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SYMMETRY GROUPS OF MARKOV PROCESSES

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We prove that if G is a subgroup of the (time-change) symmetry group of a Markov process X_t which is transitive and has a compact isotropy subgroup, then after a time change, X_t becomes G-invariant. The symmetry groups of diffusion processes are discussed in more detail. We show that if the generator of X_t is the Laplacian with respect to the intrinsic metric, then X_t has the best invariance property.

1. Introduction. In this paper, a Markov process X_t is assumed to be a standard process. This is a strong Markov process with right-continuous paths and quasileft continuity and with a separable locally compact Hausdorff state space M. See [2] for the general theory of standard processes. Recall that a Markov process X_t is really a family $\{X_t^x,\ x \in M\}$ of processes, where X_t^x is the process starting at x. If g is a transformation on M, that is, g: $M \to M$ is a homeomorphism, then $g(X_t)$ is also a Markov process. The transformation g is said to be an invariance transformation of X_t if the process $g(X_t^x)$ is identical in law with $X_t^{g(x)}$ and it is said to be a symmetry transformation of X_t if $g(X_t^x)$ is identical in law with $X_t^{g(x)}$ after a time change. We will give a precise definition of time changes later. Both invariance transformations and symmetry transformations form groups, which will be called, respectively, the invariance group and the symmetry group of X_t. The invariance group is obviously contained in the symmetry group. It is also clear that if two Markov processes differ by a time change, then they have the same symmetry group. We will use the symbols $\mathrm{Inv}(X_t)$ and $\mathrm{Sym}(X_t)$ to denote, respectively, the invariance group and the symmetry group of the process X_t.

The symmetry group was studied in Glover's recent paper [3]; see also [4] and [9] for related discussion. Among other things, he proved that if G is a subgroup of $\mathrm{Sym}(X_t)$, which is transitive on M and has a trivial isotropy subgroup, then after a time change, X_t becomes a process Y_t which is G-invariant, that is, $G \subset \mathrm{Inv}(Y_t)$. Recall that a transformation group G is said to be transitive on M if for any $x, y \in M$, there is $g \in G$ such that $g(x) = y$ and the isotropy subgroup G_p of G at some point $p \in M$ is the subgroup of G consisting of transformations which fix p.

The present paper is motivated by the above result. Our main result (Theorem 1) extends Glover's result to the case when the isotropy subgroup G_p of G is assumed to be compact instead of trivial. The extension makes possible the application of the theorem to more interesting examples. We note here that

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our technical assumptions (see the statement of Theorem 1) are stronger than those used in [3]. This will enable us to avoid some difficult measure theoretical arguments.

Consider a family of Markov processes which differ from one another by time changes. These processes have the same symmetry group and also the same potential theory. A potential theory can be characterized by the total collection of harmonic functions (see the next section for a precise definition of harmonic functions). Note that two Markov processes have the same potential theory if and only if they differ by a time change. This is a direct consequence of the Blumenthal–Getoor–McKean theorem (see Chapter 5 of [2]). Thus, we can talk about the symmetry group of a given potential theory. In the probabilistic approach to potential theory, we study a Markov process which has the desired potential theory. We have many choices for the Markov process to be used. However, it seems that the best choice should be the one with the best invariance property, that is, the one with the largest invariance group. The question is whether such a process exists. By our result, if the symmetry group is transitive on M and has a compact isotropy subgroup, then there is a process in the family whose invariance group is the common symmetry group, so it has the largest invariance group. We note here that the symmetry group of a Brownian motion on a Euclidean space does not have a compact isotropy subgroup, but we will see that its invariance group is still the largest among all time-changed processes.

Some basic definitions are disposed of in the next section. In Section 3, we state and prove the main result, Theorem 1, for the existence of invariant processes. Section 4 contains several examples of invariant processes. We will see that they may not be the processes with which we are most familiar.

In Section 5, we discuss symmetry groups of diffusion processes. This is closely connected with the invariance theory of differential operators. Some results may be known under a different context. We show that if the generator of X_t is the Laplacian with respect to its intrinsic metric, then X_t has the largest invariance group, provided that the dimension is not equal to 2 and the metric is complete. The symmetry groups for Brownian motions on Euclidean spaces, spheres and hyperbolic spaces are determined in Section 6.

2. Harmonic functions and time changes. Let D be an open subset of M. A nonnegative function h defined in M is said to be harmonic in D if for any $x \in D$ and for any compact subset K of D,

$$E\{h(X^x(T_{K^c}))\} = h(x),$$

where T_{K^c} is the first hitting time of K^c, the complement of K. For typographical convenience, we may write $X(t)$ for X_t. Note that the harmonicity of h in D may depend on its values outside the closure of D, since X_t is not assumed to be continuous.

Let $H(D)$ be the space of all nonnegative functions on M which are harmonic in D and let $\mathscr{H}$ be the set of all homeomorphisms: $M \to M$. Since two processes which differ by a time change have the same hitting distributions, by the Blumenthal–Getoor–McKean theorem (Chapter 5 of [2]), we obtain the following characterization of the symmetry group $\mathrm{Sym}(X_t)$:

$$(1)\quad \mathrm{Sym}(X_t) = \{g \in \mathscr{H}; h \circ g \in H(g^{-1}(D)) \text{ for } h \in H(D) \text{ and any open } D\}.$$

REMARK 1. If X_t is a diffusion process on a differentiable manifold M whose generator has smooth coefficients (see [6] for the definition of diffusion processes on manifolds), then $\mathrm{Sym}(X_t)$ contains only differentiable transformations. To see this, we observe that any open subset D of M with smooth boundary has a smooth Poisson kernel. So any function h which is harmonic in D is smooth in D. If $g \in \mathrm{Sym}(X_t)$, then $h' = h \circ g$ is harmonic in $g^{-1}(D)$ and so is smooth there. For any fixed point $x \in D$, by adjusting the boundary values or even the open set D, we can obtain a harmonic h whose gradient at x has any desired direction. This means that we can solve for g from a system of equations $h' = h \circ g$. Hence, g is smooth.

