + \|\zeta_1-\zeta_2\|\\ &\qquad \times\sum_{+,-}\left[p(\delta,x,\pm 1) + p(t,x,\pm 1)\right.\\ &\qquad\qquad \left.+\int_0^{t-\delta}p'_1(t-s,x,\pm 1)\,ds\right], \end{aligned} \tag{32}$$

where $p'_1(\tau,x,y) = \partial p(\tau,x,y)/\partial\tau$. Taking into account that

$$\sup_{x,y} p(\tau,x,y) \le \text{const}/\sqrt{\tau}, \qquad \sup_{x,y} p'_1(\tau,x,y) \le \text{const}/\tau^{3/2},$$

and choosing $\delta = \min(t, \sqrt{\|\zeta_1-\zeta_2\|/C})$, we obtain from (32) that $\|w_1 - w_2\| \le \text{const}\cdot\sqrt{C\|\zeta_1-\zeta_2\|}$. □

THEOREM 5. *Assume that the family* $\zeta_t^\varepsilon = (\int_0^t \xi_+(s/\varepsilon)\,ds, \int_0^t \xi_-(s/\varepsilon)\,ds)$, $\varepsilon \downarrow 0$, *in the space* $\mathbb{C}_{0T}$ *has the action functional* $\varepsilon^{-1}S^\zeta(\varphi_+, \varphi_-)$. *The action functional for the field* $u^\varepsilon(t,x)$ *in the space* $C_{[0,T]\times[-1,1]}$ *when* $\varepsilon \downarrow 0$ *is equal to* $\varepsilon^{-1}S^u(g)$, $g \in C_{[0,T]\times[-1,1]}$, *where*

$$S^u(g) = \begin{cases} S^\zeta\left(\frac{\partial g}{\partial x}(s,1), -\frac{\partial g}{\partial x}(s,-1)\right), & \text{if } \frac{\partial g}{\partial t} = \frac{D}{2}\frac{\partial^2 g}{\partial x^2} + f(x,g) \\ & \text{for } 0 < t \le T,\ |x| < 1, \text{ and the functions} \\ & \frac{\partial g}{\partial x}(t, \pm 1) \text{ are absolutely continuous}, \\ \pm\infty, & \text{for the rest of } C_{[0,T]\times[-1,1]}. \end{cases}$$

PROOF. Since we assume in this paper that $|\xi_\pm(t)| \le C$, it follows from Lemma 9 that the mapping $\zeta^\varepsilon \to w^\varepsilon$ is continuous in uniform topology in $\mathbb{C}_{0T}$ and in $C_{[0,T]\times[-1,1]}$. Because of the last statement of Lemma 2, the mapping $w^\varepsilon \to u^\varepsilon$: $C_{[0,T]\times[-1,1]} \to C_{[0,T]\times[-1,1]}$ is also continuous. Thus the mapping $\zeta^\varepsilon \to u^\varepsilon$ is a continuous mapping from $\mathbb{C}_{0T}$ to $C_{[0,T]\times[-1,1]}$. Then the statement of Theorem 5 follows from Theorem 3.3.1 of [4]. □

5. Examples.

EXAMPLE 1. Let $\xi_t = (\xi_+(t), \xi_-(t))$ be the two-dimensional diffusion process in a domain $D \subset R^2$, corresponding to a second-order elliptic operator L with reflection in the direction of the co-normal on the boundary ∂D of D. We assume that the coefficients of the operator L are smooth enough and that the domain D is bounded and has smooth boundary. Denote by $p(t, x^1, x^2)$, $x^i = (x_+^i, x_-^i)$, the transition density, and by $m(x)$, $x \in D \cup \partial D$, the invariant density of the process ξ_t. We take $m(x)$ as the initial density of ξ_t; then the process ξ_t is stationary. Assume that $E\xi_t = \int x m(x)\,dx = 0$. The correlation matrix is given as follows:

$$\begin{aligned} K_{\lambda\mu}(\tau) &= E\xi_\lambda(t)\xi_\mu(t+\tau) \\ &= \int_D\int_D x_\lambda^1 x_\mu^2 p(\tau, x^1, x^2) m(x^1)\,dx^1\,dx^2, \qquad \lambda, \mu \in \{+,-\},\ \tau \ge 0, \end{aligned}$$

$$K_{++}(\tau) = K_{++}(-\tau), \qquad K_{--}(\tau) = K_{--}(-\tau), \qquad K_{+-}(\tau) = K_{-+}(-\tau),$$
$$K_{-+}(\tau) = K_{+-}(-\tau).$$

