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THE LAW OF THE ITERATED LOGARITHM FOR INDEPENDENT RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

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Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of independent real random variables, where $\mathbb{N}^d$ is the d-dimensional lattice. In this paper, the law of the iterated logarithm is established for such a field of random variables. Theorem 1 brings into focus a connection between a certain strong law of large numbers and the law of the iterated logarithm. A general technique is developed by which one can derive the strong law of large numbers and the law of the iterated logarithm, exploiting the convergence rates in the weak law of large numbers in Theorem 2. In Theorem 3, we use Gaussian randomization techniques to obtain the law of the iterated logarithm which generalizes Wittmann's result.

1. Introduction. The strong laws of large numbers (SLLN) for independent random variables (real or Banach space valued) with multidimensional indices have been investigated over the last few years. See, for example, Gut (1978, 1980), Li (1990), Mikosch (1984), Mikosch and Norvaisa (1987) and Smythe (1973), among others. The laws of the iterated logarithm have been investigated in the past for independent identically distributed (iid) random variables (real or Banach space valued) with multidimensional indices. See Li and Wu (1989), Morrow (1981) and Wichura (1973). The present paper is concerned with providing some general conditions under which the law of the iterated logarithm holds for independent random variables (real or Banach space valued) with multidimensional indices. We need some notation before explaining the basic theme of this paper.

Let $\mathbb{N}^d$ be the set of all d-dimensional lattice points with positive integers as components, where $d \geq 1$ is a fixed integer. Let $\mathbb{N}^d$ be equipped with the coordinate-wise partial order $\leq$. Points in $\mathbb{N}^d$ are denoted by $\bar{m}, \bar{n}$, etcetera. For $\bar{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$, we define $|\bar{n}| = \prod_{i=1}^{d} n_i$. Further, $\bar{n} \to \infty$ means that $|\bar{n}| \to \infty$ and is denoted by $\lim_{\bar{n} \in \mathbb{N}^d}$. The limit superior of a field $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, of real numbers is defined by $\inf_{r \geq 1} \sup_{|\bar{n}| \geq r} a_{\bar{n}}$ and is denoted by $\limsup_{n \in \mathbb{N}^d} a_{\bar{n}}$. The limit inferior is defined in analogous fashion. For $\bar{n}, \bar{m}$ in $\mathbb{N}^d$ with $\bar{n} \leq \bar{m}$, the set $A(\bar{n}, \bar{m}) = \{\bar{k} \in \mathbb{N}^d; \bar{n} \leq \bar{k} \leq \bar{m}\}$ is called a rectangle. Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of real random variables. For any $A \subseteq \mathbb{N}^d$, set

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$S_A = \sum_{\bar{n} \in A} X_{\bar{n}}$, $|A|$ = cardinal number of A, $(\bar{n}) = \{\bar{m} \in \mathbb{N}^d;\ \bar{m} \le \bar{n}\}$ and $S_{\bar{n}} = S_{(\bar{n})}$. Note that $|\bar{n}| = |(\bar{n})|$. Let $L(x) = \log(\max\{e, x\})$ and $L_2(x) = L(L(x))$, $x \ge 0$.

In order to bring into focus the essentials of this paper, we begin with a description of Wichura's (1973) work. Let $X, X_{\bar{n}}, \bar{n} \in \mathbb{N}^d$, be a field of iid real random variables. Then

$$\limsup_{\bar{n} \in \mathbb{N}^d} |S_{\bar{n}}| / \left(2|\bar{n}| L_2(|\bar{n}|)\right)^{1/2} = \sqrt{d}\,\sigma \quad \text{a.s.,} \tag{1.1}$$

if and only if

$$EX = 0, \qquad EX^2 = \sigma^2 \quad \text{and} \quad EX^2 \left(L(|X|)\right)^{d-1} / L_2(|X|) < \infty. \tag{1.2}$$

Recently, Li and Wu (1989) obtained a similar result in the context of separable Banach space valued random variables. Our main aim in this paper is to develop some techniques by which one can establish the strong law of large numbers and the law of the iterated logarithm for independent random variables.

The basic framework under which we work in this paper is described as follows. Let $X_{\bar{n}}, \bar{n} \in \mathbb{N}^d$, be a field of real independent random variables not necessarily identically distributed. Let $EX_{\bar{n}} = 0$ and $EX_{\bar{n}}^2 < \infty$ for each $\bar{n} \in \mathbb{N}^d$, and $B_{\bar{n}} = \sum_{\bar{k} \le \bar{n}} EX_{\bar{k}}^2 \to \infty$ as $|\bar{n}| \to \infty$. In analogy with (1.1), we wish to have

$$\limsup_{\bar{n} \in \mathbb{N}^d} |S_{\bar{n}}| / \sqrt{2 B_{\bar{n}} L_2(B_{\bar{n}})} = \sqrt{d} \quad \text{a.s.,} \tag{1.3}$$

under some adequate moment conditions on the random variables. The following example indicates that (1.3) is very unlikely to be fulfilled even under very strong conditions. We take $d = 2$, and the independent random variables satisfy $P(X_{(1,n)} = 1) = P(X_{(1,n)} = -1) = 1/2 = P(X_{(n,1)} = 1) = P(X_{(n,1)} = -1)$, $X_{\bar{n}} = 0$ a.s. for all other $\bar{n}$. Obviously, $X_{\bar{n}}, \bar{n} \in \mathbb{N}^2$, is a field of uniformly bounded independent nonidentically distributed real random variables with $EX_{\bar{n}} = 0$ for every $\bar{n} = (n_1, n_2) \in \mathbb{N}^d$ and $B_{\bar{n}} = \sum_{\bar{k} \le \bar{n}} EX_{\bar{k}}^2 = n_1 + n_2 - 1 \to \infty$ as $|\bar{n}| \to \infty$. By the classical law of the iterated logarithm, we have

