The Annals of Probability
1992, Vol. 20, No. 2, 724–745

EQUILIBRIUM BEHAVIOR OF THE SEXUAL REPRODUCTION PROCESS WITH RAPID DIFFUSION[1]

BY CHRIS NOBLE

Oberlin College

The sexual reproduction process is a reaction–diffusion type interacting particle system which exhibits phase transition. In this paper an upper bound is given for the critical value of the reaction rate and the equilibrium density is determined for all noncritical values of the reaction rate.

The method of renormalized bond construction is the basis of our proof. Results for the hydrodynamic limit of the process are employed and some results for the associated nonlinear partial differential equation are developed to build the elements of this construction.

1. Introduction. In this paper we consider a family of continuous time Markov processes parameterized by a birth rate λ and a scale parameter ε. The state at time t is denoted ${}_\varepsilon\xi_t \subset \varepsilon\mathbf{Z}$. ${}_\varepsilon\xi_t(x)$ will denote the indicator of the event $\{x \in {}_\varepsilon\xi_t\}$. We think of the points in ${}_\varepsilon\xi_t$ as being occupied by a single particle; other points in the lattice are vacant. The particles have independent exponentially distributed lifespans. When two particles are in adjacent lattice sites, they produce offspring at rate λ; hence the process is called sexual reproduction. The new particle is sent to either site adjacent to the parent pair with equal probability. If the site of the birth is already occupied, then the new particle coalesces with the particle occupying the site and the birth has no effect on the state of the process. The particles move according to the laws of the simple exclusion process, with "stirrings" at rate ε^{-2} for each pair of adjacent sites. At the time of a stirring the particles (if any) in the affected sites jump to the other site in the pair. Although a stirring does not affect the state of the process when both sites are occupied, it will be useful to think of the two particles as changing places in order to make use of properties of the random walk. The result of the stirring is that each particle performs a simple random walk with steps at rate $2\varepsilon^{-2}$. Two particles are prevented from occupying a site simultaneously, so the random walks are not independent.

With this choice of the lattice spacing and stirring rate, the process simplifies considerably in the limit as ε tends to zero. This is the hydrodynamic limit. The properties of this limit have been studied for a number of models. This particular model is referred to as "binary production" in an overview of hydrodynamic limits ([7]). As a consequence of Theorem 1 in [2], if the

Received August 1990; revised February 1991.

[1]Partially supported by Cornell University and by the Army Research Office through the Mathematical Sciences Institute at Cornell University.

AMS 1980 *subject classifications.* Primary 60K35; secondary 35K57, 60F99.

Key words and phrases. Interacting particle system, hydrodynamic limit, equilibrium, reaction-diffusion equation.

distributions of the initial configurations ${}_{\varepsilon}\xi_t$ satisfy: (i) $\{x \in {}_{\varepsilon}\xi_0\}$ are independent events; and (ii) $P(x \in {}_{\varepsilon}\xi_0) = m_0(x) \in C^3$ with bounded derivatives; then for fixed t, when $x_\varepsilon \to x$,

$$\lim_{\varepsilon \to 0} P\{x_\varepsilon \in {}_{\varepsilon}\xi_t\} = m(t,x), \tag{1.1}$$

where $m(t,x)$ is the unique solution of

$$\frac{\partial m}{\partial t} = \frac{\partial^2 m}{\partial^2 x} - m + \lambda m^2(1-m) \tag{1.2}$$

with $m(0,x) = m_0(x)$.

If ${}_{\varepsilon}\xi_0$ satisfies (i) and (ii) above, then using PM to indicate product measure, we will write

$${}_{\varepsilon}\xi_0 = PM(m_0(x)). \tag{1.3}$$

The second derivative term in (1.2) is the result of the random walk of each particle converging to Brownian motion. The $-m$ appears because the number of deaths is proportional to the population. The number of births is proportional to the number of occupied pairs of adjacent sites with a vacant neighboring site. Because the simple exclusion process has product measure as its equilibrium [9], in the limit as the rate of stirrings tends to infinity, the sites are independent. Thus, the rate at which new particles appear is proportional to the square of the density of occupied sites times the density of vacant sites: $m^2(1-m)$.

In this paper we will use the hydrodynamic limit results in [2] to prove a theorem about the equilibria of the particle system with small positive ε. The most fundamental question we can ask about the asymptotic behavior of the particle system is whether or not there exist nonzero equilibria. Since the state in which all sites are vacant is absorbing, it is an equilibrium for all values of the parameters. We will be concerned with the existence of other equilibria.

We call the process starting with all sites occupied ${}_{\varepsilon}\xi_t^1$. (${}_{\varepsilon}\xi_0^1 = PM(1)$). Since the system is attractive, $P\{A \subset {}_{\varepsilon}\xi_t^1\}$ is decreasing in t for any set A, so it converges. (See Chapter 2 of [8].) There exists a nonzero equilibrium if and only if

$$\lim_{t \to \infty} P\{x \in {}_{\varepsilon}\xi_t^1\} > 0. \tag{1.4}$$

When this is the case we say that the particle system survives. If it does not survive then it dies out.

For each $\varepsilon > 0$, let $\lambda_c(\varepsilon) = \inf\{\lambda\colon {}_{\varepsilon}\xi_t^1 \text{ survives}\}$. The following simple calculation will show that

$$\lambda_c(\varepsilon) \geq 2 \quad \text{for all } \varepsilon > 0. \tag{1.5}$$

The distribution of sites in ${}_{\varepsilon}\xi_t^1$ is translation invariant. When each $a_k \in \{0,1\}$, let

$$\begin{aligned} &\mu_t(a_{-m},\dots,a_{-1},\underline{a_0},a_1,\dots,a_n) \\ &\quad = P\{{}_{\varepsilon}\xi_t^1(x+k\varepsilon) = a_k \text{ for } -m \leq k \leq n\} \end{aligned} \tag{1.6}$$

denote the finite-dimensional distributions at time t. Then

$$(1.7)\quad \begin{aligned} \frac{d}{dt}\mu_1(\underline{1}) &= -\mu_t(\underline{1}) + \frac{\lambda}{2}\mu_t(1,1,\underline{0}) + \frac{\lambda}{2}\mu_t(\underline{0},1,1) \\ &\quad + \frac{1}{2}\varepsilon^{-2}\left[\mu_t(\underline{0},1) - \mu_t(0,\underline{1}) + \mu_t(1,\underline{0}) - \mu_t(\underline{1},0)\right]. \end{aligned}$$

Because the system is symmetric and translation invariant,

$$(1.8)\quad \begin{aligned} &\mu_t(\underline{0},1) = \mu_t(0,\underline{1}) = \mu_t(1,\underline{0}) = \mu_t(\underline{1},0), \\ &\mu_t(1,\underline{1},0) = \mu_t(0,\underline{1},1) = \mu_t(\underline{0},1,1) = \mu_t(1,1,\underline{0}) \quad \text{for all } t. \end{aligned}$$

Clearly

$$\mu_t(\underline{1}) \geq \mu_t(1,\underline{1},0) + \mu_t(0,\underline{1},1) = 2\mu_t(\underline{0},1,1).$$

So

$$(1.9)\quad \mu_t(\underline{0},1,1) = \mu_t(1,1,\underline{0}) \leq \tfrac{1}{2}\mu_t(\underline{1}).$$

Combining (1.7), (1.8) and (1.9), we obtain

$$\frac{d}{dt}\mu_t(\underline{1}) \leq \mu_t(\underline{1})\left[-1 + \frac{\lambda}{2}\right],$$

from which it is clear that if $\lambda < 2$, then

$$0 = \lim_{t\to\infty}\mu_t(\underline{1}) = \lim_{t\to\infty}\mathrm{P}\{x \in {}_\varepsilon\xi_t^1\}.$$

Through a comparison with the contact process, it is not hard to show that $\lambda_c(\varepsilon) < \infty$ for any $\varepsilon > 0$, so the particle system exhibits a phase transition. That is, the existence of a nonzero equilibrium depends on the value of the parameter λ. An immediate consequence of the theorem is

$$(1.10)\quad \limsup_{\varepsilon\downarrow 0}\lambda_c(\varepsilon) \leq 4.5.$$

As the comparison with the contact process gives a uniform upper bound for $\lambda_c(\varepsilon)$ when ε is bounded away from zero, this result implies the existence of a uniform bound on $\lambda_c(\varepsilon)$ for all ε and the existence of nonzero equilibria for sufficiently large values of λ. Our result will also give an estimate for the density of particles at equilibrium.

