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NONLINEAR MARKOV RENEWAL THEORY WITH STATISTICAL APPLICATIONS[1]

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An analogue of the Lai–Siegmund nonlinear renewal theorem is proved for processes of the form $S_n + \xi_n$, where $\{S_n\}$ is a Markov random walk. Specifically, $Y_0, Y_1, \dots$ is a Markov chain with complete separable metric state space; $X_1, X_2, \dots$ is a sequence of random variables such that the distribution of X_i given $\{Y_j, j \geq 0\}$ and $\{X_j, j \neq i\}$ depends only on Y_{i-1} and Y_i; $S_n = X_1 + \cdots + X_n$; and $\{\xi_n\}$ is slowly changing, in a sense to be made precise below. Applications to sequential analysis are given with both countable and uncountable state space.

1. Introduction. Let $Y_0, Y_1, \dots$ be a Markov chain with complete separable metric state space E. Let $X_1, X_2, \dots$ be a sequence of (real) random variables with the property that the conditional distribution of X_i given $\{Y_j, j \geq 0\}$ and $\{X_j, j \neq i\}$ depends only on Y_{i-1} and Y_i. [For example, set $X_i = f(Y_{i-1}, Y_i)$, where $f\colon E \times E \to \mathbb{R}$ is a measurable function.] If $S_0 = 0$,

$$S_n = X_1 + \cdots + X_n, \qquad n \geq 1, \tag{1}$$

and

$$\tau_a = \inf\{n \geq 1 \colon S_n > a\}, \qquad a \geq 0, \tag{2}$$

then the renewal theorem of Kesten (1974), Theorem 1 below, gives conditions under which the excess over the boundary, $S_{\tau_a} - a$, converges in distribution as $a \to \infty$. The process $\{S_n\}$ will be called a Markov random walk.

The main result of this paper is a nonlinear version of the above result. Specifically, in Theorem 3, conditions are given on a sequence of random variables $\{\xi_n\}$ so that $Z_{t_a} - a$ converges in distribution as $a \to \infty$, where $Z_0 = 0$,

$$Z_n = S_n + \xi_n \tag{3}$$

and

$$t_a = \inf\{n \geq 1 \colon Z_n > a\}. \tag{4}$$

As a first step in proving Theorem 3, the convergence in Kesten's theorem is shown to hold uniformly on compact sets; this is Theorem 2 below.

A similar generalization of Blackwell's renewal theorem is proved in Lai and Siegmund (1977). There it is shown that if $X_1, X_2, \dots$ is a sequence of

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independent and identically distributed random variables with finite mean and nonarithmetic distribution, and if $\xi_1, \xi_2, \dots$ are slowly changing, that is,

$$\frac{1}{n}\max\{|\xi_1|, \dots, |\xi_n|\} \to 0 \quad \text{in probability as } n \to \infty$$

and

$$\lim_{\delta \to 0} \sup_{n \geq 1} P\Big\{ \max_{0 \leq k \leq n\delta} |\xi_{n+k} - \xi_n| \geq \varepsilon \Big\} = 0, \qquad \forall\, \varepsilon > 0,$$

then $Z_{t_a} - a$ has the same limiting distribution as $S_{\tau_a} - a$. This result has important applications, especially in sequential analysis; see Woodroofe (1982) and Siegmund (1985). In a recent paper, Woodroofe (1990) extends the Lai–Siegmund result to a wider range of processes.

Two applications of Theorem 3 are given below. The first, in which the state space E is countable, concerns approximating the error probabilities in a sequential probability ratio test with an underlying biased-coin collection scheme. The second application, with uncountable state space, involves a first-order autoregressive model.

2. Statement of principal results. Let (E, d) be a complete separable metric space, and let $\mathscr{E}$ be the Borel sigma algebra on E. Let Q be a probability transition kernel on $(E, \mathscr{E})$, that is, let Q be a function from $E \times \mathscr{E}$ into [0, 1] such that:

for fixed $A \in \mathscr{E}$, $Q(\cdot\,; A)$ is a measurable function and
for fixed $y \in E$, $Q(y; \cdot\,)$ is a probability measure on $(E, \mathscr{E})$.

Let $Y_0, Y_1, \dots$ denote a (homogeneous) Markov chain with transition function Q, so that for $y \in E$, $A \in \mathscr{E}$ and $n, k \geq 1$,

$$Q^k(y; A) = P\{Y_{n+k} \in A | Y_n = y\}.$$

Let $X_1, X_2, \dots$ be a sequence of (real-valued) random variables with the property that for $x, y \in E$ and Borel sets A,

$$P\{X_n \in A | Y_{n-1} = x, Y_n = y, Y_i, i \neq n-1, n, X_j, j \neq n\} = F(A|x, y),$$

where $F(\cdot\,|x, y)$ is a probability distribution independent of n.

Let $\mathbb{N}$ denote the nonnegative integers and let $\mathscr{B}$ denote the Borel sigma algebra of subsets of $\mathbb{R}$. It is assumed that $\{Y_n\}$ and $\{X_n\}$ are defined as coordinate functions on the canonical probability space $(\Omega, \mathscr{F})$, that is, $\Omega = (E \times \mathbb{R})^{\mathbb{N}}$, $\mathscr{F} = (\mathscr{E} \times \mathscr{B})^{\mathbb{N}}$ and if $\omega = \{(\omega_n(1), \omega_n(2))\}_{n \in \mathbb{N}}$, then $Y_n(\omega) = \omega_n(1)$ and $X_{n+1}(\omega) = \omega_n(2)$. For $y \in E$, P_y represents the unique probability measure on $(\Omega, \mathscr{F})$ pertaining to paths with $Y_0 = y$, so that for $n \geq 1$, $A_i \in \mathscr{E}$ and $B_j \in \mathscr{B}$,

$$\begin{aligned} &P_y\{Y_i \in A_i, 0 \leq i \leq n, X_j \in B_j, 1 \leq j \leq n\} \\ &\quad = 1_{A_0}(y) \int_{A_1} Q(y; dy_1) \cdots \int_{A_n} Q(y_{n-1}; dy_n) \int_{B_1} F(dz_1|y, y_1) \\ &\qquad \cdots \int_{B_n} F(dz_n|y_{n-1}, y_n); \end{aligned}$$

see Revuz (1975) for details of this construction. Finally, define S_n and τ_a by (1) and (2) above. In Kesten (1974), it is shown that Y_{τ_a} and $S_{\tau_a} - a$ have a joint limiting distribution as $a \to \infty$ under conditions (K1)–(K4) given below.

Kesten's conditions. Here and below, a.s. stands for a.e.$[P_y]$ for each y. For $f\colon (E \times \mathbb{R})^{\mathbb{N}} \to \mathbb{R}$ and $\delta > 0$, define

$$f^{\delta}(y_0, s_0, y_1, s_1, \dots) = \lim_{m\to\infty} \sup\{f(y'_0, s'_0, y'_1, s'_1, \dots)\colon d(y_i, y'_i) + |s_i - s'_i| < \delta \ \forall\, i \le m\}.$$

This definition will be used in condition K4. Also, for $y \in E$ and $\eta > 0$, let $B(y; \eta) = \{z \in E\colon d(y, z) < \eta\}$.