$$\begin{aligned} &\limsup_{\bar{n} \in \mathbb{N}^d} |S_{\bar{n}}| / \sqrt{2 B_{\bar{n}} L_2(B_{\bar{n}})} \\ &= \limsup_{(n+m) \to \infty} \left| \sum_{i=1}^{n} X_{(1,i)} + \sum_{j=2}^{m} X_{(j,1)} \right| / \sqrt{2(n+m-1) L_2(n+m-1)} \\ &= 1 \ne \sqrt{2} \quad \text{a.s.} \end{aligned} \tag{1.4}$$

In view of the failure of the validity of (1.3), we will only study under what conditions

$$\limsup_{\bar{n} \in \mathbb{N}^d} |S_{\bar{n}}| / \sqrt{2 B_{\bar{n}} L_2(B_{\bar{n}})} < \infty \quad \text{a.s.} \tag{1.5}$$

holds.

The basic idea in our treatment is to seek a connection between a certain strong law of large numbers and the law of the iterated logarithm. To motivate the result, we begin with the following simple observation. Let $X_1, X_2, \dots$ be a sequence of iid real random variables with $EX_1 = 0$. Then

$$(1.6)\quad \left(\limsup_{n\to\infty}(1/n)\sum_{k\le n} X_k^2\right)^{1/2} = \limsup_{n\to\infty}\left|\sum_{k\le n} X_k\right|/(2nL_2 n)^{1/2} \quad \text{a.s.}$$

In the case of independent random variables with multidimensional indices, we demonstrate that

$$(1.7)\qquad \limsup_{\bar{n}\in\mathbb{N}^d}\sum_{\bar{k}\le\bar{n}} X_{\bar{k}}^2/B_{\bar{n}} < \infty \quad \text{a.s.}$$

essentially implies that

$$(1.8)\qquad \limsup_{\bar{n}\in\mathbb{N}^d}\left|\sum_{\bar{k}\le\bar{n}} X_{\bar{k}}\right|/(2B_{\bar{n}}L_2 B_{\bar{n}})^{1/2} < \infty \quad \text{a.s.}$$

See Theorem 1.

Some comments are in order on (1.7) and (1.8). There is a world of difference between the cases $d = 1$ and $d > 1$. In the iid case, when $d = 1$, (1.7) and (1.8) are equivalent. But when $d > 1$, (1.8) need not imply (1.7) even in the iid case. Smythe (1973) gave necessary and sufficient conditions for the validity of (1.7) in the iid case. These conditions are different from (1.2).

We also develop a general technique by which one can derive the strong law of large numbers and the law of the iterated logarithm exploiting the convergence rates in the weak law of large numbers. See Theorem 2.

Further, we use Gaussian randomization techniques to obtain the law of the iterated logarithm which generalizes Wittmann's (1985) result. See Theorem 3.

For the development of results in this paper, we need some concepts from Mikosch and Norvaisa (1987). Let $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of positive numbers such that $\lim_{\bar{n}\in\mathbb{N}^d} a_{\bar{n}} = \infty$. This field of numbers is said to have the "star" property if there exists a sequence D_k, $k \ge 1$, of finite subsets of $\mathbb{N}^d$ such that $D_k \uparrow \mathbb{N}^d$ and satisfies the following.

Condition A. Set $D_0 = \{(1,1,\dots,1)\}$ and $I_k = D_k - D_{k-1}$, $k \ge 1$. If $\bar{n} \in I_k$, then $(\bar{n}) \subseteq D_k$.

Condition B. There are constants $b > 1$, $c_1, c_2 > 0$ such that for every $\bar{n} \in I_k$, the relation $c_1 b^k \le a_{\bar{n}} \le c_2 b^k$ holds.

Condition C. For every $k \ge 1$, there exist disjoint rectangles $E_{k,r}$ and an appropriate index set R_k such that $I_k = D_k - D_{k-1} = \bigcup_{r\in R_k} E_{k,r}$.

Condition D. There are constants $c_3 > 0$ and $s \ge 0$ such that $\#R_k \le c_3 k^s$ for all $k \ge 1$.

CONDITION E. $v_0 = \limsup_{k\to\infty} \max_{\bar{n}\in I_k} b^{-k/2}\sum_{i=1}^{k} b^{i/2}\,\#\{r \in R_i;\ E_{i,r} \cap (\bar{n}) \neq \varnothing\} < \infty$.

Conditions A, B and C come from Mikosch and Norvaisa (1987). Condition E is a minor variation of Condition D of Mikosch and Norvaisa [(1987), page 243]. For examples and interpretation of Conditions C and E, see Mikosch and Norvaisa (1987). The fields of numbers having the star property are very broad and include the following:

1. $a_{\bar{n}} = |\bar{n}|^{\gamma}\varphi(|\bar{n}|)$, $\bar{n} \in \mathbb{N}^d$, where $\varphi(\cdot)$ is a nondecreasing slowly varying function and $\gamma > 0$.
2. $d = 1$ and $a_n \uparrow \infty$.

2. Preliminaries. In this section, we establish some lemmas which will be useful in the proofs of the main results in the next section.

