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THE A.S. BEHAVIOR OF THE WEIGHTED EMPIRICAL PROCESS AND THE LIL FOR THE WEIGHTED TAIL EMPIRICAL PROCESS

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The tail empirical process is defined to be for each $n \in \mathbb{N}$, $w_n(t) = (n/k_n)^{1/2}\alpha_n(tk_n/n)$, $0 \le t \le 1$, where α_n is the empirical process based on the first n of a sequence of independent uniform $(0,1)$ random variables and $\{k_n\}_{n=1}^{\infty}$ is a sequence of positive numbers with $k_n/n \to 0$ and $k_n \to \infty$. In this paper a complete description of the almost sure behavior of the weighted empirical process $a_n\alpha_n/q$, where q is a weight function and $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers, is established as well as a characterization of the law of the iterated logarithm behavior of the weighted tail empirical process w_n/q, provided $k_n/\log\log n \to \infty$. These results unify and generalize several results in the literature. Moreover, a characterization of the central limit theorem behavior of w_n/q is presented. That result is applied to the construction of asymptotic confidence bands for intermediate quantiles from an arbitrary continuous distribution.

1. Introduction. Let $U_1, U_2, \ldots$ be a sequence of independent uniform $(0,1)$ random variables and for each $n \in \mathbb{N}$, let

$$F_n(t) = n^{-1}\sum_{i=1}^{n} 1_{[0,t]}(U_i), \qquad 0 \le t \le 1,$$

be the empirical distribution function based on the first n of these random variables. The uniform empirical process will be written as

$$\alpha_n(t) = n^{1/2}\big(F_n(t) - t\big), \qquad 0 \le t \le 1.$$

Let $\{k_n\}_{n=1}^{\infty}$ denote a sequence of numbers such that for all $n \in \mathbb{N}$, $0 < k_n < n$, $k_n/n \to 0$ and $k_n \to \infty$. Now define the tail empirical process in terms of the sequence $\{k_n\}_{n=1}^{\infty}$, by

$$w_n(t) = (n/k_n)^{1/2}\alpha_n(tk_n/n), \qquad 0 \le t \le 1.$$

Let $\mathbb{Q}$ be the class of weight functions defined by

$$\mathbb{Q} = \{q\colon [0,1] \to [0,\infty)\colon q \text{ continuous and strictly increasing}\}.$$

The aim of this paper is threefold. First, we will present a complete description of the almost sure behavior of the weighted empirical process

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$a_n\alpha_n/q$, where $q \in \mathbb{Q}$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers. Second, under the additional assumption that $k_n/\log\log n \to \infty$, we will characterize the almost sure behavior of $(2\log\log n)^{-1/2}w_n/q$, $q \in \mathbb{Q}$; that is, we will obtain necessary and sufficient conditions for the law of the iterated logarithm (LIL) for the weighted tail empirical process. Both results, which may be considered as final, unify and generalize several results in the literature. The method of proof, which is similar for both results, is classical: it consists of using a sharp exponential inequality in conjunction with a maximal inequality. This method is new, however, in this type of theorem. Third, necessary and sufficient conditions for the central limit theorem (CLT) for w_n/q, $q \in \mathbb{Q}$, are presented. That result easily yields asymptotic confidence bands for intermediate quantiles from an arbitrary continuous distribution.

The statement and proof of the result for $a_n\alpha_n/q$ are deferred to Section 2. The strong and weak limiting behavior of w_n/q are presented in Sections 3 and 4, respectively. All the results are followed by a thorough discussion.

2. The a.s. behavior of the weighted empirical process. In this section the subject of our study is the process $a_n\alpha_n/q$. Introduce a subclass of $\mathbb{Q}$ by

$$\mathbb{Q}_0 = \{q \in Q: q/I^{1/2} \text{ nonincreasing}\},$$

where I denotes the identity function. Furthermore, write

$$A(q) = \int_0^{e^{-e}} 1/(q^2(t)\log\log(1/t))\,dt$$

and $\|f\| = \sup_{0<t<1}|f(t)|$ for a function f: $(0,1) \to \mathbb{R}$. The following law of the iterated logarithm is well known.

FACT 1 [James (1975)]. Let $q \in \mathbb{Q}_0$. If $A(q) < \infty$, then almost surely the sequence $\{(2\log\log n)^{-1/2}\alpha_n/q\}_{n=1}^{\infty}$ is relatively compact in $\mathbb{B}$ with the set of limit points equal to $\{f/q: f \in \mathbb{F}\}$. Conversely, if $A(q) = \infty$, then

$$\limsup_{n\to\infty}(2\log\log n)^{-1/2}\|\alpha_n/q\| = \infty \quad \text{a.s.}$$

(Here $\mathbb{B}$ denotes the space of bounded real-valued functions on $[0,1]$ with the sup-norm and $\mathbb{F}$ denotes the set of absolutely continuous functions $f \in \mathbb{B}$ such that $f(0) = f(1) = 0$ and $\int_0^1 (f'(t))^2\,dt \le 1$.)

