ON THE LAW OF THE ITERATED LOGARITHM FOR MARTINGALES

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The Kolmogorov law of the iterated logarithm fails when the boundedness condition on the increments is relaxed. In this paper, we consider this in the martingale setting and establish a lower bound, extending a result known in the independent case.

1. Introduction. Let \( \{X_i, \ i \geq 1\} \) be a sequence of independent random variables with \( EX_i = 0 \) and \( EX_i^2 < \infty \), for \( i = 1, 2, \ldots \). Define \( s_n^2 = \sum_{i=1}^{n} EX_i^2 \) and suppose that \( s_n^2 \to \infty \) as \( n \to \infty \). Kolmogorov’s law of the iterated logarithm (LIL) [Kolmogorov (1929)] states that if

\[
|X_n| \leq c_n s_n \left( \log \log s_n \right)^{-1/2} \quad \text{a.s.,}
\]

for constants \( c_n \to 0 \) as \( n \to \infty \), then

\[
\lim_{n \to \infty} \sup_n S_n / \left( 2s_n^2 \log \log s_n \right)^{1/2} = 1 \quad \text{a.s.,}
\]

where \( S_n = \sum_{i=1}^{n} X_i \) and \( \log x = \log(\log x) \).

If the Kolmogorov condition (1.1) is weakened so that \( c_n \) is replaced by a constant \( c > 0 \), then the result (1.2) fails in general. This has been shown by Marcinkiewicz and Zygmund (1937), Feller (1943) and Weiss (1959).

Upper and lower bounds for \( \lim \sup_{n \to \infty} S_n / \left( 2s_n^2 \log \log s_n \right)^{1/2} \) in this case have been derived by Tomkins (1978) and Teicher (1979). In particular, it follows from their results that

\[
0 < \lim_{n \to \infty} \sup_n S_n / \left( 2s_n^2 \log \log s_n \right)^{1/2} < \infty \quad \text{a.s.}
\]

The second inequality in (1.3) was derived earlier by Egorov (1969).

A martingale analogue of the Kolmogorov law of the iterated logarithm was first established by Stout (1970). In the supermartingale case analogous to the weakened condition, a finite upper bound was derived by Fisher (1986), extending an earlier and more restricted result of Stout [(1974), Theorem 5.4.1].

In this paper we establish a lower bound in the martingale setting. A consequence is that (1.3) is extended to the martingale case.

Section 2 of this paper consists of a statement of the main result and a discussion of it. Section 3 consists of the proof of the theorem.

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2. Statement of theorem and remarks. Let \( \{U_n, \mathcal{F}_n, n \geq 1\} \) be a martingale defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \{\mathcal{F}_n, n \geq 1\} \) is an increasing sequence of sub-\(\sigma\)-fields of \( \mathcal{F} \). Let \( \{X_i, i \geq 1\} \) be the martingale difference sequence defined by \( X_i = U_i - U_{i-1} \) (define \( U_0 = 0 \)). Suppose that \( \mathbb{E}[X_0^2|\mathcal{F}_{i-1}] < \infty \), for \( i \geq 1 \) (let \( \mathcal{F}_0 = \{\phi, \Omega\} \)), and define \( s_n^2 = \sum_{k=1}^{n} \mathbb{E}[X_k^2|\mathcal{F}_{k-1}] \), for \( n \geq 1 \). For convenience, we define \( \varphi(x) = (2 \log_2(x^2 + e^2))^{1/2} \) and \( \eta(x) = (2x \log_2(x \vee e^2))^{1/2} \), for \( x > 0 \).

**Theorem 1.** Let \( \{U_n, \mathcal{F}_n, n \geq 1\} \) be a martingale described with the preceding notation. Assume that \( s_n^2 \to \infty \) a.s. as \( n \to \infty \) and that

\[
|X_i| \leq K_i s_i / \varphi(s_i) \quad \text{a.s.,}
\]

where \( K_i \) is an \( \mathcal{F}_{i-1} \)-measurable function for each integer \( i \geq 1 \) with

\[
\limsup_{i \to \infty} K_i < K,
\]

for \( K > 0 \) an arbitrary constant.