Let us recall briefly the definition of time changes for Markov processes; see [2], Chapter 5 for details. A real-valued process A_t is said to be an additive functional of X_t if it is adapted to the filtration generated by X_t and satisfies

$$A_0 = 0 \quad \text{and} \quad A_{t+s} = A_t + A_s \circ \theta_t,$$

where θ_t is the shift operator associated with X_t. In this paper, we will assume that an additive functional has continuous strictly increasing paths. Let τ_t be the inverse of A_t considered as a function of t. τ_t is called a time-change process of X_t. It is easy to show that $X(\tau_t)$, which has the same physical paths as X_t but runs according to a different clock, is also a Markov process. A Markov process Y_t is said to differ from X_t by a time change if it is identical in law with $X(\tau_t)$ for some time-change process τ_t of X_t.

Let $a(x)$ be a positive Borel function on M and $A_t = \int_0^t a(X_s)\,ds$. Then A_t is an additive functional and $a(x)$ will be called the density of A_t. Note that the density $a(x)$ of an additive functional A_t is assumed to be strictly positive. If τ_t is the inverse of A_t, then the time-changed process $X(\tau_t)$ is said to be obtained from X_t by the time change with density $a(x)$. Sometimes it would be convenient to write A_t^x and τ_t^x for A_t and τ_t restricted on the paths of X_t starting from x. If A_t has a density $a(x)$, then $A_t^x = \int_0^t a(X_s^x)\,ds$.

If $g \in \mathrm{Sym}(X_t)$, then the process $g(X_t)$ is identical in law with $X(\tau_t^g)$ for some time-change process τ_t^g. Precisely, this means that for any $x \in M$, the process $g(X_t^x)$ is identical in law with $X^{g(x)}(\tau_t^{g,g(x)})$, where $\tau_t^{g,y}$ is the restriction of τ_t^g to the paths of X_t^y (i.e., the paths of X_t starting at y). Let A_t^g be the associated additive functional. If $a(x)$ is the density of A_t^g, it will be called the time-change density of g for X_t.

REMARK 2. Let G be a subgroup of $\mathrm{Sym}(X_t)$. It would be useful to know whether there is a positive continuous function $a(x,g)$ on $M \times G$ such that for each $g \in G$, $a(x,g)$ is the time-change density of g for X_t. However, for this to make sense, we need first to equip G with a topology.

Let G be a transformation group on M. It is said to be a topological transformation group on M if the maps $(g,h)\to gh$ from $G\times G$ into G, $g\mapsto g^{-1}$ from G into G and $(x,g)\mapsto g(x)$ from $M\times G$ into M are continuous. By [1], if M is locally connected, then the compact-open topology on G is the weakest topology on G so that G is a topological transformation group. When M is a manifold, it may be possible to equip G with a manifold structure so that it becomes a Lie transformation group (this means that the three maps above are smooth). The topology of a Lie transformation group may be stronger than the compact-open topology. There is at most one topology on G to make it a Lie transformation group ([8]).

Let X_t be a diffusion process on a manifold M. Assume that the generator L of X_t has smooth coefficients and for any $x\in M$, there is $f\in C_c^\infty$, the space of smooth functions on M with compact support, such that $Lf(x)\neq 0$. If G is a Lie transformation group on M which is contained in $\mathrm{Sym}(X_t)$, then there is a positive smooth function $a(x,g)$ on $M\times G$ such that for each $g\in G$, $a(x,g)$ is the time-change density of g for X_t.

To prove this, first observe that $g(X_t)$ is a diffusion process X'_t with generator L^g defined by

$$L^g f(x) = L(f\circ g)(g^{-1}(x)). \tag{2}$$

By the martingale characterization of diffusion processes (see Definition 6.1 in Chapter IV of [6]),

$$Lf(x) = \lim_n \frac{E[f(X^x(T_n))] - f(x)}{E[T_n]}$$

and

$$L^g f(x) = \lim_n \frac{E[f(X'^x(T'_n))] - f(x)}{E[T'_n]},$$

where T_n and T'_n are, respectively, the exit times of X_t and X'_t from open sets U_n which shrink to x. Since X_t and X'_t differ only by a time change, $E[f(X^x(T_n))] = E[f(X'^x(T'_n))]$. By our assumptions, $\exists\, f\in C_c^\infty$ with $Lf(x)\neq 0$, and it follows that $\lim_n E[T_n]/E[T'_n]$ exists and is finite. On the other hand, $L^g f'(x)\neq 0$ for some $f'\in C_c^\infty$, so this limit is strictly positive. Now we can define

$$a(x,g) = \lim_n E[T_n]/E[T'_n] = Lf(x)/L^g f(x).$$

It is clear that $a(x,g)$ is a positive smooth function on $M\times G$ whose definition is independent of $f\in C_c^\infty$. We have $L^g = a(x,g)^{-1}L$.

For fixed $g\in G$, let $a(x)=a(x,g)$, A_t be the additive functional of X_t with density $a(x)$ and τ_t be the inverse of A_t. Check that $\tau_t = \int_0^t a(X(\tau_s))^{-1}\,ds$.