LEMMA 1. *Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of independent symmetric real random variables and $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of positive numbers having the star property. Then*

$$\limsup_{\bar{n}\in\mathbb{N}^d} |S_{\bar{n}}|/\sqrt{2a_{\bar{n}}L_2(a_{\bar{n}})} < \infty \quad a.s., \tag{2.1}$$

if and only if

$$\sup_{k\geq 1}\max_{r\in R_k} |S_{E_{k,r}}|/\big(2b^kL_2(b^k)\big)^{1/2} < \infty \quad a.s. \tag{2.2}$$

PROOF. By the Borel–Cantelli lemma, (2.2) is equivalent to the fact that for some $\lambda > 0$,

$$\sum_{k\geq 1}\sum_{r\in R_k} P\Big(|S_{E_{k,r}}|/\big(2b^kL_2(b^k)\big)^{1/2} \geq \lambda\Big) < \infty, \tag{2.3}$$

which is a Prohorov-type condition. Using a similar argument as in Theorem 2.2 of Mikosch and Norvaisa [(1987), page 244], one can show the equivalence of (2.1) and (2.3). □

LEMMA 2. *Let T be a countable index set and T_n, $n \geq 1$, a partition of T such that $\#T_n \leq cn^s$ for all $n \geq 1$ for some $c > 0$ and $s \geq 0$. Let g_t, $t \in T$, be a family of independent identically distributed $N(0,1)$ random variables. Then*

$$E\sup_{n\geq 1}\max_{t\in T_n} |g_t|/\sqrt{2L(n)} < \infty. \tag{2.4}$$

PROOF. Note that (2.4) is equivalent to

$$\sum_{m\geq 1} P\Big(\sup_{n\geq 1}\max_{t\in T_n}|g_t|/\sqrt{2L(n+2)} \geq km\Big) < \infty, \tag{2.5}$$

where $k \geq (3+s)^{1/2}$. For every $m \geq 1$, we have

$$\begin{aligned} &P\Big(\sup_{n\geq 1}\max_{t\in T_n}|g_t|/\sqrt{2L(n+2)} \geq km\Big)\\ &\quad\leq \sum_{n\geq 1} 2(\#T_n)\int_{km\sqrt{2L(n+2)}}^{\infty}(1/\sqrt{2\pi})\exp\{-x^2/2\}\,dx\\ &\quad\leq c\sum_{n\geq 1} n^s\exp\{-k^2m^2L(n+2)\}\\ &\quad\leq c\sum_{n\geq 1} 1/(n+1)^{k^2m^2-s} \leq c\big(1/2^{k^2m^2-s-1}\big). \end{aligned} \tag{2.6}$$

Obviously, $\sum_{m\geq 1}(1/2^{k^2m^2-s-1}) < \infty$. □

Recently, Ledoux and Talagrand (1988) obtained necessary and sufficient conditions that a Banach space valued random variable X satisfy the bounded law of the iterated logarithm and the compact law of the iterated logarithm, respectively. The gaussian randomization technique was basic in the proofs of their main results. Modifying their argument to suit our needs, we have the following result.

LEMMA 3. *Let T be a countable index set and R_n, $n \geq 1$, be a sequence of index sets such that $\{T_{n,r};\ r \in R_n,\ n \geq 1\}$ forms a partition of T; that is, $\bigcup_{n\geq 1}\bigcup_{r\in R_n} T_{n,r} = T$ and $T_{n,r}\cap T_{m,s} = \varnothing$ for $m \neq n$ or $m = n$ and $r \neq s$. Let X_t, $t \in T$, be a family of zero mean independent real random variables and $g_{n,r}$, $r \in R_n$, $n \geq 1$, be a family of independent standard normal random variables such that these two families are independent. Then*

$$\begin{aligned} &E\sup_{n\geq 1}\sup_{r\in R_n}\Big|\sum_{t\in T_{n,r}} X_t\Big|\\ &\qquad\leq cE\sup_{n\geq 1}\sup_{r\in R_n}|g_{n,r}|\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2}, \end{aligned} \tag{2.7}$$

for some constant $c > 0$.

PROOF. Let g_t, $t \in T$, be a family of independent standard normal random variables which is independent of X_t, $t \in T$. Denote by E_g and E_X expectations with respect to the independent families g_t, $t \in T$, and X_t, $t \in T$, of random variables, respectively. Using an argument similar to the one used on page 1251 and then Lemma 3.3 in Ledoux and Talagrand (1988), we have

$$
\begin{aligned}
E\sup_{n\ge1}\sup_{r\in R_n}&\Big|\sum_{t\in T_{n,r}} X_t\Big| \\
&\le 2\big(E|g_t|\big)^{-1}E\sup_{n\ge1}\sup_{r\in R_n}\Big|\sum_{t\in T_{n,r}} g_tX_t\Big| \\
&= 2\big(E|g_t|\big)^{-1}E_XE_g\sup_{n\ge1}\sup_{r\in R_n}\Big|\sum_{t\in T_{n,r}} g_tX_t\Big| \\
&\le C\sqrt{2\pi}\,E_X\Bigg(\sup_{n\ge1}\sup_{r\in R_n}E_g\Big|\sum_{t\in T_{n,r}} g_tX_t\Big| \\
&\qquad\qquad +E\sup_{n\ge1}\sup_{r\in R_n}|g_{n,r}|\bigg(E_g\Big|\sum_{t\in T_{n,r}} g_tX_t\Big|^2\bigg)^{1/2}\Bigg) \\
&\le C\sqrt{2\pi}\,E_X\Bigg(\sup_{n\ge1}\sup_{r\in R_n}\bigg(E_g\Big(\sum_{t\in T_{n,r}} g_tX_t\Big)^2\bigg)^{1/2} \\
&\qquad\qquad +E\sup_{n\ge1}\sup_{r\in R_n}|g_{n,r}|\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2}\Bigg) \qquad (2.8)\\
&= C\sqrt{2\pi}\,E\Big(\sup_{n\ge1}\sup_{r\in R_n}\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2} \\
&\quad + C\sqrt{2\pi}\,E\sup_{n\ge1}\sup_{r\in R_n}|g_{n,r}|\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2} \\
&\le 6CE\sup_{n\ge1}\sup_{r\in R_n}|g_{n,r}|\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2},
\end{aligned}
$$

