The Annals of Probability
1992, Vol. 20, No. 3, 1310–1340

LIMIT THEOREMS FOR THE FRONTIER OF A ONE-DIMENSIONAL BRANCHING DIFFUSION

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Let R_t be the position of the rightmost particle at time t in a time-homogeneous one-dimensional branching diffusion process. Let $\gamma(\alpha, t)$ be the αth quantile of R_t under P^0, where P^x denotes the probability measure of the branching diffusion process starting with a single particle at position x. We show that $\gamma(\alpha, t)$ is a limiting quantile of R_t under P^x in the sense that $\lim_{t \to \infty} P^x\{R_t \le \gamma(\alpha, t)\}$ exists for all $\alpha \in (0, 1)$ and all $x \in \mathbb{R}$. If the underlying diffusion is recurrent, we show that, after an appropriate rescaling of space, the P^x distribution of $R_t - t$ converges weakly to a nontrivial limiting distribution w_x.

0. Introduction. The simplest example of a branching diffusion process in one dimension is *branching Brownian motion*, defined as follows. Starting at time $t = 0$ and position $x \in \mathbb{R}$, a particle begins a Brownian motion $X_1(t)$. At a random time T, independent of the motion $X_1(t)$ and with the unit exponential distribution, the particle undergoes a binary fission, creating a daughter particle, which begins its own Brownian motion $X_2(t)$ starting at $(T, X_1(T))$. Each particle repeatedly undergoes binary fissions following (independent) exponentially distributed gestation periods, creating new particles which behave as the original. At any given time $t \ge 0$, the state of the process is specified by the positions $(X_j(t))_{1 \le j \le N(t)}$ of the particles in existence at time t, indexed according to the order of birth.

A remarkable feature of the branching Brownian motion is that the distribution of the right frontier $R_t = \max(X_1(t), \dots, X_{N(t)}(t))$ is asymptotically a "travelling wave" with velocity $\sqrt{2}$. In particular, if $\gamma(1/2, t)$ is the median of the distribution of R_t under P^0 (P^x denotes the probability measure governing the process when the initial point is x), then

$$\lim_{t \to \infty} \gamma(1/2, t)/t = \sqrt{2} \tag{0.1}$$

and

$$\lim_{t \to \infty} P^0\{R_t \le \gamma(1/2, t) + y\} = w_0(y), \tag{0.2}$$

where $w_0(y)$ is a proper, continuous c.d.f. (cf. [5]). There is also a conditional

Received March 1989; revised January 1991.
[1]Supported by NSF Grants DMS-90-005118 and DMS-86-57071.
AMS 1980 *subject classifications*. Primary 60J80; secondary 60G55, 60F05.
Key words and phrases. Branching diffusion process, travelling wave, extreme value distribution.

analogue of (1.2):

$$\lim_{r\to\infty}\lim_{t\to\infty} P^0\{R_t \le \gamma(1/2,t) + y|\mathscr{F}_r\} = \exp\{-Ze^{-\sqrt{2}y}\} \tag{0.3}$$

for a certain r.v. Z valued in $(0,\infty)$, where $\mathscr{F}_r = \sigma((X_j(s))\colon s \le r)$ (cf. [2]). This exhibits the travelling wave $w_0(y)$ as a translation mixture of the extreme value law $\exp\{-e^{-\sqrt{2}y}\}$.

The purpose of this paper is to study the distribution of R_t for a more general class of one-dimensional branching diffusion processes in which the motions of individual particles are governed by a (more or less) arbitrary diffusion law (see below) and the rate of fission is position-dependent (as in [3] and [4]). It is clear that (0.2) cannot hold in this generality, because the local drift, diffusion and fission rate coefficients may vary wildly at ∞. Nevertheless, we shall prove that the distribution of R_t varies regularly in time in the sense that

$$\lim_{t\to\infty} P^x\{R_{t-s} \le \gamma(\alpha,t)\} = g(\alpha,s,x) \tag{0.4}$$

exists for all $\alpha \in (0,1)$ and $s, x \in \mathbb{R}$, where $\gamma(\alpha,t)$ is the αth quantile of the distribution of R_t under P^0 (Theorem 3.2). Thus, although $R_t - \gamma(1/2,t)$ may not converge in law as $t \to \infty$, the quantiles $\gamma(\alpha,t)$ change with t in a somewhat regular manner. Furthermore, we shall prove that if $\lim_{s\to\infty} g(1/2,s,x) = 1$ and $\lim_{s\to-\infty} g(1/2,s,x) = 0$, which is always the case if the underlying diffusion is recurrent, then there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} P^x\{f(R_t) \le t + y\} = g(1/2,y,x), \qquad \forall\, x,y \in \mathbb{R}. \tag{0.5}$$

Thus, the rescaled branching diffusion exhibits the travelling wave phenomenon (Theorem 3.6).