The key to the asymptotic behavior of the particle system for small ε lies in the behavior of solutions of (1.2). We first examine solutions when $m(0,x)$ is a constant. It is easy to see that in this case the solutions are constant for all time and (1.2) is reduced to

$$(1.11)\quad \frac{dm}{dt} = -m + \lambda m^2(1-m).$$

If $\lambda < 4$, then $dm/dt < 0$ for all positive m and $\lim_{t\to\infty} m(t) = 0$. If $\lambda = 4$, then $\frac{1}{2}$ is a double root of the polynomial in (1.11) and if $\lambda > 4$, then 0, ρ_c, ρ_f will be, in increasing order, the values of m for which $dm/dt = 0$. It is elementary to show that

$$(1.12)\qquad \lim_{t\to\infty} m(t) = \begin{cases} \rho_f, & \text{if } m(0) > \rho_c; \\ \rho_c, & \text{if } m(0) = \rho_c; \\ 0, & \text{if } m(0) < \rho_c. \end{cases}$$

Although the occupation density of the particle system with small ε is approximately $m(t)$ at finite values of t, our theorem will show that these limits do not fit the asymptotic behavior of the particle system. As ε tends to zero, the equilibrium density of the particle system approaches 0 or ρ_f. But in contrast to solutions of (1.11), 0 is the limiting density whenever $\lambda < 4.5$. Furthermore, in contrast with (1.12), for fixed λ the equilibrium density of the particle system is the same for all initial configurations with positive density.

THEOREM. *Fix any $\delta > 0$ and let ${}_\varepsilon\xi_0 = PM(\delta)$.*

(i) *If $\lambda > 4.5$, then*

$$\liminf_{\varepsilon\to 0}\ \liminf_{t\to\infty} \mathrm{P}\{x \in {}_\varepsilon\xi_t\} \ge \rho_f.$$

(ii) *If $\lambda < 4.5$, then*

$$\lim_{\varepsilon\to 0}\ \limsup_{t\to\infty} \mathrm{P}\{x \in {}_\varepsilon\xi_t\} = 0.$$

In [3] results are obtained about the solutions of (1.2) which provide an indication of why 4.5 is the critical value for λ. They demonstrate the existence of traveling wave front solutions of (1.2). These solutions take the form $m(t,x) = w(x - \alpha t)$, where w is a monotone function tending to ρ_f at $-\infty$ and to 0 at $+\infty$. They prove that the wave speed α has the same sign as

$$(1.13)\qquad \int_0^{\rho_f} -x + \lambda x^2(1-x)\,dx.$$

This expression is positive if $\lambda > 4.5$ and negative if $\lambda < 4.5$.

In [5], Fife and MacLeod go on to show that when the wave speed is positive, the solution of (1.2) with initial value $m(0,x) = \rho_f 1_{[-L,L]}(x)$ converges to a diverging pair of wavefronts if L is large enough. Similarly, if the wave speed is negative and the initial value is $m(0,x) = \rho_f[1 - 1_{[-L,L]}(x)]$, then the interval on which the solution is less than δ will expand with time for any $\delta > 0$. In other words, only one of the steady states given in (1.12) is stable under all compact perturbations. We will develop results for the differential equation (1.2) with boundary conditions. Our results are inspired by the work of Fife and MacLeod, but the proofs do not rely upon their results.

To prove part (i) of our theorem we show that when suitable time and space scales are chosen, the process dominates oriented percolation with parameter p close to one. Specifically, we show that if the particle system starts with a large region in which the particle density is nearly the value of the stable state, then with high probability that region will expand and produce two regions with higher density. The proof of part (ii) is similar, but the roles of vacant and occupied sites are interchanged.

The second part of the theorem shows that if $\lambda < 4.5$, then the density of particles in equilibrium tends to 0 as ε tends to 0. Since the steady state of the hydrodynamic limit is zero and the zero equilibrium is absorbing, it seems unlikely that a very low but positive density could be stable when ε is small. This leads us to the following:

CONJECTURE. $\liminf_{\varepsilon \downarrow 0} \lambda_c(\varepsilon) \geq 4.5$.

Associated with this particle system is a dual process which is constructed on the same probability space as the particle system. The dual will be used extensively in our proofs. (See [6] for a general discussion of particle systems and their duals.) In Section 2, we construct several varieties of the process and its dual. Section 3 is devoted to developing useful results about the differential equations. Those results are applied to the particle system and the main Theorem is proved in Section 4.

2. Construction of the process and its dual. The basic idea for the construction uses a graphical representation that allows the construction of several versions of the process with any initial configuration on the same probability space. We construct the process as follows. Independently, for each $x \in \varepsilon\mathbf{Z}$, let δ_x, β_x^r, β_x^l and $\sigma_{x,x-\varepsilon}$ be independent Poisson point processes with rates 1, $\lambda/2$, $\lambda/2$ and ε^{-2}, respectively. These point processes represent the times of deaths, births, and stirrings. For $t \in \delta_x$, ${}_\varepsilon\xi_t(x) = 0$. For $t \in \beta_x^r$, there will be a new particle created at site x if it is vacant and the two sites to its right, $x + \varepsilon$ and $x + 2\varepsilon$, are both occupied. Similarly, births occur at x when $t \in \beta_x^l$, x is vacant and $x - \varepsilon$ and $x - 2\varepsilon$ are both occupied. A stirring occurs at time $t \in \sigma_{x,x-\varepsilon}$; the values of ${}_\varepsilon\xi_t(x)$ and ${}_\varepsilon\xi_t(x-\varepsilon)$ will switch if they are different. The Markov property of the Poisson point process ensures that ${}_\varepsilon\xi_t$ is a Markov process in the space of subsets of $\varepsilon\mathbf{Z}$. We will use Ω for the space of realizations of the point processes and ω to represent an element of this space. In [6], it is proven that the evolution of the process can be computed from this construction for all initial configurations for almost all ω.

The same construction will be used to formulate a dual process of the particle system in the manner of [6]. For fixed $T > 0$ and for each $x \in \varepsilon\mathbf{Z}$, the state of the dual process at time t will be denoted ${}_\varepsilon\tilde{\xi}_t^{x,T}$. The state of the dual process is a finite collection of finite subsets of $\varepsilon\mathbf{Z}$. Each set in the collection represents a potential set of ancestors for a particle at the site x at time T. If B is a set among the collection ${}_\varepsilon\tilde{\xi}_t^{x,T}$ and each site of B is occupied at time $T - t$, then the particles occupying these sites will produce a particle at x at time T. In other words, for every $t \leq T$ and for each realization ω, the process and dual satisfy

$$(2.1) \qquad x \in {}_\varepsilon\xi_T \Leftrightarrow B \subset {}_\varepsilon\xi_{T-t} \quad \text{for some } B \in {}_\varepsilon\tilde{\xi}_t^{x,T}.$$

Thus, for fixed ω, it is precisely the initial configurations in which at least one of the sets in ${}_\varepsilon\tilde{\xi}_T^{x,T}$ is occupied which give rise to the event $\{{}_\varepsilon\tilde{\xi}_t^{x,T} = 1\}$.

The dual is formulated as follows. We start with ${}_\varepsilon\tilde{\xi}_0^{x,T} = \{x\}$. If $T - t \in \beta_y^r$ (respectively, β_y^l), then for each set B in ${}_\varepsilon\tilde{\xi}_t^{x,T}$ with $y \in B$, we construct a set B' which is obtained from B by removing y and adding $\{y + \varepsilon, y + 2\varepsilon\}$ (respectively, $\{y - \varepsilon, y - 2\varepsilon\}$). So if either B or B' is occupied at time $(T - t)^-$, then B will be occupied at time $T - t$. Thus ${}_\varepsilon\tilde{\xi}_{t+}^{x,T}$ is obtained from ${}_\varepsilon\tilde{\xi}_t^{x,T}$ by adding the set B'. If $T - t \in \delta_y$, then ${}_\varepsilon\tilde{\xi}_{t+}^{x,T}$ will be obtained from ${}_\varepsilon\tilde{\xi}_t^{x,T}$ by removing all sets B containing y from ${}_\varepsilon\tilde{\xi}_t^{x,T}$. This is because a particle at site y at time $(T - t)^-$ cannot possibly contribute in producing particles after time $T - t$. If y was a nonessential site for the existence of a particle at site x at time T, then ${}_\varepsilon\tilde{\xi}_t^{x,T}$ will contain a set which does not contain y. Finally, if $y \in B \in {}_\varepsilon\tilde{\xi}_t^{x,T}$ and $T - t \in \sigma_{y,z}$ or $T - t \in \sigma_{z,y}$, then we will replace B with a set B' which is obtained from B by interchanging y and z. Because of the homogeneity of the point processes, the distribution of ${}_\varepsilon\tilde{\xi}_t^{x,T}$ is independent of the choice of T $(\geq t)$.