Let $Y_t = X(\tau_t)$. Then

$$f(Y_t) - f(Y_0) - \int_0^t (a^{-1}L) f(Y_s)\, ds$$

$$= f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s)\, d\tau_s = f(X_{\tau_t}) - f(X_0) - \int_0^{\tau_t} Lf(X_s)\, ds$$

is a martingale. This implies that Y_t is a diffusion process on M with generator $a(x)^{-1}L$. Hence, Y_t is identical in law with $g(X_t)$. Our claim is proved.

3. Existence of invariant processes. Let X_t be a Markov process on M and G be a topological transformation group on M which is contained in $\mathrm{Sym}(X_t)$. We will say that G has a continuous time-change density for X_t if there is a positive continuous function $a(x, g)$ on $M \times G$ such that for each $g \in G$, $a(x, g)$ is the time-change density of g for X_t. By Remark 2, if X_t is a diffusion process and G is a Lie transformation group which is contained in $\mathrm{Sym}(X_t)$, then G has a continuous time-change density $a(x, g)$ for X_t, which is in fact smooth. We note that if G does not have a continuous time-change density for X_t, it may have one for X_t after a time change.

For a topological transformation group G on M, the map $\Phi\colon G \to M$ defined by $\Phi(g) = g(p)$ for some fixed $p \in M$ is continuous. If G is transitive on M, then Φ is surjective and it induces naturally a continuous bijection $\overline{\Phi}$: $G/G_p \to M$, where G/G_p is equipped with the quotient topology induced from the natural projection $G \to G/G_p$. In general, $\overline{\Phi}^{-1}$ is not continuous. It is continuous if Φ is an open map, that is, if Φ maps open sets of G into open sets of M. This is equivalent to saying that Φ maps any neighborhood of the identity element of G into a neighborhood of p. We will say that G is regularly transitive on M if G is transitive on M and if Φ is an open map. Note that our definition is in fact independent of the choice of the fixed point $p \in M$. By [1], if G is a transitive Lie transformation group on a manifold M, then it is regularly transitive on M. We note that if G is a regularly transitive topological transformation group on M, then $\overline{\Phi}$: $G/G_p \to M$ is a homeomorphism.

THEOREM 1. *Let X_t be a Markov process and G be a topological transformation group which is contained in* $\mathrm{Sym}(X_t)$. *Assume that G has a continuous time-change density for X_t, it is regularly transitive on M and its isotropy subgroup G_p at some point $p \in M$ is compact. Then there is a Markov process Y_t which differs from X_t by a time change and whose invariance group* $\mathrm{Inv}(Y_t)$ *contains G, that is, Y_t is G-invariant.*

PROOF. A point $x \in M$ is said to be trap of X_t if $P(\forall\ t, X_t^x = x) = 1$. It is easy to see that if $g \in \mathrm{Sym}(X_t)$, then g maps the set of traps onto the set of traps. Therefore, by restricting X_t to the complement of the set of traps, we may assume that X_t has no traps. Then, by the strong Markov property, for any $x \in M$, the exit time of X_t^x from the single point set $\{x\}$ is finite. Hence, we can find a sequence of open neighborhoods U_n of x such that if T_n is the exit time of X_t^x from U_n, then $P(T_n < \infty) \to 1$.

Since G has a continuous time-change density for X_t, there is a positive continuous function $a(x, g)$ on $M \times G$ such that for each $g \in G$, $a(x, g)$ is the time-change density of g for X_t. We will first prove the following formula.

(3) $\quad \forall\, x \in M$ and $g, h \in G, \qquad a(x, gh) = a(g^{-1}(x), h)a(x, g).$

If two processes X_t and Y_t are identical in law, we will write $X_t \approx Y_t$. Recall that

$$h\left(X_t^{h^{-1}(x)}\right) \approx X^x\left(\tau_t^{h,x}\right),$$

where $\tau_t^{h,x}$ is the inverse of $A_t^{h,x} = \int_0^t a(X_s^x, h)\, ds$ as a function of t. We have

$$X^x\left(\tau_t^{gh,x}\right) \approx gh\left(X_t^{h^{-1}g^{-1}(x)}\right) \approx g\left(X^{g^{-1}(x)}\left(\tau_t^{h,g^{-1}(x)}\right)\right).$$

Since

$$g\left(X_t^{g^{-1}(x)}\right) \approx X^x\left(\tau_t^{g,x}\right),$$

if we let

$$A'_t = \int_0^t a\left(g^{-1}\left(X_{\tau_s^{g,x}}^x\right), h\right) ds$$

and let τ'_t be the inverse of A'_t, then

$$g\left(X^{g^{-1}(x)}\left(\tau_t^{h,g^{-1}(x)}\right)\right) \approx X^x\left(\tau_{\tau'_t}^{g,x}\right).$$

Hence,

$$X^x\left(\tau_t^{gh,x}\right) \approx X^x\left(\tau_{\tau'_t}^{g,x}\right).$$

Let Z_t and Z'_t be, respectively, the left-hand side and the right-hand side of the above. Fix an open neighborhood U of x and let T and u be, respectively, the exit times of X_t^x and Z_t from U. Then $\tau_u^{gh,x} = T$, so $u = A_T^{gh,x}$. Similarly, if we let v be the exit time of Z'_t from U, then $\tau^{g,x}(\tau'_v) = T$, $\tau'_v = A_T^{g,x}$ and $v = A'(A_T^{g,x})$. Since $Z_t \approx Z'_t$, we have $u \approx v$ and $A_T^{gh,x} \approx A'(A_T^{g,x})$. But

$$A_T^{gh,x} = \int_0^T a(X_s^x, gh)\, ds,$$

$$\begin{aligned} A'\left(A_T^{g,x}\right) &= \int_0^{A_T^{g,x}} a\left(g^{-1}\left(X_{\tau_s^{g,x}}^x\right), h\right) ds = \int_0^T a\left(g^{-1}\left(X_t^x\right), h\right) dA_t^{g,x} \\ &= \int_0^T a\left(g^{-1}\left(X_t^x\right), h\right) a\left(X_t^x, g\right) dt. \end{aligned}$$