We shall also prove an analogue of the conditional law (0.3):

$$\lim_{r\to\infty}\lim_{t\to\infty} P^x\{R_{t-s} \le \gamma(\alpha,t)|\mathscr{F}_r\} = Y_{\alpha,s} \tag{0.6}$$

exists a.s. (P^x) for all $\alpha \in (0,1)$ and $s, x \in \mathbb{R}$ [cf. (4.2)]. Moreover, for each α, the random function $Y_{\alpha,s}$ assumes one of the following forms a.s. (P^x):

$$Y_{\alpha,s} = \begin{cases} 1, & \text{if } s > U_\alpha, \\ 0, & \text{if } s < U_\alpha, \end{cases} \tag{0.7}$$

$$Y_{\alpha,s} = \begin{cases} 1, & \text{if } s < U_\alpha, \\ 0, & \text{if } s > U_\alpha, \end{cases} \tag{0.8}$$

or

$$Y_{\alpha,s} = \exp\{-Z_\alpha e^{-C_\alpha s}\}, \tag{0.9}$$

where C_α is a real constant and U_α and Z_α are random variables satisfying $-\infty \le U_\alpha \le \infty$ and $0 \le Z_\alpha \le \infty$ (Theorem 5.1). For a given value of α, only one of the forms (0.7)–(0.9) can occur with positive P^x-probability. In cases (0.7) and (0.8) the behavior of R_t is ultimately predictable in the sense that the observed quantile $Q_t = \inf\{\alpha: R_t \le \gamma(\alpha, t)\}$ stabilizes as $t \to \infty$ (Proposition 5.2). The random variables U_α and $\log Z_\alpha$ may be thought of as random stabilization times. In Section 7 we shall present examples to show that each of the three types of possible behavior (0.7), (0.8) and (0.9) actually occurs. The reader should perhaps consult these examples before reading Sections 2–6.

The travelling wave phenomenon (0.2) occurs for many branching diffusions other than branching Brownian motion (cf. e.g., [3] and [4]; see also Examples 7.2, 7.3 and 7.5). In Section 6 we investigate the implications of our general results (0.4)–(0.9) for such processes. We will show that if (0.2) occurs, then the quantiles $\gamma(\alpha, t)$ must move linearly in t (as $t \to \infty$). Furthermore, we will show that the representations (0.7)–(0.9) simplify in this case by finding relations among the quantities U_α, Z_α and C_α for different α. We shall also give a simple sufficient condition for (0.9), and thus for the wave front to be a translation mixture of extreme value distributions $\exp\{-e^{-Cy}\}$ (Proposition 6.5). Finally, we call the reader's attention to Example 7.2, which exhibits a peculiar feature. In this example the underlying particle motion is the standard Ornstein–Uhlenbeck process and the fission rate is 1; under any P^x,

$$R_t - \sqrt{t} \to_{\mathscr{D}} 0.$$

This shows that R_t, suitably recentered, may converge in distribution even when R_t does not grow at a linear rate. However, Theorem 3.6 implies that if $R_t/t \to_P 0$ and $R_t - \gamma(1/2, t)$ converges in distribution, then the limit distribution must be degenerate.

1. Branching diffusion processes. The individual particles in our processes will move according to a conservative, nonsingular diffusion process in $(-\infty, \infty)$. In particular, there are no killings and no shunts ([1], Chapters 3 and 4). A conservative, nonsingular diffusion in $(-\infty, \infty)$ is determined by its scale function $S(x)$ and its speed measure $\mu(dx)$ ([1], Section 4.2); we assume for simplicity that μ has no atoms. An important fact that we will use repeatedly is that for a one-dimensional diffusion process with no shunts the transition probabilities $\underline{P}(t, x, dy)$ satisfy

$$\underline{P}(t,x,dy) = p(t,x,y)\mu(dy),$$
$$p(t,x,y) > 0, \qquad \forall\, t > 0, \forall\, x, y \in \mathbb{R}$$

(cf. [1], Section 4.11 and Problem 4.11.5). Since $\mu(J) > 0$ for every nonempty, open interval J, it follows that $\underline{P}(t, x, J) > 0$, $\forall\, t > 0$, $x \in \mathbb{R}$ and J open and nonempty. Recall also that diffusion processes have continuous sample paths —this is crucial for many of our arguments.

