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A NOTE ON PLANAR BROWNIAN MOTION

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The joint density of the total winding and the radius of a planar Brownian motion is calculated, by solving the associated backward equation. An explicit expression for the distribution of the hitting time of an angular barrier is shown; in particular, the behavior of the tail of the distribution is determined.

1. Introduction. Consider a planar Brownian motion $Z_t = X_t + iY_t$ starting at some point $z_0 \neq 0$. Let $R_t = |Z_t|$ and θ_t be the total winding of the a.s. continuous path $\{Z_s;\ s \le t\}$ about zero. The continuous process (R_t, θ_t) can be thought of as a diffusion on the Riemann surface of the logarithm (since Z does not hit 0 a.s.), identified with the half upper plane divided into vertical strips of width 2π. Each one of these strips is sent into $\mathbb{R}^2 \setminus 0$ via the map $(r, \theta) \to (r\cos\theta, r\sin\theta)$, which introduces local coordinates. As discussed in Theorem 1, (R_t, θ_t) is governed by $\frac{1}{2}\Delta$, which yields (in the given coordinates), that the density

$$(1.1)\quad p(t, r, \theta; \rho, \alpha) = \mathbf{P}\{R_t \in dr,\ \theta_t \in d\theta | R_0 = \rho,\ \theta_0 = \alpha\}\ 1/(r\,dr\,d\theta)$$

satisfies the equation

$$(1.2)\qquad \partial_t u = \frac{1}{2}\left(\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u\right), \qquad t, r > 0,\ \theta \in (-\infty, \infty).$$

As a first result (Theorem 1) we obtain p by finding a fundamental solution of (1.2) by standard methods [with the aid of Gradshteyn and Ryzhik's tables (1980)]. Once this density is known, an application of the reflection principle to θ_t permits computation of the joint distribution of (R_t, θ_t) and T_β [the exit time of θ_t from the region $(0, \beta)$]. This is done in Theorem 2. The asymptotic behavior of $\mathbf{P}\{T_\beta > t\}$ as t goes to infinity is determined as a corollary; in particular, the result of Spitzer (1958) asserting that $\mathbf{E}T_\beta$ is finite if and only if $\beta < \pi/2$ follows directly from it.

While we were writing this note, we learned of an article of Yor (1980), where he shows, using martingale methods, that $\mathbf{E}\{\exp(iv(\theta_t - \theta_0)) | R_t = r, R_0 = \rho\} = (I_{|v|}(\rho r/t))/(I_0(\rho r/t))$. [In fact, this formula has a longer history; see Yor (1980) and Pitman and Yor (1986) for details and further references]. Since the distribution of R_t is known, the joint density p can be deduced. We think, however, that it has some interest to consider the extended equation, since it is very simple and may be useful in other cases.

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2. Results and proofs. Our results will be expressed in terms of the modified Bessel functions I_v. Several representations of them are known. See Gradshteyn and Ryzhik (1980) for a brief account, or Watson (1966) for details. We write down some of them, just for easy reference.

Representations of the Bessel function I_v. We shall only consider $I_v(x)$ for $x, v \in \mathbb{R}$, $x > 0$. Then,

$$I_v(x) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} \exp(x\cosh\omega - v\omega)\,d\omega. \tag{2.1}$$

Taking as the path of integration three sides of the rectangle with corners at $\infty - \pi i$, $-\pi i$, πi and $\infty + \pi i$, the previous expression yields

$$I_v(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos v\theta\,d\theta - \frac{\sin v\pi}{\pi}\int_0^{\infty} e^{-x\cosh t - vt}\,dt. \tag{2.2}$$

Finally, the series expansion is

$$I_v(x) = \sum_{k=0}^{\infty}\frac{(x/2)^{v+2k}}{k!\Gamma(v+k+1)}. \tag{2.3}$$

Now, we can state:

THEOREM 1. *The fundamental density defined in* (1.1) *is given by*

$$p(t,r,\theta;\rho,\alpha) = \frac{1}{\pi t}\exp\left(-\frac{r^2+\rho^2}{2t}\right)\int_0^{\infty}\cos v(\theta-\alpha)\,I_v\left(\frac{\rho r}{t}\right)dv. \tag{2.4}$$