Then there exists a positive constant \( \varepsilon(K) \) so that

\[
\limsup_{n \to \infty} U_n / s_n \varphi(s_n) \geq \varepsilon(K) \quad \text{a.s.}
\]

In particular, one can take \( \varepsilon(K) \) as

\[
\varepsilon(K) = h_K^{-1}(1) \wedge (1/81K),
\]

where \( h_K(x) = x^2 + 12K^{1/2}x^{5/2}, x > 0 \).

**Remark 1.** In the martingale analogue of the Kolmogorov LIL established by Stout (1970), condition (2.2) is replaced by the assumption \( K_i \to 0 \) as \( i \to \infty \). The lower half of this result follows from Theorem 1 by observing that \( h_K^{-1}(1) \uparrow 1 \) as \( K \to 0 \).

**Remark 2.** As noted in Stout (1970), the hypothesis that \( K_i \) is a random variable rather than simply a constant means a less restrictive hypothesis than the classical one when Theorem 1 is applied in the independent case.

**Remark 3.** An immediate consequence of Theorem 1 is that

\[
\limsup_{n \to \infty} U_n / s_n \varphi(s_n) > 0 \quad \text{a.s.}
\]

This, in combination with Lemma 1, results in the conclusion that

\[
0 < \limsup_{n \to \infty} U_n / s_n \varphi(s_n) < \infty \quad \text{a.s.},
\]

extending what has been proved in the independent case.

3. Proof of main result. The proof of Theorem 1 makes use of two results that we list as Lemma 1 and Lemma 2.
Lemma 1. Assume the hypothesis of Theorem 1. Then there exists a constant \( \lambda(K) \), \( 0 < \lambda(K) < \infty \), so that

\[
\limsup_{n \to \infty} \frac{U_n}{s_n \varphi(s_n)} \leq \lambda(K) \quad a.s.
\]

Proof. This is an immediate corollary of Fisher [(1986), Theorem 1]. \( \square \)

Lemma 2 is a large deviation result for martingales derived by Freedman (1975). We adopt his notation for the following definitions.

Let \( a \) and \( b \) be positive numbers. Define \( \sigma_b = \inf(n: s_n^2 > b) \) if such \( n \) exists and \( \sigma_b = \infty \) otherwise. Let

\[
L(b) = \text{ess sup}_{\omega} \sup_{n \leq \sigma_b(\omega)} |X_n(\omega)|.
\]

Let \( A \) and \( B \) be the events defined as

\[
A = \{U_n \geq a \text{ for some } n \text{ such that } s_n^2 < b\}
\]

and

\[
B = \left\{ \sup_n s_n^2 < b \right\}.
\]

Lemma 2. Let \( 0 < \delta \leq \frac{1}{3} \). Suppose \( L(b) \) is finite and satisfies the conditions

\[
b/a > 9L(b)/\delta^2
\]

and

\[
a^2/b > (16/\delta^2) \log(64/\delta^2).
\]

Then

\[
P(A \cup B) \geq \frac{1}{2} \exp\left[-\frac{1}{2}(a^2/b)(1 + 4\delta)\right].
\]

Proof. See Freedman [(1975), Proposition 2.4]. \( \square \)

Proof of Theorem 1. Let \( r > 1 \). Define \( t_k = \sup\{n: s_n^2 \leq r^k\} \), where \( k \geq 1 \) is an integer. Since \( s_n \to \infty \) a.s., \( t_k \) is a well-defined stopping time relative to \( \{\mathcal{F}_i, i \geq 1\} \).

Consider the martingale \( (U^{(k)}_n, \mathcal{F}^{(k)}_n, n \geq 0) \), where \( U^{(k)}_n = U_{t_k+n} - U_{t_k} \) and \( \mathcal{F}^{(k)}_n = \mathcal{F}_{t_k+n} \). [Recall that if \( \tau \) is a stopping time relative to \( \{\mathcal{F}_i, i \geq 1\} \), then by \( \mathcal{F}_\tau \), is meant the \( \sigma \)-field of events \( A \in \mathcal{F}_\tau \) such that \( A \cap \{\tau = i\} \in \mathcal{F}_i \) for all integers \( i \geq 1 \).]