Fact 1 gives a complete description of the almost sure behavior of $a_n\alpha_n/q$ as long as $A(q) < \infty$: we immediately have that $a_n(\log\log n)^{1/2} \to 0$ implies $\lim_{n\to\infty} a_n\|\alpha_n/q\| = 0$ a.s., whereas $a_n(\log\log n)^{1/2} \to \infty$ implies $\limsup_{n\to\infty} a_n\|\alpha_n/q\| = \infty$ a.s. So for the present investigation we may and will restrict ourselves to the case $A(q) = \infty$ and because of the converse part of James' result we will assume in addition that $a_n(\log\log n)^{1/2} \to 0$.

Now we are prepared to present the results of this section. For notational convenience we set $b_n = q^{-1}(a_n/n^{1/2})$ and work with the sequence $\{b_n\}_{n=1}^{\infty}$.

Define the following conditions:

$$n^{1/2}q(b_n)(\log\log n)^{1/2} \to 0, \qquad n \to \infty, \tag{2.1}$$

$$nb_n \downarrow, \tag{2.2}$$

$$n^c b_n \uparrow \quad \text{for some (large) } c > 1, \tag{2.3}$$

$$\text{(2.4) or } \begin{cases} \text{(a)} & q/I^{1/2} \uparrow \text{ and } q/I \downarrow \\ \text{(b)} & q/I^{1/2} \downarrow \text{ and } q/I^{\eta} \uparrow \text{ for some } \eta > 0. \end{cases}$$

THEOREM 1. *Assume* $q \in \mathbb{Q}$ *and* $A(q) = \infty$.

(i) *If* $\sum b_n = \infty$ *and* (2.4) *holds, then*

$$\limsup_{n\to\infty} n^{1/2}q(b_n)\|\alpha_n/q\| = \infty \quad a.s. \tag{2.5}$$

(ii) *If* $\sum b_n < \infty$ *and* (2.1)–(2.4) *hold, then*

$$\lim_{n\to\infty} n^{1/2}q(b_n)\|\alpha_n/q\| = 0 \quad a.s. \tag{2.6}$$

COROLLARY. *If* $q \in \mathbb{Q}$, $A(q) = \infty$, (2.1), *with* $b_n = (n(\log n)^{1+\varepsilon})^{-1}$, *holds for any* $\varepsilon > 0$ *and* (2.4) *holds, then*

$$\limsup_{n\to\infty} \frac{-\log\left(nq^{-1}\left(\dfrac{1}{n^{1/2}\|\alpha_n/q\|}\right)\right)}{\log\log n} = 1 \quad a.s.$$

DISCUSSION. Theorem 1 has been proved in the literature for various special choices of the sequence $\{b_n\}_{n=1}^{\infty}$ or the weight function q. Csáki (1975, 1982) proved the result for $q = I^{1/2}$, Shorack and Wellner (1978) for $q = I$ and Mason (1981) for $q = I^{\alpha}$, $1/2 < \alpha < 1$. Moreover, Lai (1974) and Wellner (1977, 1978) established the theorem for $a_n = n^{-1/2}$, Andersen, Giné and Zinn (1988) for $a_n = n^{\beta-1/2}$, $0 < \beta < 1/2$, and Einmahl and Mason (1989) for $a_n = n^{\beta-1/2}/(\log\log n)^{\beta}$, $0 < \beta < 1/2$. However, the general result, that is, leaving both $\{b_n\}_{n=1}^{\infty}$ and q arbitrary, seemed inaccessible. Note that it is pointless to consider the process $a_n\alpha_n/q$ itself instead of $a_n\|\alpha_n/q\|$, since it follows from the theorem that no proper standardization of the process is possible. It is also pointless to consider weight functions which are "infinitely smaller" than the identity function I, since $\lim_{t\downarrow 0} q(t)/t = 0$ trivially implies $\|\alpha_n/q\| = \infty$. Finally, note that the most interesting and hardest part of the theorem concerns the weight functions satisfying (2.4b) and $A(q) = \infty$. These weight functions bridge the gap between James' (1975) LIL and Csáki's (1975, 1982) theorem. The corollary is the appropriate generalization of Corollary 3.2 in Csáki (1975).

Before we present the proof of Theorem 1 we state, for convenient reference later on, a number of facts.

FACT 2 [Csáki (1975, 1982)]. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, with $\sum b_n < \infty$ and $nb_n \downarrow$, then

$$\lim_{n\to\infty} (nb_n)^{1/2}\|\alpha_n/I^{1/2}\| = 0 \quad \text{a.s.}$$

FACT 3 [Shorack and Wellner (1986), page 446]. Let $0 < a \le b/2 < b \le 1/2$. Then for $q \in \mathbb{Q}$,

$$P\Big(\sup_{a\le t\le b} |\alpha_n(t)|/q(t) \ge \lambda\Big) \le 8\int_{a/2}^{b} \frac{1}{t} \exp\left(\frac{-\lambda^2}{128}\frac{q^2(t)}{t}\psi\left(\frac{\lambda q(t)}{n^{1/2}t}\right)\right) dt, \qquad \lambda > 0.$$

(Here ψ: $[0,\infty) \to (0,\infty)$ is defined by

$$\psi(x) = 2x^{-2}\{(1+x)\log(1+x) - x\};$$

we will use the properties: (a) $\psi \downarrow$, (b) $x \ge 10$ implies $\psi(x) \ge (\log x)/x$.)