PROOF. The backward equation for $p(t, r, \theta; \rho, \alpha)$ is

$$\partial_t u = \frac{1}{2}\left(\partial_\rho^2 u + \frac{1}{\rho}\partial_\rho u + \frac{1}{\rho^2}\partial_\alpha^2 u\right). \tag{2.5}$$

Indeed, the proof that the backward equation for the density of Z_t is the heat equation is based on a local argument about the initial point [see, for instance, Feller (1971)] and yields, in polar coordinates, (2.5). But the argument still applies if one considers the total angle instead of identifying points with equal angle modulo 2π. The periodic boundary conditions that arise in this last case disappear and we are left with equation (2.5), for $\rho > 0$, and $\alpha \in (-\infty, +\infty)$.

By symmetry, p is an even function of the difference $\omega = \theta - \alpha$ (that will be denoted again by p). We shall consider (2.5) in this variable and $\alpha \in [0, 2\pi)$, so it corresponds with the usual polar coordinate of the initial point of Z_t. The corresponding forward equation is (1.2), so p also satisfies

$$\partial_t u = \frac{1}{2}\left(\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\omega^2 u\right). \tag{2.6}$$

Let us call $\mathscr{M} = (0, \infty) \times (-\infty, \infty)$. Recall that the volume element that corresponds to $\mathscr{M}$ in our case is $r\,dr\,d\omega$. We shall find a fundamental solution of this last equation, that is, a function $u(t, r, \omega; \rho) \in \mathbf{C}^\infty[(0,\infty) \times \mathscr{M}]$ satisfying it and such that for any given open set $\mathbf{U} \subset \mathscr{M}$,

$$\lim_{t \downarrow 0} \int_{\mathbf{U}} u(t, r, \omega; \rho) r\, dr\, d\omega = \mathbb{1}_{\mathbf{U}}(\rho, 0), \tag{2.7}$$

where $\mathbb{1}_{\mathbf{U}}$ denotes the indicator function of the set $\mathbf{U}$. Now, denote $\hat{u}(t, r, v; \rho) = \int_{-\infty}^{\infty} e^{iv\omega} u(t, r, \omega; \rho)\, d\omega$. From (2.5) and (2.6),

$$\partial \hat{u}_t = \frac{1}{2}\left(\partial_r^2 \hat{u} + \frac{1}{r}\partial_r \hat{u} - \frac{v^2}{r^2}\hat{u}\right) = \frac{1}{2}\left(\partial_\rho^2 \hat{u} + \frac{1}{\rho}\partial_\rho \hat{u} - \frac{v^2}{\rho^2}\hat{u}\right). \tag{2.8}$$

Performing the separation of variables $\hat{u} = T(t)F(r, \rho, v)$, one gets

$$2T' = \lambda_0 T, \tag{2.9a}$$

$$\partial_r^2 F + \frac{1}{r}\partial_r F - \frac{v^2}{r^2}F = \lambda_0 F, \tag{2.9b}$$

$$\partial_r^2 F + \frac{1}{\rho}\partial_\rho F - \frac{v^2}{\rho^2}F = \lambda_0 F, \tag{2.9c}$$

where λ_0 is the separation constant. Observe that $\int_{-\infty}^{\infty} p(t, r, \omega; \rho)\, d\omega = \int_0^{2\pi} \varphi(t, r, \omega; \rho)\, d\omega$, where

$$\varphi(t, r, \omega; \rho) = \frac{1}{2\pi t}\exp\left(-\frac{r^2 + \rho^2}{2t} + \frac{\rho r}{t}\cos\omega\right), \tag{2.10}$$

that is, the density $P\{R_t \in dr,\ \tilde{\theta}_t \in d\omega | R_0 = \rho,\ \theta_0 = 0\}\ 1/(r\,dr\,d\omega)$ $\tilde{\theta}_t$ being now the usual polar angle of Z_t. Then, $\hat{p}$ remains bounded as $t \to \infty$ (for fixed positive r, ρ) which justifies taking $\lambda_0 = -\lambda \le 0$. In this case, (2.9b) and (2.9c) are Bessel equations with solution of the form $AJ_v(\lambda^{1/2} r) J_v(\lambda^{1/2}\rho)$, where A may depend on v and λ. (Recall that p should be regular as $r, \rho \to 0$.) Putting all this together, one obtains that