Individual particles reproduce as follows. The initial particle, moving along its trajectory $X_1(t)$, produces offspring at a random time T_1, where

$$P(T_1 > t | X_1(s), s \ge 0) = \exp\left\{-\int_0^t \beta(X_1(s))\, ds\right\}$$

and $\beta(x) \ge 0$ is a continuous function. Observe that T_1 may be ∞ with positive probability. Conditional on the path $X_1(s)$, $s \geq 0$, and the value of T_1, the number of offspring produced at time T_1 is governed by a probability distribution $\{p_n(X_1(T_1))\}_{n \geq 1}$, where $p_n(x)$ are continuous functions of x satisfying $\sum_{n=1}^{\infty} p_n(x) = 1$. The original particle and each of the offspring produced at time T_1 then follow (conditionally) independent paths governed by the law of the underlying diffusion, and obey the same reproduction law as the original particle. We make no assumptions about $\beta(x)$ and $\{p_n(x)\}_{n \geq 1}$ *except* that there are no "explosions," that is, the number $N(t)$ of particles born before time t is finite with probability 1. If $\beta(x)\sum_{n=1}^{\infty} np_n(x) < C$, $\forall\, x \in \mathbb{R}$, then the fact that $e^{-Ct}N(t)$ is a nonnegative supermartingale implies that there are no explosions.

We will generally consider only branching diffusion processes initiated by a single particle located at position x at time $t = 0$; the notation P^x will be used for the probability measure governing the process. (Sometimes we will let the initial point be a random variable with distribution ν, in which case P^{ν} will denote the probability measure.) The state of the process at time t consists of the locations $(X_j(t))$ of the particles in existence at t, $j = 1, 2, \dots, N(t)$. In some arguments we will need several copies of the branching diffusion process, for example, $(X_j(t))$ and $(\tilde{X}_j(t))$; in such cases we will use the same notational convention for all random variables associated with the processes, for example, $\tilde{N}_j(t)$ is the number of particles in $(\tilde{X}_j(t))$ at time t and $\tilde{R}_j(t) = \max(\tilde{X}_1(t), \dots, \tilde{X}_{\tilde{N}(t)}(t))$. Whenever several branching diffusion processes occur in the same context, they will always have the same diffusion law $(S(x), \mu(dx))$ and reproduction law $(\beta(x), \{p_n(x)\})$, although they may have different initial points. Sometimes it will be convenient to let a branching diffusion process begin at a time t other than 0.

Conditional on its history up to time s, the future of a branching diffusion process $(X_j(t))$ after s consists of a superposition of $N(s)$ independent branching diffusion processes begun at positions $X_1(s)$, $X_2(s), \dots, X_{N(s)}(s)$ at time s. This is the *Markov property* for branching diffusion processes. The *strong Markov property* also holds; this says the same thing as the Markov property, but with the fixed time s replaced by a finite stopping time τ. In some situations, for example, coupling arguments, $\sigma((X_j(s)),\ s \leq t)$ is not the natural filtration. We define an *admissible filtration* to be a filtration $(\mathscr{F}_t)_{t \geq 0}$ such that $(X_j(t))$ is adapted to $(\mathscr{F}_t)$ and the strong Markov property holds, that is, for any stopping time $\tau < \infty$ the distribution of $(X_j(t + \tau),\ t \geq 0)$ conditional on $\mathscr{F}_\tau$ is the same as that of $N(\tau)$ independent branching diffusion processes begun at $X_1(\tau), \dots, X_{N(\tau)}(t)$.

2. Comparison principles. Let $v(x)$ be a Borel measurable function of $x \in \mathbb{R}$ such that $0 \leq v \leq 1$; for $t \geq 0$, $x \in \mathbb{R}$ define

$$u(t, x) = E^x \prod_{j=1}^{N(t)} v\big(X_j(t)\big). \tag{2.1}$$

LEMMA 2.1. *$u(t, x)$ is a jointly continuous function of $(t, x) \in (0, \infty) \times \mathbb{R}$. Moreover, if v is continuous at x, then u is continuous at $(0, x)$.*

PROOF. The joint continuity of u in (t, x) for $t > 0$ follows from a simple coupling argument, since $0 \le v \le 1$. [If independent branching diffusion processes $(X_j(s))$ and $(X'_j(s))$ are started at x and x', respectively, then with high probability the paths $X_1(s)$ and $X'_1(s+\varepsilon)$ will meet at some $s \ll t$ before a fission has occurred on either path, provided $|x - x'|$ and $|\varepsilon|$ are small. On this event the processes may be coupled; hence the products in the definitions of $u(t,x)$ and $u(t+\varepsilon, x')$ are equal with high probability, and thus $|u(t,x) - u(t+\varepsilon, x')|$ is small.]