It is easy to see that as ε tends to zero, the dual process converges weakly to a limiting dual which we denote by $\tilde{\eta}_t^{x,T}$. In this dual, particles have independent Brownian paths; they die at rate 1 and "have parents" at rate λ. In this dual more than one particle at a time can occupy a point in space; so it will not suffice to consider only the locations of particles. The state of this dual at any time will be a finite collection of finite sets of particles, each particle having a location in $\mathbf{R}$. Analogous to the changes occurring in ${}_\varepsilon\tilde{\xi}_t^{x,T}$, at the time of the death of a particle, every set containing that particle is removed from $\tilde{\eta}_t^{x,T}$. At a birth time, for each set containing the "child" particle, a new set is added to $\tilde{\eta}_t^{x,T}$ in which the particle is replaced by the two parent particles. Both parents initially have the same location as the child.

This limiting dual has a special relationship with the solution $m(t,x)$ of (1.2). For $f \in C^3\colon \mathbf{R} \to [0,1]$, with bounded derivatives, we can think of $f(x)$ as the probability that site x is occupied at time 0. Then a set B is occupied if all the points in B are occupied. It follows from the convergence of the duals that when ${}_\varepsilon\xi_0 = PM(f(x))$ and $x_\varepsilon \to x$,

$$(2.2) \qquad \lim_{\varepsilon \downarrow 0} \mathrm{P}\{x_\varepsilon \in {}_\varepsilon\xi_t\} = \mathrm{P}\big\{\text{there exists } B \in \tilde{\eta}_t^{x,t}\colon B \text{ is occupied}\big\}.$$

Theorem 1 of [2] implies that the left-hand side of (2.2) is the solution $m(t,x)$ of (1.2) with initial value $m(0,x) = f(x)$. (The product measure condition on ${}_\varepsilon\xi_0$ satisfies all the hypotheses of that theorem in the most straightforward way.)

Ultimately, the building blocks for the proof of the theorem will be events which occur in a strip of the form $J_\alpha(L) = \{(t,x)\colon |x - \alpha t| \leq L\}$. [We will write simply J for $J_\alpha(L)$ when no confusion will result.] We construct the duals ${}_{J,\varepsilon}\tilde{\xi}_t^{x,T}$ and ${}_J\tilde{\eta}_t^{x,T}$ just as we did ${}_\varepsilon\tilde{\xi}_t^{x,T}$ and $\tilde{\eta}_t^{x,T}$ except that particles stick to the boundary of J. That is, after a particle first exits from J it no longer obeys any of the rules established in the original graphic representation, but moves deterministically, sticking to the border of J. Particles on the border of J never have parents nor die, so that once a set in the dual has such a particle, it and all sets arising out of it will always contain a particle (or site) on the border. The dual may have many particles in a single location on the border of J.

We will construct two versions of the forward time process using ${}_{J,\varepsilon}\tilde{\xi}_t^{x,T}$. The first one will be denoted ${}_{J,\varepsilon}\xi_t^0$. In this process particles die at the time of their first exit from J. In this case the border sites of J are never in ${}_{J,\varepsilon}\xi_t^0$. The relation in (2.1) is satisfied by ${}_{J,\varepsilon}\xi_t^0(x)$ and ${}_{J,\varepsilon}\tilde{\xi}_t^{x,T}$.

In the second version of the process, the sites outside of J are always occupied. So if a site outside of J is involved in a stirring or a birth, the result is the same as any other stirring or birth involving an occupied site. We denote this process ${}_{J,\varepsilon}\xi_t^1$ and once again the relation in (2.1) is satisfied.

The random elements used in the construction of ${}_J\tilde{\eta}_t^{x,T}$, Poisson point processes and Brownian motions, provide a probability measure on the set of duals. Let us denote the set of realizations of the dual ${}_J\tilde{\eta}_t^{x,t}$ by ${}_J\Theta_{x,t}$ and the measure by μ (suppressing the parameters). For $\theta \in {}_J\Theta_{x,t}$, $\hat{\theta}$ will be all the points in the union of the sets in θ.

For any dual $\theta \in {}_J\Theta_{x,t}$, the probability that there exists a set $B \in \theta$ which is occupied at time 0 can be expressed as a polynomial p_θ in the variables $oc(y) = \mathrm{P}\{y \text{ is occupied at time } 0\}$ with y ranging over $\hat{\theta}$. We can see this using the inclusion and exclusion formula. If $\theta = \{A_1, \dots, A_n\}$, then

$$p_\theta = \sum_{i=1}^{n} \Big(\prod_{y \in A_i} oc(y) \Big) - \sum_{i \neq j} \Big(\prod_{y \in A_i \cup A_j} oc(y) \Big) + \cdots + (-1)^{n-1} \prod_{y \in \cup A_i} oc(y).$$

From the definition of p_θ, it is clear that p_θ must be increasing in each variable when the variables have values in $(0, 1)$.

We define the semigroup

$$(2.3) \qquad {}_JT_t f(x) = \int_{{}_J\Theta_{x,t}} p_\theta\{f(y) : y \in \hat{\theta}\} \mu(d\theta).$$

If $m(0, x)$ is the probability that x is occupied, then

$$(2.4) \qquad \begin{aligned} m(t,x) &= {}_JT_t m(0,x) \\ &= \mathrm{P}\{\text{there exists } B \in \tilde{\eta}_t^{x,t} : B \text{ is occupied}\} \end{aligned}$$

satisfies (1.2) with constant boundary conditions on J. It is now easy to prove monotonicity of solutions of (1.2).

LEMMA 2.1. *If $u_i(t, x)$, $i = 1, 2$, satisfy* (1.2) *with constant boundary conditions on J and $u_1(0, x) \geq u_2(0, x)$ for all x, then $u_1(t, x) \geq u_2(t, x)$ for all (t, x).*

PROOF. $p_\theta\{u_1(0, y): y \in \hat{\theta}\} \geq p_\theta\{u_2(0, y): y \in \hat{\theta}\}$ for all θ and the lemma follows from (2.3) and (2.4). □

3. The differential equations. In order to describe the behavior of the particle system on the strips $J_\alpha(L)$, it will be useful to examine the solutions of the limiting differential equation. The central issue will be whether or not these solutions tend to zero as time tends to infinity. This will provide an indication (and ultimately contribute to the proof) of whether or not the particle system survives.

We start by examining solutions of (1.2) which are constant in time. To this end, consider the initial value problem for the ordinary differential equation

$$\frac{d^2u}{dx^2} = f(u), \qquad u(0) = b, \qquad u'(0) = 0, \tag{3.1}$$

where $f(u) = u - \lambda u^2(1-u)$. Let $u_b(x)$ be the unique solution of (3.1). In our first lemma we will show that when λ is large there exist choices of b such that the shape of the solution of (3.1) is a simple hump which is positive on a compact set.

Recall that $\rho_f = \frac{1}{2} + \sqrt{\frac{1}{4} - (1/\lambda)}$ is the largest root of f when $\lambda > 4$.

Lemma 3.1. *If $\lambda > 4.5$, then there exists $\rho_F \in (0, \rho_f)$ such that if $\rho_F < b < \rho_f$, then $u_b(x_0) = 0$ for some $x_0 > 0$.*

Proof. From (3.1),

$$u''_b \cdot u'_b = f(u_b) \cdot u'_b.$$

Integrating both sides we obtain

$$\tfrac{1}{2}[u'_b(x)]^2 = F(u_b(x)) - F(b),$$

where $F(x) = \int_0^x f(t)\,dt = (\lambda/4)x^4 - (\lambda/3)x^3 + \frac{1}{2}x^2$.

If we assume $\rho_c < b < \rho_f$ (ρ_c is the smaller positive root of f), then $f(b) < 0$ and $u''_b(0) < 0$. Since $u'_b(0) = 0$, this implies that for some $\delta > 0$, $u'_b(x) < 0$ when $0 < x < \delta$. So in this interval,

$$u'_b(x) = -\sqrt{2F(u_b(x)) - 2F(b)}\,. \tag{3.2}$$

Now we choose ρ_F such that

Condition 1. $F(x) \geq F(\rho_F)$ when $x \leq \rho_F$.

Condition 2. F is decreasing on (ρ_F, ρ_f).

These criteria are realized by taking $\rho_F \geq q = \frac{2}{3} - \sqrt{\frac{4}{9} - (2/\lambda)}$, the unique solution in $(0, \rho_f)$ of $F(x) = F(0)$. Here we see the reason for the hypothesis $\lambda > 4.5$. Only when $\lambda > 4.5$ can we choose ρ_F to satisfy these conditions.

Now we can show that $u'_b(x) < 0$ for all $x > 0$. If this were not the case, we could let $z = \min\{x > 0\colon u'_b(x) = 0\}$. Since $u'_b(x)$ is continuous, z is well defined unless $u'_b(x) < 0$ for all $x > 0$. Now $u'_b(x) < 0$ if $0 < x < z$, but this implies that $u_b(z) < u_b(0) = b$ and (3.2) implies that $u'_b(z) \neq 0$. Therefore, z is infinity and $u'_b(x) < 0$ for all positive x.