$$\int_0^\infty \exp\left(\frac{-\lambda t}{2}\right) A(\lambda, v) J_v(\lambda^{1/2} r) J_v(\lambda^{1/2}\rho)\, d\lambda \tag{2.11}$$

is a solution of (2.8), where A has to be determined from (2.7). But it is known [see Gradshteyn and Ryzhik (1980), formula 6.615] that

$$\int_0^\infty \exp\left(\frac{-\lambda t}{2}\right) J_v(\lambda^{1/2} r) J_v(\lambda^{1/2}\rho)\, d\lambda = \frac{2}{t} I_v\left(\frac{\rho r}{t}\right)\exp\left(-\frac{\rho^2 + r^2}{2t}\right). \tag{2.12}$$

Also, from the asymptotic expansion of I_v [see Watson (1966)], one knows that $(2\pi\rho r/t)^{1/2} I_v(\rho r/t) e^{-\rho r/t} \to_{t \to 0} 1$, which yields that $(r/t) I_v(\rho r/t) \times \exp - (\rho^2 + r^2)/2t \to_{t \to 0} \delta(r - \rho)$. Taking $A(\lambda, v) = \frac{1}{2}$, one finds (after inverting the Fourier transform) that the right-hand side (r.h.s.) of (2.4) is a formal solution of the forward equation (1.2) satisfying (2.7). It is easily checked that it is, in fact, a solution. Indeed, from (2.1) it follows that the r.h.s. of (2.12) is C^∞ as a function of t and r; it is clear from the left-hand side (l.h.s.) that it solves in (2.8). Taking now as the path of integration in (2.1) three sides of the rectangle with corners at $\infty - \pi i$, $a - \pi i$, $a + \pi i$ and $\infty + \pi i$, for some positive a, it follows at once that $I_v(x) \in L^1(dv)$, for fixed positive x. Fourier inversion is then justified. Moreover, $v^2 I_v(x)$ also belongs to $L^1(dv)$, which permits us to conclude that we have a true solution. In order to see that it is, in fact, the density p, it is enough to see that it is positive and that it is the smallest elementary solution of (2.6). [See McKean, (1969).]

Now, the positivity follows from the formula

$$\begin{aligned}\int_0^\infty I_v(x)\cos v\omega\, dv &= \frac{1}{2}e^{r\cos\omega}\mathbb{1}_{(-\pi,\pi)}(\omega)\\ &\quad - \frac{1}{2\pi}\int_0^\infty e^{-r\cosh t}\left[\frac{\pi+\omega}{(\pi+\omega)^2+t^2} + \frac{\pi-\omega}{(\pi-\omega)^2+t^2}\right]dt,\end{aligned} \tag{2.13}$$

which is obtained after some calculations from (2.2).

Also, it is known [see Gradshteyn and Ryzhik (1980), formula 8.511-5] that

$$\begin{aligned}&\frac{1}{\pi t}\exp\left(-\frac{r^2+\rho^2}{2}\right)\sum_{k\in\mathbb{Z}}\int_0^\infty \cos[v(\omega+2k\pi)]\,I_v\left(\frac{r\rho}{t}\right)dv\\ &\quad = \frac{1}{2\pi t}\exp\left(-\frac{r^2+\rho^2}{2t} + \left(\frac{\rho r}{t}\right)\cos\omega\right).\end{aligned} \tag{2.14}$$

Observe that the right-hand side is the density (2.10) (of the polar coordinates of Z_t), which is the smallest solution of (2.6) with periodic boundary condition at $[0, 2\pi]$ and pole at $(\rho, 0)$. Then the r.h.s. of (2.4) is the smallest elementary solution of (2.6), as desired. □

REMARKS. From (2.13) we have for p a formula in terms of elementary functions (recall that $\omega = \theta - \alpha$):

$$\begin{aligned}p(t,r,\omega;\rho) &= \frac{1}{\pi t}\exp\left(-\frac{r^2+\rho^2}{2t}\right)\\ &\quad\times\left\{\frac{1}{2}e^{r\cos\omega}\mathbb{1}_{(-\pi,\pi)}(\omega)\right.\\ &\quad\left. - \frac{1}{2\pi}\int_0^\infty e^{-r\cosh t}\left[\frac{\pi+\omega}{(\pi+\omega)^2+t^2} + \frac{\pi-\omega}{(\pi-\omega)^2+t^2}\right]dt\right\}.\end{aligned} \tag{2.15}$$