for some constant $C > 0$. The last inequality follows from

$$
\begin{aligned}
E\sup_{n\ge1}\sup_{r\in R_n}&|g_{n,r}|\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2} \\
&= E_XE_g\sup_{n\ge1}\sup_{r\in R_n}|g_{n,r}|\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2} \\
&\ge E_X\sup_{n\ge1}\sup_{r\in R_n}\big(E|g_{n,r}|\big)\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2} \\
&= \sqrt{2/\pi}\,E\sup_{n\ge1}\sup_{r\in R_n}\Big(\sum_{t\in T_{n,r}} X_t^2\Big)^{1/2}.
\end{aligned}
$$

□

LEMMA 4. *Let T, $T_{n,r}$ and R_n be the same as in Lemma 3. Let X_t, $t \in T$, be a family of real independent random variables such that for every $t \in T$, $|X_t| \le C$, a.s., where C is a constant. Then*

$$\sup_{n\ge 1}\sup_{r\in R_n}\left|\sum_{t\in T_{n,r}} X_t\right| < \infty \quad a.s., \tag{2.9}$$

if and only if for every $\alpha > 0$,

$$E\left(\sup_{n\ge 1}\sup_{r\in R_n}\left|\sum_{t\in T_{n,r}} X_t\right|\right)^{\alpha} < \infty. \tag{2.10}$$

PROOF. It is sufficient to prove (2.9) ⇒ (2.10). In fact, by a maximum inequality in Hoffmann-Jørgensen (1974), it is easy to prove that under (2.9), (2.10) is equivalent to

$$E\Big(\sup_{t\in T}|X_t|\Big)^{\alpha} < \infty. \tag{2.11}$$

From the assumption that $|X_t| \le C$ a.s. for all t, (2.11) holds. □

The following lemma is a general version of (3.3), the fundamental inequality, in Hoffmann-Jørgensen (1974).

LEMMA 5. *Let X_n, $n \ge 1$, be a sequence of real independent symmetric random variables, and $S_n = \sum_{i=1}^n X_i$, $n \ge 1$. Then for every integer $j \ge 1$, there exists C_j, $D_j > 0$ (depending only on j) such that for any $t \ge 0$,*

$$P(|S_n| \ge 2jt) \le C_j P\Big(\max_{1\le i\le n}|X_i| \ge t\Big) + D_j\big(P(|S_n| \ge t)\big)^j. \tag{2.12}$$

PROOF. For $j = 1$, (2.12) is trival and we take $C_1 = D_1 = 1$. Assume that (2.12) holds for some j with relative numbers C_j and D_j. Then using (3.3) of Hoffmann-Jørgensen (1974) for $j + 1$ we have

$$\begin{aligned}
&P(|S_n| \ge 2(j+1)t)\\
&\quad\le P(|S_n| \ge t + t + 2jt)\\
&\quad\le P\Big(\max_{1\le k\le n}|X_k| \ge t\Big) + 4P(|S_n| \ge t)P(|S_n| \ge 2jt)\\
&\quad\le P\Big(\max_{1\le k\le n}|X_k| \ge t\Big)\\
&\qquad + 4P(|S_n| \ge t)\Big(C_j P\Big(\max_{1\le k\le n}|X_k| \ge t\Big) + D_j\big(P(|S_n| \ge t)\big)^j\Big)\\
&\quad\le (1 + 4C_j)P\Big(\max_{1\le k\le n}|X_k| \ge t\Big) + 4D_j\big(P(|S_n| \ge t)\big)^{j+1}.
\end{aligned}$$

Hence we can take $C_{j+1} = 1 + 4C_j$ and $D_{j+1} = 4D_j$. □

3. Main results. It is known that there is a close relationship between the strong law of large numbers and the law of the iterated logarithm. The following result shows that the study of the law of the iterated logarithm for independent random variables $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, with respect to $\sqrt{2a_{\bar{n}} L_2(a_{\bar{n}})}$, $\bar{n} \in \mathbb{N}^d$, can be reduced to a study of the strong law of large numbers for $X_{\bar{n}}^2$, $\bar{n} \in \mathbb{N}^d$, with respect to $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, in some cases. This result seems to be new even for the case $d = 1$.

THEOREM 1. *Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of real independent random variables and $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of positive numbers having the star property. If $S_{\bar{n}}/\sqrt{2a_{\bar{n}} L_2(a_{\bar{n}})}$, $\bar{n} \in \mathbb{N}^d$, is stochastically bounded, that is, bounded in probability, and*

$$\limsup_{\bar{n} \in \mathbb{N}^d} \sum_{\bar{k} \le \bar{n}} X_{\bar{k}}^2 / a_{\bar{n}} < \infty \quad a.s., \tag{3.1}$$

then

$$\limsup_{\bar{n} \in \mathbb{N}^d} |S_{\bar{n}}| / \sqrt{2a_{\bar{n}} L_2(a_{\bar{n}})} < \infty \quad a.s. \tag{3.2}$$