Let v be continuous at x, and let ε and $|x' - x|$ be small. If $(X_j(s))$ is a branching diffusion process started at x', then with high probability no fission occurs by time ε and $X_1(\varepsilon)$ is near x, which implies $|v(X_1(\varepsilon)) - v(x)|$ is small and therefore that $u(\varepsilon, x') \approx v(x)$. □

LEMMA 2.2. *For every $x \in \mathbb{R}$, $t \ge s$,*

$$u(t,x) = E^x \prod_{j=1}^{N(s)} u\big(t-s, X_j(s)\big). \tag{2.2}$$

Furthermore, if $Y_t(s) = \prod_{j=1}^{N(s)} u(t-s, X_j(s))$ for $0 \le s \le t$, then $Y_t(s)$ is a martingale relative to any admissible filtration $(\mathscr{F}_s)_{s\ge 0}$, under any P^x, $x \in \mathbb{R}$.

PROOF. By (2.1),

$$Y_t(s) = \prod_{j=1}^{N(s)} E^{X_j(s)} \prod_{i=1}^{N_j(t-s)} v\big(X_{ij}(t)\big),$$

where $X_{ij}(t)$, $i = 1, \dots, N_j(t-s)$, denote the positions at time t of the progeny of the particle at $X_j(s)$ at time s. By the Markov property of $(X_j(t))$ (conditional on $\mathscr{F}_s$ the future has the same law as an aggregation of $N(s)$ independent branching processes, started at $X_1(s), \dots, X_{N(s)}(s)$),

$$Y_t(s) = E^x\left(\prod_{j=1}^{N(t)} v\big(X_j(t)\big) \middle| \mathscr{F}_s \right).$$

Thus, $Y_t(s)$ is a martingale, and (2.2) follows from $u(t,x) = Y_t(0) = E^x(Y_t(s))$. □

Let A be an open subset of $(0,\infty) \times \mathbb{R}$ and let $(X_j(t))$ be a branching diffusion process started at some $x \in \mathbb{R}$. Define a new process $(\tilde{X}_j(t))$ by "freezing" any particle in $(X_j(t))$ the instant it hits A, not allowing it any further movement or reproduction. Thus, let $\tau_j = \inf\{t\colon (t, X_j(t)) \in A\}$ and define $X_j^*(t) = X_j(t \wedge \tau_j)$; then $(\tilde{X}_j(t))$ is the subset of $(X_j^*(t))$ obtained by deleting the path of any particle j' born of a particle j after time τ_j (see Figure 1). Let $\tilde{N}(t)$ be the number of particles in the collection $(\tilde{X}_j(t))$ at time t.

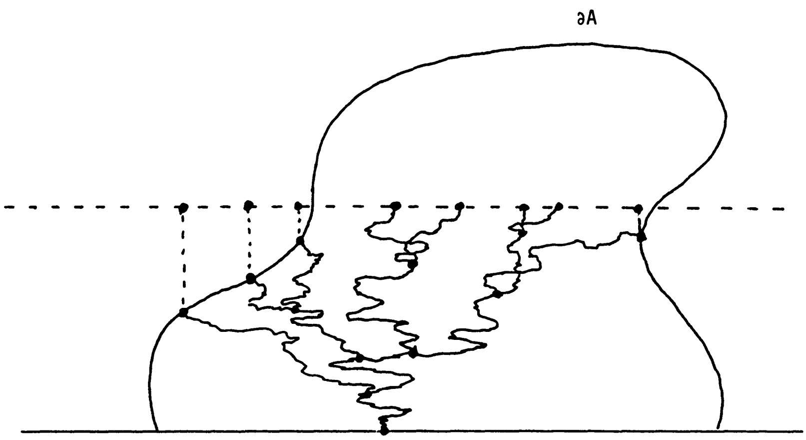

FIG. 1.