It is easy to check that the strong mixing coefficient $\alpha^*(\tau)$ for the process ξ_t decreases exponentially fast and thus

$$\int_0^\infty \tau^{k-1}\alpha^*(\tau)\,d\tau < \infty$$

for any $k > 1$. Then we conclude from Theorem 1 that the solution $u^\varepsilon(t,x)$ of problem (2) with $\xi_{\pm}(t)$ described above converges in $C_{[0,T]\times[-1,1]}$ in probability to the solution $u(t,x)$ of problem (4) when $\varepsilon \downarrow 0$.

The normalized difference $v^{\varepsilon}(t,x) = \varepsilon^{-1/2}(u^{\varepsilon}(t,x) - u(t,x))$, according to the results of Section 2, converges weakly in the space $\hat{C}_a$ with $a(x) = (1-x^2)^{\alpha}$ for any $\alpha \in (0,1)$ to the solution of problem (14), where $a_{\lambda\mu} = \int_{-\infty}^{\infty} K_{\lambda\mu}(\tau)\,d\tau$.

To describe the large deviations for the field $u^{\varepsilon}(t,x)$, we need first of all to recall results on large deviations for the family $\zeta_t^{\varepsilon} = \int_0^t \xi_{s/\varepsilon}\,ds$, $\varepsilon \downarrow 0$ (see [4], Chapter 7). Consider the eigenvalue problem

$$L\varphi(x) + (\beta, x)\varphi(x) = \lambda\varphi(x), \qquad x \in D, \qquad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial D} = 0.$$

Here $\beta = (\beta_1, \beta_2)$ is a parameter, $n = n(x)$ is the co-normal corresponding to the operator L. Let $\lambda = \lambda(\beta)$ be the eigenvalue corresponding to the positive eigenfunction. Such an eigenvalue is simple, real and continuously differentiable in β (we assume that the coefficients of L and the boundary of the domain D are smooth enough and that D is bounded). One can check that $\lambda(\beta)$ is convex. Denote by $L(\alpha)$ the Legendre transformation of $\lambda(\beta)$: $L(\alpha) = \sup_{\beta}((\alpha,\beta) - \lambda(\beta))$, $\alpha \in R^2$. The action functional for the family ζ_t^{ε}, $t \in [0,T]$, $\varepsilon \downarrow 0$, in $\mathbb{C}_{0T}$ is equal to $\varepsilon^{-1}\int_0^T L(\dot{\varphi}_+(s), \dot{\varphi}_-(s))\,ds$ for absolutely continuous φ, and equal to $+\infty$ for the rest of $\mathbb{C}_{0T}$.

The action functional for the process $\varepsilon^{-\kappa}\zeta_t^{\varepsilon}$, $0 < \kappa < 1/2$, in the space $\mathbb{C}_{0T}$ is $\varepsilon^{2\kappa-1}S_{0T}^{\kappa}(\varphi)$, where

$$S_{0T}^{\kappa}(\varphi) = \frac{1}{2}\int_0^T \sum_{\lambda,\mu} a^{\lambda\mu}\dot{\varphi}_{\lambda}(s)\dot{\varphi}_{\mu}(s)\,ds$$

for $\varphi_s = (\varphi_+(s), \varphi_-(s))$ absolutely continuous, and $S_{0T}^{\kappa}(\varphi) = +\infty$ for the rest of $\mathbb{C}_{0T}$; $a^{\lambda\mu} = (a_{\lambda\mu})^{-1}$, $\lambda, \mu \in (+,-)$.