PROOF. We can assume, without loss of generality, that each $X_{\bar{n}}$ is symmetric. By (3.1), there exists $C_1 > 0$ such that

$$\sum_{\bar{n} \in \mathbb{N}^d} P\left(|X_{\bar{n}}|^2 > C_1 a_{\bar{n}}\right) < \infty. \tag{3.3}$$

Let $Y_{\bar{n}} = X_{\bar{n}} I(|X_{\bar{n}}| \le \sqrt{C_1 a_{\bar{n}}})$, $Z_{\bar{n}} = X_{\bar{n}} - Y_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$. From (3.3), we have

$$\lim_{\bar{n} \in \mathbb{N}^d} \sum_{\bar{k} \le \bar{n}} Z_{\bar{k}} / \sqrt{2a_{\bar{n}} L_2(a_{\bar{n}})} = 0 \quad \text{a.s.}, \tag{3.4}$$

and

$$\limsup_{\bar{n} \in \mathbb{N}^d} \sum_{\bar{k} \le \bar{n}} Y_{\bar{k}}^2 / a_{\bar{n}} < \infty \quad \text{a.s.} \tag{3.5}$$

Let $g_{n,r}$, $r \in R_n$, $n \ge 1$, be independent standard normal random variables independent of $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$. Since $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, has the star property, using Lemmas 2–4 and (3.5), we have

$$\begin{aligned} &E\left(\sup_{n \ge 1} \max_{r \in R_n} \left| \sum_{k \in E_{n,r}} Y_{\bar{k}} \right| / \left(2b^n L_2(b^n)\right)^{1/2}\right) \\ &\le CE\left(\sup_{n \ge 1} \max_{r \in R_n} |g_{n,r}| \left(\sum_{k \in E_{n,r}} Y_{\bar{k}}^2 \right)^{1/2} / \left(2b^n L_2(b^n)\right)^{1/2}\right) \\ &\le CE\left(\sup_{n \ge 1} \max_{r \in R_n} |g_{n,r}| / \left(2L_2(b^n)\right)^{1/2}\right) E\left(\sup_{n \ge 1} \max_{r \in R_n} \sum_{k \in E_{n,r}} Y_{\bar{k}}^2 / b^n\right)^{1/2} \\ &< \infty. \end{aligned} \tag{3.6}$$

By Lemma 1,

$$\limsup_{\bar{n}\in\mathbb{N}^d}\left|\sum_{\bar{k}\le\bar{n}} Y_{\bar{k}}\right|/\sqrt{2a_{\bar{n}}L_2(a_{\bar{n}})} < \infty \quad \text{a.s.} \tag{3.7}$$

Combining (3.7) and (3.4), we have (3.2). □

REMARK 1. If each $X_{\bar{n}}$ is symmetric, then the condition that $S_{\bar{n}}/\sqrt{2a_{\bar{n}}L_2(a_{\bar{n}})}$, $\bar{n}\in\mathbb{N}^d$, be stochastically bounded can be eliminated in the above theorem.

REMARK 2. If, in addition, $X_{\bar{n}}$, $\bar{n}\in\mathbb{N}^d$, satisfy $EX_{\bar{n}}=0$, $EX_{\bar{n}}^2<\infty$ for each $\bar{n}$ and $a_{\bar{n}}=B_{\bar{n}}=\sum_{\bar{k}\le\bar{n}}EX_{\bar{k}}^2\to\infty$ as $\bar{n}\to\infty$, then it follows that $S_{\bar{n}}/\sqrt{2a_{\bar{n}}L_2(a_{\bar{n}})}$, $\bar{n}\in\mathbb{N}^d$, is stochastically bounded. In this case, we conjecture that (3.1) can be weakened to

$$\limsup_{\bar{n}\in\mathbb{N}^d}\sum_{\bar{k}\le\bar{n}} X_{\bar{k}}^2/B_{\bar{n}}L_2(B_{\bar{n}}) < \infty \quad \text{a.s.} \tag{3.8}$$

REMARK 3. An analogue of Theorem 1 can be established for separable Banach spaces of type 2.

In the following theorem, we provide a technique by which we can get the strong law of large numbers and the law of the iterated logarithm by utilizing the convergence rates in the weak law of large numbers. This technique has some intrinsic merit and is useful to deal with multidimensional indices in practice.

THEOREM 2. *Let $X_{\bar{n}}$, $\bar{n}\in\mathbb{N}^d$, be a field of real independent random variables, and $a_{\bar{n}}$, $\bar{n}\in\mathbb{N}^d$, be a field of positive numbers having the star property.*
(i) *If there exist $\lambda_1,\lambda_2>0$ such that*

$$P\big(|S_{\bar{n}}|/a_{\bar{n}}\ge\lambda_2\big)=O\Big((\log a_{\bar{n}})^{-\lambda_1}\Big) \quad as\ \bar{n}\to\infty, \tag{3.9}$$

then

$$\limsup_{\bar{n}\in\mathbb{N}^d}|S_{\bar{n}}|/a_{\bar{n}}<\infty \quad a.s., \tag{3.10}$$

if and only if for some $\lambda_3>0$,

$$\sum_{\bar{n}\in\mathbb{N}^d}P\big(|X_{\bar{n}}|\ge\lambda_3 a_{\bar{n}}\big)<\infty. \tag{3.11}$$