The restriction of u_b to the positive half line is a monotone function, so it has an inverse, ϕ_b, which is defined on some interval of the form $(a, b]$:

$$\phi'_b(u_b(x)) = -\frac{1}{\sqrt{2F(u_b(x)) - 2F(b)}}.$$

Hence

$$\phi_b(a) = \int_b^a \frac{-dy}{\sqrt{2F(y) - 2F(b)}}. \tag{3.3}$$

The integrand is bounded for y outside a neighborhood of b and near b it is $[2f(b)(y-b) + O(y-b)^2]^{-1/2}$, which is integrable. Hence $\phi_b(a) < \infty$ for all $a < b$ and, in particular,

$$x_0 = \phi_b(0) = \int_0^b \frac{dy}{\sqrt{2F(y) - 2F(b)}} < \infty. \tag{3.4}$$ □

Before proceeding with the next lemma, we will impose one more condition on ρ_F:

CONDITION 3. $f'(x) > 0$ when $\rho_F < x < \rho_f$.

We see that this condition is satisfied if $\rho_F \geq r = \frac{1}{3} + \sqrt{\frac{1}{9} - (1/3\lambda)}$, the largest root of f'. Choosing $\rho_F = \max(q, r)$ will satisfy Conditions 1, 2 and 3. In the next lemma we refine our description of the hump solution of (3.1). The hump is a butte with a flat top. As $b \uparrow \rho_f$, the width of the flat top increases without bound, but the width of the sloping sides of the butte is bounded.

LEMMA 3.2. *Given* $0 < \delta < \rho_f - \rho_F$, *for* $b \in [\rho_F + \delta, \rho_f)$:

(i) $\phi_b(b - \delta)$ *is an increasing function of* b;
(ii) $\lim_{b \uparrow \rho_f} \phi_b(b - \delta) = \infty$;
(iii) $\phi_b(0) - \phi_b(b - \delta)$ *is a bounded function of* b.

PROOF. For $\rho_f > b \geq \rho_F + \delta$, from (3.3) we have

$$\begin{aligned} \phi_b(b-\delta) &= \frac{1}{\sqrt{2}} \int_{b-\delta}^{b} \frac{dy}{\sqrt{F(y) - F(b)}} \\ &= \frac{1}{\sqrt{2}} \int_0^{\delta} \frac{dz}{\sqrt{F(b-z) - F(b)}}. \end{aligned} \tag{3.5}$$

So

$$\frac{d\phi_b(b-\delta)}{db} = \frac{1}{2\sqrt{2}}\int_0^{\delta}\frac{f(b)-f(b-z)\,dz}{[F(b-z)-F(b)]^{3/2}} > 0,$$

the last inequality being guaranteed by condition 3. Thus the interval on which the solution is greater than $b-\delta$ is increasing in width as b increases.

Now $F(b-z)-F(b) = -zf(b) + z^2\frac{1}{2}f'(\zeta)$ for some $\zeta \in (b-z, b)$. We let

$$S = \sup_{\rho_F \le \zeta \le \rho_f} f'(\zeta).$$

Then (3.5) yields

$$\phi_b(b-\delta) \ge \frac{1}{\sqrt{2}}\int_0^{\delta}\frac{dz}{\sqrt{-zf(b)+z^2(1/2)S}},$$

and an application of Fatou's lemma gives (ii):

$$\lim_{b\uparrow\rho_f}\phi_b(b-\delta) \ge \int_0^{\delta}\frac{dz}{z\sqrt{S}} = \infty.$$

Finally, using (3.3) again, we have

$$\phi_b(0)-\phi_b(b-\delta) = \frac{1}{\sqrt{2}}\int_0^{b-\delta}\frac{dy}{\sqrt{F(y)-F(b)}}.$$

Conditions 1 and 2 together imply that $F(b-\delta) \le F(y)$ when $y \le b-\delta$ and condition 3 implies that $F(b-\delta)-F(b)$ is minimal when $b=\rho_f$ so

$$\phi_b(0)-\phi_b(b-\delta) \le \frac{1}{\sqrt{2}}\frac{\rho_f-\delta}{\sqrt{F(\rho_f-\delta)-F(\rho_f)}}. \qquad \square$$

The solutions of (3.1) give us "stable" solutions of (1.2) with Dirichlet boundary conditions on $J(L) = [0,\infty)\times[-L, L]$, where $L=\phi_b(0)$ is given by (3.4). By stable we mean that the solution viewed in J is constant in time. For our present choice of J, this simply means that the time derivative is zero. As a consequence of Lemma 3.2, for any given $\delta_0 > 0$, we can choose the initial value b such that the solution $u_b(x)$ of (3.1) satisfies

$$u_b(x) > (\rho_f-\delta_0) \quad \text{when } |x| < L(1-\delta_0). \tag{3.6}$$

The next step will be to demonstrate the existence of "stable" solutions of (1.2) with Dirichlet boundary conditions on $J_\alpha(L) = \{(t,x)\colon |x-\alpha t| \le L\}$ when α is small. In this context, stable means that $u(t, x-\alpha t) = u(0,x)$ for all t. This usage agrees with the previous one in the case $\alpha = 0$. First we observe that if v satisfies

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - \alpha\frac{\partial v}{\partial x} - f(v), \tag{3.7}$$

with Dirichlet boundary conditions on $J_0(L) = [0, \infty) \times [-L, L]$ and

$$u(t, x) = v(t, x + \alpha t), \tag{3.8}$$

then u satisfies (1.2) with Dirichlet boundary conditions on $J_\alpha(L)$. Furthermore, if v is a stable solution on $J_0(L)$, then u is stable on $J_\alpha(L)$. A stable solution, v, satisfies

$$0 = \ddot{v} - \alpha \dot{v} - f(v). \tag{3.9}$$

This equation can be expressed as a system of equations in $\dot{v}$, v and α as

$$\begin{pmatrix} \dot{v} \\ v \\ \alpha \end{pmatrix}' = \begin{pmatrix} \alpha \dot{v} + f(v) \\ \dot{v} \\ 0 \end{pmatrix}. \tag{3.10}$$

We know that (3.10) has a solution for initial conditions $\dot{v}(0) = 0$, $v(0) = b$, $\alpha(0) = 0$ when $\rho_F < b < \rho_f$. A standard result in the theory of differential equations is that solutions of such a system vary continuously with the initial data in the interior of the domain of the solution (see, e.g., Section 32.2 of [1]). Therefore, the system will have a solution uniformly within δ_0 of $u_b(x)$ on $[-\phi(-\delta_0), \phi(-\delta_0)]$ for small enough α. We fix α sufficiently small that this is the case. For simplicity we shift the solution v_α of (3.9) so that the interval over which it is positive is symmetric about 0 and let $L(\alpha, b)$ be the right endpoint of this interval. We call this shifted solution v_1 and note that it is a stable solution of (3.7) with Dirichlet boundary conditions on $J_0(L(\alpha, b))$ and is therefore also a stable solution of (1.2) with Dirichlet boundary conditions on $J_\alpha(L(\alpha, b))$.

Now that we have stable solutions, we would like to do a little better. In the next lemma, we find solutions which are increasing in time.

LEMMA 3.3. *Let $v(t, x)$ be a nontrivial stable solution of* (1.2) *with Dirichlet boundary conditions on $J_\alpha(L)$. Let $M > L$ and let $u(t, x)$ be a solution of the problem on $J_\alpha(M)$ with initial value $u(0, x) = v(0, x)$. Then for all $t \geq 0$ and $s > 0$, we have $u(t + s, x + \alpha(t + s)) > u(t, x + \alpha t)$ whenever $|x| < M$.*

PROOF. To simplify notation, let $I = J_\alpha(L)$ and let $J = J_\alpha(M)$. There is a joint realization of ${}_I\tilde{\eta}^s$ and ${}_J\tilde{\eta}^s$ such that every set $A \in {}_I\tilde{\eta}_t^{x,T}$ which contains no boundary points of I is also in ${}_J\tilde{\eta}_t^{x,T}$. But with positive probability, ${}_J\tilde{\eta}_t^{x,T}$ contains a set entirely in I which is not a set in ${}_I\tilde{\eta}_t^{x,T}$. (This event requires only that a Brownian motion move outside of I and back in.) Since $u(0, x)$ is zero on the boundary of I,

$$\begin{aligned} u(s, x) &= \mathrm{P}\{\text{for some } B \in {}_J\tilde{\eta}_s^{x,s},\ B \text{ is occupied at time } 0\} \\ &= \mathrm{P}\{\text{for some } B \in {}_J\tilde{\eta}_s^{x,s},\ B \subset I \text{ and } B \text{ is occupied at time } 0\} \\ &> \mathrm{P}\{\text{for some } B \in {}_I\tilde{\eta}_s^{x,s},\ B \subset I \text{ and } B \text{ is occupied at time } 0\} \\ &= \mathrm{P}\{\text{for some } B \in {}_I\tilde{\eta}_s^{x,s},\ B \text{ is occupied at time } 0\} \\ &= v(s, x). \end{aligned}$$