Now we can describe large deviations of order 1 as $\varepsilon \downarrow 0$ for the field $u^{\varepsilon}(t,x)$. The action functional for the family $u^{\varepsilon}(t,x)$, $\varepsilon \downarrow 0$, in the space $C_{[0,T]\times[-1,1]}$, according to Theorem 5, has the form $\varepsilon^{-1}S^u(g)$, $g \in C_{[0,T]\times[-1,1]}$, where

$$S^u(g) = \begin{cases} \int_0^T L\left(\frac{\partial g}{\partial x}(s,1), -\frac{\partial g}{\partial x}(s,-1)\right), & \text{if } \frac{\partial g}{\partial t} = D\frac{\partial g^2}{\partial x^2} + f(x,g) \\ & \text{for } 0 < s \le T,\ |x| < 1, \text{ and the functions} \\ & \frac{\partial g}{\partial x}(s, \pm 1),\ 0 \le s \le T, \text{ are absolutely continuous,} \\ +\infty, & \text{for the rest of } C_{[0,T]\times[-1,1]}. \end{cases}$$

Using this result and taking into account that the couple $(u^{\varepsilon}(t,\cdot), \xi_t)$ is a Markov process in the functional space, we can describe transitions between different stable stationary solutions of the nonperturbed problem. We can also consider the exit problems and asymptotic behavior of the invariant measure of the perturbed problem when $\varepsilon \downarrow 0$ in a way similar to the finite-dimensional case (see [4]).

To describe the deviations $u^{\varepsilon}(t,x)$ from $u(t,x)$ of order ε^{κ}, $0 < \kappa < 1/2$, we should check the conditions of Lemma 8. Consider the semigroup P_{zf}^t acting in the space C_D of continuous functions on $D \cup \partial D$ with values in R^1:

$$\left(P_{zf}^t \varphi\right)(x) = E_x \varphi(\xi_t) \exp\left[z \int_0^T f(\xi_s)\, ds\right], \qquad t \geq 0,$$

where $f(x)$, $x \in D \cup \partial D$, is a smooth enough function such that $\int_D f(x) m(x)\, dx = 0$. This semigroup has a positive eigenfunction $\varphi_z(x)$, and the corresponding eigenvalue has the form $\exp\{t\mu(zf)\}$, where $\mu(zf) = \mu$ is the first eigenvalue of the problem

$$L\psi(x) + zf(x)\psi(x) = \mu\psi(x), \qquad x \in D, \qquad \left.\frac{\partial \psi}{\partial n}\right|_{\partial D} = 0 \tag{33}$$

(see, e.g., [4], Chapter 7). The first eigenvalue μ is real and simple and has two continuous derivatives in z. The eigenfunction $\varphi_z(x)$ is differentiable in z. It is easy to check that $\mu(0) = 0$ and $\varphi_0(x) \equiv 1$. Then, differentiating (33) in z, we have

$$L\psi_0'(x) = -f(x) + \mu_0', \qquad x \in D, \qquad \left.\frac{\partial \psi_0'}{\partial n}\right|_{\partial D} = 0. \tag{34}$$

Problem (34) is solvable only if the right-hand side of (34) is orthogonal to the invariant density $m(x)$. Since we assumed that $\int_D f(x) m(x)\, dx = 0$, we get $\mu_0' = 0$. Now taking into account that $\mu(0) = \mu'(0) = 0$, we have

$$\mu(zf) \leq \frac{c|z|^2}{2} \tag{35}$$

for some constant C and $|z| < 1$. Using this bound we conclude that the conditions of Lemma 8 are fulfilled in this example:

$$E \exp\left\{z \int_t^{t+\tau} \frac{f(\xi(s/\varepsilon))}{\sqrt{\varepsilon}}\, ds\right\} = \int_D m(x) \left(P_{z\sqrt{\varepsilon} f}^{\tau/\varepsilon} 1\right)(x)\, dx$$

$$\leq \text{const} \cdot \exp\left\{\frac{\tau}{\varepsilon} \mu\left(z\sqrt{\varepsilon}\right)\right\} \leq \text{const} \cdot \exp\left\{\frac{c|z|^2}{2} \tau\right\} \tag{36}$$

for $|z| < 1/\sqrt{\varepsilon}$. As the function $f = f(x_+, x_-)$, we should take $f = x_+$ or $f = x_-$.