(ii) *If there exists $\delta<1$ such that for every $\varepsilon>0$,*

$$P\big(|S_{\bar{n}}|/a_{\bar{n}}\ge\varepsilon\big)=O\Big((\log a_{\bar{n}})^{-\varepsilon^{\delta}}\Big) \quad as\ \bar{n}\to\infty, \tag{3.12}$$

then

$$\lim_{\bar{n}\in\mathbb{N}^d}S_{\bar{n}}/a_{\bar{n}}=0 \quad a.s., \tag{3.13}$$

if and only if

$$\sum_{\bar{n}\in\mathbb{N}^d} P\big(|X_{\bar{n}}| \geq \varepsilon a_{\bar{n}}\big) < \infty \quad \textit{for every } \varepsilon > 0. \tag{3.14}$$

PROOF. We will only give the proof of (ii) since the proof of (i) is similar except for the necessary modifications. We can assume, without loss of generality, that each $X_{\bar{n}}$ is symmetric. As in Lemma 1, (3.13) is equivalent to

$$\lim_{k\to\infty} \max_{r\in R_k} |S_{E_{k,r}}|/b^k = 0 \quad \text{a.s.} \tag{3.15}$$

But (3.15) is equivalent to

$$\sum_{k\geq 1} \sum_{r\in R_k} P\big(|S_{E_{k,r}}|/b^k \geq \varepsilon\big) < \infty \tag{3.16}$$

for any $\varepsilon > 0$. For fixed $\varepsilon > 0$, choose $j = j(\varepsilon)$ such that $\alpha = \alpha(\varepsilon) = (\varepsilon/2c_2)^\delta j^{1-\delta} > 1 + s$ (since $\delta < 1$), where c_2 is the constant in Condition B and s is the constant in Condition D. Denote $\varepsilon_1 = \varepsilon/(2j)$. By Lévy's inequality and (3.12), we have

$$P\big(|S_{E_{k,r}}|/b^k \geq \varepsilon_1\big) = O\big(k^{-(\varepsilon_1/c_2)^\delta}\big)$$

holds uniformly for $r \in R_k$ as $k \to \infty$. By Lemma 5, we have uniformly for $r \in R_k$,

$$\begin{aligned} &P\big(|S_{E_{k,r}}|/b^k \geq \varepsilon\big) \\ &\quad\leq C_j P\Big(\max_{\bar{n}\in E_{k,r}} |X_{\bar{n}}| \geq \varepsilon_1 b^k\Big) \\ &\qquad + D_j\Big(P\big(|S_{E_{k,r}}|/b^k \geq \varepsilon_1\big)\Big)^j \\ &\quad\leq C_j \sum_{\bar{n}\in E_{k,r}} P\big(|X_{\bar{n}}| \geq (\varepsilon_1/c_2)a_{\bar{n}}\big) + O(k^{-\alpha}) \quad \text{as } k\to\infty. \end{aligned} \tag{3.17}$$

From (3.14), we have

$$\begin{aligned} &\sum_{k\geq 1} \sum_{r\in R_k} \sum_{\bar{n}\in E_{k,r}} P\big(|X_{\bar{n}}| \geq (\varepsilon_1/c_2)a_{\bar{n}}\big) \\ &\quad= \sum_{\bar{n}\in\mathbb{N}^d} P\big(|X_{\bar{n}}| \geq (\varepsilon_1/c_2)a_{\bar{n}}\big) < \infty. \end{aligned} \tag{3.18}$$

Since $\#R_k \leq c_3 k^s$ by Condition D, we have

$$\begin{aligned} &\sum_{k\geq 1} \sum_{r\in R_k} \Big\{P\big(|S_{E_{k,r}}|/b^k \geq \varepsilon_1\big)\Big\}^j \\ &\quad= \sum_{k\geq 1} O(k^{-\alpha+s}) < \infty. \end{aligned} \tag{3.19}$$

Thus we have proved that (3.12) and (3.14) imply (3.13). By the Borel–Cantelli lemma, obviously, (3.13) implies (3.14). □

REMARK. The condition that $\delta < 1$ in Theorem 2(ii) is very crucial. For example, suppose $X, X_1, X_2, \dots$ are iid standard normal random variables. Let $a_n = (2nL_2n)^{1/2}$, $n \geq 1$. Then for any $\varepsilon > 0$,

$$P\left(\left|\sum_{k=1}^{n} X_k\right|/a_n \geq \varepsilon\right) = P\left(|X|/(2L_2n)^{1/2} \geq \varepsilon\right) = O\left((\log a_n)^{-\varepsilon^2}\right) \quad \text{as } n \to \infty.$$

Thus (3.12) is satisfied with $\delta = 2$. Evidently, (3.14) is true. However, it is well known that

$$\limsup_{n \to \infty} \left(\sum_{k=1}^{n} X_k\right)/a_n = 1 \neq 0 \quad \text{a.s.}$$

Theorem 2 is rather a general result. An analogue of Theorem 2 can be established for separable Banach spaces. No geometric conditions are required. We examine how the conclusion (1.1) can be drawn from Theorem 2. Let $X, X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of iid real random variables such that $EX = 0$, $EX^2 < \infty$ and $EX^2L(|X|)^{d-1}/L_2(|X|) < \infty$. If $d \geq 2$, by a standard argument (cutting methods and exponential inequalities) it follows that for some $\lambda > 0$,

$$P\left(|S_{\bar{n}}|/\left(2|\bar{n}|L_2(|\bar{n}|)\right)^{1/2} \geq \lambda\right) = O\left((\log|\bar{n}|)^{-1/2}\right) \quad \text{as } \bar{n} \to \infty. \tag{3.20}$$

Thus we see that (3.9) is satisfied. (3.11) follows from $EX^2(L|X|)^{d-1}/L_2(|X|) < \infty$. Consequently, (3.10) holds true with $a_{\bar{n}} = (2|\bar{n}|L_2(|\bar{n}|))^{1/2}$. Thus (1.1) follows in a weak form. Theorem 2 is unable to identify concretely the limit superior.