FACT 4 [cf. James (1975) and Einmahl (1987), page 20]. Write $n_k = 2^k$, $k \in \mathbb{N}$; let $0 < a < b \le 1/2$ and $q \in \mathbb{Q}_0$. Then we have for all $k \in \mathbb{N}$ and $\lambda > (8b)^{1/2}/q(b)$:

$$P\Big(\max_{n_k < n \le n_{k+1}} \sup_{a\le t\le b} |\alpha_n(t)|/q(t) \ge \lambda\Big) \le 2P\Big(\sup_{a\le t\le b} |\alpha_{n_{k+1}}(t)|/q(t) \ge \lambda/8^{1/2}\Big).$$

PROOF OF THEOREM 1. (i) Assume $\sum b_n = \infty$ and (2.4). We have for any $\delta > 0$, $P(U_n \le \delta b_n \text{ i.o.}) = 1$ and

$$n^{1/2}q(b_n)\|\alpha_n/q\| \ge n^{1/2}q(b_n)/\big(2n^{1/2}q(U_n)\big).$$

Hence

$$\limsup_{n\to\infty} n^{1/2}q(b_n)\|\alpha_n/q\| \ge \limsup_{n\to\infty} q(b_n)/\big(2q(\delta b_n)\big) \ge \delta^{-\eta}/2 \quad \text{a.s.}$$

Letting $\delta \downarrow 0$ completes the proof of this part.

(ii) Assume $\sum b_n < \infty$ and (2.1)–(2.4). It is easily seen, since $\sum Mb_n < \infty$ for any $M > 1$ and since $q/I \downarrow$, that it suffices to show for some $K \in (0,\infty)$,

$$\limsup_{n\to\infty} n^{1/2}q(b_n)\|\alpha_n/q\| \le K \quad \text{a.s.}$$

Note that $P(\min_{1\le i\le n} U_i \le 2b_n \text{ i.o.}) = 0$. Hence

$$\limsup_{n\to\infty} n^{1/2}q(b_n) \sup_{0<t<2b_n} |\alpha_n(t)|/q(t) \le \limsup_{n\to\infty} nq(b_n)\frac{2b_n}{q(2b_n)}$$

$$\le \limsup_{n\to\infty} 2nb_n = 0 \quad \text{a.s.}$$

So it remains to prove that

$$\limsup_{n\to\infty} n^{1/2}q(b_n) \sup_{2b_n\le t<1} |\alpha_n(t)|/q(t) \le K \quad \text{a.s.}$$

First assume (2.4a) holds. Then

$$n^{1/2}q(b_n) \sup_{2b_n\le t<1} |\alpha_n(t)|/q(t) \le n^{1/2}q(b_n) \sup_{b_n\le t<1} \frac{|\alpha_n(t)|}{t^{1/2}} \frac{t^{1/2}}{q(t)}$$

$$\le (nb_n)^{1/2}\|\alpha_n/I^{1/2}\|.$$

An application of Fact 2 completes the proof for this case.

Now assume (2.4b) holds. Because of (2.1) and Fact 1, it suffices to show for some small $\delta > 0$,

$$\limsup_{n\to\infty} n^{1/2}q(b_n) \sup_{2b_n\le t\le\delta} |\alpha_n(t)|/q(t) \le K \quad \text{a.s.}$$

From Fact 4 and the Borel–Cantelli lemma, it is enough to prove, for $n_k = 2^k$, that

$$(2.7) \quad \sum_k P\left(\sup_{2b_{n_{k+1}}\le t\le\delta} |\alpha_{n_{k+1}}(t)|/q(t) \ge K/\left((8n_{k+1})^{1/2}q(b_{n_k})\right)\right) < \infty.$$

Now we use Fact 3 to obtain sharp upper bounds for the probabilities in (2.7), assuming that k is large. We get

$$P\left(\sup_{2b_{n_{k+1}}\le t\le\delta} |\alpha_{n_{k+1}}(t)|/q(t) \ge K/\left((8n_{k+1})^{1/2}q(b_{n_k})\right)\right)$$

$$\le 8\int_{b_{n_{k+1}}}^{\delta} \frac{1}{t} \exp\left(\frac{-K^2q^2(t)}{1024n_{k+1}q^2(b_{n_k})t} \psi\left(\frac{Kq(t)}{8^{1/2}n_{k+1}q(b_{n_k})t}\right)\right) dt.$$

Define $t_k \in (0, \delta]$ by $Kq(t_k)/(8^{1/2}n_{k+1}q(b_n)t_k) = 10 \vee e^{1/\eta}$; if no solution in $(0, \delta]$ exists set $t_k = \delta$. We then have, if K is such that $K^2q^2(\delta)\psi(10 \vee e^{1/\eta}) \ge 3 \cdot 2048\delta$,

$$\int_{t_k}^{\delta} \frac{1}{t} \exp\left(\frac{-K^2q^2(t)}{1024n_{k+1}q^2(b_{n_k})t} \psi\left(\frac{Kq(t)}{8^{1/2}n_{k+1}q(b_{n_k})t}\right)\right) dt$$

$$\le \int_{t_k}^{\delta} \frac{1}{t}\, dt \exp\left(\frac{-K^2q^2(\delta)\psi(10 \vee e^{1/\eta})}{2048n_kq^2(b_{n_k})\delta}\right) \le \log \frac{1}{t_k} \exp\left(-\frac{3}{n_kq^2(b_{n_k})}\right)$$

$$\le \log n_k \exp(-3\log\log n_k) = (\log n_k)^{-2},$$

which is summable.