Hence,

$$\int_0^T a(X_s^x, gh)\, ds \approx \int_0^T a\left(g^{-1}\left(X_s^x\right), h\right) a\left(X_s^x, g\right) ds.$$

Since $a(x, g)$ is continuous in x and X_t has no traps, we can find a sequence of open neighborhoods U_n of x shrinking to x and having the following property. Let T_n be the exit time of X_t^x from U_n. Then $P(T_n < \infty)$ tends to 1 and

$$a(x, gh)T_n + R_n \approx a(g^{-1}(x), h)a(x, g)T_n + R'_n,$$

where the random variables R_n and R'_n satisfy $R_n \leq T_n/n$ and $R'_n \leq T_n/n$. The equation (3) follows from the above relation.

Next, we show that

$$\forall\, g \in G_p, \qquad a(p, g) = 1. \tag{4}$$

Since G_p is compact, there is a left-invariant finite Haar measure μ on G_p. Integrating both sides of (3) with respect to $\mu(dh)$ and letting $x = p$ and $g \in G_p$, we obtain

$$\int a(p, h)\mu(dh) = \int a(p, gh)\mu(dh) = \int a(g^{-1}(p), h)a(p, g)\mu(dh)$$
$$= \int a(p, h)\mu(dh)a(p, g).$$

This proves (4).

Since G is transitive on M, for any $x \in M$, $\exists\ g \in G$ such that $g(p) = x$. Define

$$a(x) = a(x, g), \quad \text{where } g \in G \text{ satisfies } g(p) = x. \tag{5}$$

By (3) and (4), the definition of $a(x)$ is independent of the choice of g. We show that $a(x)$ is continuous on M. The function $c(g)$ defined by $c(g) = a(g(p), g)$ is continuous on G. Check that $c(gh) = c(g)$ for $g \in G$ and $h \in G_p$. Therefore, c induces naturally a continuous function $\bar{c}$ on G/G_p such that $\bar{c}(gG_p) = c(g)$ for any $g \in G$. Letting $\overline{\Phi}$: $G/G_p \to M$ be defined as in the paragraph before Theorem 1, we see that $a(x) = \bar{c} \circ \overline{\Phi}^{-1}(x)$. This shows that $a(x)$ is continuous on M.

Let $A_t = \int_0^t a(X_s)\, ds$, $\tau_t = A_t^{-1}$, the inverse of A_t considered as a function of t, and let $Y_t = X(\tau_t)$. We claim that $G \subset \mathrm{Inv}(Y_t)$.

Fix $g \in G$. We want to show $g(Y_t^x) \approx Y_t^{g(x)}$ for any $x \in M$. Since $g(X_t^x) \approx X^{g(x)}(\tau_t^{g, g(x)})$, we have

$$g(Y_t^x) = g\left(X_{\tau_t^x}^x\right) \approx X^{g(x)}\left(\tau_{\tau'_t}^{g, g(x)}\right),$$

where τ'_t is the inverse of

$$A'_t = \int_0^t a\left(g^{-1}\left(X^{g(x)}\left(\tau_s^{g, g(x)}\right)\right)\right) ds.$$

Since $Y_t^{g(x)} = X^{g(x)}(\tau_t^{g(x)})$, it suffices to show

$$X^{g(x)}\left(\tau_{\tau'_t}^{g, g(x)}\right) = X^{g(x)}\left(\tau_t^{g(x)}\right),$$

or equivalently, $\tau^{g, g(x)}(\tau'_t) = \tau_t^{g(x)}$. We have

$$\tau^{g, g(x)}(\tau'_t) = \tau^{g, g(x)}\left(A_t'^{-1}\right) = \left(A^{g, g(x)}\right)^{-1}\left(A_t'^{-1}\right) = \left(A' \circ A^{g, g(x)}\right)_t^{-1}.$$

Since $\tau_t^{g(x)}$ is the inverse of $A_t^{g(x)}$, it now suffices to prove $A_t^{g(x)} = A'(A_t^{g,g(x)})$.

$$A'\left(A_t^{g,g(x)}\right) = \int_0^{A_t^{g,g(x)}} a\left(g^{-1}\left(X^{g(x)}\left(\tau_s^{g,g(x)}\right)\right)\right) ds = \int_0^t a\left(g^{-1}\left(X_u^{g(x)}\right)\right) dA_u^{g,g(x)}$$

$$= \int_0^t a\left(g^{-1}\left(X_u^{g(x)}\right)\right) a\left(X_u^{g(x)}, g\right) du.$$

Let $z = X_u^{g(x)}$ and choose $h \in G$ such that $h(p) = g^{-1}(z)$. We have

$$a(g^{-1}(z))a(z,g) = a(g^{-1}(z), h)a(z,g) = a(z, gh) = a(z).$$

Therefore,

$$A'\left(A_t^{g,g(x)}\right) = \int_0^t a\left(X_u^{g(x)}\right) du = A_t^{g(x)}.$$

The theorem is proved.

REMARK 3. The proof of Theorem 1 shows that the G-invariant process Y_t is obtained from X_t by the time change with density $a(x)$ which is given by (5).

REMARK 4. We will sketch a simpler proof of the above theorem when X_t is a diffusion process on a manifold M whose generator L has smooth coefficients and G is a transitive Lie transformation group on M which is contained in $\mathrm{Sym}(X_t)$ and which has a compact isotropy subgroup G_p.

