

## BROWNIAN MOTION IN DENJOY DOMAINS<sup>1</sup>

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A planar domain whose boundary  $E$  lies in the real line is called a Denjoy domain. In this article we consider some geometric properties of Brownian motion in such a domain. The first result is that if  $E$  has zero length ( $|E| = 0$ ), then there is a set  $F \subset E$  of full harmonic measure such that every Brownian path which exits at  $x \in F$  hits both  $(-\infty, x)$  and  $(x, \infty)$  with probability 1, verifying a conjecture of Burdzy. Next we show that if  $\dim(E) < 1$ , then almost every Brownian path forms infinitely many loops separating its exit point from  $\infty$  and we give an example to show  $\dim(E) < 1$  cannot be replaced by  $|E| = 0$ .

**1. Introduction.** Suppose  $E$  is a closed proper subset of  $\mathbb{R}$  and let  $\Omega = \mathbb{R}^2 \setminus E$ . Such a domain is called a Denjoy domain. The purpose of this article is to consider two properties of Brownian motion in Denjoy domains. The first concerns the way in which a Brownian path hits the real line immediately before hitting  $E$ . The second concerns the probability that a Brownian path will form a “loop” separating one component of  $E$  from another. The proofs are mainly classical potential theory and real analysis. Our first result is the following theorem.

**THEOREM 1.** *Suppose  $\Omega = \mathbb{R}^2 \setminus E$  is a Denjoy domain. Then for almost every  $x \in E$  (with respect to harmonic measure,  $\omega$ ) and every  $\varepsilon > 0$ , a Brownian motion in  $\Omega$  conditioned to exit at  $x$  will hit the interval  $[x - \varepsilon, x)$  with probability 1 if and only if it hits the interval  $(x, x + \varepsilon]$  with probability 1.*

Theorem 1 says that a Brownian path does not have a preferred “direction of approach” to  $x$ . Let  $\mathcal{P}_E^x$  denote the cone of positive harmonic functions on  $\Omega$  which vanish on  $E \setminus \{x\}$  and at  $\infty$ . In [3] Benedicks showed that  $\mathcal{P}_E^x$  always has dimension 1 or 2. The proof of Theorem 1 will show that whether the path hits the intervals with probability less than or equal to 1 depends on whether  $\mathcal{P}_E^x$  has dimension 1 or 2; for a.e.  $x$  in the first case a Brownian path hits both  $[x - \varepsilon, x)$  and  $(x, x + \varepsilon]$  almost surely and in the second it does not. In Section 3 we will state and prove a version of this when  $E \subset \mathbb{R}^n$ ,  $\Omega = \mathbb{R}^{n+1} \setminus E$  and the line segments are replaced by cones.

In [3] Benedicks gives several criteria for determining the dimension of  $\mathcal{P}_E^x$ . For example,  $\mathcal{P}_E^x$  is one dimensional if every  $h \in \mathcal{P}_E^x$  is symmetric with

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respect to  $R$  and satisfies  $h(y) = o(|x - y|^{-1})$  and  $\dim(\mathcal{P}_E^x) = 2$  iff there is an  $h \in \mathcal{P}_E^x$  satisfying  $h(x, t) \geq C/t$  for some  $C > 0$  and  $t$  small. Also  $\dim(\mathcal{P}_E^x) = 2$  implies  $\omega([x - t, x + t]) = O(t)$ , but  $\dim(\mathcal{P}_E^x) = 1$  implies  $\omega([x - t, x + t])/t \rightarrow \infty$ . If  $|E| = 0$ , that is,  $E$  has zero Lebesgue measure, then it is easy to see that the set  $F$  of  $x \in E$  for which  $\dim(\mathcal{P}_E^x) = 2$  has harmonic measure 0. This is because by Benedicks' theorem and the Vitali covering lemma, we can cover  $F$  by intervals  $\{I_j\}$  such that  $\omega(I_j) \leq C|I_j|$  and  $\sum |I_j|$  is as small as we wish. Thus if  $|E| = 0$  the first alternative holds in Theorem 1 for  $\omega$  a.e  $x \in E$ .

Theorem 1 can also be stated in terms of a Cauchy process on the real line (see [15] and [18]). This is a process  $C_s$  with independent increments given by

$$P(C_{s+t} - C_s \in A) = \int_A \frac{t dx}{\pi(t^2 + x^2)}.$$

This process is not continuous and can also be defined as follows. If  $B^1$  and  $B^2$  are independent one-dimensional Brownian motions and  $T_s = \inf\{t \geq 0: B_t^2 = s\}$ , then  $C_s = B^1(T_s)$ . The trace of this process is the same as that of a two-dimensional Brownian motion hitting the real line, only time has been rescaled. Thus Theorem 1 says that if  $E$  has zero length and  $x \in E$  is the point where the process  $C_s$  first hits  $E$ , then almost surely the process hits every interval of the form  $[x - \varepsilon, x]$  and  $(x, x + \varepsilon]$ . This had been conjectured by Burdzy.

For  $x \in \mathbb{R}$  and  $t > 0$  let  $Q_t$  denote the square  $(x - t, x + t) \times (-t, t)$  and let  $\beta(x, t) = \omega(x, \partial Q_t, Q_t \setminus E)$ , that is,  $\beta$  is the harmonic measure at  $x$  of the sides of  $Q_t$  in  $Q_t \setminus E$ . Benedicks proved that  $\dim(\mathcal{P}_E^x) = 2$  if and only if for any  $0 < \alpha < 1$ ,

$$(1.1) \quad \int_{-1}^1 \beta(x + t, \alpha|t|) \frac{dt}{|t|} < \infty.$$

Benedicks' theorem is only stated in [3] for the point  $x = \infty$ , but this case easily implies the result stated above. We will show that a Brownian motion conditioned to exit at  $x$  will hit  $(x, x + \varepsilon)$  with probability 1 if and only if

$$\int_0^1 \beta(x + t, \alpha t) \frac{dt}{t} = \infty$$

[and similarly for  $(x - \varepsilon, x)$ ] so Theorem 1 reduces to proving the following theorem.

**THEOREM 2.** *For almost every  $x \in E$  (with respect to harmonic measure on  $\Omega$ )*

$$(1.2) \quad \int_0^1 \beta(x + t, \alpha t) \frac{dt}{t} < \infty$$

*if and only if*

$$(1.3) \quad \int_0^1 \beta(x - t, \alpha t) \frac{dt}{t} < \infty.$$

Now we come to the second topic of this paper. Suppose  $E \subset \mathbb{R}^2$  is compact. We will say a Brownian path  $\gamma$  in  $\Omega = \mathbb{R}^2 \setminus E$  separates  $E$  if there are

$0 \leq s < t < \tau$  ( $\tau$  is the time the path hits  $E$ ) such that  $\gamma(s) = \gamma(t)$  and so that the closed curve  $\Gamma = \gamma([s, t])$  separates  $E$  (i.e., has points of  $E$  in more than one of its complementary components). We will also refer to this as “making a loop around  $E$ .” A path is said to surround a point  $x \in E$  if there are  $s, t$  such that  $x$  is in a bounded component of  $\mathbb{R}^2 \setminus \Gamma$ . We shall call a set  $E$  *Brownian disconnected* if almost every Brownian path surrounds its exit point. This is stronger than saying  $E$  is a.s. separated by Brownian paths. Indeed any compact set can be given the latter property by adding a countable set [e.g., construct a set  $\{z_n\}$  such that for every  $z \notin E$  there is an  $n$  with  $|z - z_n| \leq \text{dist}(z, E)/2$ ].

Burdzy and Lyons have pointed out that if the set  $E$  has small enough Hausdorff dimension (definition in Section 4), then Brownian motion necessarily separates  $E$ . To see this, consider a Brownian path starting at distance  $\varepsilon$  from the origin and let it run until the first time it hits the unit circle. Let  $P(\varepsilon)$  denote the probability that the origin and  $\infty$  are in the same connected component of the path’s complement, and suppose that it satisfies  $P(\varepsilon) \leq C\varepsilon^\alpha$  for some fixed  $\alpha > 0$ . If  $E$  has finite  $\alpha$ -dimensional Hausdorff measure, then an application of the Borel–Cantelli lemma shows that  $E$  is necessarily separated by Brownian paths hitting it. It is easy to see that  $P(\varepsilon) \leq \varepsilon^\alpha$  for some  $\alpha$  and in [8] Burdzy and Lawler show that  $P(\varepsilon) \leq \varepsilon^{\pi^{-2}}$ . Thus  $\text{dim}(E) < \pi^{-2}$  implies  $E$  is almost surely separated. It is also true that if  $\text{dim}(E) < \frac{1}{2}$ , then  $E$  is Brownian disconnected (see Section 6).