Finally, consider

$$\int_{b_{n_{k+1}}}^{t_k} \frac{1}{t} \exp\left(\frac{-K^2 q^2(t)}{1024 n_{k+1} q^2(b_{n_k}) t} \psi\left(\frac{Kq(t)}{8^{1/2} n_{k+1} q(b_{n_k}) t} \right) \right) dt$$

$$\leq \int_{b_{n_{k+1}}}^{t_k} \frac{1}{t} \exp\left(\frac{-Kq(t)}{512 q(b_{n_k})} \log\left(\frac{Kq(t)}{8^{1/2} n_{k+1} q(b_{n_k}) t} \right) \right) dt. \tag{2.8}$$

From $q/I^\eta \uparrow$, we see that

$$q \log\left(\frac{Kq}{8^{1/2} n_{k+1} q(b_{n_k}) I} \right) \uparrow \quad \text{on } (0, t_k].$$

Hence, using (2.3) and (2.4b) and taking K large enough, the right-hand side of (2.8) is bounded from above by

$$\int_{b_{n_{k+1}}}^{t_k} \frac{1}{t}\, dt \exp\left(\frac{-K}{512 \cdot 2^{c/2}} \log\left(\frac{K}{(8 \cdot 2^c)^{1/2} n_{k+1} b_{n_{k+1}}} \right) \right)$$

$$\leq \log \frac{1}{b_{n_{k+1}}} (n_{k+1} b_{n_{k+1}})^2.$$

So it remains to show that this last expression is summable in k. Using (2.3), we see that it suffices to prove that

$$\sum_k k (n_k b_{n_k})^2 < \infty.$$

Observe that $\Sigma_n b_n < \infty$ and $b_n \downarrow$ imply $\Sigma_k n_k b_{n_k} < \infty$. But $\Sigma_k n_k b_{n_k} < \infty$ and $n_k b_{n_k} \downarrow$ imply $k n_k b_{n_k} \to 0$, $k \to \infty$, and hence $k(n_k b_{n_k})^2 \leq n_k b_{n_k}$ for large k. This completes the proof. □

PROOF OF THE COROLLARY. Apply Theorem 1 with $b_n = (n \log n)^{-1}$ and $b_n = (n(\log n)^{1+\varepsilon})^{-1}$, $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ gives the desired result. □

3. The LIL for the weighted tail empirical process. In this section we will deal with the process $(2 \log\log n)^{-1/2} w_n/q$. For the sequence $\{k_n\}_{n=1}^\infty$ define condition

(C) $\quad 0 < k_n < n, \quad k_n/n \downarrow 0, \quad k_n \uparrow \quad \text{and} \quad k_n/\log\log n \to \infty.$

For the unweighted process ($q \equiv 1$), the following is known.

FACT 5 [Mason (1988)]. Let $\{k_n\}_{n=1}^\infty$ satisfy condition (C). Then almost surely the sequence $\{(2 \log\log n)^{-1/2} w_n\}_{n=1}^\infty$ is relatively compact in $\mathbb{B}$ with the set of limit points equal to $\mathbb{K}$. [Here $\mathbb{K}$ is the non–tied-down version of $\mathbb{F}$, i.e., apart from $f(1) = 0$, $\mathbb{K}$ satisfies the same conditions as $\mathbb{F}$ does.]

It is the purpose of this section to establish the analogue of James' (1975) LIL for the tail empirical process, that is, to give necessary and sufficient conditions for the LIL for the weighted tail empirical process. For $q \in \mathbb{Q}$ define the following condition:

for any $M > 0$ and n large, the function

$$q \log\left(\left(\frac{\log\log n}{k_n}\right)^{1/2} \frac{q}{I}\right)$$

attains, on

$$\left[q^{-1}\left(\frac{M}{(k_n \log\log n)^{1/2}}\right), \frac{2\log\log n}{k_n}\right], \tag{3.1}$$

its minimum in one of the two endpoints;
moreover, if this is the right endpoint, then for large n,

$$q^{-1}\left(\frac{1}{(k_n \log\log n)^{1/2}}\right) \geq n^{-c}$$

for some (large) $c > 1$.

THEOREM 2. *Let $\{k_n\}_{n=1}^{\infty}$ satisfy condition* (C) *and assume $q \in \mathbb{Q}_0$.*

(i) *If*

$$\sum \frac{k_n}{n} q^{-1}\left(\frac{M}{(k_n \log\log n)^{1/2}}\right) < \infty$$

for all $M > 0$ and (3.1) *holds, then almost surely the sequence $\{(2\log\log n)^{-1/2} w_n/q\}_{n=1}^{\infty}$ is relatively compact in $\mathbb{B}$ with the set of limit points equal to $\{f/q\colon f \in \mathbb{K}\}$.*

(ii) *If*

$$\sum \frac{k_n}{n} q^{-1}\left(\frac{M}{(k_n \log\log n)^{1/2}}\right) = \infty$$

for some $M > 0$, then almost surely the sequence $\{(2\log\log n)^{-1/2} w_n/q\}_{n=1}^{\infty}$ fails to be relatively compact in $\mathbb{B}$.