LEMMA 2.3. *For any $t > 0$, $x \in \mathbb{R}$,*

$$u(t,x) = E^x \prod_{j=1}^{\tilde{N}(t)} u\big(t - (\tau_j \wedge t), \tilde{X}_j(t \wedge \tau_j)\big). \tag{2.3}$$

PROOF. It follows from Lemma 2.2 that, for any stopping time $\nu \leq t$,

$$u(t,x) = E^x \prod_{j=1}^{N(\nu)} u\big(t - \nu, X_j(\nu)\big). \tag{2.4}$$

Let $0 \leq \nu_1 \leq \nu_2 \leq \cdots$ be the successive times at which paths in the collection $(X_j(t))$ reach (the boundary of) A. At time ν_1 one of the particles, say, the ith, has reached ∂A. Consider (2.4) with $\nu = \nu_1 \wedge t$; freeze the factor $u(t - \nu, X_i(\nu))$ corresponding to the particle that has just reached ∂A, then apply (2.4) to each of the other factors with $\nu = \nu_2 \wedge t$. Proceeding recursively through $\nu_3 \wedge t$, $\nu_4 \wedge t, \ldots,$ at each step freezing the factor corresponding to any particle that has reached ∂A, we obtain (2.3). □

Let $v_1(x)$ and $v_2(x)$ be Borel measurable functions of $x \in \mathbb{R}$ satisfying $0 \leq v_i \leq 1$, and let $u_1(t,x)$ and $u_2(t,x)$ be defined by (2.1) with $v = v_1$ and $v = v_2$, respectively.

LEMMA 2.4 (Majorization principle). *If $v_1(x) \geq v_2(x)$ for each $x \in \mathbb{R}$, then $u_1(t,x) \geq u_2(t,x)$ for each $(t,x) \in (0,\infty) \times \mathbb{R}$. If in addition $v_1(x) > v_2(x)$ for every x in some nonempty open interval, then $u_1(t,x) > u_2(t,x)$ for every $(t,x) \in (0,\infty) \times \mathbb{R}$.*

PROOF. By definition, $u_i(t,x) = E^x \prod_{j=1}^{N(t)} v_i(X_j(t))$. If $v_1 \geq v_2$, then the integrand for $i = 1$ is greater than or equal to that for $i = 2$, so $u_1 \geq u_2$. To prove the second statement, we will show that if $v_1 > v_2$ in the interval (a, b), then for every $(t, x) \in (0, \infty) \times \mathbb{R}$ there is positive P^x-probability that

$$\prod_{j=1}^{N(t)} v_1(X_j(t)) > \prod_{j=1}^{N(t)} v_2(X_j(t)).$$

Let $A = \{N(t) = 1\}$. Then $P^x(A|\mathscr{G}_1) > 0$, where $\mathscr{G}_1 = \sigma(X_1(t): t \geq 0)$ [this follows from the fact that the birth rate function $\beta(x)$ is bounded on compact intervals]. Let $B = \{X_1(t) \in (a, b)\}$. Then $P^x(B) > 0$ (cf. [1], Problem 5, Section 4.11). Since $B \in \mathscr{G}_1$ it follows that $P^x(A \cap B) > 0$. □

LEMMA 2.5 (Sign-change lemma). *If*

$$v_2(x) \geq v_1(x), \qquad \forall\, x > x_0, \tag{2.5}$$

and

$$v_2(x) \leq v_1(x), \qquad \forall\, x < x_0, \tag{2.6}$$

then

$$u_2(t, x_1) > u_1(t, x_1) \quad \Rightarrow \quad u_2(t, x) > u_1(t, x), \qquad \forall\, x \geq x_1, \tag{2.7}$$

and

$$u_2(t, x_2) < u_1(t, x_2) \quad \Rightarrow \quad u_2(t, x) < u_1(t, x), \qquad \forall\, x \leq x_2. \tag{2.8}$$

PROOF. It suffices to prove (2.7). Assume $v_2(x_0) < v_1(x_0)$. The same argument with slight modification works when $v_2(x_0) \geq v_1(x_0)$.

Define $A = \{(s, x): 0 < s < t \text{ and } u_1(t - s, x) > u_2(t - s, x)\}$; by Lemma 2.1, A is an open subset of $(0, \infty) \times \mathbb{R}$.