Since v is a stable solution, $v(s,x) = v(0, x - \alpha s)$. But u and v have the same initial values, $v(s,x) = u(0, x - \alpha s)$ and this proves the lemma in the case $t = 0$. The generalization is straightforward. Since the particle system is attractive,

$$\begin{aligned} u(t+s, x+\alpha(t+s)) &= {}_J T_t u(s, x+\alpha(t+s)) \\ &> {}_J T_t u(0, x+\alpha t) \\ &= u(t, x+\alpha t). \end{aligned} \qquad \square$$

Furthermore, since $u(t, x+\alpha t)$ is increasing in t, $w(x) = \lim_{t\to\infty} u(t, x + \alpha t)$ is well defined and

$$\begin{aligned} {}_J T_s w(x) &= \int_{{}_J\Theta_{x,s}} p_\theta\Big\{\lim_{t\to\infty} u(t, y+\alpha t)\Big\}\mu(d\theta) \\ &= \int_{{}_J\Theta_{x,s}} \lim_{t\to\infty} p_\theta\{u(t, y+\alpha t)\}\mu(d\theta) \\ &= \lim_{t\to\infty} \int_{{}_J\Theta_{x,s}} p_\theta\{u(t, y+\alpha t)\}\mu(d\theta) \\ &= \lim_{t\to\infty} {}_J T_s u(t, x+\alpha t) \\ &= \lim_{t\to\infty} u(t+s, x+\alpha(t+s)) \\ &= w(x). \end{aligned}$$

Hence w is a stable solution.

So far we have developed the differential equation tools which we will use in proving part (i) of the theorem. In summary, we have constructed a solution to (1.2) which is "tied down" to zero along the sides of $J_\alpha(M)$ and is strictly increasing in time along each ray of slope $1/\alpha$ inside the strip. In Section 4 we will use this result to show that a large isolated clump of particles will, with probability close to 1, produce a clump slightly shifted sometime later. For part (ii) of the theorem we will need a similar result to show that a large, mostly vacant region surrounded by particles will, with high probability, produce a similar zone slightly shifted sometime later. For part (ii) of the theorem we will need a solution of (1.2) which is "tied up" to 1 along the sides of the strip and is decreasing in time. We start by again considering solutions of (3.1).

LEMMA 3.4. *If $\lambda < 4.5$, then there exists a $\rho_F > 0$ such that if $0 < b < \rho_F$, then $u_b(x_1) = 1$ for some $x_1 > 0$.*

PROOF. The proof parallels the proof of Lemma 3.1. The conditions for choosing ρ_F become:

CONDITION 1′. $F(x) \geq F(\rho_F)$ when $x \geq \rho_F$.

CONDITION 2′. $F(x)$ is increasing on $(0, \rho_F)$.

If $\lambda \le 4$, then F is monotone and we can take $\rho_F = 1$. If $4 < \lambda < 4.5$, then these conditions are satisfied by choosing ρ_F less than or equal to q, the unique solution in $(0, \rho_f)$ of $F(x) = F(\rho_f)$. This time $u'_b(x) > 0$ for $x > 0$ and the end result of a set of computations identical to those in the proof of Lemma 3.1 is

$$x_1 = \phi_b(1) = \int_b^1 \frac{dy}{\sqrt{2F(y) - 2F(b)}} < \infty. \tag{3.11}$$ □

The condition corresponding to Condition 3 is:

CONDITION 3′. $f'(x) > 0$ when $0 < x < \rho_F$.

This condition is always satisfied when $\lambda < 3$ and when $3 \le \lambda < 4.5$, it is satisfied by choosing $\rho_F \le r = \frac{1}{3} - \sqrt{\frac{1}{9} - (1/3\lambda)}$, the smallest root of f'.

We fix $\rho_f = \min\{q, r\}$ so that Conditions 1′, 2′ and 3′ are satisfied. Then the solutions to (3.1) when $\lambda < 4.5$ and $b < \rho_F$ look like basins. In the next lemma we show that the width of the flat basin bottom increases to infinity as b tends to 0 while the width of the sloping sides of the basin remains bounded.

LEMMA 3.5. *Given $\delta < \rho_F$, for $b \in (0, \rho_F - \delta]$,*

(i) *$\phi_b(b + \delta)$ is decreasing in b;*
(ii) *$\lim_{b \downarrow 0} \phi_b(b + \delta) = \infty$;*
(iii) *$\phi_b(1) - \phi_b(b + \delta)$ is a bounded function of b.*

PROOF. The computations are identical to those in the proof of Lemma 3.2 with only typographic changes. $b - \delta$ becomes $b + \delta$ and equation (3.11) is used in place of (3.4) so the inequality in (3.5) gets reversed. Throughout the argument, ρ_f is replaced by 0 and $\phi(0)$ by $\phi(1)$. □

Now we have "stable" solutions of (1.2) which are 1 on the boundary of $J(L)$. Given $\delta_0 > 0$, we can choose b such that $u_b(x)$ satisfies

$$u_b(x) < \delta_0 \quad \text{when } |x| \le L(1 - \delta_0). \tag{3.12}$$

As before we obtain a stable solution on $J_\alpha(L(\alpha, b))$ and now this solution, v_1, is equal to 1 on the sides of the strip. We continue by converting Lemma 3.3 to suit our present objectives with:

LEMMA 3.6. *Let $v(t, x)$ be a stable solution of* (1.2) *which is* 1 *outside of $J_\alpha(L)$. Let $M > L$ and let $u(t, x)$ be a solution of* (1.2) *which is* 1 *outside $J_\alpha(M)$ and has initial value $u(0, x) = v(0, x)$. Then for all $t \ge 0$ and for all $s > 0$, $u(t + s, x + \alpha(t + s)) < u(t, x + \alpha t)$ when $|x| < M$.*

PROOF. Let $I = J_\alpha(L)$ and $J = J_\alpha(M)$. We again use the joint realization of ${}_I\tilde{\eta}_t^{x,T}$ and ${}_J\tilde{\eta}_t^{x,T}$. This time we note that for every set A in ${}_J\tilde{\eta}_t^{x,T}$, there is a set B in ${}_I\tilde{\eta}_t^{x,T}$ such that all y in B which are not boundary points of I are also points in A. If $\{A_1, \dots, A_k\}$ are all the sets in ${}_J\tilde{\eta}_t^{x,T}$ corresponding in this way to a particular B in ${}_I\tilde{\eta}_t^{x,T}$, then P{for some $i \le k$, A_i is occupied} $\le$ P{B is occupied}. Furthermore, with positive probability, ${}_I\tilde{\eta}_t^{x,T}$ contains a set B (necessarily containing a point on the boundary of I) for which there is no corresponding set A in ${}_J\tilde{\eta}_t^{x,T}$. Thus,

$$\begin{aligned} u(s,x) &= \mathrm{P}\{\text{for some } A \in {}_J\tilde{\eta}_s^{x,s},\ A \text{ is occupied at time } 0\} \\ &\le \mathrm{P}\{\text{for some } B \in {}_I\tilde{\eta}_s^{x,s} \text{ which has a corresponding } A \in {}_J\tilde{\eta}_s^{x,s}, \\ &\qquad\qquad B \text{ is occupied}\} \\ &< \mathrm{P}\{\text{for some } B \in {}_I\tilde{\eta}_s^{x,s},\ B \text{ is occupied}\} \\ &= v(s,x) \\ &= v(0, x-\alpha s) \\ &= u(0, x-\alpha s). \end{aligned}$$

And

$$\begin{aligned} u(t+s, x+\alpha(t+s)) &= {}_JT_t u(s, x+\alpha(t+s)) \\ &< {}_JT_t u(0, x+\alpha t) \\ &= u(t, x+\alpha t). \qquad \square \end{aligned}$$

4. The distribution of particles. This section will start with the proof of part (i) of the theorem. After proving part (i) we will go back and indicate the necessary modifications to give part (ii), but will avoid duplicating the details of the arguments. For part (i) we use the results from the differential equations to show that if $\lambda > 4.5$, then with high probability, a large cluster of particles will form two such clusters in a fixed amount of time. We will then be able to use a result of Durrett for 1-dependent oriented percolation to show that these clusters percolate on a lattice of appropriate scale. The lower bound on the equilibrium density will follow from the percolation of these dense clusters.

In the following definitions and lemmas we break up the strip $J_\alpha(M) = J$ into substrips in such a way that both the number of lattice sites in each substrip and the number of substrips tend to infinity as ε tends to zero. We show that with high probability the density of particles increases over a fixed span of time in every substrip when ${}_{J,\varepsilon}\xi_0 = PM(v_1)$.