The theorem has the following neat corollary.

COROLLARY. *Let $\{k_n\}_{n=1}^{\infty}$ satisfy condition* (C), *let $q \in \mathbb{Q}_0$ and assume in addition $q/I^{\eta} \uparrow$ for some $\eta > 0$.*

(i) *If*

$$\sum \frac{k_n}{n} q^{-1}\left(\frac{1}{(k_n \log\log n)^{1/2}}\right) < \infty,$$

then almost surely the sequence $\{(2\log\log n)^{-1/2} w_n/q\}_{n=1}^{\infty}$ *is relatively compact in* $\mathbb{B}$ *with the set of limit points equal to* $\{f/q\colon f \in \mathbb{K}\}$.

(ii) *If*

$$\sum \frac{k_n}{n} q^{-1}\left(\frac{1}{(k_n \log\log n)^{1/2}}\right) = \infty,$$

then for any $0 < \delta < 1$,

$$\limsup_{n\to\infty} \sup_{0<t\le\delta} (2\log\log n)^{-1/2} |w_n(t)|/q(t) = \infty \quad a.s.$$

DISCUSSION. Theorem 2 is the generalization of Corollary 5 in Mason (1988) to arbitrary q; of course, it also generalizes Theorem 1 in Einmahl and Mason (1988). It is striking that in Theorem 2 the class of weight functions for which there is a functional LIL depends on the sequence $\{k_n\}_{n=1}^{\infty}$, whereas in its weak analogue (Theorem 3 in the next section), the class of weight functions for which the functional CLT holds does not depend on $\{k_n\}_{n=1}^{\infty}$. On the other hand, under the assumption of the corollary, it is easily shown that the summability condition does not depend on $\{k_n\}_{n=1}^{\infty}$ for the special, but interesting, choice $k_n = n^{\alpha}$, $0 < \alpha < 1$. In fact, from some analysis it follows that in this case $\Sigma(k_n/n)q^{-1}((k_n \log\log n)^{-1/2}) < \infty$ is equivalent to the old James condition $A(q) < \infty$ (see Fact 1). Finally, it should be noted that the weight functions $(\log(1/I))^{-\beta}$, $\beta > 0$, satisfy the first part of condition (3.1) and that, if $\beta \le 1$, $\Sigma(k_n/n)q^{-1}(M(k_n \log\log n)^{-1/2}) < \infty$, all $M > 0$, for all $\{k_n\}_{n=1}^{\infty}$ satisfying condition (C). Loosely speaking, this shows that Theorem 2 works on the whole range of sequences $\{k_n\}_{n=1}^{\infty}$, that is, from sequences slightly smaller than n, down to sequences slightly larger than $\log\log n$.

PROOF OF THEOREM 2. This theorem is an easy consequence of the following proposition combined with Fact 5. □

PROPOSITION. *Let* $\{k_n\}_{n=1}^{\infty}$ *satisfy condition* (C) *and let* $q \in \mathbb{Q}_0$. *If*

$$\sum \frac{k_n}{n} q^{-1}\left(\frac{M}{(k_n \log\log n)^{1/2}}\right) < \infty \quad \text{for all } M > 0, \tag{3.2}$$

and (3.1) *holds, then for every* $\varepsilon > 0$ *there exists a* $0 < \delta < 1$ *such that*

$$\limsup_{n\to\infty} \sup_{0<t\le\delta} \frac{|w_n(t)|}{(\log\log n)^{1/2} q(t)} \le \varepsilon \quad a.s. \tag{3.3}$$

If the summation in (3.2) *is infinite for some* $M > 0$, *then for any* $0 < \delta < 1$ *the* lim sup *in* (3.3) *is greater than or equal to* $1/(2M)$ *a.s. If, in addition,* $\lim_{t\downarrow 0} t^{1/2}/q(t) > 0$, *then for any* $0 < \delta < 1$ *the* lim sup *in* (3.3) *is infinite a.s.*

PROOF. Write

$$r_n = r_n(M) = \frac{k_n}{n} q^{-1}\left(\frac{M}{(k_n \log\log n)^{1/2}}\right) \quad \text{and} \quad v_n = \frac{n}{k_n} r_n. \tag{3.4}$$

We first prove the second part. We have $P(U_n \le r_n(M) \text{ i.o.}) = 1$ for the M for which the summation in (3.2) is infinite. Hence for any $0 < \delta < 1$,

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{0<t\le\delta} \frac{|w_n(t)|}{(\log\log n)^{1/2} q(t)} \\
&\quad = \limsup_{n\to\infty} \sup_{0<s\le\delta k_n/n} \frac{\left(\frac{n}{k_n}\right)^{1/2} |\alpha_n(s)|}{(\log\log n)^{1/2} q\left(s\frac{n}{k_n}\right)} \\
&\quad \ge \limsup_{n\to\infty} \frac{\left(\frac{n}{k_n}\right)^{1/2} n^{1/2} \frac{1}{2n}}{(\log\log n)^{1/2} \frac{M}{(k_n \log\log n)^{1/2}}} = \frac{1}{2M} \quad \text{a.s.}
\end{aligned}$$