We also note that, if $d \geq 2$, the Marcinkiewicz–Zygmund strong law of large numbers established in Gut (1978) and Smythe (1973) follows as a consequence of Theorem 2. Moreover, following the same ideas as in Theorem 2, one can give an alternative and simple proof of Theorem 1 of Li and Wu (1989).

We come back to our original problem. Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of independent real random variables such that $EX_{\bar{n}} = 0$, $EX_{\bar{n}}^2 < \infty$ for each $\bar{n}$, and $B_{\bar{n}} = \sum_{\bar{k} \leq \bar{n}} EX_{\bar{k}}^2 \to \infty$ as $\bar{n} \to \infty$. There are at least two ways to obtain (3.9): (1) cutting methods and exponential inequalities; (2) approximation by the central limit theorems. The following corollaries exemplify each of these ways separately.

COROLLARY 1. *Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of real independent random variables with $EX_{\bar{n}} = 0$ for each $\bar{n}$. If there exists $r, \lambda > 0$ such that*

$$\sup_{\bar{n} \in \mathbb{N}^d} EX_{\bar{n}}^2\left(L(|X_{\bar{n}}|)\right)^r < \infty \tag{3.21}$$

and

$$\sum_{\bar{n} \in \mathbb{N}^d} P\left(|X_{\bar{n}}| \geq \lambda\left(2|\bar{n}|L_2(|\bar{n}|)\right)^{1/2}\right) < \infty, \tag{3.22}$$

then

$$\limsup_{\bar{n}\in\mathbb{N}^d} |S_{\bar{n}}|/\big(2|\bar{n}|L_2(|\bar{n}|)\big)^{1/2} < \infty \quad a.s. \tag{3.23}$$

If, in addition,

$$\liminf_{\bar{n}\in\mathbb{N}^d} B_{\bar{n}}/|\bar{n}| > 0, \tag{3.24}$$

then

$$0 < \limsup_{\bar{n}\in\mathbb{N}^d} |S_{\bar{n}}|/\big(2|\bar{n}|L_2(|\bar{n}|)\big)^{1/2} < \infty \quad a.s. \tag{3.25}$$

COROLLARY 2. *Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of independent real random variables and $a_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of positive numbers having the star property. If there exists $\lambda_1, \lambda_2 > 0$ such that*

$$\sup_{x\in R} \left|P\left(S_{\bar{n}}/\sqrt{a_{\bar{n}}} \le x\right) - \Phi(x)\right| = o\left((\log a_{\bar{n}})^{-\lambda_1}\right) \quad as\ \bar{n} \to \infty \tag{3.26}$$

and

$$\sum_{\bar{n}\in\mathbb{N}^d} P\left(|X_{\bar{n}}| \ge \lambda_2 a_{\bar{n}}\right) < \infty, \tag{3.27}$$

then

$$\limsup_{\bar{n}\in\mathbb{N}^d} |S_{\bar{n}}|/\sqrt{2a_{\bar{n}}L_2(a_{\bar{n}})} < \infty \quad a.s. \tag{3.28}$$

The following is a generalization of the Kolmogorov law of the iterated logarithm.

COROLLARY 3. *Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of independent real random variables such that $EX_{\bar{n}} = 0$, $EX_{\bar{n}}^2 < \infty$ for each $\bar{n}$ and $B_{\bar{n}} = \sum_{\bar{k}\le\bar{n}} EX_{\bar{k}}^2 \to \infty$ as $\bar{n} \to \infty$. Suppose $\sqrt{B_{\bar{n}}L_2(B_{\bar{n}})}$, $\bar{n} \in \mathbb{N}^d$, has the star property. If*

$$|X_{\bar{n}}| = o\left(\sqrt{B_{\bar{n}}/L_2(B_{\bar{n}})}\right) \quad a.s., \tag{3.29}$$

then

$$1 \le \limsup_{\bar{n}\in\mathbb{N}^d} |S_{\bar{n}}|/\sqrt{2B_{\bar{n}}L_2(B_{\bar{n}})} < \infty \quad a.s. \tag{3.30}$$

Wittmann (1985) obtained a rather general law of the iterated logarithm when $d = 1$. The following theorem has resemblance to Wittmann's result, but our method of proof is entirely different from Wittmann's method. Even in the classical case of $d = 1$, we can construct an example which satisfies the assumptions of our Theorem 3, but an assumption in Wittmann's result fails.

THEOREM 3. *Let $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, be a field of independent real random variables such that $EX_{\bar{n}} = 0$, $EX_{\bar{n}}^2 < \infty$ for each $\bar{n}$ and $0 < B_{\bar{n}} = \sum_{\bar{k}\le\bar{n}} EX_{\bar{k}}^2 \to \infty$*

as $\bar{n} \to \infty$. *If* $B_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, *has the star property and for some* $1 \le p \le 2$,

$$\sum_{\bar{n} \in \mathbb{N}^d} E|X_{\bar{n}}^2 - EX_{\bar{n}}^2|^p / \left(2B_{\bar{n}} L_2(B_{\bar{n}})\right)^p < \infty, \tag{3.31}$$

then

$$\limsup_{\bar{n} \in \mathbb{N}^d} |S_{\bar{n}}| / \sqrt{2B_{\bar{n}} L_2(B_{\bar{n}})} < \infty \quad a.s. \tag{3.32}$$