The idea of a Brownian path separating the boundary is related to another problem: characterizing the compact sets  $E$  in  $\mathbb{R}^2$  such that  $E \cap \partial\Omega$  has zero harmonic measure in  $\Omega$  for any simply connected  $\Omega$ . We shall call such a set a SC-null set. If  $\Omega = \mathbb{R}^2 \setminus E$  and Brownian paths separate  $E$  a.s., then  $E$  has this property, since Brownian paths will have to hit  $\partial\Omega \setminus E$  a.s. before hitting  $E$ . Conversely, if  $E$  is a set which a Brownian motion is unlikely to separate, then  $\Omega = \mathbb{R}^2 \setminus E$  should “look” simply connected. The result that  $\text{dim}(E) < \frac{1}{2}$  implies  $E$  is Brownian disconnected is the analog of a result of Beurling that if  $\Omega$  is simply connected and  $E \subset \partial\Omega$  satisfies  $\Lambda_{1/2}(E) = 0$ , then  $\omega(E) = 0$  (here  $\Lambda_\alpha$  denotes  $\alpha$ -dimensional Hausdorff measure and  $\omega$  is harmonic measure on  $\Omega$ ). The result of Beurling has been dramatically improved by Makarov [21] who showed that if  $\Lambda_h(E) = 0$ , then  $\omega(E) = 0$ , where  $\Lambda_h$  is the Hausdorff measure corresponding to the function

$$h(t) = t \exp\left(C\sqrt{\log(1/t)\log\log\log(1/t)}\right)$$

and that this is sharp except for the choice of  $C > 0$ . In particular, this shows that  $\text{dim}(E) < 1$  implies  $E$  is SC-null. Therefore it is appropriate to conjecture that  $E$  is Brownian disconnected whenever  $\text{dim}(E) < 1$ . Our next result is to verify this for Denjoy domains.

**THEOREM 3.** *If  $E \subset \mathbb{R}$  and  $\text{dim}(E) < 1$ , then  $E$  is Brownian disconnected.*

If a path makes infinitely many loops around  $x$ , then it certainly hits both  $(x - \varepsilon, x)$  and  $(x, x + \varepsilon)$  for every  $\varepsilon$  so Theorem 3 is consistent with Theorem

1. Indeed, on the basis of Theorem 1 one might think that “ $\dim(E) < 1$ ” in Theorem 3 could be replaced by “ $|E| = 0$ ,” but this is not the case.

**THEOREM 4.** *There is a  $E \subset \mathbb{R}$  with  $|E| = 0$  which is not almost surely separated by Brownian motion.*

This example points out a distinction between SC-null sets and Brownian separated sets. This is because a theorem of Øksendal [25] says that a subset of  $\mathbb{R}$  is SC-null iff it has zero length, so the set constructed in Theorem 4 is SC-null even though it is not Brownian separated. Øksendal had conjectured that a subset of a rectifiable curve is SC-null iff it has zero length and this is proven in [5]. It follows from an example of Lavrentiev [20] that a general set with zero  $\Lambda_1$  measure need not be SC-null. Also see [19] and [22].

The term “Denjoy domain” is based on the theorem of Denjoy [13] that if  $E \subset \mathbb{R}$ , then  $\mathbb{C} \setminus E$  supports a nonconstant bounded holomorphic function iff  $E$  has positive Lebesgue measure. Denjoy domains provide interesting examples of multiply connected plane domains which are easier to deal with than a general domain because of the symmetry with respect to  $\mathbb{R}$ . See, for example, the papers of Rubel and Ryff [27], Carleson [10], Garnett and Jones [17] and Zinsmeister [30]. Some papers involving Brownian motion and the Martin boundary of Denjoy domains include [1], [3], [7], [11], [16] and [28]. Some related results are also discussed in [4] and [23]. Classical estimates on harmonic measure and problems related to harmonic measure on simply connected domains are discussed in the excellent survey [2].

In the next section we prove Theorems 1 and 2. In Section 3 we will state the higher-dimensional version of this result (Theorem 5) and indicate the (minor) modifications needed in the proof. In Section 4 we prove Theorem 3 and in Section 5 we prove Theorem 4. We finish with some remarks in Section 6.

**2. Proof of Theorems 1 and 2.** First some notation. As usual,  $B(x, r)$  denotes the disk of radius  $r$  centered at  $x$  and  $C$  will denote a constant (whose value may change from line to line). For  $E \subset \mathbb{R}$ ,  $|E|$  denotes the Lebesgue measure of  $E$ . If  $I$  is an interval,  $\lambda I$  denotes the concentric interval of length  $\lambda|I|$ .  $E$  will be a compact subset of the line and  $\Omega = \mathbb{R}^2 \setminus E$  its complement.  $\omega(z, F, \Omega)$  denotes the harmonic measure of the set  $F \cap \partial\Omega$  with respect to point  $z \in \Omega$ . For convenience we will often only write  $\omega(F, \Omega)$  or  $\omega(F)$  when the basepoint or domain are clear from context. The letters  $z$  and  $w$  will represent points in  $\mathbb{R}^2$  while  $x$  and  $y$  will denote real numbers. Thus for a function  $h$  defined on the plane, we will write either  $h(z)$  or  $h(x, y)$ . The notation  $a \sim b$  means that the ratio  $a/b$  is bounded and bounded away from 0. Given a point  $x \in E$ , the function  $h(z)$  will denote an element of  $\mathcal{P}_E^x$ . Brownian motion conditioned to exit at  $x \in E$  is the process with transition probability  $P_t^\Omega(z, dw)h(w)/h(z)$ , where  $P_t^\Omega$  is the transition probability for ordinary Brownian motion in  $\Omega$ . We may always assume  $h$  is symmetric with

respect to  $\mathbb{R}$  (under this assumption the conditioned process is uniquely defined). See [14, 26] for details.

Our first goal is to deduce Theorem 1 from Theorem 2. The main part of this argument is to establish the following lemma.

LEMMA 1. *Suppose that  $0 \in E$ ,  $I = [r, 2r]$ ,  $\tilde{I} = [r/2, 4r]$  and let  $u(z)$  denote the probability that a Brownian path starting at  $z$  and conditioned to exit at  $\{0\}$  will hit  $I$ . If  $|z| \geq 8r$ , then*

$$C^{-1} \int_I \beta(t, \alpha t) \frac{dt}{t} \leq u(z) \leq C \int_{\tilde{I}} \beta(t, \alpha t) \frac{dt}{t}.$$

*If we only consider  $|z| = 8r$  and let  $u$  be the probability of hitting  $I$  before leaving  $B(0, 16r) \setminus B(0, r/4)$ , we get the same estimate (except for the value of  $C$ ).*

The proof of Lemma 1 will be given at the end of this section. Clearly the right-hand side implies that the probability that a Brownian path starting at  $z$  (with  $|z| \geq 8r$ ) will hit the interval  $(0, r)$  is less than

$$C \int_0^{2r} \beta(t, \alpha t) \frac{dt}{t}.$$

Thus the finiteness of this integral implies that  $(0, r)$  will be hit with probability less than 1 if  $r$  is small.

On the other hand, suppose the integral in (1.2) diverges. Then we can choose a sequence of dyadic intervals  $\{I_j\} = \{r_j, 2r_j\}$  such that  $r_j \geq 64r_{j+1}$  but such that the sum of the corresponding  $\beta$ -integrals diverges. By considering the first time the Brownian path hits  $8r_j$  and using the extended Borel–Cantelli lemma [6], Corollary 5.29, we see that the path must hit infinitely many of the  $\{I_j\}$  (the last sentence of the lemma gives us an estimate of the probability of hitting  $I_j$  which does not depend on whether we hit any other interval).

A path conditioned to exit at  $x$  will hit  $(x, x + \varepsilon)$  almost surely iff the integral in (1.2) is infinite. For such a point Theorem 2 implies the integral (1.3) also diverges (except for an exceptional set of harmonic measure 0). Hence the path also hits  $(x - \varepsilon, x)$  with probability 1. Thus Theorem 2 implies Theorem 1.  $\square$

Now we turn to the proof of Theorem 2. It is enough to show that if the integral in (1.2) is finite on a set of positive harmonic measure, then the integral in (1.3) is finite on some subset of positive measure. Fix a point in  $\Omega$ , say  $\infty$ , and let  $\omega$  be harmonic measure on  $\partial\Omega$  with respect to this point. Let  $\delta_1, \delta_2$  and  $\delta_3$  be small positive numbers to be chosen later. Let

$$E_0 = \left\{ x \in E: \int_0^1 \beta(x + t, \alpha t) \frac{dt}{t} < \infty \right\}.$$

By hypothesis  $E_0$  has positive harmonic measure. Choose  $\varepsilon_1 > 0$  so that

$$E_1 = \left\{ x \in E_0 : \int_0^{\varepsilon_1} \beta(x+t, \alpha t) \frac{dt}{t} < \delta_1 \right\}$$

also has positive harmonic measure. Since points of density have full measure (e.g., [29], Theorem 10.49), there is an  $\varepsilon_2$  such that

$$E_2 = \left\{ x \in E_1 : \frac{\omega(E_1 \cap I)}{\omega(E \cap I)} > 1 - \delta_2, x \in I, |I| \leq \varepsilon_2 \right\}$$

has positive measure. Similarly,

$$E_3 = \left\{ x \in E_2 : \frac{\omega(E_2 \cap I)}{\omega(E \cap I)} > 1 - \delta_3, x \in I, |I| \leq \varepsilon_3 \right\}$$

has positive harmonic measure for some  $\varepsilon_3 > 0$ . (Passing to points of density twice may seem redundant, but is necessary for the proof of Lemma 3.) Finally, if we set

$$E_4 = \{ x \in E_3 : \exists y \in E_2, x - \varepsilon_3 < y < x - \varepsilon_4 \},$$

this set has positive harmonic measure for some  $\varepsilon_4 > 0$ . Otherwise,  $E_3$  would be the union of a set of zero measure and a countable set (right endpoints of intervals in  $\mathbb{R} \setminus E_2$ ).