CONDITION K1. There exists a probability measure φ on $\mathscr{E}$ which is invariant for Q, that is, for all $A \in \mathscr{E}$,

$$\varphi(A) = \int \varphi(dy) Q(y; A).$$

In addition, for all open A with $\varphi(A) > 0$,

$$(5) \qquad P_y\{Y_n \in A \ \exists\ n \ge 0\} = 1 \quad \text{for all } y \in E.$$

CONDITION K2.

$$\int E_y |X_1| \varphi(dy) < \infty,$$

$$\mu := \int E_y(X_1) \varphi(dy) > 0,$$

$$\lim_{n\to\infty} \frac{1}{n} S_n = \mu \quad \text{a.s.}$$

CONDITION K3. There exists a sequence $\{\zeta_\nu\} \subset \mathbb{R}$ such that the group generated by $\{\zeta_\nu\}$ is dense in $\mathbb{R}$ and such that for each ζ_ν and $\delta > 0$, there exists a $z = z(\nu, \delta) \in E$ with the following property: For each $\varepsilon > 0$, there is an $A \in \mathscr{E}$ with $\varphi(A) > 0$, integers m_1, m_2 and an $\eta \in \mathbb{R}$ such that for each $y \in A$,

$$P_y\big\{d(Y_{m_1}, z) < \varepsilon,\ |S_{m_1} - \eta| \le \delta\big\} > 0$$

and

$$P_y\big\{d(Y_{m_2}, z) < \varepsilon,\ |S_{m_2} - \eta - \zeta_\nu| \le \delta\big\} > 0.$$

CONDITION K4. For each $y \in E$ and $\delta > 0$, there is a $b_0 = b_0(y, \delta)$ such that for all product measurable functions $f\colon (E \times \mathbb{R})^{\mathbb{N}} \to \mathbb{R}$ and for all z with $d(y, z) < b_0$,

$$E_y f(Y_0, S_0, Y_1, S_1, \dots) \le E_z f^{\delta}(Y_0, S_0, Y_1, S_1, \dots) + \delta \sup|f|$$

and

$$E_z f(Y_0, S_0, Y_1, S_1, \dots) \le E_y f^\delta(Y_0, S_0, Y_1, S_1, \dots) + \delta \sup|f|.$$

REMARK 1. Note that (5) is only required to hold for φ-positive *open* sets A and φ is required to be a *probability* measure, so that Condition K1 is different than Harris recurrence. Kesten (1974) also proves that Y_{τ_a} and $S_{\tau_a} - a$ have joint limiting distribution K under alternate conditions which require positive Harris recurrence [see Nummelin (1984)] for the Markov chain $\{Y_n\}$, but do not require a continuity condition like Condition K4. Theorem 3 may be modified to hold under this alternate set of conditions; the uniformity result in Theorem 2, however, does not hold in this case, and thus uniform convergence must be added as a hypothesis in Theorem 3. (The alternate conditions may be easier to verify in some cases, but periodicity in the Markov chain $\{Y_n\}$ is disallowed, ruling out examples like that given in Section 4.)

REMARK 2. Condition K4 is trivially true if E is discrete.

REMARK 3. Theorems similar to Theorem 1 are proved in Orey (1961), Jacod (1971), and Athreya, McDonald and Ney (1978). Kesten's version is used here because it does not require that the $\{X_n\}$ process be positive.

The limiting distribution. In order to define the limiting distribution in Theorem 1, it is necessary to introduce a two-sided process $\{Y_n', X_{n+1}'\}_{n \in \mathbb{Z}}$ associated with the original process $\{Y_n, X_{n+1}\}_{n \in \mathbb{N}}$. The process is defined on the probability space $(\Omega', \mathscr{F}', P')$, where $\Omega' = (E \times \mathbb{R})^{\mathbb{Z}}$ and $\mathscr{F}' = (\mathscr{E} \times \mathscr{B})^{\mathbb{Z}}$. For $\omega' \in \Omega'$, let $Y_n'(\omega') = \omega_n'(1)$ and $X_{n+1}'(\omega') = \omega_n'(2)$, where $\omega' = \{\omega'(1), \omega'(2)\}_{n \in \mathbb{Z}}$, and define the probability P' by

$$\begin{aligned} P'\{Y_{k+i}' \in A_i, 0 \le i \le n, X_{k+i}' \in B_i, 1 \le i \le n\} \\ = \int_{A_0} \varphi(y_0) \int_{A_1} Q(y_0; dy_1) \cdots \int_{A_n} Q(y_{n-1}; dy_n) \\ \times \int_{B_1} F(dz_1 | y_0, y_1) \int_{B_2} F(dz_2 | y_1, y_2) \cdots \int_{B_n} F(dz_n | y_{n-1}, y_n) \end{aligned}$$

for $A_i \in \mathscr{E}$, $B_i \in \mathscr{B}$ and $k \in \mathbb{Z}$. This is a standard method of constructing a two-sided process; for details, see Kesten [(1974), page 367] or Doob [(1953), page 456].

Now define

$$S_n' = \begin{cases} \sum_{i=1}^{n} X_n', & \text{if } n > 0; \\ 0, & \text{if } n = 0; \\ -\sum_{i=n+1}^{0} X_n', & \text{if } n < 0, \end{cases}$$

and define the measure ψ on $\mathscr{E}$ by

$$\psi(A) = P'\Big\{\sup_{n<0} S_n' < 0, Y_0' \in A\Big\}.$$

Then the limiting distribution K is given by

$$K(A \times (r,\infty)) = \frac{1}{\mu}\int_E \psi(dz)\int_{E\times(0,\infty)} P_z\{Y_{\tau_0} \in dw, S_{\tau_0} \in d\lambda\}$$
$$\times \int_0^\lambda 1_{(A\times(r,\infty))}(\omega, s)\, ds.$$

THEOREM 1 [Kesten (1974)]. *Assume that Conditions* K1–K4 *are satisfied. Then for any starting point* $y \in E$, $(Y_{\tau_a}, S_{\tau_a} - a)$ *has joint limiting distribution* K. *In particular, for any* $y \in E$ *and* $r > 0$,

$$\lim_{a\to\infty} P_y\{S_{\tau_a} - a > r\} = \frac{1}{\mu}\int \psi(dz)\int_r^\infty (\lambda - r) P_z\{S_{\tau_0} \in d\lambda\}.$$

The following result, a strengthened version of Kesten's theorem, is required for the proof of Theorem 3. The result may also be of independent interest. The proof of Theorem 2 is given in Section 3.

THEOREM 2. *Assume Conditions* K1–K4. *For every* $A \in \mathscr{E}$ *and* $r > 0$ *such that* $A \times (r,\infty)$ *is a continuity set for* K, *every compact set* $C \subset E$, *and each* $\varepsilon > 0$, *there is an* $a_0 < \infty$ *such that for all* $a > a_0$,

$$\sup_{y\in C} |P_y\{Y_{\tau_a} \in A, S_{\tau_a} - a > r\} - K(A \times (r,\infty))| < \varepsilon,$$

that is, the convergence in Theorem 1 *holds uniformly* (*in* y) *on compact sets.*

The main result of this paper is presented next. Let $\xi_1, \xi_2, \dots$ be a sequence of random variables and define Z_n and t_a by (3) and (4). The smoothness conditions on the $\{\xi_n\}$ process are similar to those used by Lai and Siegmund (1977).