We conclude from Lemma 8 and Theorem 4 that the action functional for the family of fields $(1/\varepsilon^\kappa)(u^\varepsilon(t,x) - u(t,x)) = v_\kappa^\varepsilon(t,x)$, $0 < \kappa < 1/2$, in the space $\hat{C}_a$, $a = (1 - x^2)^\alpha$, $0 < \alpha < 1$, has the form $\varepsilon^{2\kappa - 1} S^{u,\kappa}(g)$, where

$$S^{u,\kappa}(g) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^T \sum_{\lambda,\mu} a^{\lambda\mu} \lambda\mu \frac{\partial g(s,\lambda)}{\partial x} \frac{\partial g(s,\mu)}{\partial x}\, ds, \\ \qquad \text{if } \dfrac{\partial g}{\partial t} = \dfrac{D\partial^2 g}{\partial x^2} + f'(x,u) g \text{ for } t \in [0,T],\ |x| < 1, \\ \qquad \text{and } \dfrac{\partial g}{\partial x}(t, \pm 1) \text{ are absolutely continuous}, \\ +\infty, \qquad \text{for the rest of } \hat{C}_a. \end{cases}$$

EXAMPLE 2. Let $\xi_t = (\xi_+(t), \xi_-(t))$ be a Markov process with the finite state space $\{(b_k^+, b_k^-),\ k = 1, \dots, N\}$; $Q = (q_{ij})$ is the transition intensities matrix $P\{\xi_{t+\Delta} = (b_j^+, b_j^-) | \xi_t = (b_i^+, b_i^-)\} = q_{ij} \cdot \Delta + o(\Delta)$, $\Delta \downarrow 0$, $i \neq j$, $q_{ii} = -\sum_{j: j \neq i} q_{ij}$. We assume that $q_{ij} > 0$ for $i \neq j$ and denote by $\{q_i\}_1^N$ the stationary distribution of the process. If we take $\{q_i\}$ as the initial distribution for ξ_t it will be a stationary process. We assume that $E\xi_t = (\sum_1^N q_i b_i^+, \sum_1^N q_i b_i^-) = 0$. The correlation function has the form $K_{\lambda\mu}(\tau) = \sum_{i,j=1}^N q_i p(\tau, i, j) b_i^\lambda b_j^\mu$, where $p(\tau, i, j)$ is the transition probabilities matrix, $\tau \geq 0$; $K_{\lambda\mu}(\tau)$ is defined for $\tau < 0$ as in Example 1. Denote $a_{\lambda,\mu} = \int_{-\infty}^{\infty} K_{\lambda\mu}(\tau)\, d\tau$. From the assumption $q_{ij} > 0$ one can deduce that $|a_{\lambda\mu}| < \infty$ and that the strong mixing coefficient $\alpha^*(\tau)$ decreases exponentially fast. Then, from the results of Sections 1 and 2, we conclude that $u^\varepsilon(t,x) \to u(t,x)$ in probability in the space $C_{[0,T]\times[-1,1]}$ when $\varepsilon \downarrow 0$, and that $v^\varepsilon(t,x) = \varepsilon^{-1/2}(u^\varepsilon(t,x) - u(t,x))$ converges weakly in the space $\hat{C}_a$, $a = (1-x^2)^\alpha$, $0 < \alpha < 1$, to the generalized solution $v(t,x)$ of problem (14).