Note that the last statement of this part is easy, since in that case $q \le cI^{1/2}$ for some $c > 0$, which implies

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{0<t\le\delta} \frac{|w_n(t)|}{(\log\log n)^{1/2} q(t)} \\
&\quad \ge \limsup_{n\to\infty} \sup_{0<s\le(n\log n)^{-1}} \frac{|\alpha_n(s)|}{(c\log\log n)^{1/2} s^{1/2}} = \infty \quad \text{a.s.}
\end{aligned}$$

Now we prove the first part. Assume (3.1) holds and $\Sigma r_n(M) < \infty$ for all $M > 0$. Note that $P(\min_{1\le i\le n} U_i \le 4r_n \text{ i.o.}) = 0$. Hence, since $q/I^{1/2} \downarrow$,

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{0<t<4v_n} \frac{|w_n(t)|}{(\log\log n)^{1/2} q(t)} \\
&\quad = \limsup_{n\to\infty} \sup_{0<s<4r_n} \frac{\left(\frac{n}{k_n}\right)^{1/2} |\alpha_n(s)|}{(\log\log n)^{1/2} q\left(s\frac{n}{k_n}\right)} \\
&\quad \le \limsup_{n\to\infty} \left(\frac{n}{k_n \log\log n}\right)^{1/2} \sup_{0<s<4r_n} \frac{n^{1/2} s}{q\left(s\frac{n}{k_n}\right)} \\
&\quad \le \limsup_{n\to\infty} \frac{4nr_n}{M} = 0 \quad \text{a.s.},
\end{aligned} \tag{3.5}$$

where for the last equality it is used that $r_n \downarrow$ and $\Sigma r_n < \infty$.

Hence it remains to show that for every small $\varepsilon > 0$ there exists a $0 < \delta < 1$ such that

$$\limsup_{n\to\infty} \sup_{4v_n \le t \le \delta} \frac{\left(\frac{n}{k_n}\right)^{1/2} \left|\alpha_n\left(t\frac{k_n}{n}\right)\right|}{(\log\log n)^{1/2} q(t)} \le \varepsilon \quad \text{a.s.} \tag{3.6}$$

Note that $\sum q^{-1}((n \log\log n)^{-1/2}) < \infty$, since $q^{-1}/I^2 \uparrow$, and hence $q^{-1}(t)/t^2 \to 0$, $t \downarrow 0$, which in turn implies $q(t)/t^{1/2} \to \infty$, $t \downarrow 0$. We will use this several times. From the change of variables $tk_n/n_j = s$, we see that (3.6) is equivalent to

$$\limsup_{n\to\infty} \sup_{4r_n \le s \le \delta k_n/n} \frac{\left(\frac{n}{k_n}\right)^{1/2} |\alpha_n(s)|}{(\log\log n)^{1/2} q\left(s\frac{n}{k_n}\right)} \le \varepsilon \quad \text{a.s.}$$

Applying Fact 4, we obtain from the Borel–Cantelli lemma that it suffices to prove for $n_j = 2^j$,

$$\sum_j P\left(\sup_{4r_{n_{j+1}} \le s \le \delta k_{n_j}/n_j} \frac{|\alpha_{n_{j+1}}(s)|}{q\left(s\frac{n_j}{k_{n_j}}\right)} \ge \varepsilon \left(\frac{\log\log n_j}{8}\right)^{1/2} \left(\frac{k_{n_{j+1}}}{n_{j+1}}\right)^{1/2} \right) < \infty. \tag{3.7}$$

Now we use Fact 3 and take j sufficiently large. The probability in (3.7) is then bounded from above by

$$8\int_{2r_{n_{j+1}}}^{\delta k_{n_j}/n_j} \frac{1}{s} \exp\left(\frac{-\varepsilon^2 \log\log n_j k_{n_{j+1}} q^2\left(s\frac{n_j}{k_{n_j}}\right)}{1024 n_{j+1} s} \right.$$

$$\left. \times \psi\left(\frac{\varepsilon}{n_{j+1}} \left(\frac{(\log\log n_j) k_{n_{j+1}}}{8}\right)^{1/2} \frac{q\left(s\frac{n_j}{k_{n_j}}\right)}{s} \right) \right) ds.$$

From the change of variables $sn_j/k_{n_j} = t$, we obtain as an upper bound

$$8\int_{v_{n_{j+1}}}^{\delta} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 \log\log n_j q^2(t)}{2048 t} \psi\left(\frac{(\log\log n_j)^{1/2} q(t)}{k_{n_{j+1}}^{1/2} t} \right) \right) dt.$$