We will show that the integral (1.3) is finite a.e. on the set  $E_4$ . It is convenient to record the following simple lemmas which will be proven at the end of this section.

LEMMA 2. *There exists  $C > 0$  so that if  $s \leq t \leq 2s$ , then  $\beta(x, t) \leq \beta(x, s) \leq C\beta(x, t)$ .*

LEMMA 3. *There exists  $C, \delta > 0$  so that if  $I \subset J$  and*

$$\int_I \beta(x, |I|) \frac{dx}{|I|} \leq \delta,$$

then

$$\omega(E \cap I, \Omega) \geq C(|I|/|J|)\omega(J, \Omega \setminus J) \geq C(|I|/|J|)\omega(E \cap J, \Omega).$$

*In particular, for any  $\eta > 0$ , there are  $\delta, C > 0$  such that if  $\delta_1 \leq \delta$ ,  $x \in E_1$ ,  $J = [x, x+r]$ ,  $r \leq \varepsilon_1$  and  $I \subset J$  has length at least  $\eta|J|$ , then  $\omega(E \cap I, \Omega) \geq C\omega(E \cap J, \Omega)$ .*

LEMMA 4. *For any  $\eta > 0$  there is a  $\delta > 0$  such that if  $\delta_1, \delta_2, \delta_3 \leq \delta$ ,  $x \in E_4$  and  $0 < r \leq \varepsilon_4$ , then there is a  $y \in E_2$  such that  $x - r \leq y \leq x - r(1 - \eta)$ .*

Assuming these for the moment, let us complete the proof of Theorem 2. By Lemma 2 it is clear that the convergence of the integrals in question does not depend on  $\alpha$ . For convenience we set  $\alpha = \frac{1}{2}$ . Without loss of generality we may also assume that  $E_0 \subset [-1, 1]$ . Choose an integer  $N$  so that  $2^{-N} \leq \varepsilon_4$  and for

$n \geq N$  let  $\mathcal{E}_n$  denote the collection of dyadic intervals in  $[-1, 1]$  of length  $2^{-n}$  which meet  $E_4$ . Let  $\mathcal{E} = \cup \mathcal{E}_n$ . To each  $I \in \mathcal{E}_n$  we associate another interval  $J = [2^{-n}, 2^{-n+1}]$ . Note that

$$(2.1) \quad \int_{E_4} \int_0^{2^{-N+1}} \beta(x - t, \alpha t) \frac{dt}{t} d\omega(x) = \sum_{\mathcal{E}} \int_{I \cap E_4} \int_J \beta(x - t, \alpha t) \frac{dt}{t} d\omega(x),$$

and that to prove Theorem 2, it suffices to show the integral on the left is finite. Given  $I = [a, b] \in \mathcal{E}_n$ , we also associate an interval  $I' = [a', b'] = I - 4|I|$  (translate  $I$  to the left). Now observe that if  $x \in I, y \in I'$  and  $t \in J$ , then  $x - t = y + s$  for some  $s \sim t$ . Thus by Lemma 2 there is a  $C > 0$  such that

$$\frac{1}{C} \leq \frac{\beta(x - t, \alpha t)}{\beta(y + s, \alpha s)} \leq C,$$

so for any  $x \in I$  and  $y \in I'$ ,

$$\int_J \beta(x - t, \alpha t) \frac{dt}{t} \leq C \int_{J'} \beta(y + s, \alpha s) \frac{ds}{s},$$

where  $J' = [C^{-1}2^{-n}, C2^{n+1}]$  is chosen so  $t \in J$  implies  $s \in J'$ . By Lemma 4 (with small enough  $\eta$ ), there is a point  $y \in E_2$  in  $I''$ , the left half of  $I'$ . By the definition of  $E_2$ ,

$$\omega(E_1 \cap I') \geq C\omega(E \cap I') \geq C\omega(E \cap (I' \setminus I'')).$$

By Lemma 3,

$$\omega(E \cap (I' \setminus I'')) \geq C\omega(E \cap [a', b]) \geq C\omega(E \cap I) \geq C\omega(E_4 \cap I).$$

Therefore

$$\int_{E_4 \cap I} \int_J \beta(x - t, \alpha t) \frac{dt}{t} d\omega(x) \leq C \int_{E_1 \cap I'} \int_{J'} \beta(y + t, \alpha t) \frac{dt}{t} d\omega(y).$$

Moreover, any pair  $(y, t) \in I' \times J'$  is used at most a bounded number of times as  $I$  ranges over all of  $\mathcal{E}$ . Thus summing these integrals over all of  $\mathcal{E}$ , using (2.1) and the definition of  $E_1$ ,

$$\begin{aligned} \int_{E_4} \int_0^{2^{-N+1}} \beta(x - t, \alpha t) \frac{dt}{t} d\omega(x) &\leq C \sum_{\mathcal{E}} \int_{I' \cap E_1} \int_{J'} \beta(y + t, \alpha t) \frac{dt}{t} d\omega(y) \\ &\leq C \int_{E_1} \int_0^{\epsilon_4} \beta(y + t, \alpha t) \frac{dt}{t} d\omega(y) \\ &\leq C\omega(E_1)\delta_1 \\ &< \infty, \end{aligned}$$

as required. This completes the proof of Theorem 2, except for the proofs of the lemmas.  $\square$

**PROOF OF LEMMA 1.** We begin by reviewing some facts about harmonic measure on  $\Omega$  from [3]. Let  $x \in \mathbb{R}, t > 0, J = (x - t, x + t), Q = J \times (-t, t)$

and  $K$  the two sides of  $Q$  parallel to  $\mathbb{R}$ . Then [3], Lemma 7, says that  $\omega(x, K, Q \setminus E) \geq \frac{1}{2}\omega(x, \partial Q, Q \setminus E)$  (just as we would expect). Let  $L$  be the vertical segment with endpoints  $x, z_0 = (x, t)$  and suppose  $f$  is a bounded, positive, symmetric, harmonic function on  $\Omega$  which vanishes on  $E \cap Q$ . We claim that

$$\max_{w \in L} f(w) \leq Cf(z_0).$$

On the top half of  $L$  this follows from Harnack's inequality. For  $w$  in the bottom half the maximum principle gives an arc  $\gamma$  connecting  $w$  to  $\partial Q$  on which  $f(z) \geq f(w)$ . By symmetry we may assume  $\gamma$  lies in the upper half plane  $H$  so

$$f(z_0) \geq f(w)\omega(z_0, \gamma, H \setminus \gamma) \geq Cf(w),$$

as desired.

We now prove the lemma. The function  $u$  is  $h$ -harmonic, that is, it is of the form  $u = v/h$ , where  $v$  is a harmonic function in  $\Omega$  and is given by the formula [14, page 672],

$$(2.2) \quad u(z) = \frac{v(z)}{h(z)} = \int_I \frac{h(w)}{h(z)} d\omega(z, \Omega_0),$$

where  $\Omega_0 = \Omega \setminus I$  (one can also check that  $v$  is harmonic and has the correct boundary values). We will first prove the case  $z = z_0 = (0, 8r)$ . Consider a point  $x \in \mathbb{R}$  and let  $Q$  be the square with center  $x$  and sidelength  $2\alpha|x|$  [i.e., the square in the definition of  $\beta(x, \alpha|x|)$ ]. Harnack's inequality and the preceding paragraph imply that  $h(z) \leq Ch(z_0)$  for  $z \in \partial Q$  and  $h(z) \geq Ch(z_0)$  for  $z$  on  $K$ , the "top" and "bottom" sides of  $Q$ . Thus, using [3], Lemma 7,

$$(2.3) \quad h(x, 0) = \int_{\partial Q} h(w) d\omega(x, Q \setminus E) \sim h(z_0)\beta(x, \alpha|x|).$$