For definiteness we define the strip width $M = (1-\delta_0)^{-1}L(\alpha, b)$. [$L(\alpha, b)$ is defined in the discussion preceding Lemma 3.3.] The fixed time span will be $T \ge M/\alpha$, then the strips $J_\alpha(M)$ and $J_{-\alpha}(M)$ slant in opposite directions and at time T cover intervals which intersect in at most a point.

We will partition J into n (depending on ε) substrips of equal width which we call $J_1, \dots, J_n$. The choice of n will be determined later, but it will satisfy $n \to \infty$ and $n\varepsilon \to 0$ as $\varepsilon \to 0$. Each substrip J_k is covered by a larger strip $\bar{J}_k$ which is the union of the J_i which are a distance not greater than $n^{-1/4} \cdot M$

from J_k. The density of particles in the kth substrip at time T is given by

$$ {}_{\varepsilon}D_T(k) = \frac{\left(\sum_{\{x:(T,x)\in J_k\}}{}_{\varepsilon}\xi_T(x)\right)}{\left(\sum_{\{x:(T,x)\in J_k\}}1\right)}. \tag{4.1} $$

If $v(t,x)$ is the solution of (1.2) with Dirichlet boundary conditions on J and initial value $v_1(x)$ [the stable solution on $J_\alpha(\alpha, b)$], we let $v_2(x) = v(T, x + \alpha T)$, the solution at T translated back over the initial interval of J. Let

$$ \delta = \inf_{|x|\le M\sqrt{1-\delta_0}} |v_2(x) - v_1(x)| $$

be the minimal separation between v_1 and v_2 over an interval containing the support of v_1, and let

$$ b_k = \sup_{\{x:(0,x)\in \bar{J}_k\}} v_1(x) + \delta/4. $$

The $\{b_k\}$ will be target values for ${}_{\varepsilon}D_T(k)$. It will later be important that these densities dominate $v_1(x)$ over a region much wider than the width of the strip J_k. Since the width of $\bar{J}_k$ shrinks to 0 as $\varepsilon \to 0$, for sufficiently small ε,

$$ b_k \cdot 1_{\{x:(0,x)\in \bar{J}_k\}}(x) \le v_2(x) - \delta/2. $$

In the proof of the next lemma we will use the following consequence of Proposition 3.4 of [2]: When ${}_{\varepsilon}\xi_0$ has any product measure, then for each t,

$$ \operatorname{Cov}\{{}_{\varepsilon}\xi_t(x), {}_{\varepsilon}\xi_t(y)\} \to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly in } \{(x,y): x \ne y\}. \tag{4.2} $$

We define

$$ c_\varepsilon(t) = \sup_{x\ne y} \operatorname{Cov}\{{}_{\varepsilon}\xi_t(x), {}_{\varepsilon}\xi_t(y)\}. \tag{4.3} $$

LEMMA 4.1. *There exist constants n_ε such that*

$$ \lim_{\varepsilon\to 0} n_\varepsilon^{-1} + \varepsilon n_\varepsilon = 0 $$

and such that when n_ε is the number of subintervals,

$$ \lim_{\varepsilon\to 0} \mathrm{P}\{{}_{\varepsilon}D_T(k) > b_k \text{ for all } k\} = 1. \tag{4.4} $$

REMARK. The conclusion $\lim_{\varepsilon\to 0} n_\varepsilon^{-1} + \varepsilon n_\varepsilon = 0$ will ensure that the number of subintervals and the number of sites in each subinterval tend to infinity.

PROOF. Let $q = 2M/\varepsilon n_\varepsilon$, the average number of sites in a subinterval. Then

$$\operatorname{Var}\{{}_\varepsilon D_T(k)\} = q^{-2}\left[\sum_{(T,x)\in J_k} \operatorname{Var}{}_\varepsilon\xi_T(x) + \sum_{\substack{(T,x_1),(T,x_2)\in J_k\\ x_1\neq x_2}} \operatorname{Cov}({}_\varepsilon\xi_T(x_1), {}_\varepsilon\xi_T(x_2))\right]. \tag{4.5}$$

Since ${}_\varepsilon\xi_T(x)$ has values in $\{0, 1\}$, it has variance not greater than $\frac{1}{4}$. Using the quantities $c_\varepsilon(t)$ defined in (4.3), we obtain

$$\operatorname{Var}\{{}_\varepsilon D_T(k)\} \le \frac{1}{4q} + c_\varepsilon(T).$$

So by Chebyshev's inequality,

$$\mathrm{P}\{{}_\varepsilon D_T(k) < b_k\} \le \left[\mathrm{E}\{{}_\varepsilon D_T(k)\} - b_k\right]^{-2}\left(\frac{1}{4q} + c_\varepsilon(T)\right).$$

Clearly

$$\mathrm{E}\{{}_\varepsilon D_T(k)\} \ge \inf_{\{x:\,(0,x)\in J_k\}} \mathrm{E}\{{}_\varepsilon\xi_T(x + \alpha T)\}.$$

Applying the result in (2.2), we see that for ε sufficiently small,

$$\begin{aligned} \mathrm{E}\{{}_\varepsilon D_T(k)\} &\ge \inf_{\{x:\,(0,x)\in J_k\}} \left\{v_2(x) - \frac{\delta}{4}\right\} \\ &\ge \inf_{\{x:\,(0,x)\in \bar{J}_k\}} \left\{v_2(x) - \frac{\delta}{4}\right\} \\ &\ge b_k + \frac{\delta}{4}. \end{aligned} \tag{4.6}$$

Hence

$$\mathrm{P}\{{}_\varepsilon D_T(k) < b_k\} \le \left(\frac{\delta}{4}\right)^{-2}\left(\frac{1}{4q} + c_\varepsilon(T)\right);$$

and for ε sufficiently small,

$$\mathrm{P}\{{}_\varepsilon D_T(k) > b_k \text{ for all } k\} \ge 1 - n_\varepsilon\left(\frac{\delta}{4}\right)^{-2}\left(\frac{1}{4q} + c_\varepsilon(T)\right). \tag{4.7}$$

The proof will be complete if we choose n_ε so that the remainder term in (4.7), n_ε^{-1} and $\varepsilon n_\varepsilon$ all tend to zero as ε tends to zero. Recalling that

$q = O(1/\varepsilon n_\varepsilon)$ and that $c_\varepsilon(T) \to 0$ as $\varepsilon \to 0$, we can see that it will suffice to choose n_ε such that n_ε^{-1}, $\varepsilon(n_\varepsilon)^2$ and $n_\varepsilon c_\varepsilon(T)$ all tend to 0. The choice

$$n_\varepsilon = \min\left\{\exp\left(\sqrt{-\log c_\varepsilon(T)}\right), \varepsilon^{-1/3}\right\} \tag{4.8}$$

satisfies these conditions. □

Let G (for good) be the set of configurations satisfying

$$_\varepsilon D_0(k) > b_k \quad \text{for all } k. \tag{4.9}$$

In the next lemma we show that starting the process from any configuration in G cannot be much worse for the survival of the process than having product measure v_1 a short time later. In particular, we will compare $PM(v_1)$ with the configuration at time $\tau = n_\varepsilon^{-1/2}$ of the process starting with a configuration in G. The comparison is made over sets sufficiently small to be likely candidates for sets in $_\varepsilon\tilde{\xi}_T^{x,T}$. Given $\gamma > 0$, choose N such that P{the dual without deaths has cardinality greater than N at time T} $< \gamma$.

LEMMA 4.2.

$$\liminf_{\varepsilon\to 0}\left\{\inf_{_\varepsilon\xi_0\in G}\ \inf_{\{x_1,\dots,x_m\}}\left[E\left\{\prod_{k=1}^m {_\varepsilon\xi_\tau}(x_k)|_\varepsilon\xi_0\right\} - \prod_{k=1}^m v_1(x_k)\right]\right\} \geq 0,$$

where $\inf_{\{x_1,\dots,x_m\}}$ *is taken over all* $m \leq N$ *and all sets of* m *distinct points in* $[-M, M]$.