Let \( X^{(k)}_n = U^{(k)}_n - U^{(k)}_{n-1} \) for \( n \geq 1 \) and

\[
(s^{(k)}_n)^2 = \sum_{i=1}^{n} E\left[\left(X^{(k)}_i\right)^2 | \mathcal{F}^{(k)}_{i-1}\right].
\]
Define

\[ Y_n(k) = X_n^{(k)} I \left( \bigcap_{i=1}^{n} (K_i^{(k)} \leq K) \right), \]

for \( n \geq 1 \), where \( K_i^{(k)} = K_{i_k+i} \). (The notation \( I(A) \) denotes the indicator function of the event \( A \).)

Define \( V_n^{(k)} = \sum_{i=1}^{n} Y_i^{(k)} \). Then \( \{ V_n^{(k)}, \mathcal{F}_n^{(k)}, n \geq 1 \} \) is a martingale. This follows from the fact that \( \{ X_n^{(k)}, \mathcal{F}_n^{(k)}, n \geq 1 \} \) is a martingale difference sequence and \( I(\bigcap_{i=1}^{n} (K_i^{(k)} \leq K)) \) is \( \mathcal{F}_n^{(k)} \)-measurable. Let

\[ (v_n^{(k)})^2 = \sum_{i=1}^{n} E \left[ Y_i^{(k)}^2 \mid \mathcal{F}_{i-1}^{(k)} \right]. \]

Assume the space \( (\Omega, \mathcal{F}) \) is sufficiently regular so that there exists a regular conditional probability \( P_k(\omega, B) \) on \( (\Omega, \mathcal{F}, P) \) given \( \mathcal{F}_{i_k} \). That is, for each \( \omega \in \Omega \), \( P_k(\omega, \cdot) \) is a probability measure on \( (\Omega, \mathcal{F}) \) and, for each fixed \( B \in \mathcal{F}, P_k(\omega, B) = P(B \mid \mathcal{F}_{i_k})(\omega) \) a.s. It follows as a consequence of a standard result on regular conditional probabilities that \( \{ V_n^{(k)}, \mathcal{F}_n^{(k)}, n \geq 1 \} \) is a martingale defined on the space \( (\Omega, \mathcal{F}, P_k(\omega, \cdot)) \) for \( \omega \in \Omega \) a.s. [see Loève (1978), Sections 29 and 30].

Following Freedman's approach [Freedman (1975), Section 6], we apply Lemma 2 to this martingale. Define the event \( A_k \) as

\[ A_k = \left\{ V_n^{(k)} > \epsilon(K)(1 - r^{-1/2}) \eta(r^{k+1}) \right\} \text{ for some } n \geq 1 \]

such that \( (v_n^{(k)})^2 < r^{k+1} - r^k \),

where \( \epsilon(k) \) is defined by (2.4).

In the notation of Lemma 2, \( a \) and \( b \) are defined as

\[ a = \epsilon(K)(1 - r^{-1/2}) \eta(r^{k+1}) \]

and

\[ b = r^{k+1} - r^k. \]

In addition, we have

\[ L(b) \leq \text{ess sup} \sup_{\omega} \sup_{0 < n \leq t_{k+1} - t_k + 1} \left| Y_n^{(k)}(\omega) \right|. \]

Let \( m = t_{k+1} \) for \( k \) sufficiently large [e.g., so that \( \varphi(r^{(k+1)/2}) > K \)]. For \( \omega \in \bigcap_{i=1}^{m+1} \left( K_i^{(k)} \leq K \right) \), it follows that

\[ s_{m+1}^2 \leq s_m^2 + K^2 s_m^2 + \varphi^2(s_m + 1). \]

By the definition of \( t_{k+1} \),

\[ s_{m+1}^2 \leq r^{k+1} / (1 - K^2 / \varphi^2(r^{(k+1)/2})). \]

Since \( s_n / \varphi(s_n) \) increases as \( n \) increases we have

\[ L(b) \leq (K / \varphi(r^{(k+1)/2})) r^{(k+1)/2} (1 - K^2 / \varphi^2(r^{(k+1)/2}))^{-1/2}. \]
Let $\delta = 3K^{1/2}e^{1/2}(K)$. It is immediate from (2.4) that $0 < \delta \leq \frac{1}{3}$. For $k$ large, elementary calculations verify that the remaining hypotheses of Lemma 2 described by (3.3) and (3.4) are satisfied.