If  $\omega_0$  denotes the harmonic measure on  $\Omega_0$  with respect to the point  $z_0$ , then using the observation  $H \subset \Omega_0 \subset \mathbb{R}^2 \setminus I$  ( $H$  is the upper half plane), shows that on  $I$ ,

$$(2.4) \quad C^{-1} \frac{dx}{|I|} \leq d\omega_0 \leq C \left( \frac{\text{dist}(x, \mathbb{R} \setminus I)}{|I|} \right)^{-1/2} \frac{dx}{|I|}.$$

By the left-hand side of (2.4) and Harnack's inequality,

$$u(z) \geq \int_I \beta(t, \alpha t) d\omega_0(z) \geq C \int_I \beta(t, \alpha t) \frac{dt}{t}.$$

To prove the right-hand side of the inequality, let  $J_s = [s, 4s]$ . Then for  $r/2 \leq s \leq r$  we have  $I \subset J_s \subset \tilde{I}$ . So if  $u_s$  is the probability of hitting  $J_s$ , we have  $u \leq u_s$ . Set  $\Omega_s = \Omega \setminus J_s$  and let  $\chi_s$  denote the indicator function of  $J_s$



and note that for all  $t \in R$ ,

$$\frac{2}{r} \int_{r/2}^r (\text{dist}(x, \mathbb{R} \setminus J_s) / s)^{-1/2} \chi_s(t) \, ds \leq C.$$

Using this, (2.3), (2.4) and Fubini's theorem, we get

$$\begin{aligned} u(z) &\leq \frac{2}{r} \int_{r/2}^r u_s(z) \, ds \\ &\leq C \frac{2}{r} \int_{r/2}^r \left\{ \int_{J_s} \beta(t, \alpha t) \, d\omega(z, \Omega_s) \right\} ds \\ &\leq C \int_I \left\{ \frac{2}{r} \int_{r/2}^r (\text{dist}(t, \mathbb{R} \setminus J_s) / s)^{-1/2} \chi_s(t) \, ds \right\} \beta(t, \alpha t) \frac{dt}{t} \\ &\leq C \int_I \beta(t, \alpha t) \frac{dt}{t}. \end{aligned}$$

If  $|z| = 8r$ , simply note that since  $v$  vanishes on  $E \cap \{4r \leq |z| \leq 16r\}$ , the argument which proved (2.3) applies to  $v$  and shows

$$(2.5) \quad v(x, 0) \sim v(z_0) \beta(x, \alpha|x|) \sim h(z_0) u(z_0) \beta(x, \alpha|x|),$$

for, say,  $x \in S = \{6r \leq |x| \leq 10r\}$ . If  $P_w$  denotes the Poisson kernel for the upper half plane and  $|z| = 8r$ , then clearly

$$P_z(t) \leq CP_{z_0}(t) + P_z(t) \chi_S(t).$$

This, (2.5),  $\beta \leq 1$  and the facts  $h(z)u(z_0) \sim v(z)$  for  $z \in S$  and  $z = z_0$  imply

$$v(z) \leq Ch(z_0)u(z_0) + Ch(z_0)u(z_0) \int_S \beta(x, \alpha|x|) P_z(x) \, dx \leq Ch(z_0)u(z_0).$$

[Since  $h$  is unbounded, one should consider  $H \setminus B(0, \varepsilon)$  for some  $\varepsilon \ll r$  instead of  $H$ , but the argument is the same.] Reversing the roles of  $v$  and  $h$  gives  $v(z) \sim h(z)u(z_0)$  for all  $|z| = 8r$ . Thus  $u(z) \sim u(z_0)$  for such  $z$ . The case  $|z| \geq 8r$  follows immediately.

To prove the final claim, we merely repeat the proof with  $\Omega_0 = (\Omega \setminus I) \cap \{r/4 < |z| < 16r\}$ . The estimate (2.4) for  $\omega_0$  on  $I$  still holds, so the proof is unchanged.  $\square$

PROOF OF LEMMA 2. If  $s \leq t$ , then  $\beta(x, t) \leq \beta(x, s)$  by the maximum principle. If  $s \leq t \leq 2s$  let  $K$  denote the two sides of  $Q_s$  parallel to  $\mathbb{R}$  (as above). There is clearly a  $C > 0$  such that

$$\omega(z, \partial Q_t, Q_t \setminus E) > C, \quad z \in K.$$

Also by [3], Lemma 7,  $\omega(x, K, Q_s \setminus E) \geq \frac{1}{2} \omega(x, \partial Q_s, Q_s \setminus E)$ . Therefore by the

maximum principle,

$$\omega(x, \partial Q_t, Q_t \setminus E) \geq \frac{C}{2} \omega(x, \partial Q_s, Q_s \setminus E),$$

which is the desired inequality.  $\square$

PROOF OF LEMMA 3. Let  $I \subset J = [x, x + r]$  and let  $I' = \frac{1}{3}I$  be the middle third of  $I$ . By hypothesis and Lemma 2,

$$(2.6) \quad \int_{I'} \beta(x, |I'|) dx \leq C\delta|I'|.$$

By Tchebyshev's inequality there is an  $A \subset I'$  with  $|A| \geq |I'|/2$  and  $\beta(t, |I'|) \leq 2C\delta \leq \frac{1}{2}$  for  $t \in A$  (if  $\delta$  is small enough). This implies  $\omega(z, E \cap I, \Omega) \geq \frac{1}{2}$  for  $z \in A$ . Let  $\Omega_0 = \Omega \setminus J$  and note that  $d\omega_0 \geq C\omega(J) dx/|J|$  on  $I'$ . Our remark above implies  $\omega(z, E \cap I, \Omega) \geq \omega(z, A, \Omega_0)/2$  for every  $z \in \partial\Omega_0$  and hence for all  $z \in \Omega_0$  by the maximum principle. Thus

$$\omega(E \cap I, \Omega) \geq \frac{1}{2} \omega(A, \Omega_0) \geq C \frac{|A|}{|J|} \omega(J, \Omega_0) \geq C \frac{|I|}{|J|} \omega(E \cap J, \Omega),$$

which is the desired inequality. To prove the second statement, note that these hypotheses and Lemma 2 imply (2.6) with  $\delta = \delta_1$  and a constant  $C = C(\eta)$ . Thus the conclusion still holds if  $\delta$  is small enough.  $\square$

PROOF OF LEMMA 4. Suppose  $x \in E_4$ . First, we show there is a sequence  $\{z_n\} \subset E_2$  with  $z_n \rightarrow x$  and  $x - z_n \sim 2(x - z_{n+1})$ . By the definition of  $E_4$ , there is a  $y \in E_2$  with  $x - \varepsilon_3 < y < x - \varepsilon_4$ . Now set  $s = \frac{1}{2}$ ,  $\eta = \frac{1}{4}$ . Let  $w = x - s(x - y)$  and  $I = [w, w + \eta(x - y)]$ . By Lemma 3, if  $\delta_1$  is small enough, then  $\omega(E \cap I) \geq C\omega(E \cap [y, x]) > 0$ . If  $\delta_3$  is small enough, then the definition of  $E_3$  implies  $\omega(E_2 \cap I) > 0$  and so  $E_2 \cap I \neq \emptyset$ . (This is where we use the definition of  $E_3$ ; otherwise, the sequence would only lie in  $E_1$ , not in  $E_2$ .)

By induction we obtain a sequence of points  $z_1 < z_2 < \dots < x$  in  $E_2$  with  $\frac{1}{4} \leq (x - z_{n+1})/(x - z_n) \leq \frac{1}{2}$ . With  $r$  as in the lemma, choose  $n$  so that  $x - z_n > r \geq x - z_{n+1}$  and then apply the argument above with  $s = r/(x - z_n)$ ,  $y = z_n$  and the desired  $\eta$ . This gives a point  $z \in E_2$  with the correct estimate.  $\square$

**3. Higher dimensions.** For  $\theta > 0$  and  $\sigma \in \{z \in \mathbb{R}^n : |z| = 1\}$ , let  $C(\sigma, \theta)$  denote the spherical cap with center  $\sigma$  and subtended angle  $\theta$ . Let  $\Gamma(\sigma, \theta, \delta) = \{r \cdot z : 0 < r \leq \delta, z \in C(\sigma, \theta)\} \subset \mathbb{R}^n$  be the cone of radius  $\delta$  subtended by this spherical cap.