To describe the deviations of $u^\varepsilon(t,x)$ from $u(t,x)$ of order 1, denote by $\lambda = \lambda(\beta)$ the first eigenvalue of the matrix $Q^\beta = (q_{ij} + (a_i\beta_1 + b_i\beta_2)\delta_j^i)$, where $\beta = (\beta_1, \beta_2) \in R^2$, δ_j^i is the Kronecker symbol. The function $\lambda(\beta)$ is twice continuously differentiable and convex. Let $L(\alpha)$ be the Legendre transform of $\lambda(\beta)$. Then the deviations of order 1 in the space $\mathbb{C}_{0T}$ for $\zeta_t^\varepsilon = \int_0^t \xi_{s/\varepsilon}\, ds$ are defined by the action functional $\varepsilon^{-1} S_{0T}^\zeta(\varphi)$, $\varphi(s) = (\varphi_+(s), \varphi_-(s))$, where $S_{0T}^\zeta(\varphi) = \int_0^T L(\dot\varphi_+(s), \dot\varphi_-(s))\, ds$ for absolutely continuous $\varphi(s)$, and $S_{0T}^\zeta(\varphi) = +\infty$ for the rest of $\mathbb{C}_{0T}$. The action functional $S^u(g)$, $g \in C_{[0,T]\times[-1,1]}$, for the family of fields $u^\varepsilon(t,x)$ is given by Theorem 5 with the functional $S^\zeta(\varphi)$ defined above.

As was done in the previous example, one can check that the conditions of Lemma 8 are fulfilled. The corresponding action functional $S^\kappa(\varphi)$ for the family $\eta^\varepsilon = \varepsilon^{-\kappa}\int_0^t \xi_{s/\varepsilon}\, ds$ in this case has the form (see [4], Chapter 7):

$$S(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \sum_{\lambda,\mu} a^{\lambda\mu}\dot\varphi_\lambda(s)\dot\varphi_\mu(s)\, ds, & \text{if } \varphi = (\varphi_+, \varphi_-) \in \mathbb{C}_{0T}, \\ & \varphi_\pm(s) \text{ are absolutely continuous}, \\ +\infty, & \text{for the rest of } \mathbb{C}_{0T}. \end{cases}$$

Thus Theorem 4 gives the action functional for the deviations of $u^\varepsilon(t,x)$ from $u(t,x)$ of order ε^κ, $0 < \kappa < 1/2$.

EXAMPLE 3. Let $\eta_+^0, \eta_+^1, \eta_+^2, \dots$ and $\eta_-^0, \eta_-^1, \eta_-^2, \dots$ be two sequences of independent random variables with common distribution functions $F(x)$ such that $F(-a) = 0$, $F(a) = 1$ for some $a > 0$, $E\eta_\pm^k = 0$, $D\eta_\pm^k = \sigma^2$. Denote by θ the random variable distributed uniformly on the interval $[0,1]$ and independent of $\{\eta_\pm^k\}$. Put

$$\xi_\pm(t) = \eta_\pm^{k+1}, \quad \text{for } t \in [\theta + k, \theta + k + 1), \qquad k = 0, 1, \dots,$$

$$\xi_\pm(t) = \eta_\pm^0, \quad \text{for } t \in [0, \theta).$$

The process $\xi_t = (\xi_+(t), \xi_-(t))$ is stationary, $E\xi_\pm(t) = 0$, $K_{\lambda\mu}(\tau) = E\xi_\lambda(t)\xi_\mu(t+\tau)$, $\lambda, \mu \in (+, -)$, is equal to 0 for $\lambda \neq \mu$, and $K_{++}(\tau) = K_{--}(\tau)$ is the continuous piecewise linear function with vertices in the points $-1, 0, 1$ such that $K_{\lambda\lambda}(0) = \sigma^2$, $K_{\lambda\lambda}(\tau) = 0$ for $|\tau| \geq 1$. If the perturbations of the boundary conditions in problem (2) are equal to $\xi_\pm(t/\varepsilon)$, then $u^\varepsilon(t,x) \to u(t,x)$ when $\varepsilon \downarrow 0$ in the space $C_{[0,T]\times[-1,1]}$. The strong mixing coefficient $\alpha^*(\tau)$ in this case is equal to 0 for $|\tau| > 1$ and thus $\int_0^\infty \tau^{k-1}\alpha^*(\tau)\,d\tau < \infty$ for any $k > 1$. According to Theorem 3 the field $(1/\sqrt{\varepsilon})(u^\varepsilon(t,x) - u(t,x))$ converges weakly as $\varepsilon \downarrow 0$ in the space $\hat{C}_a$, $a = (1-x^2)^\alpha$, $0 < \alpha < 1$, to the Gaussian field $v(t,x)$, which is the generalized solution of problem (14), where $a_{\lambda\mu} = \sigma^2\delta_\mu^\lambda$, δ_μ^λ is the Kronecker symbol.