Define $t_j \in (0, \delta]$ by $(\log\log n_j)^{1/2} q(t_j)/(k_{n_{j+1}}^{1/2} t_j) = 10$ for large j, and observe that for δ small enough,

$$\int_{t_j}^{\delta} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 \log\log n_j q^2(t)}{2048t} \psi\left(\frac{(\log\log n_j)^{1/2} q(t)}{k_{n_{j+1}}^{1/2} t} \right) \right) dt$$

$$\leq \int_{t_j}^{\delta} \frac{1}{t}\, dt \exp\left(\frac{-\varepsilon^2 \log\log n_j q^2(\delta)}{2048\delta} \psi(10) \right) \tag{3.8}$$

$$\leq \log \frac{1}{t_j} (\log n_j)^{-3} \leq (\log n_j)^{-2}.$$

We also have for large j,

$$\int_{(\log\log n_j)/k_{n_j}}^{t_j} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 \log\log n_j q^2(t)}{2048t} \psi\left(\frac{(\log\log n_j)^{1/2} q(t)}{k_{n_{j+1}}^{1/2} t} \right) \right) dt$$

$$\leq \int_{(\log\log n_j)/k_{n_j}}^{t_j} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 (\log\log n_j)^{1/2} q(t) k_{n_j}^{1/2}}{2048} \log 10 \right) dt \tag{3.9}$$

$$\leq \log\left(\frac{k_{n_j}}{\log\log n_j} \right) \exp\left(\frac{-\varepsilon^2 (\log\log n_j)^{1/2} k_{n_j}^{1/2}}{2048} (\log 10) q\left(\frac{\log\log n_j}{k_{n_j}} \right) \right)$$

$$\leq \log n_j \exp(-3 \log\log n_j) = (\log n_j)^{-2}.$$

From (3.8) and (3.9) it follows that the proof is complete if we show that

$$\int_{v_{n_{j+1}}}^{(\log\log n_j)/k_{n_j}} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 \log\log n_j q^2(t)}{2048t} \psi\left(\frac{(\log\log n_j)^{1/2} q(t)}{k_{n_{j+1}}^{1/2} t} \right) \right) dt \tag{3.10}$$

is summable in j. The expression in (3.10) is bounded by

$$\int_{v_{n_{j+1}}}^{(2\log\log n_{j+1})/k_{n_{j+1}}} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 (\log\log n_j)^{1/2} k_{n_{j+1}}^{1/2} q(t)}{4096} \right.$$

$$\left. \times \log\left(\frac{(\log\log n_{j+1})^{1/2} q(t)}{k_{n_{j+1}}^{1/2} t} \right) \right) dt.$$

For large M this expression is in turn bounded by [use (3.1)]

$$\log \frac{1}{v_{n_{j+1}}} \exp\left(\frac{-\varepsilon^2 (\log\log n_j)^{1/2} k_{n_{j+1}}^{1/2} M}{4096 k_{n_{j+1}}^{1/2} (\log\log n_{j+1})^{1/2}} \times \log\left(\frac{(\log\log n_{j+1})^{1/2} M}{k_{n_{j+1}}^{1/2} k_{n_{j+1}}^{1/2} (\log\log n_{j+1})^{1/2} v_{n_{j+1}}} \right) \right)$$

$$\vee c \log n_{j+1} \exp\left(\frac{-\varepsilon^2 (\log\log n_j)^{1/2} k_{n_{j+1}}^{1/2}}{4096} q\left(\frac{\log\log n_{j+1}}{k_{n_{j+1}}} \right) \log 10 \right)$$

$$\leq \log \frac{1}{v_{n_{j+1}}} \exp\left(\frac{-\varepsilon^2 M}{8192} \log\left(\frac{M}{n_{j+1} r_{n_{j+1}}} \right) \right)$$

$$\vee c \log n_{j+1} \exp(-3 \log\log n_{j+1})$$

$$\leq \log \frac{1}{r_{n_{j+1}}} (n_{j+1} r_{n_{j+1}})^2 \vee c (\log n_{j+1})^{-2}.$$

Since the second term is easily seen to be summable in j, we restrict our attention to the first term. Observe that we may assume without loss of generality $r_n \geq n^{-2}$, which reduces our problem to showing that $\sum j (n_j r_{n_j})^2 < \infty$. The proof can now be completed as the proof of Theorem 1. □

PROOF OF THE COROLLARY. (i) Observe that

$$\sum (k_n/n) q^{-1}\left(M (k_n \log\log n)^{-1/2} \right) < \infty \quad \text{for all } M > 0, \text{ since } q^{-1}/I^{1/\eta} \downarrow.$$

Moreover,

$$q \log\left(\left(\frac{\log\log n}{k_n} \right)^{1/2} \frac{q}{I} \right) \uparrow \quad \text{on } \left(0, \frac{2 \log\log n}{k_n}\right],$$

and hence (3.1) is satisfied. Applying Theorem 2 completes the proof of part (i).

(ii) Note that now $\sum (k_n/n) q^{-1}(M (k_n \log\log n)^{-1/2}) = \infty$, for all $M > 0$. Hence, letting $M \downarrow 0$ in the second part of the proposition yields part (ii). □

4. The CLT for the weighted tail empirical process and confidence bands for intermediate quantiles. Here we present, without proof, the weak analogue of Theorem 2. This result may be viewed as the Chibisov–O'Reilly theorem for the tail empirical process. The sufficiency part follows easily from Corollary 4.2.1 in Csörgő, Csörgő, Horváth and Mason (1986). Also a direct proof (with the aid of Fact 3) of that part is rather straightforward. The proof of the necessity part is standard. Note that the conditions on $\{k_n\}_{n=1}^{\infty}$ are milder than those in Theorem 2.