**THEOREM 5.** *Suppose  $E \subset \mathbb{R}^n$  is a proper closed subset and let  $\Omega = \mathbb{R}^{n+1} \setminus E$ . Then for almost every  $x \in E$  (with respect to harmonic measure) a Brownian motion conditioned to exit at  $x$  hits every cone  $\Gamma(\sigma, \theta, \delta)$  with probability 1 if and only if it hits any such cone with probability 1. In particular, this happens if  $\Lambda_n(E) = 0$ .*

Benedicks' results all hold in  $\mathbb{R}^n$  except that the integral in (1.1) is replaced by

$$(3.1) \quad \int_{|y| \leq 1} \beta(x + y, \alpha|y|) \frac{dy}{|y|^n}.$$

The argument to deduce Theorem 1 from Theorem 2 goes through just as before, so to deduce the theorem all we need do is show that

$$(3.2) \quad \int_{\Gamma(\sigma, \theta, 1)} \beta(x + y, \alpha|y|) \frac{dy}{|y|^n} < \infty$$

on a set of positive harmonic measure implies (3.1) is finite on a subset of positive harmonic measure. The proof goes exactly as before until the definition of  $E_4$ . We now set

$$E_4 = \{x \in E_3 : \exists y \in E_3 \cap (x - \Gamma(\sigma, \theta/2, \varepsilon_3)), |y - x| > \varepsilon_4\}.$$

To prove  $E_4$  has positive harmonic measure, note that if  $x \in E_3$  is not in  $E_4$  for any choice of  $\varepsilon_4$ , then  $x$  is the vertex of a cone in  $\mathbb{R}^n \setminus E_3$ . The set of such points is well known to have sigma finite  $(n - 1)$ -dimensional Hausdorff measure and so has zero Newtonian capacity in  $\mathbb{R}^{n+1}$ . Therefore  $E_4$  must have positive harmonic measure for some  $\varepsilon_4$ . The proof now proceeds as before, replacing intervals by cones or cubes. The only other remark which is needed is that

$$\int_{r \leq |y| \leq 2r} \beta(x + y, \alpha|y|) \frac{dy}{|y|^n} \leq M \int_{y \in \Gamma(\sigma, \theta, 2Cr) \setminus B(x, r)} \beta(z + y, \alpha|y|) \frac{dy}{|y|^n}$$

for some  $C, M > 0$  and any  $z \in x - \Gamma(\sigma, \theta/2, Cr)$  with  $|z - x| \geq r$ . Details are left to the reader.

**4. Proof of Theorem 3.** Given an increasing continuous function  $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $h(0) = 0$ , we set

$$\Lambda_h(E) = \lim_{\delta \rightarrow 0} \left( \inf \left\{ \sum h(r_j) : E \subset \cup B(x_j, r_j), r_j \leq \delta \right\} \right).$$

This is called the Hausdorff measure associated to  $h$  and for  $h(t) = t^\alpha$  we simply write  $\Lambda_\alpha$ . See [9] for details. We also define

$$\dim(E) = \inf \{ \alpha : \Lambda_\alpha(E) = 0 \}.$$

To prove Theorem 3, we will show that there is a subset  $F \subset E$  of positive harmonic measure which satisfies

$$\omega(B(x, r) \cap F) \leq Cr^{1-\nu}$$

for every  $x \in F$ , where  $C$  depends only on  $\nu$ . Then for any covering  $\{B(x_j, r_j)\}$  of the set  $F$  by disks, we have

$$0 < \omega(F) \leq \sum \omega(B(x_j, r_j)) \leq C \sum r_j^{1-\nu},$$

which implies  $\dim(E) \geq \dim(F) \geq 1 - \nu$ . Taking  $\nu \rightarrow 0$ , we obtain Theorem 3.

The idea is that if  $x \in E$  is a point around which Brownian motion is unlikely to loop, then  $\Omega$  must look simply connected near  $x$ , that is,  $E$  must look like a line, a ray or a union of two rays. On such domains harmonic measure looks like linear measure except near the endpoint of a ray where it has a square-root-type singularity. A typical point will not lie too near the end of a ray, however, so harmonic measure will look usually like arclength.

To make this precise, let  $F \subset E$  denote a subset of positive harmonic measure such that for each point  $x \in F$  a Brownian path conditioned to exit at  $x$  loops around  $x$  with probability less than 1. By passing to a subset, we may also assume that given  $\varepsilon > 0$  (to be fixed later) there is a small  $\varepsilon_0 > 0$  such that a Brownian path starting at a point  $z = (x, y)$  with  $x \in F$  and  $|y| < \varepsilon_0$  and conditioned to exit at  $x$  has probability less than  $\varepsilon$  of making a loop around  $x$ . Given a  $\delta_1 > 0$  (to be fixed later), define

$$F_1 = \left\{ x \in F: \frac{\omega(F \cap I)}{\omega(E \cap I)} > 1 - \delta_1, |I| \leq \varepsilon_1, x \in I \right\}.$$

We assume  $\varepsilon_1 < \varepsilon_0$  is chosen so this set has positive measure.

An interval  $I$  will be called “good” if  $|I| \leq \varepsilon_0$  and  $\frac{1}{3}I$  contains a point of  $F_1$ . Divide  $I$  into  $N$  subintervals,  $\{I_j\}$ , all with length  $N^{-1}|I| \leq |I_j| \leq CN^{-1}|I|$ . Suppose  $\eta > 0$  (to be fixed later) and call such a subinterval “bad” if  $\omega(I_j) \geq \omega(I)N^{\eta-1}$  (i.e., if it has more harmonic measure than expected, assuming  $\omega$  looks like linear measure). Note that, despite the names, an interval may be both good and bad.

LEMMA 5. *Given  $\delta$  there is an integer  $N$  and  $\delta_1, \varepsilon > 0$  such that the following holds. Suppose  $I$  is a good interval and that the  $\{I_j\}$  are defined as above. Let  $K$  denote the union of the “bad” subintervals. Then  $\omega(K) \leq \delta\omega(I)$ . If  $I_j$  is a bad subinterval which hits  $F_1$ , then  $\omega(I_j) \leq \omega(I)N^{-1/4}$ .*

[The assumption that  $I$  is good is necessary, as can be seen by considering  $E = [0, \infty)$ ,  $I = [-1, N^{-1}]$ , that is, an interval which is mostly empty. Assuming  $I$  contains points of  $F_1$  near its center eliminates this problem.]

PROOF OF LEMMA 5. To prove Lemma 5, we may assume  $I = [-1, 1]$ . Suppose that  $[-2, 2]$  has been divided into disjoint subintervals  $\{I_j\}$  with  $N^{-1} \leq |I_j| \leq CN^{-1}$ . For each  $j$  let

$$\beta_{I_j} = \beta_j = \int_{I_j} \beta(x, |I_j|) \frac{dx}{|I_j|}.$$

Let  $\Omega_j = \Omega \setminus (\mathbb{R} \setminus 3I_j)$ . We claim that if  $B_{\pm} = B(\pm i, \frac{1}{2})$ , then

$$(4.1) \quad \omega(i, \partial B_-, \Omega_j \setminus B_-) \geq C\beta_j/N^2.$$

To see this, suppose  $I_j = [a, b]$  and let  $z_j = ((a + b)/2, |I_j|)$  and  $B_j = B(\bar{z}_j, |I_j|/2)$ . Using the definition of  $\beta$ , [3, Lemma 7], symmetry and the fact

that  $\omega(z_j, I_j, H) \geq C > 0$ ,

$$\omega(z_j, B_j, \Omega_j \setminus B_j) \geq C \int_{I_j} \beta(x, |I_j|) d\omega(z_j, H) \geq C \int_{I_j} \beta(x, |I_j|) \frac{dt}{|I_j|} = C\beta_j.$$

By standard estimates for the upper half plane  $H$ , we get  $\omega(i, I_j, H) \sim \omega(z_j, B_+, H) \sim N^{-1}$ . Harnack's inequality, symmetry and the Markov property of Brownian motion imply (4.1).

As in (2.2), the probability of hitting  $B_-$  by a path starting at  $i$  and conditioned to exit at some  $x \in E$  is given by  $(h \in \mathcal{P}_E^x)$ ,

$$\frac{1}{h(i)} \int_{\partial B_-} h(w) d\omega(i, \Omega_j \setminus B_-) \geq \frac{\omega(i, \partial B_-, \Omega_j \setminus B_-)}{h(i)} \inf_{w \in B_-} h(w).$$

Since by Harnack and the symmetry of  $h$ ,  $h(w) \sim h(i)$  for  $w \in B_-$ , we get the same estimate (except for  $C$ ) for conditioned Brownian motion as for unconditioned Brownian motion.