It is easy to check that the conditions of Lemma 8 are fulfilled in this example. We conclude that the action functional for the family of fields $v_\kappa^\varepsilon(t,x) = \varepsilon^{-\kappa}(u^\varepsilon(t,x) - u(t,x))$ in the space $\hat{C}_a$, $a = (1-x^2)^\alpha$, $0 < \alpha < 1$, is given by Theorem 4, where (see [4], Chapter 7)

$$S_{0T}(\varphi) = \begin{cases} \dfrac{1}{2\sigma^2}\displaystyle\int_0^T \left[\dot\varphi_+^2(s) + \dot\varphi_-^2(s)\right] ds, & \\ & \text{if } \varphi_\pm(s) \text{ are absolutely continuous,} \\ -\infty, & \text{for the rest of } \mathbb{C}_{0T}. \end{cases}$$

To describe the deviations of $u^\varepsilon(t,x)$ from $u(t,x)$ of order 1 when $\varepsilon \downarrow 0$, denote

$$H(\beta) = \ln \int_{-a}^{a} e^{\beta x}\, dF(x), \qquad L(\alpha) = \sup_\beta((\alpha,\beta) - H(\beta)), \qquad \alpha, \beta \in R^2.$$

Then, as follows from the results of Chapter 7 in [4], the action functional for $u^\varepsilon(t,x)$ in $C_{[0,T]\times[-1,1]}$ is given by Theorem 5, where $S^\zeta(\varphi)$, $\varphi = (\varphi_+, \varphi_-) \in \mathbb{C}_{0T}$, is defined as follows:

$$S^\zeta(\varphi) = \begin{cases} \displaystyle\int_0^T [L(\dot\varphi_+^2(s)) + L(\dot\varphi_-^2(s))]\, ds, & \text{if } \varphi_\pm(s) \text{ are absolutely continuous,} \\ +\infty, & \text{for the rest of } \mathbb{C}_{0T}. \end{cases}$$

REFERENCES

[1] Borodin, A. (1977). A limit theorem for solutions of differential equations with random right-hand side. *Theory Probab. Appl.* **22** 482–497.

[2] Faris, W. and Jona-Lasinio, G. (1982). Large deviations for a non-linear heat equation with noise. *J. Phys. A* **15** 3025–3055.

[3] Freidlin, M. (1988). Random perturbations of reaction–diffusion equations: The quasi-deterministic approach. *Trans. Amer. Math. Soc.* **305** 665–697.

[4] Freidlin, M. and Wentzell, A. (1984). *Random Perturbations of Dynamical Systems*. Springer, New York.

[5] Khas'minskii, R. (1974). A limit theorem for solutions of differential equations with random right-hand sides. *Theory Probab. Appl.* **19** 487–505.

[6] PARDOUX, E. (1985). Asymptotic analysis of a semi-linear PDE with wide-band noise disturbances. In *Stochastic Space–Time Models* (L. Arnold and P. Kotelenez, eds.) 227–242. Reidel, Dordrecht.
[7] SOWERS, R. (1992). Large deviations for a reaction–diffusion equation with non-Gaussian perturbations. *Ann. Probab.* **20** 504–537.
[8] ZABCHYK, J. (1989). On large deviations for stochastic evolution equations. *Proc. Sixth IFIP Conference on Stochastic Systems and Optimization, Warsaw, 1988.*

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