CONDITION C1. For each $n \geq 1$, ξ_n is $\mathscr{F}_n$-measurable, where $\mathscr{F}_n = \sigma(Y_0, \dots, Y_n, X_1, \dots, X_n)$.

CONDITION C2. $P_y\{(1/n)\max_{1\leq k\leq n} |\xi_k| > \varepsilon\} \to 0$ as $n \to \infty$ for each $\varepsilon > 0$ and $y \in E$.

CONDITION C3. For every $\varepsilon > 0$ and $y \in E$, there is a $\delta > 0$ such that

$$P_y\Big\{\max_{0\leq k\leq n\delta} |\xi_{n+k} - \xi_n| > \varepsilon\Big\} < \varepsilon \quad \text{for all } n \geq 1.$$

THEOREM 3. *Assume that Conditions* K1–K4 *and* C1–C3 *are satisfied. Then for any starting point* $y \in E$, $(Y_{t_a}, Z_{t_a} - a)$ *has the same limiting distribution as* $(Y_{\tau_a}, S_{\tau_a} - a)$. *In particular, for any* $y \in E$ *and* $r > 0$,

$$\lim_{a \to \infty} P_y\{Z_{t_a} - a > r\} = \frac{1}{\mu} \int \psi(dz) \int_r^{\infty} (\lambda - r) P_z\{S_{\tau_0} \in d\lambda\}.$$

REMARK 4. Condition C2 holds if $(\xi_n/n) \to_{\text{a.s.}} 0$ and Condition C3 holds if $\xi_n \to_{\text{a.s.}} c$ for some finite constant c; see Woodroofe [(1982), page 41].

REMARK 5. If $\{\xi_n\}_{n \geq 1}$ and $\{\zeta_n\}_{n \geq 1}$ satisfy Condition C3 and are tight, then $\{\xi_n \zeta_n\}_{n \geq 1}$ satisfies Condition C3; see Woodroofe [(1982), Lemma 1.4].

REMARK 6. Notice that the limiting distribution K does not depend on the starting point y of the Markov chain.

3. Proofs of Theorems 2 and 3. Throughout this section, Conditions K1–K4 will be in force. For $B \subseteq E$ and $\delta > 0$, let B^{δ} denote the open δ-halo around B, that is, $B^{\delta} = \{z \in E\colon d(z, B) < \delta\}$, and let $B^{-\delta} = \{z \in E\colon d(z, B^c) > \delta\}$, where B^c is the complement of B. Also, let $R_a^0 = S_{\tau_a} - a$, $a \geq 0$. The following lemma will be used in proving Theorem 2.

LEMMA 1. *For each* $y \in E$ *and* $\delta > 0$, *there is a* $b_0 = b_0(y, \delta)$ *such that whenever* $a > \delta$ *and* $z \in B(y; b_0)$,

$$\begin{aligned} P_z\{Y_{\tau_{a-\delta}} \in A^{-\delta}, R_{a-\delta}^0 > r + 2\delta\} - \delta &\leq P_y\{Y_{\tau_a} \in A, R_a^0 > r\} \\ &\leq P_z\{Y_{\tau_{a+\delta}} \in A^{\delta}, R_{a+\delta}^0 > r - 2\delta\} + \delta \end{aligned} \tag{6}$$

for all $A \in \mathscr{E}$ *and* $r > 2\delta$.

PROOF. Fix $y \in E$ and $\delta > 0$ and let $b_0 = b_0(y, \delta)$ be the constant given in Condition K4. Let $A \in \mathscr{E}$, $r > 2\delta$ and $N \in \mathbb{N}$, define the function h_N: $(E \times \mathbb{R})^{\mathbb{N}} \to \mathbb{R}$ by

$$h_N(y_0, s_0, y_1, s_1, \dots) = \sum_{n=1}^{N} 1\{s_k \leq a, k < n, s_n > a + r, y_n \in A\},$$

and note that for $z \in E$, $E_z h_N(Y_0, S_0, Y_1, S_1, \dots) = P_z\{Y_{\tau_a} \in A, \ R_a^0 > r, \tau_a \leq N\}$. It will be shown next that for each $z \in E$,

$$E_z h_N^{\delta}(Y_0, S_0, Y_1, S_1, \dots) \leq P_z\{Y_{\tau_{a+\delta}} \in A^{\delta}, R_{a+\delta}^0 > r - 2\delta, \tau_{a+\delta} \leq N\}. \tag{7}$$

Since h_N does not depend on the values of $y_{N+1}, s_{N+1}, y_{N+2}, s_{N+2}, \dots$,

$$h_N^{\delta}(y_0, s_0, y_1, s_1, \dots) \leq \sum_{n=1}^{N} 1\{s_k \leq a + \delta, k < n, s_n > a + r - \delta, y_n \in A^{\delta}\}.$$

Hence

$$\begin{aligned} &E_z h_N^\delta(Y_0, S_0, Y_1, S_1, \dots) \\ &\quad \le E_z \sum_{n=1}^{N} 1\{S_k \le a + \delta,\, k < n,\, S_n > a + r - \delta,\, Y_n \in A^\delta\} \\ &\quad = P_z\{Y_{\tau_{a+\delta}} \in A^\delta,\, R_{a+\delta}^0 > r - 2\delta,\, \tau_{a+\delta} \le N\}, \end{aligned}$$

establishing (7).

So by Condition K4 and (7), for every $z \in B(y; b_0)$,

$$\begin{aligned} P_y\{Y_{\tau_a} \in A,\, R_a^0 > r,\, \tau_a \le N\} &\le P_z\{Y_{\tau_{a+\delta}} \in A^\delta,\, R_{a+\delta}^0 > r - 2\delta,\, \tau_{a+\delta} \le N\} + \delta \\ &\le P_z\{Y_{\tau_{a+\delta}} \in A^\delta,\, R_{a+\delta}^0 > r - 2\delta\} + \delta. \end{aligned}$$

Now let $N \to \infty$. Since the right-hand side of the above inequality does not depend on N, the second inequality in (6) is proved. The first may be proved similarly. □

PROOF OF THEOREM 2. Fix $A \in \mathscr{E}$ and $r > 0$ such that $A \times (r, \infty)$ is a continuity set for K, fix $\varepsilon > 0$ and let $\delta > 0$ be such that $A^\delta \times (r - 2\delta, \infty)$ and $A^{-\delta} \times (r + 2\delta, \infty)$ are continuity sets for K, $K(A^\delta \times (r - 2\delta, \infty)) - K(A^{-\delta} \times (r + 2\delta, \infty)) < \varepsilon/4$ and $\delta < \varepsilon/12$. Fix a compact set $C \subseteq E$. There exist $y_1, y_2, \dots, y_m \in E$ such that $C \subseteq \bigcup_{i=1}^m B(y_i, b_i)$, where $b_i = b_0(y_i, \delta)$ as in Condition K4. By Theorem 1, for each $i = 1, \dots, m$, there exists an $a_i < \infty$ such that for all $a \ge a_i$,

$$(8) \qquad |P_{yi}\{Y_{\tau_{a+\delta}} \in A^\delta,\, R_{a+\delta}^0 > r - 2\delta\} - K(A^\delta \times (r - 2\delta, \infty))| < \delta$$

and

$$(9) \qquad |P_{yi}\{Y_{\tau_{a-\delta}} \in A^{-\delta},\, R_{a-\delta}^0 > r + 2\delta\} - K(A^{-\delta} \times (r + 2\delta, \infty))| < \delta.$$