Thus the probability that a conditioned Brownian path starting at  $i$  will cross  $\mathbb{R}$  through  $3I_j$ , hit  $B_-$ , then cross  $\mathbb{R}$  through  $3I_k$  and hit  $B_+$  is at least  $C\beta_j\beta_k N^{-4}$ . Hence for any  $\beta > 0$  there is an  $\varepsilon = \varepsilon(\beta, N) > 0$  so that any two intervals  $I_j, I_k$  with non-overlapping triples and  $\beta_j, \beta_k \geq \beta$  cannot be separated by a point of  $F$  (otherwise the probability of looping around this point would be too large). Also, if  $\beta_j \leq \beta$  is small enough, then  $I_j$  must hit  $F$  [because  $\beta_j$  small implies  $\omega(\frac{1}{3}I_j)$  has harmonic measure at least comparable to  $\omega(I)/N$  (Lemma 3) and so if  $\delta_1$  in the definition of  $F_1$  is small enough,  $\frac{1}{3}I_j$  will contain a point of  $F$ ]. Therefore the triples of two intervals with  $\beta_j \geq \beta$  cannot be separated by one with  $\beta_k \leq \beta$ . Let  $J$  be the minimal interval in  $[-2, 2]$  containing all the intervals with  $\beta_j \geq \beta$ . By the remarks above and the assumption that all the intervals have size approximately  $N^{-1}$ , we see that every  $I_j$  in  $J$  has  $\beta_j \geq \beta$  except possibly for a bounded number near each endpoint of  $J$  (i.e., there is a  $C$  such that  $CI_j \subset J$  implies  $\beta_j \geq \beta$ ). In particular,  $J$  does not hit  $F$ , except possibly within distance  $C/N$  of its endpoints. Since  $F_1 \cap [-\frac{1}{3}, \frac{1}{3}] \neq \emptyset$  we can deduce  $[-1, 1] \setminus J$  has length at least  $\frac{1}{6}$  (if  $N$  is large).

Now we verify that the collection of bad intervals in  $I \setminus J$  has small harmonic measure. Let  $K = [-2, 2] \setminus J$ ,  $\Omega_0 = \Omega \setminus K$  and let  $\omega_0$  denote harmonic measure on  $\Omega_0$  (also with respect to  $\infty$ ).  $K$  consists of either one or two intervals,  $K_1, K_2$ , at least one of length greater than or equal to 1 and the other (if it exists) of length greater than or equal to  $N^{-1}$  (since it is a union of  $I_j$ 's). Since  $|I \setminus J| \geq \frac{1}{6}$ ,  $\omega_0(K) \sim \omega_0(I \setminus J) \sim \omega_0(I)$  (this is where we need that  $I$  is "good"). Furthermore, since  $\beta$  is small on most of  $I \setminus J$ ,  $\omega(I) \sim \omega_0(I)$  (Lemma 3). Also note [as in (2.4)]

$$\begin{aligned} (4.2) \quad d\omega_0 &\leq C\omega_0(I) \left( \frac{\text{dist}(x, \mathbb{R} \setminus K)}{|K_i|} \right)^{-1/2} \frac{dx}{|K_i|} \\ &= \omega_0(I) f(x) dx, \quad x \in K_i, i = 1, 2, \end{aligned}$$

and  $f \in L^p(K, dx)$  for  $p < 2$ . In particular, it is in  $L^{4/3}(K, dx)$  with norm at most  $\max_i |K_i|^{-1/4} \leq N^{1/4}$ . We claim that for an interval  $I_j \subset [-1, 1] \setminus J$ , then  $\omega(I_j) \leq 2\omega_0(3I_j)$ . Clearly  $\omega_0(I_j) \leq C\omega_0(I)/N$  and the definition of  $\beta(x, t)$  implies that the probability of first hitting  $\partial\Omega_0 \setminus (\partial\Omega \cup 3I_j)$  and then hitting  $I_j$  is less than

$$\int_{K \setminus 3I_j} \beta(x, N^{-1}) d\omega_0(x).$$

Hölder’s inequality (with  $p = 4, q = \frac{4}{3}$ ) and the fact  $\beta \leq 1$  give

$$\begin{aligned} \int_K \beta(x, N^{-1}) d\omega_0(x) &\leq C\omega_0(I) \int_K \beta(x, N^{-1}) f(x) dx \\ &\leq C\omega_0(I) \left( \int_K \beta(x, N^{-1})^4 dx \right)^{1/4} \left( \int_K f(x)^{4/3} dx \right)^{3/4} \\ &\leq C\omega_0(I) \left( \int_K \beta(x, N^{-1}) dx \right)^{1/4} \max_{i=1,2} |K_i|^{-1/4} \\ &\leq C\omega_0(I) \left( \sum_{I_j \subset K} \beta_j |I_j| \right)^{1/4} N^{1/4} \\ &\leq C\omega_0(I) \beta^{1/4} N^{1/4} \\ &\leq \omega_0(I_j) \end{aligned}$$

if  $\beta \leq N^{-5}/C$ . Thus  $\omega(I_j) \leq 2\omega_0(3I_j)$ .

Suppose  $r > 0$  (to be chosen below) and suppose  $\text{dist}(I_j, J) \geq r \geq N^{-1}$ . Since  $I_j \subset I \setminus J$  it is in a component of  $K$  of length at least 1, so by (4.2),

$$\omega(I_j) \leq C\omega_0(3I_j) \leq C\omega_0(I)|I_j|r^{-1/2} \leq C\omega(I)N^{-1}r^{-1/2} \leq \omega(I)|I_j|^{1-\eta}$$

if  $N$  is large enough (depending on  $C, r, \eta$ ). If  $\text{dist}(I_j, J) \leq r$ , then  $\omega_0(I_j) \leq C\omega(I)|I_j|^{1/2} \leq \omega(I)|I_j|^{1/4}$  (if  $N$  is large enough, depending on  $C$ ). Moreover, the total harmonic measure of the intervals satisfying  $\text{dist}(I_j, J) \leq r$  is less than  $C\sqrt{r}$  which is less than  $\delta$  (if  $r$  is small enough).

Now we show that  $J$  must have small harmonic measure. If  $I_1, I_2$  are the intervals of length  $r$  at the ends of  $J$ , then these have small harmonic measure by an argument similar to the previous paragraph. The rest of  $J$  contains no points of  $F$  (if  $N^{-1} \ll r$ ) so must have harmonic measure less than  $\delta_1\omega(I)$  by the definition of  $F_1$  and the fact that  $I$  contains a point of  $F_1$ . This completes the proof of the lemma.  $\square$

Now we use the lemma to deduce Theorem 3. We claim that there is an interval  $I_0$ , a collection of subintervals  $\{I_j^n\}$  and a set  $F_2 \subset F_1 \cap I_0$  so that the following conditions hold. First  $F_2$  has positive harmonic measure. For each  $n$  the  $\{I_j^n\}$  are disjoint (except for endpoints),  $\cup_j I_j^n = I_0$ , and  $N^{-n}|I_0| \leq |I_j^n| \leq 20N^{-n}|I_0|$ . Furthermore, the intervals are nested (i.e.,  $I_j^n \subset I_k^{n-1}$  for some  $k$ ).

Finally,  $I_j^n \cap F_2 \neq \emptyset$  implies  $\frac{1}{3}I_j^n \cap F_1 \neq \emptyset$  (i.e., if an interval hits  $F_2$  it must be good). Assuming all this for the moment, we continue with the proof.

By rescaling we assume  $I_0 = [0, 1]$ . Given  $\delta, \eta$  (to be chosen below), let  $N, \varepsilon, \delta_1$  be as in Lemma 5. Consider the process which successively chooses subintervals of the current interval according to their relative harmonic measures and stops whenever an interval not hitting  $F_2$  is chosen. The process is naturally parameterized by  $I_0$ . Let  $x \in F_2$  (so we never stop) and suppose  $I^n$  is the  $n$ th-generation interval containing  $x$ . Let  $T_n(x)$  be the number of bad intervals in  $\{I^1, \dots, I^n\}$ . We claim that for  $\omega$  a.e.  $x$ ,  $T_n(x) \leq 3\delta n$  for all  $n$  large enough (depending on  $x$ ). This is almost the strong law of large numbers except that we do not quite have independent events. We shall prove it below using martingale convergence. Given the claim, we easily see that if  $I^n$  is an  $n$ th-generation interval hitting  $F_2$ ,

$$\omega(I^n) \leq \omega(I_0) N^{-n3\delta(1/4) - n(1-3\delta)(1-\eta)} \equiv \omega(I_0) N^{-(1-\nu)n}.$$

Since  $|I^n| = N^{-n}$  this implies  $\omega(I^n) \leq \omega(I^0)|I^n|^{1-\nu}$  and clearly if  $\eta$  and  $\delta$  are small, then  $\nu$  is also small. If  $J$  is an arbitrary interval, choose  $k$  so that  $N^{-k} \leq |J| < N^{-k+1}$ . Then  $J$  intersects at most  $N + 1$  intervals  $\{I_j\}$  of the  $n$ th-generation so

$$\omega(J) \leq \sum \omega(I_j) \leq (N + 1)|J|^{1-\nu}.$$

Thus we have a set  $F \subset E$  of positive  $\omega$  measure and a  $\nu > 0$  such that if  $J$  is any small enough interval,  $\omega(J \cap F) \leq C|J|^{1-\nu}$ . By our earlier remarks we see that  $\dim(F) \geq 1 - \nu$ .