A point $x \in M$ is a trap of X_t if and only if $Lf(x) = 0$ for any $f \in C_c^\infty$. It follows that the set of traps is a closed subset of M. Since a symmetry transformation leaves the set of traps invariant, by restricting X_t on the complement of the set of traps, we may assume that X_t has no trap point. Then by Remark 2, there is a positive smooth function $a(x, g)$ on $M \times G$ such that for any $g \in G$, $a(x, g)$ is the time-change density of g for X_t.

The differential operator L is said to be G-invariant if $L^g = L$ for any $g \in G$, where L^g is defined by (2). It is clear that X_t is G-invariant if and only if L is G-invariant. In order to find a G-invariant process Y_t which differs from X_t by a time change, it suffices to find a G-invariant differential operator L' such that $L' = bL$ for some positive function b, because then the diffusion process Y_t generated by L' is G-invariant and differs from X_t by the time change with density b^{-1}.

For $g \in G$, the generator of $g(X_t)$ is given by (2). Since $g(X_t)$ differs from X_t by the time change with density $a(x, g)$,

$$\forall\, x \in M \text{ and } \forall\, g \in G, \qquad L(f \circ g)(g^{-1}(x)) = a(x, g)^{-1} Lf(x).$$

Let $x = p$ and $g \in G_p$ and integrate the above equation with respect to a finite left-invariant Haar measure $\mu(dg)$ on G_p. We obtain $H(f) = CLf(p)$, where C is a positive constant and $H(f)$ is defined by $\int L(f \circ h)(p)\mu(dh)$. We can show that $H(f \circ h) = H(f)$ for any $h \in G_p$, so that

$$\forall\, h \in G_p, \qquad L(f \circ h)(p) = Lf(p).$$

Now for any $x \in M$, choose $g \in G$ such that $g(p) = x$. Define $L'f(x) = L(f \circ g)(p)$. The operator L' is well defined. We check that L' is G-invariant. Since

$$L'f(x) = L(f \circ g)(g^{-1}(x)) = a(x, g)^{-1} Lf(x),$$

we have $L' = b(x)L$, where $b(x) = a(x, g)^{-1}$ and $g \in G$ is chosen so that $g(p) = x$.

REMARK 5. We will not consider the uniqueness of invariant processes Y_t in general and will be content with a simple uniqueness result for diffusion processes. In the sequel, a diffusion process is assumed to have a generator with smooth coefficients. Let X_t be a diffusion process on a manifold M and let G be a transitive Lie transformation group on M which is contained in $\mathrm{Sym}(X_t)$. If Y_t is a G-invariant diffusion which differs from X_t by a time change, then Y_t is unique up to a time change with constant rate. This means that if Z_t is a G-invariant diffusion process which differs from X_t by a time change, then Z_t differs from Y_t by a time change with constant density.

Let L and L' be, respectively, the generators of Y_t and Z_t. Fix $p \in M$. We have $L'f(p) = \lambda Lf(p)$ for any $f \in C_c^\infty$, where $\lambda > 0$ is a constant independent of f, because Y_t and Z_t differ by a time change. The G-invariance of Y_t and Z_t implies the G-invariance of their generators, so for any $g \in G$,

$$L'f(g(p)) = L'(f \circ g)(p) = \lambda L(f \circ g)(p) = \lambda Lf(g(p)).$$

Since G is transitive on M, we see that Z_t differs from Y_t by a time change with constant rate.

4. Examples of invariant processes. By Theorem 1, if the symmetry group $G = \mathrm{Sym}(X_t)$ is transitive and has a compact isotropy subgroup, then there is a G-invariant process Y_t which has the same potential theory as X_t. One may think that the processes which people are most familiar with should already have this invariance property and Theorem 1 is not needed to produce a process with better invariance. As we will see in Examples 2 and 3, this is not always the case. The processes which one is most familiar with may not have the best possible invariance. The processes which do have the best invariance as guaranteed by Theorem 1 can be quite interesting.

EXAMPLE 1. Let S be the group of transformations on the Euclidean space R^d generated by translations, orthogonal transformations and dilations. A transformation g on R^d is called a dilation if it fixes some point p and maps any other point x into $p + c(x - p)$, where c is a fixed positive number. It is easy to see that S is contained in the symmetry group of Brownian motion on R^d. We will see in Section 6 that it is in fact the symmetry group of Brownian motion. If we let $G = S$, then G does not have a compact isotropy subgroup. If we let G be the group of Euclidean motions, that is, the group generated by translations and orthogonal transformations, then G_p is compact for any $p \in R^d$ and the process Y_t obtained from Theorem 1 is just the Brownian motion on R^d.

EXAMPLE 2. Let $M = R^d - \{o\}$, where o is the origin of R^d and let G be the transformation group on R^d generated by rotations, reflections and dilations which fix o. The action of G leaves M invariant, so G can be considered as a transformation group on M. Let X_t be either a Brownian motion or a symmetric stable process of index α $(0 < \alpha < 2)$ on $M = R^d - \{o\}$. Note that the point o will never be hit by either a Brownian motion or a symmetric stable process, so our processes on M are well defined. It is easy to see that if g is a rotation or a reflection about o, then $g(X_t) \approx X_t$. By the scaling property of Brownian motions and symmetric stable processes, if g is the dilation $x \mapsto cx$ for some constant $c > 0$, then $g(X_t) \approx X(c^\alpha t)$, where $\alpha = 2$ if X_t is a Brownian motion. It follows that G is contained in $\mathrm{Sym}(X_t)$ and it has a continuous time-change density for X_t which is given by

$$a(x, g) = \|x\|^\alpha / \|g(x)\|^\alpha,$$

where $\|x\|$ is the Euclidean norm and $\alpha = 2$ if X_t is a Brownian motion.