Let $a_0 = \max\{a_1, \dots, a_m\}$ and fix $z \in C$. Then $z \in B(y_i, b_i)$ for some $i = 1, 2, \dots, m$, so by Lemma 1, (8) and (9),

$$P_z\{Y_{\tau_a} \in A,\, R_a^0 > r\} \le K(A^\delta \times (r - 2\delta, \infty)) + 2\delta$$

and

$$P_z\{Y_{\tau_a} \in A,\, R_a^0 > r\} \ge K(A^{-\delta} \times (r + 2\delta, \infty)) - 2\delta,$$

whenever $a \ge a_0$. Repeated application of the triangle inequality shows that for such a, $|P_z\{Y_{\tau_a} \in A,\, R_a^0 > r\} - K(A \times (r, \infty))| < \varepsilon$. □

For the rest of this section, all of the assumptions of Theorem 3 are in force. In the following, $[x]$ denotes the greatest integer less than or equal to x. Define

$$\begin{aligned} M_a &= \left[\frac{a}{\mu}\right], \\ R_a &= Z_{t_a} - a, \\ K_a(y; A \times (r, \infty)) &= P_y\{Y_{\tau_a} \in A,\, S_{\tau_a} - a > r\} \end{aligned}$$

for $a > 0$, $y \in E$, $r > 0$ and $A \in \mathscr{E}$.

LEMMA 2. *For all $y \in E$ and $\varepsilon > 0$,*

$$P_y\left\{\left|\frac{t_a}{M_a} - 1\right| > \varepsilon\right\} \to 0 \quad \text{as } a \to \infty.$$

An analogous result, in the iid case, is proved in Woodroofe [(1982), Lemma 4.1]. Lemma 2 may be proved by replacing P by P_y everywhere in Woodroofe's proof.

LEMMA 3. *For all $\varepsilon > 0$ and $y \in E$, there is a compact set $C \in \mathscr{E}$ such that*

$$P_y\{Y_n \in C\} > 1 - \varepsilon \quad \text{for all } n \geq 0. \tag{10}$$

PROOF. Fix $\varepsilon > 0$ and $y \in E$ and let $\varepsilon_k = (\varepsilon/2^{k+1})$ for $k \geq 1$. By Condition K4, for fixed but arbitrary $k \geq 1$, there is a $b_k > 0$ such that for all $A \in \mathscr{E}$ and all $n \geq 1$,

$$Q^n(x; A) \leq Q^n(y; A^{\varepsilon_k}) + \varepsilon_k \quad \text{whenever } x \in B(y; b_k). \tag{11}$$

Let

$$G_k = B(y; b_k)$$

and

$$\delta_k = \varepsilon_k \varphi(G_k).$$

Since E is separable, there is a sequence $A_{k1}, A_{k2}, \dots$ of open $1/k$ spheres which cover E. If i_k is large enough so that $\varphi(\bigcup_{i \leq i_k} A_{ki}) > 1 - \delta_k$, then using (11) and the fact that φ is invariant for Q,

$$\begin{aligned} 1 - \delta_k &\leq \int \varphi(dx) Q^n\left(x; \bigcup_{i \leq i_k} A_{ki}\right) \\ &\leq \varphi(G_k)\left[Q^n\left(y; \bigcup_{i \leq i_k} A_{ki}^{\varepsilon_k}\right) + \varepsilon_k\right] + (1 - \varphi(G_k)), \end{aligned}$$

so that

$$Q^n\left(y; \bigcup_{i \leq i_k} A_{ki}^{\varepsilon_k}\right) \geq \frac{\varphi(G_k) - \delta_k}{\varphi(G_k)} - \varepsilon_k = 1 - 2\varepsilon_k.$$

If C is the closure of the set $\bigcap_{k \geq 1} \bigcup_{i \leq i_k} A_{ki}^{\varepsilon_k}$, then C is totally bounded (and hence compact) and $Q^n(y; C) > 1 - \varepsilon$ for all n. □

The proof of Theorem 3 is modelled after the proof of a nonlinear version of Blackwell's renewal theorem given in Lai and Siegmund (1977); the main novelty here is that the position of the Markov chain $\{Y_n\}$ at the time of conditioning enters the argument. This is where Lemma 3 and Theorem 2 enter the picture: By Lemma 3, the Markov chain may be constrained to lie in a compact set with high probability; Theorem 2 then guarantees uniform convergence to the limiting distribution.

PROOF OF THEOREM 3. Fix $A \in \mathscr{E}$, $r > 0$, $y \in E$ and $\varepsilon < r/2$ such that $A \times (r, \infty)$ and $A \times (r - 2\varepsilon, \infty)$ are continuity sets for K. For this ε and y, let δ be as in Condition C3 and let C be a compact set for which (10) holds. For $a > 0$, define

$$N' = N'(a) = \left[\frac{(1 - (\delta/4))a}{\mu}\right], \qquad N'' = \left[\frac{(1 + (\delta/4))a}{\mu}\right].$$

Observe that for all sufficiently large a,

$$(1 + \delta)N' > N''. \tag{12}$$

Also, by Lemma 2,

$$P_y\{N' < t_a < N''\} \to 1 \quad \text{as } a \to \infty. \tag{13}$$

Below, it will be necessary that $a - Z_{N'} \to \infty$ as $a \to \infty$. Thus, define

$$B_a = \left\{\max_{1 \le k \le N'} Z_k \le a - \sqrt{a}\right\} = \left\{t_{a - \sqrt{a}} > N'\right\}, \qquad a > 0,$$

and note that on B_a, $a - Z_{N'} > \sqrt{a} \to \infty$ as $a \to \infty$. Also, by Lemma 2,

$$P_y\{B_a^c\} \to 0 \quad \text{as } a \to \infty. \tag{14}$$

Then by Theorem 2 and the definition of B_a, there exists an $a_0 < \infty$ such that for all $a > a_0$,

$$|K_{a - Z_{N'} + \varepsilon}(Y_{N'}; A \times (r - 2\varepsilon, \infty)) - K(A \times (r - 2\varepsilon, \infty))| < \varepsilon \quad \text{on } B_a \cap \{Y_{N'} \in C\}. \tag{15}$$

Finally, define

$$D_a = B_a \cap \left\{Y_{N'} \in C,\, N' < t_a < N'',\, \max_{1 \le n \le N'' - N'} |\xi_{N'+n} - \xi_{N'}| \le \varepsilon\right\}.$$

Then for sufficiently large a,

$$P_y\{D_a^c\} < 4\varepsilon \tag{16}$$

by (12), (13), (14), Condition C3 and Lemma 3.