THEOREM 3. *Let* $\{k_n\}_{n=1}^{\infty}$ *be as in Section* 1 *and let* $q \in \mathbb{Q}$. *If*

$$\int_0^1 \frac{1}{t} \exp\left(\frac{-\lambda q^2(t)}{t}\right) dt < \infty \quad \text{for all } \lambda > 0, \tag{4.1}$$

then there exists a sequence $\{W_n\}_{n=1}^{\infty}$ *of standard Wiener processes such that*

$$\|(w_n - W_n)/q\| \to_P 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Conversely, (4.2) *holding true for some sequence* $\{W_n\}_{n=1}^{\infty}$ *of standard Wiener processes implies* (4.1).

This theorem has an interesting application to the construction of asymptotic confidence bands for intermediate quantiles. The result is given in the following corollary, but first some notation has to be introduced. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with common distribution function F. For $0 \le t < 1$ define the quantile function Q by

$$Q(t) = \inf\{x: F(x) \ge t\},$$

and for each integer $n \ge 1$ and $0 < t < 1$ the empirical quantile function by

$$Q_n(t) = X_{k,n}, \qquad (k-1)/n < t \le k/n, \qquad k = 1, \ldots, n,$$

where $X_{1,n} \le \cdots \le X_{n,n}$ are the order statistics based on $X_1, \ldots, X_n$. For $t \le 0$, set $Q_n(t) = -\infty$.

COROLLARY. *Let* F *be continuous,* $\{k_n\}_{n=1}^{\infty}$ *as in Section* 1 *and let* $q \in \mathbb{Q}$. *If* (4.1) *holds, then we have for any* $0 < \alpha < 1$,

$$P\left(Q_n\left(\frac{tk_n}{n}\left(1 - \frac{cq(t)}{tk_n^{1/2}}\right)\right) \le Q\left(\frac{tk_n}{n}\right) < Q_n\left(\frac{tk_n}{n}\left(1 + \frac{cq(t)}{tk_n^{1/2}}\right)\right), 0 < t \le 1\right)$$
$$\to 1 - \alpha \quad \text{as } n \to \infty,$$

where, with W *a standard Wiener process,* $c := c(\alpha, q)$ *is defined by*

$$P\left(\sup_{0<t\le 1} |W(t)|/q(t) \ge c\right) = \alpha.$$

DISCUSSION. It should be emphasized that continuity is the only condition on F which is needed in the corollary. Usually results of this type are established under additional (mostly extreme value type) conditions. Recall that even for the definition of the standardized quantile process, which is often used to generate confidence bands for quantiles, one has to assume that F has a density. See, for example, Corollary 3.2 in Einmahl (1991), where asymptotic confidence bands for intermediate quantiles are derived along the above lines. Hence the present corollary improves the results there as far as the conditions on F (and also on q) are considered.

PROOF OF THE COROLLARY. From Theorem 3 it follows that with probability tending to $1 - \alpha$, $n \to \infty$,

$$-c \leq \frac{n}{k_n^{1/2}}\left(F_n\left(\frac{tk_n}{n}\right) - \frac{tk_n}{n}\right)\Big/ q(t) < c, \qquad 0 < t \leq 1.$$

Rewriting this yields

$$(4.3)\quad \frac{tk_n}{n}\left(1 - \frac{cq(t)}{tk_n^{1/2}}\right) \leq F_n\left(\frac{tk_n}{n}\right) < \frac{tk_n}{n}\left(1 + \frac{cq(t)}{tk_n^{1/2}}\right), \qquad 0 < t \leq 1.$$

Let $\overline{Q}_n$ [set $\overline{Q}_n(s) = 0$ for $s \leq 0$] be the uniform quantile function and observe that for $0 < s, t < 1$ we have

$$\{F_n(t) < s\} = \{\overline{Q}_n(s) > t\}.$$

Hence (4.3) can be written as

$$(4.4)\quad \overline{Q}_n\left(\frac{tk_n}{n}\left(1 - \frac{cq(t)}{tk_n^{1/2}}\right)\right) \leq \frac{tk_n}{n} < \overline{Q}_n\left(\frac{tk_n}{n}\left(1 + \frac{cq(t)}{tk_n^{1/2}}\right)\right), \qquad 0 < t \leq 1.$$

Now assuming without loss of generality that $X_i = Q(U_i)$ and noting that $Q_n = Q \circ \overline{Q}_n$ and that Q is strictly increasing on $(0, 1)$ (F is continuous), we see that (4.4) is equivalent to

$$Q_n\left(\frac{tk_n}{n}\left(1 - \frac{cq(t)}{tk_n^{1/2}}\right)\right) \leq Q\left(\frac{tk_n}{n}\right) < Q_n\left(\frac{tk_n}{n}\left(1 + \frac{cq(t)}{tk_n^{1/2}}\right)\right), \qquad 0 < t \leq 1.$$

This completes the proof. □

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