PROOF. Clearly we can assume that $x_i \in [-L(\alpha, b), L(\alpha, b)]$, for otherwise $v_1(x_i) = 0$. Let $s_1(t), \dots, s_m(t)$ be the positions of m particles moving according to the simple exclusion process with $s_i(0) = x_i$. Then

$$\begin{aligned} &E\left\{\prod_{k=1}^m {_\varepsilon\xi_\tau}(x_k)|_\varepsilon\xi_0\right\} \\ &\quad\geq \mathrm{P}\{s_i(\tau) \in {_\varepsilon\xi_0} \text{ for } i = 1,\dots,m\} \\ &\qquad - \mathrm{P}\{s_i(t) \notin J_\alpha(M) \text{ for some } i \text{ and for some } t < \tau\} \\ &\qquad - \mathrm{P}\{\text{minimum of } m \text{ exponential lifespans is less than } \tau\}. \end{aligned} \tag{4.10}$$

The last term is bounded above by $N\tau$. The second term on the right is uniformly bounded for all $\{x_1, \dots, x_m\}$ by a term of order $N \cdot \exp(-\tau^{-1})$. Let $r_1(t), \dots, r_m(t)$ be independent random walks with $r_i(0) = x_i$. From Proposition 3.4 of [3], if

$$\sup_{\substack{A\in\varepsilon\mathbf{Z}\\ x_1,\dots,x_m}} |P\{s_i(\tau) \in A; i = 1,\dots,m\} - \mathrm{P}\{r_i(\tau) \in A; i = 1,\dots,m\}| = E_\varepsilon,$$

then $\lim_{\varepsilon \to 0} E_\varepsilon = 0$. Thus, for a new E_ε incorporating the error terms in (4.10),

$$\inf_{\{x_1, \dots, x_m\}} \left[\mathrm{E}\left\{ \prod_{k=1}^{m} {}_\varepsilon\xi_\tau(x_k) |_\varepsilon\xi_0 \right\} - \prod_{k=1}^{m} v_1(x_k) \right]$$

$$(4.11) \qquad \geq \inf_{\{x_1, \dots, x_m\}} \left[\mathrm{P}\{r_i(\tau) \in_\varepsilon \xi_0; i = 1, \dots, m\} - \prod_{k=1}^{m} v_1(x_k) - E_\varepsilon \right]$$

$$= \inf_{\{x_1, \dots, x_m\}} \left[\prod_{k=1}^{m} \mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0\} - \prod_{k=1}^{m} v_1(x_k) - E_\varepsilon \right].$$

For $x_k \in J_i$,

$$\mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0\} \geq \mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0 | r_k(\tau) \in \bar{J}_i\} - \mathrm{P}\{r_k(\tau) \notin \bar{J}_i\}.$$

This last probability is uniformly bounded for all x_i by a term of order $n_\varepsilon^{-1/4}$. So for some new E_ε, the right-hand side of (4.10) is bounded below by

$$(4.12) \qquad \inf_{x_1, \dots, x_m} \left[\prod_{k=1}^{m} \mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0 | r_k(\tau) \in \bar{J}_i\} - E_\varepsilon \right].$$

Now

$$\mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0 | r_k(\tau) \in \bar{J}_i\} \geq \inf_{J_j \subset \bar{J}_i} \mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0 | r_k(\tau) \in J_j\},$$

and the conditional distribution of $r_k(\tau)$ in J_j converges uniformly in j to a uniform distribution as $\varepsilon \to 0$. So for any $\delta_0 > 0$, when ε is sufficiently small,

$$(4.13) \qquad \mathrm{P}\{r_k(\tau) \in_\varepsilon \xi_0 | r_k(\tau) \in \bar{J}_i\} \geq b_i - \delta_0 \geq v_1(x_k) + \delta/4 - \delta_0.$$

Thus, for $\delta_0 < \delta/8$, the expression in (4.11) is bounded below by $\delta/8 - E_\varepsilon$, concluding the proof. □

In the next lemma we combine the results of Lemmas 4.1 and 4.2 to show that if ${}_{J,\varepsilon}\xi_0 \in G$, then at time $T' = T + \tau$, with high probability, ${}_{J,\varepsilon}\xi_{T'} \in G$. For convenience, we define $L = \alpha T'$, $J = J_\alpha(L)$ and $D_\varepsilon(k)$ will be the density of particles in ${}_{J,\varepsilon}\xi_{T'}$ in the kth subinterval of $[0, 2L]$. The number of subintervals is n_ε given by (4.8).

LEMMA 4.3. *Given $\delta_0 > 0$, for ε sufficiently small,*

$$\inf_{{}_{J,\varepsilon}\xi_0 \in G} P\{D_\varepsilon(k) > b_k \text{ for all } k |_{J,\varepsilon}\xi_0\} > 1 - \delta_0.$$

PROOF. For $x \in (0, L)$ and ${}_{J,\varepsilon}\xi_0 \in G$,

$$\begin{aligned}
P\{x \in {}_{J,\varepsilon}\xi_{T'} | {}_{J,\varepsilon}\xi_0\} &= P\{\text{for some } B \in {}_{J,\varepsilon}\tilde{\xi}_T^{x,T'},\ B \subset {}_{J,\varepsilon}\xi_\tau | {}_{J,\varepsilon}\xi_0\} \\
&\geq P\Big\{|{}_{J,\varepsilon}\tilde{\xi}_T^{x,T'}| \leq N \text{ and for some } B \in {}_{J,\varepsilon}\tilde{\xi}_T^{x,T'},\ B \subset {}_{J,\varepsilon}\xi_\tau | {}_{J,\varepsilon}\xi_0\Big\} \\
&\geq P\Big\{|{}_{\varepsilon}\tilde{\xi}_T^{x,T'}| \leq N \text{ and for some } B \in {}_{J,\varepsilon}\tilde{\xi}_T^{x,T'}, \\
&\qquad\qquad B \subset {}_{J,\varepsilon}\xi_\tau,\ |{}_{J,\varepsilon}\xi_\tau = PM(v_1)\Big\} \qquad (4.14)\\
&\geq P\{\text{for some } B \in {}_{J,\varepsilon}\tilde{\xi}_T^{x,T'},\ B \subset {}_{J,\varepsilon}\xi_\tau | {}_{J,\varepsilon}\xi_\tau = PM(v_1)\} \\
&\quad - P\Big\{|{}_{\varepsilon}\tilde{\xi}_T^{x,T}| > N\Big\} \\
&\geq P\{x \in {}_{J,\varepsilon}\xi_T | {}_{J,\varepsilon}\xi_0 = PM(v_1)\} - \gamma \\
&\geq v_2(x) - \delta/8 - \gamma.
\end{aligned}$$

If γ is chosen less than $\delta/8$, then the last expression in (4.14) is at least $v_1(x) + 3\delta/4 \geq b_j + \delta/4$ when $x \in J_j$. Thus $E\{{}_{\varepsilon}D_{T'}(j)\} \geq b_j + \delta/4$ and our choice of n_ε and Lemma 4.1 give the desired result. □

We have now shown that with high probability, a good configuration will propagate through the tube J, and create another good configuration, slightly offset in space, at time T'. This will be used in the theorem to show that good configurations percolate on a large scale space-time lattice and finally to estimate the equilibrium density of the process.

THEOREM [Part (i)]. *If* $\lambda > 4.5$, $\delta > 0$ *and* ${}_{\varepsilon}\xi_0 = PM(\delta)$, *then*

$$\liminf_{\varepsilon \to 0} \liminf_{t \to \infty} P\{x \in {}_{\varepsilon}\xi_t\} \geq \rho_f.$$

PROOF. Let $G(m,k)$ be the set of configurations at time kT' such that the jth subinterval in $[(m-1)L, (m+1)L]$ has density at least b_j for $j = 1, \dots, n$. (Here n is as given in Lemma 4.1.) The main idea in the proof is that the set of pairs (m,k) such that the process has a configuration in $G(m,k)$ will dominate a connected cluster in supercritical oriented percolation.

First, if ${}_{\varepsilon}\xi_0 = PM(\delta)$, then for each m, ${}_{\varepsilon}\xi_0 \in G(m,0)$ with positive probability. The Borel–Cantelli lemma implies that with probability 1, ${}_{\varepsilon}\xi_0 \in G(m,0)$ for infinitely many m. Therefore we will assume, without loss of generality, that ${}_{\varepsilon}\xi_0 \in G(0,0)$.