The group G is a Lie transformation group which is transitive on M and has compact isotropy subgroups. By Theorem 1, there is a G-invariant process Y_t which differs from X_t by a time change. However, X_t itself is not G-invariant. Let $a(x)$ be defined by (5) with respect to the fixed point $p = (1, 0, \dots, 0)$. We have

$$a(x) = \|x\|^{-\alpha}.$$

By Remark 3, a G-invariant process Y_t can be obtained from X_t by the time change with density $a(x)$ defined above.

EXAMPLE 3. Let D be the unit disk in R^2 and let X_t be the Brownian motion in D which is killed upon reaching the boundary of D. The family of harmonic functions of X_t is just the family of the usual harmonic functions in D. Let G be the group of transformations on D which transform harmonic functions into harmonic functions. By (1), $G = \mathrm{Sym}(X_t)$. If we regard D as the unit disk in the complex plane, by the theory of complex analysis, G consists of transformations of the following type:

$$g(z) = e^{i\theta} \frac{z - a}{1 - \bar{a} z},$$

where θ is real, $i = \sqrt{-1}$, $a, z \in D$ and $\bar{a}$ is the complex conjugate of a. We see that G is transitive on D and G_o, which consists of rotations and reflections about o, is compact. Note that the Brownian motion X_t is not G-invariant. In fact, $\mathrm{Inv}(X_t)$ coincides with G_o. By Theorem 1, there is a G-invariant process Y_t which differs from the Brownian motion X_t by a time change. By Remark 4, in order to find such a process, we need only to find a

G-invariant differential operator L which differs from the Laplacian Δ by a positive function factor. Then the diffusion process Y_t generated by L is the desired process. A direct computation shows that the operator

$$Lf(z) = (1 - |z|^2)^2 \Delta f(z) \tag{6}$$

satisfies the requirement. The intrinsic Riemannian metric (see the next section for the definition) is just the hyperbolic metric on D and the corresponding process Y_t is the Brownian motion on the hyperbolic disk D.

5. Diffusion processes. Let M be a differentiable manifold of dimension d. Diffusion processes on M are defined in [6]. In local coordinates $x_1, x_2, \ldots, x_d$, a diffusion generator is a differential operator of the form

$$L = \sum_{j,k=1}^{d} a_{jk}(x) \frac{\partial^2}{\partial x_j \, \partial x_k} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}, \tag{7}$$

where $a_{jk}(x)$ and $b_i(x)$ are smooth functions and $a_{jk}(x)$ form a nonnegative definite symmetric matrix. If it is positive definite, L and the diffusion process X_t generated by L are called nondegenerate.

The invariance group $\mathrm{Inv}(X_t)$ and the symmetry group $\mathrm{Sym}(X_t)$ of X_t will also be called the invariance group and the symmetry group of L, and will be denoted, respectively, $\mathrm{Inv}(L)$ and $\mathrm{Sym}(L)$. By Remark 1, both $\mathrm{Inv}(L)$ and $\mathrm{Sym}(L)$ are contained in $D(M)$, the set of diffeomorphisms: $M \to M$. We have

$$\mathrm{Inv}(L) = \{\phi \in D(M);\, L^{\phi} f = Lf \text{ for } f \in C_c^{\infty}\}. \tag{8}$$

To characterize $\mathrm{Sym}(L)$, we will say that two operators L and L' are equivalent if $L' = bL$ for some positive function b. It is clear that two diffusion processes differ by a time change if and only if their generators are equivalent. Therefore, two equivalent operators have the same symmetry group. We have

$$\mathrm{Sym}(L) = \{\phi \in D(M);\, L^{\phi} \text{ and } L \text{ are equivalent}\}. \tag{9}$$

In the remaining part of this paper, we will only state our results for diffusion generators and omit the obvious conclusions for the corresponding diffusion processes.

A Riemannian metric g on M is a smooth assignment of inner products $g(\cdot, \cdot)$ on the tangent spaces of M. See [7] for the concepts in differential geometry. Let

$$g_{jk}(x) = g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) \quad \text{at } x.$$

This is a positive definite matrix. Let g^{jk} be the inverse matrix of g_{jk}. The manifold M becomes a Riemannian manifold if it is equipped with a Riemannian metric. The Riemannian metric induces a metric space structure on M which is compatible with the given topology.

The measure m defined by $dm = \sqrt{G}\, dx_1 \cdots dx_d$, where $G = \det(g_{jk})$, is independent of local coordinates and is called the Riemannian measure. A differential operator L with local expression (7) is called the Laplacian if $a_{jk} = g^{jk}$ and L is symmetric with respect to m, that is, $(Lf, g)_m = (f, Lg)_m$ for $f, g \in C_c^\infty$, where $(f, g)_m = \int fg\,dm$. It follows that the Laplacian L has the following local expression:

$$(10) \qquad Lf = \sum_{j,k} \frac{1}{\sqrt{G}} \frac{\partial}{\partial x_j} \left(\sqrt{G} g^{jk} \frac{\partial}{\partial x_k} f \right).$$

If L is the Laplacian on a Riemannian manifold M, then $\mathrm{Inv}(L)$ is just the isometry group of the Riemannian metric and $\mathrm{Sym}(L)$ is a closed subgroup of the conformal group. It follows that both $\mathrm{Inv}(L)$ and $\mathrm{Sym}(L)$ are Lie transformation groups on M. Recall that the isometry group consists of transformations which leave the metric invariant and the conformal group consists of transformations which do not change angles.

Let L be a nondegenerate differential operator with local expression (7). The Riemannian metric defined by the inverse matrix g_{jk} of a_{jk} is called the intrinsic metric of L. L is said to be an intrinsic Laplacian if it is the Laplacian with respect to the intrinsic metric.