Also, the characteristic function $\mathbf{E}\{e^{i\theta_t}|\theta_0 = 0,\ R_0 = \rho\}$ can be calculated from (2.4):

$$\begin{aligned}&\mathbf{E}\{e^{i\theta_t}|\theta_0 = 0,\ R_0 = \rho\}\\ &\quad = \rho\frac{\sqrt{2\pi}}{4\sqrt{t}}\exp\left(-\frac{\rho^2}{4t}\right)\left[I_{(|v|-1)/2}\left(\frac{\rho^2}{4t}\right) + I_{(|v|+1)/2}\left(\frac{\rho^2}{4t}\right)\right].\end{aligned} \tag{2.16}$$

This formula was deduced by Spitzer (1958) and from it, he concluded at once that $2\theta_t/\log t$ converges in distribution to a Cauchy random variable as t goes to ∞. A simpler proof of Spitzer's asymptotic law can be found in Section 3 of Pitman and Yor (1986). Extensions, related developments and more references on the subject are also contained in that article and in Pitman and Yor (1989).

Consider now $0 < \alpha < \beta$ and $T_\beta = \inf\{t > 0: \theta_t \notin (0, \beta)\}$, where $\theta_0 = \alpha$. We want to calculate

$$u(t, r, \theta; \rho, \alpha) = \mathbf{P}\{T_\beta > t,\ R_t \in dr,\ \theta_t \in d\theta | R_0 = \rho,\ \theta_0 = \alpha\} \frac{1}{(r\,dr\,d\theta)}.$$

But, from the rotational symmetry of Z_t, u can be obtained by applying sucessive reflections in both barriers. This gives that

$$u(t, r, \theta; \rho, \alpha) = \sum_{k \in \mathbb{Z}} p(t, r, 2k\beta + \theta; \rho, \alpha) - p(t, r, 2k\beta - \theta; \rho, \alpha). \tag{2.17}$$

From Theorem 1, and after some manipulations of this last expression, one obtains:

THEOREM 2.

$$\begin{aligned} &u(t, r, \theta; \rho, \alpha) \\ &\quad = \frac{1}{\beta t} \exp\left(-\frac{r^2 + \rho^2}{2t}\right) \sum_{k \in \mathbb{Z}} I_{|k\pi/\beta|}\left(\frac{\rho r}{t}\right) \sin\left(\frac{k\pi\alpha}{\beta}\right) \sin\left(\frac{k\pi\theta}{\beta}\right). \end{aligned} \tag{2.18}$$

PROOF. The series in (2.18) is clearly convergent. From (2.17), the r.h.s. of (2.18) satisfies (1.2), condition (2.7) and the boundary condition $u(t, r, 0; \rho, \alpha) = u(t, r, \beta; \rho, \alpha) = 0$ for all $r, t > 0$, so it must be the desired density.

COROLLARY. *Consider the case $\alpha = \beta/2$. Then,*

$$\mathbf{P}\{T_\beta > t\} = 4 \frac{e^{-\rho^2/4t}}{\pi^{1/2}\Gamma(\pi/2\beta + 1/2)} \left(\frac{\rho^2}{8t}\right)^{\pi/2\beta} + o(t^{-\pi/2\beta}) \quad \text{as } t \to \infty. \tag{2.19}$$

PROOF. Integration of (2.18) [see Gradshteyn and Ryzhik (1980), formula 6.618-4] yields

$$\begin{aligned} &\mathbf{P}\{T_\beta > t | R_0 = \rho, \theta_0 = \alpha\} \\ &\quad = \frac{\rho e^{-\rho^2/4t}}{(2\pi t)^{1/2}} \sum_{k \in \mathbb{Z}} \frac{\sin[(2k+1)\pi\alpha/\beta]}{2k+1} \big[\mathbf{I}_{|2k+1|\pi/2\beta - 1/2}(\rho^2/4t) \\ &\qquad\qquad + \mathbf{I}_{|2k+1|\pi/2\beta + 1/2}(\rho^2/4t)\big]. \end{aligned} \tag{2.20}$$

Taking $\theta_0 = \beta/2$ and replacing each Bessel function by its series expansions (2.3), one obtains a double series that is clearly uniformly convergent for $\rho^2/8t < 1$, and (2.19) follows.

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