We will be working in the oriented percolation lattice in $\mathbf{Z}^2$, with sites (m,k), where m and k have the same parity and with directed bonds from (m,k) to $(m \pm 1, k+1)$. We choose configurations $\xi^r, \xi^1 \in G(0,0)$ to minimize the conditional probability of $G(1,1)$ [respectively, $G(-1,1)$] given ${}_{\varepsilon}\xi_0 = \xi^r$ (respectively, ${}_{\varepsilon}\xi_0 = \xi^l$). For a given $\omega \in \Omega$, we will let the bond $(m,k) \to (m+1,k+1)$ [respectively, $(m-1,k+1)$] be open if and only if the particle system restricted to the translate of the strip $J_\alpha(L)$ [respectively, $J_{-\alpha}(L)$] with ${}_{\varepsilon}\xi_{kT'} = \xi^r + mL$ (respectively, $\xi^l + mL$) has a configuration in

$G(m+1, k+1)$ [respectively, $G(m-1, k+1)$]. A site (m, k) in this lattice is "lit up" if there is a sequence of open directed bonds connecting $(0,0)$ to (m, k). Let us call this event $S(m, k)$. Because the particle system is attractive, it will stochastically dominate the lit up sites of the lattice in the sense that for any collection of pairs $(m_{1,1}, k_1), (m_{1,2}, k_1), \dots, (m_{2,1}, k_2), \dots,$

$$(4.15)\quad \mathrm{P}\left\{{}_\varepsilon\xi_{k_i T'} \in \bigcap_j G(m_{i,j}, k_i) \text{ for } i = 1, 2, \dots\right\} \geq \mathrm{P}\left\{\bigcap_{i,j} S(m_{i,j}, k_i)\right\}.$$

Whether or not the bond $(m_1, k_1) \to (m_2, k_2)$ is open depends only on the portion of the point processes in the strip connecting $(m_1 L, k_1 L)$ to $(m_2 L, k_2 L)$. Hence the states of two such bonds are independent unless the strips overlap. In other words, $\{(m_1, k_1) \to (m_2, k_2) \text{ is open}\}$ and $\{(m_3, k_3) \to (m_4, k_4) \text{ is open}\}$ are independent events unless $(m_1, k_1) = (m_3, k_3)$ or $(m_2, k_2) = (m_4, k_4)$. The lit up sites dominate a cluster in the 1-dependent oriented site percolation model. If we choose ε sufficiently small to satisfy Lemma 4.3 with $\delta_0 = \frac{1}{2}3^{-36}$, then the results of [4], Section 10, indicate that with positive probability the lit up sites form an infinite cluster. Since an infinite number of sites are lit up at time zero, percolation occurs with probability 1.

To obtain the limiting density, we need another result about supercritical oriented percolation. Let Ω_∞ be the event that the cluster of sites connected to $(0,0)$ by a sequence of open directed bonds is infinite. If the probability of an open bond is $1 - \varepsilon_0$, then

$$(4.16)\quad \lim_{k\to\infty} \mathrm{P}\{(m, k) \text{ is lit up } | \Omega_\infty\} = \nu(\varepsilon_0)$$

and

$$(4.17)\quad \lim_{\varepsilon_0 \downarrow 0} \nu(\varepsilon_0) = 1.$$

For $x \in [(k-1)L, (k+1)L]$,

$$(4.18)\quad \begin{aligned} &\liminf_{n\to\infty} \mathrm{P}\{x \in {}_\varepsilon\xi_{nT'+\tau}\} \\ &\quad \geq \liminf_{n\to\infty} \mathrm{P}\{G(n, k)\} \cdot \mathrm{P}\{x \in {}_\varepsilon\xi_\tau | {}_\varepsilon\xi_0 \in G(0, k)\}. \end{aligned}$$

From Lemma 4.2, the last term is at least $v_1(x - kL) - E_\varepsilon$, where $\lim_{\varepsilon\downarrow 0} E_\varepsilon = 0$. Thus

$$\liminf_{n\to\infty} \mathrm{P}\{x \in {}_\varepsilon\xi_{nT'+\tau}\} \geq \left[\{v_1(x - kL)\} - E_\varepsilon\right]\nu(\varepsilon).$$

But the process is homogeneous in space, so the limit is independent of x and

$$\liminf_{n\to\infty} \mathrm{P}\{x \in {}_\varepsilon\xi_{nT'+\tau}\} \geq \left[\max\{v_1(x)\} - E_\varepsilon\right]\nu(\varepsilon).$$

By the choices of b and α made in Chapter 3, $\max\{v_1(x)\}$ can be made arbitrarily close to ρ_f. So

$$(4.19)\quad \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \mathrm{P}\{x \in {}_\varepsilon\xi_{nT'+\tau}\} \geq \rho_f.$$

Our choice of T (and hence of T') was arbitrary, provided it was sufficiently large. $P\{x \in {}_{\varepsilon}\xi_t\}$ is a continuous function of t. These two facts combined with (4.19) imply by an elementary argument that

$$\liminf_{\varepsilon \to 0} \liminf_{t \to \infty} P\{x \in {}_{\varepsilon}\xi_t\} \geq \rho_f. \tag{4.20}$$ □

THEOREM [Part (ii)]. *If* $\lambda < 4.5$, *then*

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} P\{x \in {}_{\varepsilon}\xi_t\} = 0.$$

REMARK. Because the system is attractive, it will suffice to prove the theorem when ${}_{\varepsilon}\xi_0 = \varepsilon \mathbf{Z}$ $(= PM(1))$.

PROOF. For this part of the theorem, $v_1(x)$ is a stable solution of (1.2) which is 1 on the sides of J. Define $v_2(x)$ and δ as before and let $b_k = \inf_{x \in \bar{J}_k} v_1(x) - \delta/4$. Then corresponding to equation (4.4), we have

$$\lim_{\varepsilon \downarrow 0} P\{{}_{\varepsilon}D_T(k) < b_k \text{ for all } k\} = 1. \tag{4.21}$$

The steps of the proof are exactly the same as before, and the same choice of n_{ε} suffices.

Now of course the good configurations satisfy

$${}_{\varepsilon}D_T(k) < b_k \quad \text{for all } k. \tag{4.22}$$

Corresponding to Lemma 4.2, we have

$$\liminf_{\varepsilon \to 0} \left\{ \inf_{{}_{\varepsilon}\xi_0 \in G} \inf_{\{x_1, \dots, x_m\}} \left[\prod_{k=1}^{m} v_1(x_k) - E\left\{ \prod_{k=1}^{m} {}_{\varepsilon}\xi_{\tau}(x_k) | {}_{\varepsilon}\xi_0 \right\} \right] \right\} > 0.$$

Since $v_1(x_i) = 1$ if $|x_i| > L(\alpha, b)$, we again assume $|x_i| \leq L(\alpha, b)$. Then (4.10) becomes

$$\begin{aligned} &E\left\{ \prod_{k=1}^{m} {}_{\varepsilon}\xi_{\tau}(x_k) | {}_{\varepsilon}\xi_0 \right\} \\ &\quad \leq P\{s_i(\tau) \in {}_{\varepsilon}\xi_0 \text{ for } i = 1, \dots, m\} \\ &\qquad + P\{s_i(t) \notin J_{\alpha}(M) \text{ for some } i \text{ and for some } t < \tau\} \\ &\qquad + P\{\text{minimum of } m \text{ birthtimes is less than } \tau\}. \end{aligned} \tag{4.23}$$

The same computations may be carried out to yield

$$P\{r_k(\tau) \in {}_{\varepsilon}\xi_0 | r_k(\tau) \in \bar{J}_i\} \leq v_1(x_k) - \delta/4 + \delta_0 \tag{4.24}$$

in place of (4.13).

Corresponding to Lemma 4.3 we have for small ε that

$$\inf_{J, {}_{\varepsilon}\xi_0 \in G} P\{D_{\varepsilon}(k) < b_k \text{ for all } k|_{J, {}_{\varepsilon}\xi_0}\} > 1 - \delta_0.$$

The proof follows the same computational steps with inequalities reversed and plusses and minuses interchanged to yield

$$P\{x \in_{J,\varepsilon}\xi_{T'}|_{J,\varepsilon}\xi_0\} \le b_j - \delta/4$$

when $x \in J_j$.

The final steps of the proof use the same lattice and the percolation of the sites $G(m, k)$. The inequality in (4.18) gets reversed. b and α can be chosen so that $\min\{v_1(x)\}$ is arbitrarily close to 0 and (4.20) becomes

$$\limsup_{\varepsilon \to 0} \limsup_{t \to \infty} P\{x \in_\varepsilon \xi_t\} \le 0.$$

This completes the proof of the theorem. □

Acknowledgments. This work appears in my dissertation at Cornell University, 1989. I thank my advisor Rick Durrett for the problem and for much helpful advice, including suggesting the comparison with 1-dependent oriented percolation.

REFERENCES

[1] ARNOLD, V. I. (1973). *Ordinary Differential Equations*. MIT Press.

[2] DEMASI, A., FERRARI, P. A. and LEBOWITZ, J. L. (1986). Reaction–diffusion equations for interacting particle systems. *J. Statist. Phys.* **44** 589–644.

[3] DEMASI, A., IANIRO, N., PELLEGRINOTTI, A. and PRESUTTI, E. (1984). A survey of the hydrodynamical behavior of many particle systems. *Stud. Statist. Mech.* **2** 123–299.

[4] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040.

[5] FIFE, P. C. and MACLEOD, J. B. (1977). The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Rational Mech. Anal.* **65** 335–361.

[6] GRAY, L. (1986). Duality for general attractive spin systems with applications in one dimension. *Ann. Probab.* **14** 371–396.

[7] LEBOWITZ, J. L., PRESUTTI, E. and SPOHN, H. (1988). Microscopic models of hydrodynamic behavior. *J. Statist. Phys.* **51** 841–862.

[8] LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

[9] SPITZER, F. (1970). Interaction of Markov processes. *Adv. in Math.* **5** 246–290.

DEPARTMENT OF MATHEMATICS
LAWRENCE UNIVERSITY
APPLETON, WISCONSIN 54912