Next, it will be shown that for sufficiently large a,

$$P_y\{Y_{t_a} \in A,\, R_a > r\} \le K(A \times (r - 2\varepsilon, \infty)) + 5\varepsilon. \tag{17}$$

First note that $\{N' < t_a < N'',\, R_a > r\}$ may be rewritten as

$$\{t_a > N',\, S_{N'+k} - S_{N'} \le a - Z_{N'} - (\xi_{N'+k} - \xi_{N'})\ \forall\, k < n,$$
$$S_{N'+n} - S_{N'} > a + r - Z_{N'} - (\xi_{N'+k} - \xi_{N'}) \text{ for some } 1 \le n \le N'' - N'\}.$$

Using this, it is easy to see that

$$\left\{R_a > r,\, N' < t_a \le N'',\, \max_{1 \le j \le N'' - N'} |\xi_{N'+j} - \xi_{N'}| \le \varepsilon\right\}$$
$$\subseteq \{t_a > N',\, S_{N'+k} - S_{N'} \le a - Z_{N'} + \varepsilon\ \forall\, k < n,$$
$$S_{N'+n} - S_{N'} > a + r - Z_{N'} - \varepsilon \text{ for some } n \ge 1\}.$$

Thus

$$P_y\{D_a, Y_{t_a} \in A,\ R_a > r\}$$
$$\leq P_y\{B_a, Y_{N'} \in C, t_a > N', S_{N'+k} - S_{N'} \leq a - Z_{N'} + \varepsilon \ \forall\ k < n,$$
$$S_{N'+n} - S_{N'} > a + r - Z_{N'} - \varepsilon \text{ and } Y_n \in A \text{ for some } n \geq 1\}$$
$$\leq \int_{\{Y_{N'} \in C\} \cap B_a} K_{a - Z_{N'} + \varepsilon}(Y_{N'}; A \times (r - 2\varepsilon, \infty))\, dP_y.$$

That (17) holds for all sufficiently large a now follows from (15), (16) and the inequality $P_y\{Y_{t_a} \in A,\ R_a > r\} \leq P_y\{Y_{t_a} \in A,\ R_a > r, D_a\} + P_y\{D_a^c\}$. Now let $a \to \infty$, $\varepsilon \to 0$ to get

$$\limsup_{a \to \infty} P_y\{Y_{t_a} \in A,\ R_a > r\} \leq K(A \times (r, \infty))$$

for all $r > 0$. A similar argument shows that

$$\liminf_{a \to \infty} P_y\{Y_{t_a} \in A,\ R_a > r\} \geq K(A \times (r, \infty)),$$

completing the proof of Theorem 3. □

4. A biased coin design. To illustrate the use of Theorem 3 when the state space E is discrete, the theorem is applied to a sequential probability ratio test of the difference between the means of two normal populations, with an underlying biased-coin allocation scheme.

Formally, let $X_1^+, X_2^+, \ldots,$ and $X_1^-, X_2^-, \ldots$ denote independent sequences of random variables, where $X_1^+, X_2^+, \ldots$ are i.i.d. Normal$(\theta^+, 1)$, $X_1^-, X_2^-, \ldots$ are i.i.d. Normal$(\theta^-, 1)$ and let $\theta = \theta^+ - \theta^-$. It is assumed, without loss of generality, that $\theta^- = -\theta^+$. Here $X_1^+, X_2^+, \ldots$ and $X_1^-, X_2^-, \ldots$ represent the potential responses of a treatment and control group, respectively. The sequential probability ratio test alluded to above will test whether $\theta^+ \geq \theta^-$, that is, whether $\theta > 0$.

Let $0 < p < \frac{1}{2}$, and let $Y_0, Y_1, \ldots$ be a Markov chain with the following transition mechanism:

$$P\{Y_{n+1} - Y_n = 1 | Y_0, \ldots, Y_n\} = \begin{cases} p, & \text{if } Y_n > 0; \\ \frac{1}{2}, & \text{if } Y_n = 0; \\ 1 - p, & \text{if } Y_n < 0; \end{cases}$$

and

$$P\{Y_{n+1} - Y_n = -1 | Y_0, \ldots, Y_n\} = \begin{cases} 1 - p, & \text{if } Y_n > 0; \\ \frac{1}{2}, & \text{if } Y_n = 0; \\ p, & \text{if } Y_n < 0. \end{cases}$$

For $k \geq 1$, define

$$\delta_k = I\{Y_k - Y_{k-1} = 1\},$$
$$m_k = \sum_{j=1}^{k} \delta_j,$$
$$n_k = k - m_k,$$

and note that $Y_k - Y_0 = m_k - n_k$. In the statistical example, m_k and n_k represent the number of subjects among the first k assigned to the treatment and control groups, respectively. At time $k + 1$, a subject is assigned to the treatment group with probability p, $\frac{1}{2}$ or $(1 - p)$, depending on whether a majority, half or a minority of the first k subjects were assigned to the treatment group. The goal of such a design introduced by Efron, is to achieve balance while minimizing experimenter bias. For more details on this and similar designs, see Efron (1971) and Wei (1978).

Finally, define

$$Z_k = \left(\frac{m_k n_k}{k}\right)\left(\bar{X}^+_{m_k} - \bar{X}^-_{n_k}\right), \qquad k \geq 1,$$
$$T_a = \inf\{k \geq 1: |Z_k| > a\}, \qquad a \geq 1,$$
$$t_a = \inf\{k \geq 1: Z_k > a\}, \qquad a \geq 1,$$

where $\bar{X}^+_{m_k} = (1/m_k)\Sigma_{j=1}^{k}\delta_j X_j^+$ and $\bar{X}^-_{n_k} = (1/n_k)\Sigma_{j=1}^{k}(1 - \delta_j)X_j^-$. Then T_a is the stopping time of an invariant sequential probability ratio test.

It is now shown that Z_k is of the form $S_k + \xi_k$. For $k \geq 1$, define

$$X_k = \frac{1}{2}\delta_k X_k^+ - \frac{1}{2}(1 - \delta_k)X_k^-,$$
$$S_k = X_1 + \cdots + X_k,$$
$$\xi_k = \left(\frac{n_k}{k} - \frac{1}{2}\right)\sum_{j=1}^{k}\left(\delta_j X_j^+ + (1 - \delta_j)X_j^-\right).$$

Then $P\{X_{k+1} \in A | Y_k = x, Y_{k+1} = y, Y_i, i \neq k, k + 1, X_j, j \neq k\}$

$$= \begin{cases} P\{\frac{1}{2}X_k^+ \in A\}, & \text{if } y - x = 1, \\ P\{-\frac{1}{2}X_k^- \in A\}, & \text{if } y - x = -1, \end{cases}$$

and simple algebra shows that $Z_k = S_k + \xi_k$, so Z_k is in the form considered in Theorem 3.

The following lemma will be needed to prove Condition C3 for the $\{\xi_k\}$ process.