Define the event $B_k$ as

$$B_k = A_k \cup \left\{ \sup_n (u_n^{(k)})^2 < r_k^{k+1} - r_k^k \right\}.$$ 

Applying Lemma 2 we find from (3.5) and the definition of $h_K(\cdot)$ and $\varepsilon(K)$ that

$$P(B_k|\mathcal{F}_{t_k}) \geq \frac{1}{2} \exp\left[-(\log_2 r_k^{k+1})h_K(\varepsilon(K))\right] \quad \text{a.s.}$$

Since $\varepsilon(K) \leq h_K^{-1}(1)$ and $h_K(x)$ increases as $x$ increases, the inequality $h_K(\varepsilon(K)) \leq 1$ holds. Therefore the series $\sum_{k=1}^{\infty} P(B_k|\mathcal{F}_{t_k})$ diverges a.s. By Lévy’s conditional form of the Borel–Cantelli lemma [see Stout (1974), page 55] we obtain

$$P(B_k \text{ i.o.}) = 1.$$ 

For $k$ sufficiently large (depending on $\omega$), it follows from (2.2) that $Y_n^{(k)}(\omega) = X_n^{(k)}(\omega)$, for $n \geq 1$. Therefore, for each $\omega$ outside a null set and for $k$ sufficiently large (depending on $\omega$), the equalities

$$V_n^{(k)}(\omega) = U_n^{(k)}(\omega) \quad \text{and} \quad (u_n^{(k)})^2(\omega) = (s_n^{(k)})^2(\omega)$$

hold for $n \geq 1$. Since $s_n^2 \to \infty$ a.s., it follows that $(u_n^{(k)})^2 \to \infty$ a.s. Therefore,

$$P(A_k \text{ i.o.}) = 1.$$ 

This implies that

$$P(C_k \text{ i.o.}) = 1,$$

where

$$C_k = \left\{ U_n^{(k)} > \varepsilon(K)(1 - r^{-1/2})\eta(r_k^{k+1}) \text{ for some } n \geq 1 \right\}$$

such that $(s_n^{(k)})^2 < r_k^{k+1} - r_k^k$.

Applying Lemma 1 to the martingale $(-U_n, \mathcal{F}_n, n \geq 1)$ proves that there exists a finite constant $\lambda(K) > 0$ such that

$$P\left[U_{t_k} < -\lambda(K)\eta(s_{t_k}^2) \text{ i.o.}\right] = 0.$$ 

Since $\eta(s_{t_k}^2) \leq \eta(r_k)$ and $\eta(r_k^{k+1}) \geq r^{1/2}\eta(r_k)$, then, for all $r > 1$, we obtain

$$P\left[U_{t_k} < -\lambda(K)r^{-1/2}\eta(r_k^{k+1}) \text{ i.o.}\right] = 0.$$ 

Combining (3.6) and (3.7) shows that, for all $r > 1$,

$$P\left[U_n > \eta(r_k^{k+1})(\varepsilon(K)(1 - r^{-1/2}) - \lambda(K)r^{-1/2}) \text{ for some } n \text{ satisfying } t_k < n \leq t_{k+1} \text{ i.o.}\right] = 1.$$
Since $\eta(r^{k+1}) \geq \eta(s_n^2)$ for $t_k < n \leq t_{k+1}$, (3.8) implies that

$$\limsup_{n \to \infty} U_n/s_n \varphi(s_n) \geq \epsilon(K) \quad \text{a.s.}$$

\[\square\]

REFERENCES


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