Now we prove the claim that  $T_n(x) \leq 3\delta n$  for  $\omega$  a.e.  $x$ . For each  $n$  define a function  $f_n$  on  $I_0$  as follows. Let  $I$  be a  $(n - 1)$ st-generation interval. If  $I$  does not hit  $F_2$ , then  $f_n$  is 0 on  $I$ . Otherwise, let  $K$  be the union of the bad subintervals of  $I$  and let  $f_n = 1$  on  $K$  and equal  $-\omega(K)/\omega(I \setminus K) \geq -2\delta$  on  $I \setminus K$ . Doing this for every  $(n - 1)$ st-generation interval defines  $f_n$ . Clearly  $f_n$  has mean 0 (with respect to  $\omega$ ) on each  $(n - 1)$ st-generation interval and is constant on each  $n$ th-generation interval so that  $S_n = \sum_{k=1}^n f_k$  is a martingale with respect to the sigma algebras generated by the intervals  $\{I_j^n\}$ ,  $n = 1, 2, \dots$ , and the measure  $\omega$ . Since  $\|f_n\|_\infty \leq 1$ ,  $X_n = \sum_{k=1}^n f_k/k$ , is a martingale uniformly bounded in  $L^2$  (and hence in  $L^1$ ). Thus it converges a.e. ( $\omega$ ) [12, Theorem 9.4.4]. Therefore by Kronecker's lemma,

$$\frac{S_n}{n} = \sum_{k=1}^n \left(\frac{k}{n}\right) \left(\frac{f_k}{k}\right)$$

tends to 0 a.e. as  $n \rightarrow \infty$  (this is just Lebesgue dominated convergence on the integers). Thus for  $n$  large enough (depending on  $x$ ),  $S_n \leq \delta n$ . Since  $f_n = 1$  on the bad intervals and is greater than or equal to  $-2\delta$  elsewhere, we see that  $T_n - 2\delta(n - T_n) \leq \delta n$ , hence  $T_n \leq 3\delta n/(1 + 2\delta) \leq 3\delta n$ , as desired.

Finally, we must construct the set  $F_2$  and the intervals  $\{I_j^n\}$ . Given  $\delta_2$  (to be fixed below), let

$$F_2 = \left\{ x \in F_1 : \frac{\omega(F_1 \cap I)}{\omega(E \cap I)} > 1 - \delta_2, |I| \leq \varepsilon_2, x \in I \right\}.$$

Let  $I_0$  be an interval of length less than or equal to  $\varepsilon_2$  which hits  $F_2$  in positive measure. By rescaling we assume  $I_0 = [0, 1]$ . Now suppose  $I$  is an  $n$ th-generation interval which has already been constructed, that it has length  $N^{-n} \leq |I| \leq 15N^{-n}$  and that its length is an integer multiple of  $N^{-n-1}$ . Divide  $I$  into subintervals  $\{I_j\}$  of length  $N^{-n-1}$ . If we have a subinterval  $I_j$  such that  $I_j \cap F_2 \neq \emptyset$  but  $\frac{1}{3}I_j \cap F_1 = \emptyset$ , we replace  $I_j$  by the union of itself and the two adjacent intervals of the same length. The new interval  $J = 3I_j$  obviously satisfies  $\frac{1}{3}J \cap F_1 \neq \emptyset$ . If two such modified intervals overlap, then we take the union and triple it to obtain an interval at most 15 times as long as the originals. We claim that this new interval does not meet any other modified intervals (so that this procedure does not go on forever). Otherwise, there would be two distinct intervals of length  $\frac{1}{3}|I_j|$  at most distance  $30|I_j|$  apart so that neither contains a point of  $F_1$  but which are separated by a point of  $F_1$ . By an argument in the proof of Lemma 5, since the intervals are separated by a point of  $F$ , at least one of the two  $\beta$ -integrals corresponding to these intervals must be very small. This implies (Lemma 3) that the harmonic measure of the interval is bounded below by approximately  $\omega(I)/N$ . This contradicts the definition of  $F_2$  if  $\delta_2$  is small enough (say  $\delta_2 \ll N^{-1}$ ), so the overlapping does not occur if  $I$  hits  $F_2$ .

One remaining problem is that the modified intervals are not quite nested. One simple way to fix this is to modify all the preceding generations of intervals when we construct  $n$ th-generation intervals and pass to the limit. The resulting intervals are nested and intervals of the same generation still have comparable length, as desired. This completes the proof of Theorem 3.  $\square$

**5. Proof of Theorem 4.** To construct  $E$ , we first describe a method of replacing a given interval  $I$  by a union of subintervals  $\{I_n\}$ . Suppose  $q > 0$  is small and  $N$  is large. Assume that  $I = [-1, 1]$ . Let

$$\lambda = \left( \sum_{n=N}^{\infty} \frac{1}{n^2} \right)^{-1} \sim N.$$

Let  $c_0 = 0$  and for  $n > 0$  let

$$c_n = \lambda \sum_{k=N}^n \frac{1}{k^2}$$

and let  $c_n = -c_{-n}$  for  $n < 0$ . Note that the two-sided sequence  $\{c_n\}$  accumulates at  $\{-1, 1\}$ . For  $n \geq 0$  let  $\beta_n = |c_{n+1} - c_n|$  and  $J_n = (c_n - q\beta_n, c_n + q\beta_n)$  and define  $J_n$  for negative  $n$  by symmetry. Let  $F = I \setminus \cup_n J_n$ .  $F$  is compact and  $F = I \setminus \{-1\} \cup \dots \cup I_{-2} \cup I_{-1} \cup I_1 \cup \dots \cup \{1\}$ , where  $\{I_n\}$ ,  $n \neq 0$ , are the closed intervals complementary to the  $\{J_n\}$ . One easily checks that

$$(5.1) \quad |F| = \sum_n |I_n| \leq (1 - q)|I|,$$

$$(5.2) \quad \text{dist}(J_n, J_{n+1}) \geq |J_n|/Cq,$$

$$(5.3) \quad \text{dist}(J_n, \mathbb{R} \setminus I) \geq N|J_n|/Cq.$$



Now suppose we are given sequences  $\{q_n\}$ ,  $0 < q_n < 1$ , and  $\{N_n\} \subset \mathbb{N}$ . We define  $E$  as follows. Let  $E_0 = [-1, 1]$  and let  $E_1 = F$ ,  $F$  as constructed above with  $q = q_1$ ,  $N = N_1$ . In general,  $E_{n-1}$  is a compact set consisting of a countable collection of closed intervals and their accumulation points.  $E_n$  is constructed by replacing each interval by a scaled copy of  $F$  using the parameters  $q = q_n$  and  $N = N_n$ . The sets  $\{E_n\}$  are compact and nested so there is a nonempty compact set  $E$  defined by their intersection.

It is clear that  $|E| \leq \prod_{n \geq 1} (1 - q_n)$ , so that  $|E| = 0$  if  $\sum_n q_n = \infty$ . However, it is fairly easy to prove directly that  $E$  has Hausdorff dimension 1 (as required by Theorem 3) and so is not a polar set. We will show that if the numbers  $\{N_n\}$  are chosen large enough (depending on  $\{q_n\}$ ), then the probability that a Brownian path starting at the origin never separates  $E$  is larger than

$$(5.4) \quad \prod_{n > 1} (1 - Cq_n^2),$$

where  $C > 0$  is some absolute constant. Thus, to build the desired example, it is enough to take  $q_n = 1/(Cn)$ .

We now begin the proof of (5.4). First we group the open intervals of  $\mathbb{R} \setminus E$  into generations in the obvious way, that is,  $J$  is in the  $n$ th-generation iff it is a component of  $\mathbb{R} \setminus E_n$  but not of  $\mathbb{R} \setminus E_{n-1}$ . Let  $\mathcal{S}_n$  denote the collection of  $n$ th-generation intervals. Given a  $J \in \mathcal{S}_n$  with center  $x$ , we define a disk  $D = B(x, q_n^{-2}|J|)$ . Such a disk is called an “ $n$ th-generation disk.” Let  $H$  denote the upper half plane.

LEMMA 6. *Suppose  $J$  is an  $n$ th-generation interval,  $D$  is the associated disk,  $x \in J$  and  $N_n > C/(\lambda_n q_n)$ . Let  $\Omega' = D \setminus (\mathbb{R} \setminus J)$ . Then there is an absolute constant  $C$  such that*

$$\omega(x, E_n \setminus E, \Omega') \geq 1 - Cq_n^2.$$

Furthermore, if  $z = (x, y) \in \partial D$ ,  $N_n \geq q_n^{-3}$  and  $I$  is the interval of  $E_{n-1}$  containing  $x$ , then

$$\omega(z, \mathbb{R} \setminus I, H) \leq Cq_n^2.$$

The first inequality implies that a Brownian motion starting in an  $n$ th-generation interval has a probability less than or equal to  $Cq_n^2$  of hitting an  $n$ th-generation interval or  $\partial D$  before hitting a  $k$ th-generation interval for some  $k > n$ . The second inequality says that a path starting in an  $n$ th-generation disk has probability less than or equal to  $Cq_n^2$  of first hitting  $\mathbb{R}$  in a  $j$ th-generation interval for  $j < n$ . Thus with probability greater than or equal to  $1 - Cq_n^2$ , a Brownian path starting in an  $n$ th-generation interval  $I$  first hits  $\mathbb{R} \setminus I$  in a  $k$ th-generation interval for some  $k > n$  without ever having hit a  $j$ th-generation disk for any  $j < k$ .