LEMMA 4.

$$\frac{Y_k}{(\log k)^2} \to_{a.s.} 0 \quad as\ k \to \infty.$$

PROOF. Write $Y_k = Y_{k-1} + \varepsilon_k$, $k \geq 1$, let $s > 0$ and fix $y \in \mathbb{Z}$. Then for any $k \geq 1$, since $\{Y_k \geq 0\} = \{Y_{k-1} = -1, Y_k = 0\} \cup \{Y_{k-1} = 0, Y_k = 1\} \cup \{Y_{k-1} > 0\}$,

$$\begin{aligned} M_k^y(s) &:= E_y(e^{sY_k}) \\ &\leq 1 + \int_{\{Y_k \geq 0\}} e^{sY_k}\, dP_y \\ &\leq 2 + e^s + \int_{\{Y_{k-1} > 0\}} e^{sY_{k-1}} e^{s\varepsilon_k}\, dP_y \\ &\leq 2 + e^s + (pe^s + (1-p)e^{-s}) M_{k-1}^y(s). \end{aligned}$$

For sufficiently small s, $(pe^s + (1-p)e^{-s}) < 1$. Iterating the above relationship shows that for such s, $M_k^y(s)$ is uniformly bounded in k. Using a similar argument, it may be shown that for some $s < 0$, $M_k^y(s)$ is uniformly bounded in k. Thus, for some $s > 0$, $E_y(e^{s|Y_k|})$ is bounded in k. The conclusion of the lemma now follows from Chebyshev's inequality and the Borel–Cantelli lemma. □

LEMMA 5.

$$\xi_k \to_{a.s.} 0 \quad as\ k \to \infty.$$

PROOF. Use the relation $((n_k/k) - (\frac{1}{2})) = -(Y_k - Y_0)/(2k)$ and algebra to write

$$\begin{aligned} -\xi_k = {} & \frac{Y_k - Y_0}{2k} \sum_{j=1}^{k} \left(\delta_j(X_j^+ - \theta^+) + (1-\delta_j)(X_j^- - \theta^-)\right) \\ & + \frac{Y_k - Y_0}{2k} \sum_{j=1}^{k} \left(\delta_j \theta^+ + (1-\delta_j)\theta^-\right). \end{aligned}$$

The second term on the right is equal to $(2k)^{-1}\theta^+(Y_k - Y_0)^2$, which converges a.s. to 0 by Lemma 4. For the first term, note that $\sum_{j=1}^{k}(\delta_j(X_j^+ - \theta^+) + (1-\delta_j)(X_j^- - \theta^-)) = O((k \log\log k)^{1/2})$ a.s. by the law of the iterated logarithm; combining this with Lemma 4 gives the desired result. □

THEOREM 4. *If $\theta > 0$, then for each starting point $y \in \mathbb{Z}$, $(Y_{t_a}, Z_{t_a} - a)$ has limiting distribution K. In particular, for each $y \in \mathbb{Z}$ and $r > 0$,*

$$\lim_{a \to \infty} P_y\{Z_{t_a} - a > r\} = \frac{4}{\theta} \int \psi(dz) \int_r^{\infty} (\lambda - r) P_z\{S_{\tau_0} \in d\lambda\}.$$

NOTE. In this example, the measure ψ may be described explicitly. Recall from Section 2 that $\psi(A) = P'\{\sup_{n<0} S'_n < 0, Y'_0 \in A\}$, where $\{(Y'_n, X'_{n+1})\}_{n \in \mathbb{Z}}$ is the two-sided stationary process associated with $\{(Y_n, X_{n+1})\}_{n \in \mathbb{N}}$. By conditioning on the entire sequence $\{Y'_n\}_{n \in \mathbb{Z}}$, it may be shown that $\psi(A) = \varphi(A)P\{\inf_{n>0} L_n > 0\}$, where $\{L_n\}$ is a random walk with step distribution

$N(\theta^+/2, \frac{1}{4})$. Further information on the distribution of $\inf_{n>0} L_n$ is given in Feller [(1971), Chapter XII].

PROOF OF THEOREM 4. Direct calculations show that an invariant distribution for $Y_0, Y_1, \ldots$ is given by

$$\varphi(0) = \frac{2p-1}{2(p-1)},$$

$$\varphi(\pm k) = \frac{(1-2p)p^{k-1}}{4(1-p)^{k+1}}, \qquad k = 1, 2, \ldots.$$

Using this, it is easy to show that the integral in Condition K2 is finite and that

$$\mu \equiv \sum_{y=-\infty}^{\infty} E_y(X_1)\varphi(y) = \frac{\theta}{4}.$$

Also, $(S_k/k) \to_{\text{a.s.}} \mu$ by the strong law of large numbers since, by Lemma 4, $(m_k/k) \to_{\text{a.s.}} (\frac{1}{2})$ and $(n_k/k) \to_{\text{a.s.}} (\frac{1}{2})$. Thus Conditions K1 and K2 are verified. Conditions K3, K4, and C1 are clearly true (cf. Remark 2 after the statement of the conditions) and Conditions C2 and C3 follow from Lemma 5 (cf. Remark 4 after Theorem 3). Thus the theorem is proved. □

Notice that the limiting distribution in Theorem 4 depends on the value of θ. This dependence has been suppressed in order to avoid overly messy notation, but will be made explicit when necessary.

Returning to the statistical problem, let $\theta_0 > 0$ and $\theta_1 < 0$ be fixed. Let Q_0 and Q_1 be the unique probability measures under which $\theta = \theta_0$ and $\theta = \theta_1$, respectively, and note that $2(\theta_1 - \theta_0)Z_k$ is the likelihood ratio for testing $\theta = \theta_0$ versus $\theta = \theta_1$. Finally, for $i = 0, 1$, let H_i represent the limiting distribution appearing in Theorem 4, that is, for $r > 0$,

$$1 - H_i(r) = \frac{4}{\theta_i}\int \psi(dz)\int_r^{\infty}(\lambda - r)P_z\{S_{\tau_0} \in d\lambda\},$$

$$\gamma_i = \int_0^{\infty} e^{-t}H_i(dt).$$

The following corollary indicates how Theorem 4 may be used to approximate the error probabilities in the sequential test above. The proof of Corollary 1 may be found in Woodroofe [(1982), Chapter 3].

COROLLARY 1.

$$Q_0\{Z_{T_a} > a\} \sim \gamma_1 e^{-a} \quad and \quad Q_1\{Z_{T_a} < -a\} \sim \gamma_0 e^{-a}$$

as $a \to \infty$.

The special structure of the Z_n process in this example may be used to determine the constants γ_0 and γ_1 appearing in Corollary 1. It may be easily shown that for each $m \geq 1$, as $n \to \infty$,

$$(Z_{n+1} - Z_n, \dots, Z_{n+m} - Z_n) \Rightarrow (T_1, \dots, T_m),$$

where $\{T_n\}_{n \geq 1}$ is a random walk with step distribution $N(3\theta/4, \frac{1}{4})$. This allows the use of Spitzer's identity in obtaining the expressions for γ_0 and γ_1 in Proposition 1. The proof of Proposition 1 is omitted; details of a similar argument may be found in Woodroofe [(1982), pages 24–25].