Assume L is an intrinsic Laplacian and $L' = bL$. Let m and m' be, respectively, the Riemannian measures of the intrinsic metrics of L and L'. We have $dm' = b^{-d/2}\,dm$ and

$$(L'f, g)_{m'} = \int (bLf) g b^{-d/2}\,dm = \left(f, b^{d/2} L\left(g b^{(2-d)/2}\right)\right)_{m'}.$$

Hence, L' is an intrinsic Laplacian if and only if $b^{(2-d)/2}$ is a constant. The following proposition follows immediately.

PROPOSITION 1. *Assume that L is an intrinsic Laplacian and $L' = bL$ for some function $b > 0$. If $d = 2$, then L' is also an intrinsic Laplacian. If $d \neq 2$, then L' is an intrinsic Laplacian if and only if b is a constant.*

Now we give a simple characterization for a differential operator to be equivalent to an intrinsic Laplacian. A measure on M is said to be smooth if it is absolutely continuous with respect to Lebesgue measure and has a positive smooth Radon–Nikodym derivative. Although there is no natural way to define Lebesgue measure on M, it can be defined using local coordinates. Our definition of smooth measures is independent of local coordinates.

PROPOSITION 2. *Assume that $d \neq 2$ and L is a nondegenerate diffusion generator. Then L is equivalent to an intrinsic Laplacian if and only if L is symmetric with respect to some smooth measure.*

PROOF. Let m be the associated Riemannian measure of L. For any smooth function $b > 0$, bL is an intrinsic Laplacian if and only if bL is symmetric with respect to its intrinsic Riemannian measure, that is,

$$\int (bLf)gb^{-d/2}\,dm = \int f(bLg)b^{-d/2}\,dm.$$

This amounts to saying that L is symmetric with respect to the smooth measure $dn = b^{(2-d)/2}\,dm$. On the other hand, assume L is symmetric with respect to $dn = f\,dm$ for some smooth function $f > 0$. We can choose $b > 0$ such that $f = b^{(2-d)/2}$. Then bL is symmetric with respect to its intrinsic Riemannian measure $b^{-d/2}\,dm$ so it is an intrinsic Laplacian. The proposition is proved. □

PROPOSITION 3. *If L is an intrinsic Laplacian, then for any $\phi \in D(M)$, L^ϕ is also an intrinsic Laplacian.*

PROOF. Let g and m be the intrinsic Riemannian metric and Riemannian measure of L. They are transformed by ϕ in the usual way into the Riemannian metric and Riemannian measure of L^ϕ and with respect to the latter L^ϕ is symmetric. This can be verified by a simple computation using local coordinates. Hence, L^ϕ is an intrinsic Laplacian. □

PROPOSITION 4. *Assume $d \neq 2$. If L is an intrinsic Laplacian and $\phi \in$ Sym(L), then $L^\phi = cL$ for some constant c.*

PROOF. Since $\phi \in \mathrm{Sym}(L)$, $L^\phi = bL$ for some function b. By Proposition 3, bL is an intrinsic Laplacian, and by Proposition 1, b must be a constant. □

PROPOSITION 5. *Assume $d \neq 2$ and M is compact. If L is an intrinsic Laplacian, then* $\mathrm{Sym}(L) = \mathrm{Inv}(L)$.

PROOF. By Proposition 4, for any $\phi \in \mathrm{Sym}(L)$, $L^\phi = cL$. So the transformation ϕ changes the distance on M determined by the intrinsic metric of L by a factor c^{-1}. When M is compact, this is impossible unless $c = 1$. □

A Riemannian metric is said to be complete if the induced metric space structure on M is complete. It is said to be flat if all the sectional curvatures with respect to the metric vanish. A transformation ϕ on a Riemannian manifold is said to be homothetic if it transforms the given metric g into cg for some constant c. By Lemma 2 of [7], page 242, if the Riemannian metric is complete and nonflat, then any homothetic transformation is isometric. This means that the constant c above is equal to 1. The following proposition now follows directly from Proposition 4.

PROPOSITION 6. *Assume $d \neq 2$. If L is an intrinsic Laplacian with complete and nonflat metric, then* $\mathrm{Sym}(L) = \mathrm{Inv}(L)$.

THEOREM 2. *Let L be a nondegenerate diffusion generator with complete metric. Assume that for any $\phi \in \mathrm{Sym}(L)$, $L^\phi = cL$ for some constant c. If L' is equivalent to L, then $\mathrm{Inv}(L') \subset \mathrm{Inv}(L)$. Moreover, if $\mathrm{Inv}(L)$ is transitive on M, then $\mathrm{Inv}(L') = \mathrm{Inv}(L)$ if and only if L' is a constant multiple of L.*

PROOF. There is a function $b > 0$ such that $L' = bL$. Let $\phi \in \mathrm{Inv}(L') \subset \mathrm{Sym}(L') = \mathrm{Sym}(L)$. We have $L'^\phi f = L'f$, hence,

$$b(\phi^{-1}(x))L^\phi f(x) = b(x)Lf(x).$$

By assumption, $L^\phi f = cLf$ for some constant c, so $b(\phi^{-1}(x))c = b(x)$ for any $x \in M$. We need to show $c = 1$. If not, without loss of generality, we may assume $c > 1$. Fix $x_1 \in M$ and define inductively $x_n = \phi(x_{n-1})$. Let $\rho(x, y)$ be the Riemannian distance between x and y, induced by the intrinsic metric of L. Then

$$\rho(x_{n+1}, x_n) = c^{-1}\rho(x_n, x_{n-1}) = c^{-(n-1)}\rho(x_2, x_1).$$

We see that $\{x_n\}$ is a Cauchy sequence. If x_0 is its limit, then $\phi(x_0) = x_0$ and $b(x_0)c = b(x_0)$. This is impossible if $c \neq 1$. Hence, $c = 1$ and we have proved $\mathrm{Inv}(L') \subset \mathrm{Inv}(L)$. Assume $\mathrm{Inv}(L) = \mathrm{Inv}(L')$. The above argument shows that $b(\phi(x)) = b(x)$ for any $\phi \in \mathrm{Inv}(L)$. If $\mathrm{Inv}(L)$ is transitive on M, then b must be a constant. The theorem is proved. □

The following corollary follows directly from Proposition 4.