Proof. Observe that in this case $S_{\bar{n}} / \sqrt{2B_{\bar{n}} L_2(B_{\bar{n}})}$ converges to 0 in probability as $\bar{n} \to \infty$. We can assume, without loss of generality, that each $X_{\bar{n}}$ is symmetric. By Lemma 3, we have

$$\begin{aligned} &E\left(\sup_{k \ge 1} \max_{r \in R_k} |S_{E_{k,r}}| / \left(2b^k L_2(b^k)\right)^{1/2}\right) \\ &\quad \le CE\left(\sup_{k \ge 1} \max_{r \in R_k} |g_{k,r}| \left(\sum_{\bar{n} \in E_{k,r}} X_{\bar{n}}^2\right)^{1/2} \Big/ \left(2b^k L_2(b^k)\right)^{1/2}\right), \end{aligned} \tag{3.33}$$

where $g_{k,r}$, $r \in R_k$, $k \ge 1$, are independent standard normal random variables independent of $X_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$. Note that $\sum_{\bar{n} \in E_{k,r}} EX_{\bar{n}}^2 \le c_2 b^k$, $r \in R_k$, $k \ge 1$, since $B_{\bar{n}}$, $\bar{n} \in \mathbb{N}^d$, satisfies Condition B. We have

$$\begin{aligned} &E\left(\sup_{k \ge 1} \max_{r \in R_k} |g_{k,r}| \left(\sum_{\bar{n} \in E_{k,r}} X_{\bar{n}}^2\right)^{1/2} \Big/ \left(2b^k L_2(b^k)\right)^{1/2}\right) \\ &\quad \le \sqrt{2}\, E\left(\sup_{k \ge 1} \max_{r \in R_k} |g_{k,r}| / \left(2L_2(b^k)\right)^{1/2}\right) \\ &\qquad + E\left(\sup_{k \ge 1} \max_{r \in R_k} |g_{k,r}| \left|\sum_{\bar{n} \in E_{k,r}} (X_{\bar{n}}^2 - EX_{\bar{n}}^2)\right|^{1/2} \Big/ \left(2b^k L_2(b^k)\right)^{1/2}\right) \\ &\quad = I_1 + I_2, \quad \text{say.} \end{aligned} \tag{3.34}$$

From Lemma 2, $I_1 < \infty$. Since $1 \le p \le 2$, using Hölder's inequality,

$$\begin{aligned} I_2 &\le E\left(\sup_{k \ge 1} \max_{r \in R_k} g_{k,r}^2 \left|\sum_{\bar{n} \in E_{k,r}} (X_{\bar{n}}^2 - EX_{\bar{n}}^2)\right| \Big/ \left(2b^k L_2(b^k)\right)\right)^{1/2} \\ &\le \left(E\left(\sup_{k \ge 1} \max_{r \in R_k} g_{k,r}^2 \left|\sum_{\bar{n} \in E_{k,r}} (X_{\bar{n}}^2 - EX_{\bar{n}}^2)\right| \Big/ \left(2b^k L_2(b^k)\right)\right)^p\right)^{1/2p} \\ &\le C_1\left(\sum_{k \ge 1} \sum_{r \in R_k} \sum_{\bar{n} \in E_{k,r}} E|X_{\bar{n}}^2 - EX_{\bar{n}}^2|^p / \left(2b^k L_2(b^k)\right)^p\right)^{1/2p} \\ &\le C_2\left(\sum_{\bar{n} \in \mathbb{N}^d} E|X_{\bar{n}}^2 - EX_{\bar{n}}^2|^p / \left(2B_{\bar{n}} L_2(B_{\bar{n}})\right)^p\right)^{1/2p} < \infty. \end{aligned} \tag{3.35}$$

Combining (3.34), (3.35), (3.36) and Lemma 1, we see that (3.32) holds. □

COROLLARY 4. *Let X_n, $n \geq 1$, be a sequence of independent real random variables such that $EX_n = 0$, $EX_n^2 < \infty$ for each n and $B_n = \sum_{i=1}^n EX_i^2 \to \infty$ as $n \to \infty$. If*

$$\sum_{n \geq 1} E|X_n|^{2p}/(2B_n L_2(B_n))^p < \infty \tag{3.36}$$

for some $1 \leq p \leq 2$, then

$$\limsup_{n \to \infty} \left| \sum_{i=1}^n X_i \right| / \sqrt{2B_n L_2(B_n)} < \infty \quad a.s. \tag{3.37}$$

REMARK 1. An analogue of Theorem 3 can be established for separable Banach spaces of type 2.

REMARK 2. Let ε_n, $n \geq 1$, be a sequence of iid random variables with $P(\varepsilon_1 = 1) = 1/2 = P(\varepsilon_2 = -1)$, that is, a Rademacher sequence. Let $X_n = 2^{n/2}\varepsilon_n$, $n \geq 1$. Note that $EX_n = 0$ for every n and $B_n = \sum_{i=1}^n EX_i^2 = 2^{n+1} - 2 \to \infty$ and $B_{n+1}/B_n \to 2$ as $n \to \infty$. Also

$$\sum_{n \geq 1} E|X_n^2 - EX_n^2|^p/B_n^p = 0. \tag{3.38}$$

But, for any $\alpha > 0$,

$$\sum_{n \geq 1} (2B_n L_2(B_n))^{-(2+\alpha)/2} E|X_n|^{2+\alpha} = \infty. \tag{3.39}$$

In view of (3.39), Theorem 1.2 of Wittmann (1985) is no longer applicable. But by Theorem 3 above,

$$\limsup_{n \to \infty} \left| \sum_{k=1}^n X_k \right| / (2B_n L_2(B_n))^{1/2} < \infty \quad \text{a.s.}$$

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