PROPOSITION 1.

$$\gamma_i = \left(\frac{4}{3\theta_i}\right)\exp\left\{-\sum_{k=1}^{\infty}\frac{1}{k}E_i(e^{-T_k^+})\right\}, \qquad i = 0, 1,$$

where $^+$ *denotes positive part and* E_0, E_1 *denote expectation under* Q_0 *and* Q_1, *respectively.*

5. Autoregressive example. Let $Y_n = \beta Y_{n-1} + \varepsilon_n$, $n \geq 1$, where $\{\varepsilon_n\}_{-\infty < n < \infty}$ is an independent and identically distributed sequence of random variables with $E(\varepsilon_1) = 0$ and $E(\varepsilon_1^2) = 1$ and $|\beta| < 1$. Define Z_n, C_n and D_n by

$$Z_n = \frac{(\sum_{k=1}^n Y_{k-1}Y_k)^2}{2(\sum_{k=1}^n Y_{k-1}^2)} \equiv \frac{C_n^2}{2D_n}.$$

As an application of Theorem 3, conditions on the $\{\varepsilon_n\}$ sequence will be given under which $(Y_{t_a}, Z_{t_a} - a)$ converges in distribution as $a \to \infty$. Notice that $E = \mathbb{R}$ is uncountable in this example.

If it is further assumed that $\varepsilon_1 \sim N(0,1)$, then t_a is the stopping time of the repeated significance test of $\beta = 0$, which rejects $\beta = 0$ if $t_a \leq N_a$ for a suitably chosen constant N_a. Thus the limiting distribution of the overshoot $Z_{t_a} - a$ is useful in approximating the error probabilities of such a test.

It is easily seen that $(C_n/n) \to_{\text{a.s.}} (\beta/(1-\beta^2))$ and $(D_n/n) \to_{\text{a.s.}} (1/(1-\beta^2))$. Observe that Z_n may be written in the form $Z_n = ng(C_n/n, D_n/n)$, where $g(x,y) = x^2/2y$ and then expanded in a Taylor series about $(\beta/(1-\beta^2), 1/(1-\beta^2))$ to obtain

$$Z_n = \beta\sum_{k=1}^n Y_{k-1}Y_k - \frac{\beta^2}{2}\sum_{k=1}^n Y_{k-1}^2 + \xi_n, \tag{18}$$

where ξ_n is the remainder in the Taylor series expansion. After defining $X_k = \beta Y_{k-1}Y_k - (\beta^2/2)Y_{k-1}^2$ and $S_n = X_1 + \cdots + X_n$, it is clear from (18) that Z_n is in the form considered in Theorem 3.

The five Conditions K1–K3 and C1–C2 are relatively easy to verify, so only a few comments will be made on these. Conditions C3 and K4, however, are more difficult to verify, and will be examined presently.

Let $\varphi(A) = P[(\sum_{n=0}^{\infty}\beta^n\varepsilon_{-n}) \in A]$ for $A \in \mathscr{B}$. Then (K1) holds as long as the distribution of ε_1 is nonsingular; see Nummelin [(1984), pages 91, 113]. Now

$E_y(X_1) = \frac{1}{2}\beta^2 y^2$, so

$$\begin{aligned}\mu \equiv \int E_y(X_1)\varphi(dy) &= \frac{1}{2}\beta^2 \int y^2 \varphi(dy)\\ &= \frac{1}{2}\beta^2 E\left[\left(\sum_{n=0}^{\infty} \beta^n \varepsilon_{-n}\right)^2\right]\\ &= \frac{\beta^2}{2(1-\beta^2)}.\end{aligned}$$

It also may easily be shown that $\int E_y|X_1|\varphi(dy) < \infty$.

It will be shown next that $(S_n/n) \to_{\text{a.s.}} \mu$. Since $S_n = Z_n - \xi_n$ and $Z_n/n \to_{\text{a.s.}} (\beta^2/2(1-\beta^2))$, it is only necessary to show that $(\xi_n/n) \to_{\text{a.s.}} 0$. For this, note that from the Taylor expansion mentioned above,

$$\begin{aligned}\frac{\xi_n}{n} &= \frac{1}{2} g_{xx}(c_n^*, d_n^*)\left(\frac{C_n}{n} - \frac{\beta}{1-\beta^2}\right)^2\\ &\quad + g_{xy}(c_n^*, d_n^*)\left(\frac{C_n}{n} - \frac{\beta}{1-\beta^2}\right)\left(\frac{D_n}{n} - \frac{1}{1-\beta^2}\right)\\ &\quad + \frac{1}{2} g_{yy}(c_n^*, d_n^*)\left(\frac{D_n}{n} - \frac{1}{1-\beta^2}\right)^2,\end{aligned} \tag{19}$$

where c_n^* is a point between C_n/n and $\beta/(1-\beta^2)$, d_n^* is a point between D_n/n and $1/(1-\beta^2)$ and g_{xx}, g_{xy} and g_{yy} are the second partial derivatives of g. Now $(\xi_n/n) \to_{\text{a.s.}} 0$ follows from the fact that $(C_n/n) \to_{\text{a.s.}} \beta/(1-\beta^2)$ and $(D_n/n) \to_{\text{a.s.}} 1/(1-\beta^2)$. [Incidentally, by the first remark after Theorem 3, this also verifies Condition C2.]

PROPOSITION 2. *The sequence* $\{\xi_n\}$, *defined in* (18), *satisfies Condition* C3.

PROOF. Using the form of $\{\xi_n\}$ given in (19), it suffices to show that $g_{xx}(c_n^*, d_n^*)$, $g_{xy}(c_n^*, d_n^*)$, $g_{yy}(c_n^*, d_n^*)$, $n^{1/2}((C_n/n) - (\beta/(1-\beta^2)))$ and $n^{1/2}((D_n/n) - (1/(1-\beta^2)))$ satisfy Condition C3 and are tight (cf. Remark 2 after Theorem 3). The first three are easily dispensed with, since they converge a.s. to $g_{xx}(\beta/(1-\beta^2), 1/(1-\beta^2))$, $g_{xy}(\beta/(1-\beta^2), 1/(1-\beta^2))$ and $g_{yy}(\beta/(1-\beta^2), 1/(1-\beta^2))$, respectively.

For the fourth, it is well known that

$$n^{1/2}\left(\frac{C_n}{n} - \frac{\beta}{1-\beta^2}\right) \Rightarrow N(0, 1-\beta^2)$$

[see, e.g., Pollard (1984)], so $\{n^{1/2}((C_n/n) - (\beta/(1-\beta^2)))\}_{n \geq 1}$ is tight.