Suppose  $\gamma$  is a Brownian path and let  $I_1, I_2, I_3, \dots$  be the sequence of (distinct) intervals in  $\mathbb{R} \setminus E$  it hits and  $D_1, D_2, \dots$  the associated disks. Using the lemma, the Markov property of Brownian motion and induction, we see

that with probability greater than

$$\prod_{j=1}^{\infty} (1 - Cq_n^2) > 0,$$

each  $I_{n+1}$  belongs to a strictly higher generation than  $I_n$  does, that  $\gamma$  never leaves  $D_n$  after hitting  $I_n$  (so  $D_{n+1} \subset D_n$ ) and it never hits any  $D_k$ ,  $k > n$ , before hitting  $I_n$ . But such a path never separates  $E$ , for if there were  $0 \leq s < t$  such that  $\Gamma = \gamma([s, t])$  separated  $E$ ,  $\gamma$  would have to cross  $\mathbb{R}$  at least twice between times  $s$  and  $t$  and in different intervals  $I_n$  and  $I_k$  (say  $n < k$ ). If  $\gamma$  is as above, then it hits  $I_n$  before hitting  $I_k$ , never hits  $D_k$  before hitting  $I_n$  and it never leaves  $D_k$  after hitting  $I_k$ . Therefore  $\gamma(s)$  cannot equal  $\gamma(t)$ . This finishes the proof of Theorem 4 except for proving Lemma 6.

To prove the first inequality in the lemma, let  $F_n$  be the union of all  $n$ th-generation intervals. Since  $N_n > C/(\lambda_n q_n)$ , (5.3) implies there are no  $(n-1)$ st- or lower-generation intervals in  $D \cap \mathbb{R}$ . Thus it suffices to show both

$$\omega(x, \partial D, \Omega') \leq Cq_n^2, \quad \omega(x, F_n, \Omega') \leq Cq_n^2.$$

The first estimate is easy (apply the mapping  $z \rightarrow 1/z$  and compare with the complement of a line segment). To prove the second inequality, rescale so  $J = [-1, 1]$ . By the maximum principle the harmonic measure for  $\Omega'$  is dominated (on  $\mathbb{R}$ ) by the measure for the larger domain  $\tilde{\Omega} = \mathbb{R}^2 \setminus (\mathbb{R} \setminus J)$ . This domain is the image of the upper half plane by the mapping  $z \rightarrow (z+i)(z-i)/2z$  so the harmonic measure on  $\tilde{\Omega}$  can be explicitly computed. In particular, one easily shows

$$d\tilde{\omega} \sim \frac{C dx}{1+x^2}, \quad |x| \geq 2$$

(with respect to the point  $\{0\}$ ). The harmonic measure of  $F_n$  at  $x$  is dominated by a constant times the measure at 0, so it is enough to estimate it there. The set  $F_n \cap D$  is a union of intervals  $\{J_j\}$ . By (5.2) the two closest of these to  $J$  have length approximately equal to 1 and distance approximately  $1/q_n$  from  $J$ . Thus their harmonic measure is at most  $Cq_n^{-2}$ . To each of the other intervals  $J_j$ , we associate the adjacent complementary interval  $I_j$  which is closer to  $J$ .  $I_j$  has length approximately equal to  $|J_j|/q_n$ . Therefore  $\omega(I_j) \geq Cq_n^{-1}\omega(J_j)$ . Since the  $\{I_j\}$  are disjoint,

$$\omega(x, F_n, \tilde{\Omega}) \leq Cq_n \omega\left(0, \bigcup_k I_k, \tilde{\Omega}\right) \leq Cq_n \omega(0, \{|y| > C/q_n\}, \tilde{\Omega}) \leq Cq_n^2.$$

This proves the first part of the lemma.

The second estimate is a simple consequence of the well-known estimate

$$\omega(i, \mathbb{R} \setminus [-R, R], H) \leq CR^{-1}$$

(just apply the Poisson formula). Since we have  $z = (x, y)$  with  $|y| \leq q_n^{-2}|J|$

and [by (5.3)],  $\text{dist}(x, \mathbb{R} \setminus I) \geq \lambda N_n |J|q_n^{-1} - |J|q_n^{-1}$ , we get by rescaling

$$\omega(z, \mathbb{R} \setminus I, H) \leq C \frac{|y|}{\text{dist}(x, \mathbb{R} \setminus I)} \leq Cq_n^{-1}N_n^{-1} \leq Cq_n^2.$$

This proves Lemma 6 and completes the proof of Theorem 4.  $\square$

**6. Remarks.** Lemma 1 can be extended to include estimates when  $|z| \leq r/4$ . The correct estimate then becomes

$$C^{-1} \int_I \beta(t, \alpha t) \frac{dt}{t} \leq u(z) \left( \frac{h(0, |z|)}{h(0, r)} \right)^2 \leq C \int_I \beta(t, \alpha t) \frac{dt}{t}.$$

The extra factor measures how unlikely it is for the path to greatly increase its distance from the origin. Using this, one can find a necessary and sufficient condition in terms of  $\beta$ -integrals for a Brownian path conditioned to exit at 0 to loop around 0 infinitely often.

We mentioned in Section 1 that any compact set  $E$  with  $\dim(E) < \frac{1}{2}$  is Brownian disconnected. We will not give a complete proof here, but since it is not recorded elsewhere, we will sketch the idea. The proof breaks down into two lemmas. The first shows that if  $E$  is not Brownian disconnected, then there is a set  $F \subset E$  of positive harmonic measure such that  $E$  intersects “most” annuli centered at points of  $F$ . More precisely, given  $x \in \mathbb{R}^2$  and  $r, \varepsilon > 0$ , let  $A(x, r, \varepsilon) = \{(1 - \varepsilon)r < |z - x| < r\}$ . Let “cap” denote logarithmic capacity.

**LEMMA 7.** *There is a  $\delta > 0$  such that if for a.e.  $\omega$  point  $x \in E$  there are  $r_0, \varepsilon > 0$  such that  $\text{cap}(A(x, \varepsilon, r) \cap E) \leq \delta \varepsilon r$ ,  $r < r_0$ , then  $E$  is Brownian disconnected.*

**LEMMA 8.** *Given  $\nu, \delta > 0$  there exists  $\varepsilon > 0$  such that the following holds. Suppose  $x \in E$  satisfies  $\text{cap}(A(x, r, \varepsilon) \cap E) \geq \delta \varepsilon r$  for some  $\delta > 0$  and all  $r \leq r_0$ . Then  $\omega(B(x, r)) \leq Cr^{1/2-\nu}$ . The constant  $C$  depends only on  $\nu, \delta$  and  $r_0$ .*

Taken together, the two lemmas imply that any set  $E$  which is not Brownian disconnected must contain a subset of positive harmonic measure on which harmonic measure satisfies  $\omega(B(x, r)) \leq Cr^{1/2-\nu}$ . This set must have Hausdorff dimension greater than  $\frac{1}{2} - \nu$ , so it suffices to prove the lemmas.

The idea behind Lemma 7 is that if  $A(x, r, \varepsilon) \cap E$  has small capacity (compared to  $r$ ), then a Brownian path in the annulus has a small but positive chance of making a loop around the annulus and crossing itself, thus separating the point  $x$  from  $\infty$ . If there are infinitely many such annuli around  $x$ , then the probability of making such a loop is 1. Thus we need only verify the claim that if  $\delta$  is small enough, then  $\text{cap}(A(x, r, \varepsilon) \cap E) \leq \delta \varepsilon r$  implies the probability of making a loop around the annulus is greater than or equal to  $\eta(\varepsilon, \delta)$ .

To prove Lemma 8, we fix a point in  $F$ , which we may assume is 0, and change  $E$  by replacing  $A_n = A(0, (1 - \varepsilon)^n r_0, \varepsilon) \cap E$  with an interval  $[r_n, r_n + \mu] \subset \mathbb{R}^+$  of the same capacity. First we show that the harmonic measure of a small disk around 0 is not decreased by this procedure. This is a fairly standard symmetrization argument due to Beurling (see [24], Section 4.5, for example). Then we show the many small intervals can be replaced by larger intervals of the form  $[2^{-n}, (1 - \eta)2^{-n+1}]$  [ $\eta = \eta(\delta, \varepsilon)$  is small if  $\delta, \varepsilon$  are small], again without decreasing the harmonic measure of small disks around 0. Finally, we use a standard estimate on harmonic measure (e.g., Tsuji's inequality as in [2]) to show  $\omega(B(x, r)) \leq Cr^{1/2-\nu}$  where  $\nu$  is small if  $\eta$  is small.

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