COROLLARY 1. *Assume $d \neq 2$. If L is an intrinsic Laplacian with complete metric, then L has the largest invariance group in the sense that if L' is equivalent to L, then $\mathrm{Inv}(L')$ is contained in $\mathrm{Inv}(L)$. Moreover, if $\mathrm{Inv}(L)$ is transitive on M, then $\mathrm{Inv}(L') = \mathrm{Inv}(L)$ if and only if L' is a constant multiple of L.*

6. Examples of symmetry groups. In this last section, we will determine the symmetry groups for the Laplacians on the Euclidean space R^d, the sphere S^d and the hyperbolic space H^d for any dimension d. Recall that the hyperbolic space H^d can be modelled on the unit ball in R^d and has constant negative curvature. As we will see, the dimension 2 plays a special role. It seems that these groups should be well known, but we are unable to find a place where these facts are proved.

EXAMPLE 4. Let L be the Laplacian on R^d. We will show that $\mathrm{Sym}(L)$ is the transformation group generated by Euclidean motions and dilations on R^d. It then follows from Theorem 2 that for any dimension d, L has the largest invariance group in the sense of Corollary 1.

Let S be the group generated by Euclidean motions and dilations. It is easy to show that $S \subset \mathrm{Sym}(L)$. First assume $d \neq 2$. For any $\phi \in \mathrm{Sym}(L)$, by Proposition 4, $L^\phi = cL$ for some constant $c > 0$. Let ψ be the dilation which changes the distance by a factor of $1/c$. We have $L^{\phi\psi} = L$, so $\phi\psi \in \mathrm{Inv}(L) \subset S$.

This implies $\phi \in S$ and, hence, proves $\mathrm{Sym}(L) = S$. When $d = 2$, let G be the conformal group of R^2. Think of R^2 as the complex plane. By complex analysis, if $g \in G$, then $g(z)$ is either $f(z)$ or $f(\bar{z})$, where $f(z)$ is an analytic function. Since g is one-to-one from R^2 onto R^2, $f(z)$ must be a linear function. It is then easy to show that G is generated by Euclidean motions and dilations on R^2. Since $\mathrm{Sym}(L) \subset G$, we can now conclude $\mathrm{Sym}(L) = G$.

EXAMPLE 5. Let L be the Laplacian on the sphere S^d or the hyperbolic space H^d with $d \neq 2$. By Proposition 6, $\mathrm{Sym}(L) = \mathrm{Inv}(L)$, which is just the isometry group on M. The isometry groups on S^d and H^d are well understood. They can be naturally identified, respectively, with the orthogonal group $O(d+1)$ and the Lorentz group $O(d, 1)$ on R^{d+1}.

EXAMPLE 6. Let L be the Laplacian on the two-dimensional sphere S^2. We will show that $\mathrm{Sym}(L)$ coincides with the conformal group of S^2, which is generated by rotations on S^2 and those transformations which correspond to Euclidean motions and dilations on R^2 via the usual stereographic projection σ from S^2 minus the north pole onto R^2, when S^2 is regarded as embedded in R^3 as the unit sphere.

The stereographic projection σ is conformal in the sense that it does not change angles. Let $\mathrm{Con}(R^2)$ and $\mathrm{Con}(S^2)$ be, respectively, the conformal groups of R^2 and S^2. We have seen in Example 4 that $\mathrm{Con}(R^2)$ is generated by Euclidean motions and dilations on R^2. Since σ is a conformal map, $\sigma^{-1}\,\mathrm{Con}(R^2)\sigma$ as a transformation group on S^2 is contained in $\mathrm{Con}(S^2)$. Any $\phi \in \mathrm{Con}(S^2)$ has a fixed point because S^2 is compact. We can find a rotation ψ such that $\phi\psi$ fixes the north pole. Then $\sigma\phi\psi\sigma^{-1}$ is a conformal transformation on R^2, so it is contained in $\mathrm{Con}(R^2)$. We have proved that $\mathrm{Con}(S^2)$ is generated by $O(3)$ and $\sigma^{-1}\,\mathrm{Con}(R^2)\sigma$. Note that $O(3)$ is identified with the group of rotations and reflections on S^2.

Now we show $\mathrm{Sym}(L) = \mathrm{Con}(S^2)$. Since the symmetry group is always contained in the conformal group of the intrinsic metric, it suffices to show $\mathrm{Con}(S^2) \subset \mathrm{Sym}(L)$. If $\phi \in \mathrm{Con}(S^2)$, then $L^\phi = bL + U$ for some function $b > 0$ and vector field U. By Propositions 1 and 3, both bL and L^ϕ are intrinsic Laplacians. This is impossible unless $U = 0$. Therefore, $\phi \in \mathrm{Sym}(S^2)$.

EXAMPLE 7. Let L be the Laplacian on the two-dimensional hyperbolic space H^2. The group G defined in Example 3 consists of transformations on the unit disk D which transform harmonic functions into harmonic functions. This is the symmetry group of the usual Laplacian Δ in D. Since D can be identified with H^2 and L is equivalent to Δ by (6), we see that $\mathrm{Sym}(L) = G$.

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