To verify Condition C3 for this term, note that for $n, k \geq 1$,

$$\left|(n+k)^{-1/2}\left(C_{n+k} - (n+k)\left(\frac{\beta}{1-\beta^2}\right)\right) - n^{-1/2}\left(C_n - n\left(\frac{\beta}{1-\beta^2}\right)\right)\right|$$
$$\leq n^{-1/2}\left|C_{n+k} - C_n - k\left(\frac{\beta}{1-\beta^2}\right)\right|$$
$$+\left[1 - \left(\frac{n}{n+k}\right)^{1/2}\right] n^{-1/2}\left|C_n - n\left(\frac{\beta}{1-\beta^2}\right)\right|.$$

Since $\{n^{1/2}((C_n/n) - (\beta/(1-\beta^2)))\}_{n\geq 1}$ has just been shown to be tight, Condition C3 may easily be verified for the second term on the right. [For details of a similar argument see Woodroofe (1982), Example 1.8.] The first term will need to be rewritten. For this, note that for $n \geq 1$,

$$\sum_{j=1}^{n} Y_j^2 = \beta^2 \sum_{j=1}^{n} Y_{j-1}^2 + 2\beta \sum_{j=1}^{n} Y_{j-1}\varepsilon_j + \sum_{j=1}^{n} \varepsilon_j^2,$$

whence

$$\beta \sum_{j=1}^{n} Y_{j-1}^2 = \left(\frac{\beta}{1-\beta^2}\right)\sum_{j=1}^{n} \varepsilon_j^2 + 2\left(\frac{\beta^2}{1-\beta^2}\right)\sum_{j=1}^{n} Y_{j-1}\varepsilon_j$$
$$+\left(\frac{\beta}{1-\beta^2}\right)(Y_0^2 - Y_n^2).$$

Thus,

$$n^{-1/2}\left|C_{n+k} - C_n - k\left(\frac{\beta}{1-\beta^2}\right)\right|$$
$$= n^{-1/2}\left|\sum_{j=n+1}^{n+k} Y_{j-1}Y_j - k\left(\frac{\beta}{1-\beta^2}\right)\right|$$
$$= n^{-1/2}\left|\sum_{j=n+1}^{n+k} \beta Y_{j-1}^2 + \sum_{j=n+1}^{n+k} Y_{j-1}\varepsilon_j - k\left(\frac{\beta}{1-\beta^2}\right)\right|$$
$$\leq n^{-1/2}\left|\left(\frac{\beta}{1-\beta^2}\right)\sum_{j=n+1}^{n+k}(\varepsilon_j^2 - 1) + 2\left(\frac{\beta^2}{1-\beta^2}\right)\sum_{j=1}^{n} Y_{j-1}\varepsilon_j\right.$$
$$\left.+\left(\frac{\beta}{1-\beta^2}\right)(Y_0^2 - Y_n^2) + \sum_{j=n+1}^{n+k} Y_{j-1}\varepsilon_j\right|.$$

Now the second and fourth terms on the right are martingales, so Condition C3 follows from Doob's inequality; for the first term, Condition C3 follows by Kolmogorov's inequality; while for the third term, Condition C3 is clearly satisfied. Condition C3 and tightness for $\{n^{1/2}((D_n/n) - (1/(1-\beta^2)))\}_{n\geq 1}$ may be verified similarly. □

PROPOSITION 3. *Condition* K4 *is satisfied.*

PROOF. Define

$$Y_k^o = \sum_{j=1}^{k} \beta^{k-j} \varepsilon_k,$$

$$S_n^o = \beta \sum_{k=1}^{n} Y_{k-1}^o Y_k^o - \frac{1}{2}\beta^2 \sum_{k=1}^{n} \left(Y_{k-1}^o\right)^2,$$

$$S_n^{oo} = \sum_{k=1}^{n} \beta^k Y_k^o,$$

$$e_n = \sum_{k=1}^{n} \beta^{2k} \qquad \left(e_\infty = \frac{\beta^2}{1-\beta^2}\right).$$

Then $S_n = S_n^o + (\frac{1}{2}) Y_o^2 e_n + Y_0 S_n^{oo}$ and with

$$S_k(y) = S_k^o + \tfrac{1}{2} y^2 e_k + y S_k^{oo},$$

$$Y_k(y) = \beta^k y + Y_k^o, \qquad y \in \mathbb{R},\ k \geq 1,$$

the P_y distributions of $\{Y_k\}$ and $\{S_k\}$ are equal to the P_0 distributions of $\{W_k(y)\}$ and $\{S_k(y)\}$, respectively. Also note the inequalities (for $y, z \in \mathbb{R}$)

$$|Y_k(y) - Y_k(z)| \leq |y - z|,$$

$$|S_k(y) - S_k(z)| \leq \tfrac{1}{2} e_\infty |y^2 - z^2| + |y - z|\, |S_k^{oo}|.$$

Let f be a product measurable function from $(\mathbb{R} \times \mathbb{R})^{\mathbb{N}}$ into $\mathbb{R}$ and assume without loss of generality that $\sup|f| < \infty$. For notational purposes, define $f_z = f(Y_0(z), S_0(z), Y_1(z), S_1(z), \dots)$ for $z \in \mathbb{R}$ and notice that

$$E_y f(Y_0, S_0, Y_1, S_1, \dots) - E_z f^\delta(Y_0, S_0, Y_1, S_1, \dots) = \int \left(f_y - f_z^\delta\right) dP_0.$$

Now fix $\delta > 0$ and $y \in \mathbb{R}$ and let $C \in \mathbb{R}$ be so large that $P_0\{|S_n^{oo}| > C\ \exists\, n \geq 1\} < \delta/2$. For $z \in \mathbb{R}$, define

$$B_z = \left\{|Y_n(z) - Y_n(y)| + |S_n(z) - S_n(y)| \leq \delta\ \forall\, n \geq 0\right\},$$

and note that

$$P_0(B_z^c) \leq P_0\left\{|y - z| + \frac{1}{2} e_\infty |y^2 - z^2| + |y - z|\, |S_n^{oo}| > \delta\ \exists\, n \geq 1\right\}$$

$$= P_0\left\{|S_n^{oo}| > \frac{\delta - |y - z| - \frac{1}{2} e_\infty |y^2 - z^2|}{|y - z|}\ \exists\, n \geq 1\right\}.$$

If b_0 is chosen so that $((\delta - |y - z| - \frac{1}{2}e_\infty |y^2 - z^2|)/|y - z|) > C$ whenever $|y - z| < b_0$, then $P_0\{B_z^c\} < \delta/2$ whenever $|y - z| < b_0$.

For such z, since $f_y \le f_z^\delta$ on B_z,

$$\int (f_y - f_z^\delta)\, dP_0 = \int_{B_z} (f_y - f_z^\delta)\, dP_0 + \int_{B_z^c} (f_y - f_z^\delta)\, dP_0$$

$$\le 0 + 2P_0\{B_z^c\} \sup|f| \le \delta \sup|f|.$$

This verifies the first part of Condition K4; the second part may be verified similarly. □

THEOREM 5. *If the distribution of ε_1 is nonsingular, then for any starting point $y \in \mathbb{R}$, $(Y_{t_a}, Z_{t_a} - a)$ has limiting distribution K. In particular, for each $y \in \mathbb{R}$ and $r > 0$,*

$$\lim_{a \to \infty} P_y\{Z_{t_a} - a > r\} = \frac{2(1 - \beta^2)}{\beta^2} \int \psi(dz) \int_r^\infty (\lambda - r) P_z\{S_{\tau_0} \in d\lambda\}.$$

REMARK. In Section 4, Spitzer's identity is used to obtain expressions for the error probabilities which are amenable to numerical calculations. Unfortunately, carrying out such a program for the autoregressive example of Section 5 would require an analogue of Spitzer's identity for partial sum processes with dependent summands.

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