We shall prove that if $u_2(t, x_1) > u_1(t, x_1)$ and if (2.5) and (2.6) hold, then there is a continuous path $\gamma(s)$, $0 \leq s \leq t$, such that $\gamma(0) = x_1$, $\gamma(t) > x_0$ and $(s, \gamma(s)) \in A^c$, $\forall\ 0 < s < t$. Suppose not. Then every continuous path $\gamma(s)$, $0 \leq s \leq t$, such that $\gamma(0) = x_1$ must either enter A or terminate at $\gamma(t) \leq x_0$, but then Lemma 2.3 implies that $u_2(t, x_1) \leq u_1(t, x_1)$, a contradiction.

The path $\gamma(s)$ satisfies $u_2(t - s, \gamma(s)) \geq u_1(t - s, \gamma(s))$ for each $0 < s < t$. Since $\gamma(0) = x_1$ and $u_2(t, x_1) > u_1(t, x_1)$, and since u_1, u_2 are continuous, there exists $\delta > 0$ such that $u_2(t - s, \gamma(s)) > u_1(t - s, \gamma(s))$ for $0 \leq s \leq \delta$. Define $A^* = \{(s, x): 0 < s < t \text{ and } x < \gamma(s)\}$. Observe that any path $\beta(s)$ such that $\beta(0) \geq x_1$ and such that $(s, \beta(s))$ enters A^* must cross $(s, \gamma(s))$ as it enters A^* (see Figure 2). We now apply Lemma 2.3 to calculate $u_1(t, x)$, $u_2(t, x)$ for $x \geq x_1$, this time using the region A^* instead of A. Since $(\tau_j, \tilde{X}_j(\tau_j))$ is on $(s, \gamma(s))$ if $\tau_j < t$ and $X_j(t) \geq \gamma(t) > x_0$ if $t \leq \tau_j$,

$$u_2\big(t - (\tau_j \wedge t), \tilde{X}_j(\tau_j \wedge t)\big) \geq u_1\big(t - (\tau_j \wedge t), \tilde{X}_j(\tau_j \wedge t)\big), \quad \forall\ 1 \leq j \leq \tilde{N}(t).$$

Furthermore, strict inequality occurs with positive P^x-probability, because there is positive probability that $(s, X_1(s))$ crosses $(s, \gamma(s))$ for some $s \leq \delta$ ([1], Section 4.11, Problem 5). Thus, by Lemma 2.3, $u_2(t, x) > u_1(t, x)$ for $x \geq x_1$.

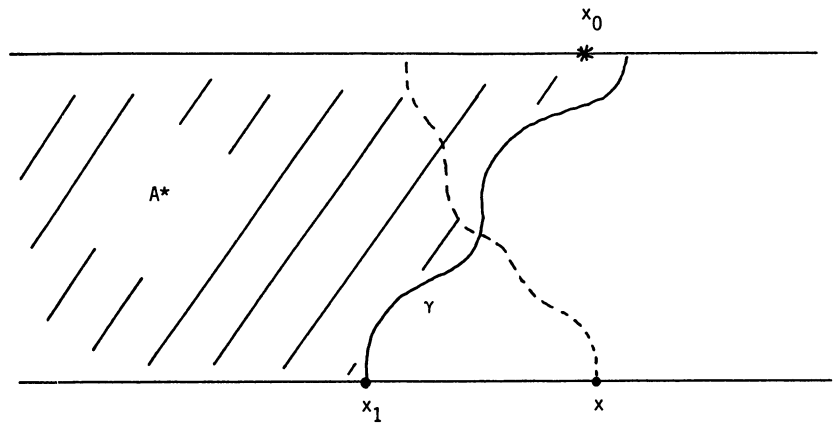

FIG. 2.

□

Lemma 2.5 states that if (2.5) and (2.6) hold, then for each $t > 0$ the difference $u_2(t, x) - u_1(t, x)$, as a function of x, has at most one sign change, from $-$ to $+$. A trivial but noteworthy consequence of Lemma 2.5 is that if (2.5) and (2.6) hold, then

$$(2.9) \quad u_2(t, x_1) \geq u_1(t, x_1) \quad \Rightarrow \quad u_2(t, x) \geq u_1(t, x), \qquad \forall\, x \geq x_1,$$

$$(2.10) \quad u_2(t, x_2) \leq u_1(t, x_2) \quad \Rightarrow \quad u_2(t, x) \leq u_1(t, x), \qquad \forall\, x \leq x_2.$$

Say that a Borel function $H: \mathbb{R} \to [0, 1]$ is a *martingale function* if, $\forall\, x \in \mathbb{R}$, $\forall\, t \geq 0$,

$$(2.11) \qquad H(x) = E^x \prod_{j=1}^{N(t)} H(X_j(t)).$$

Note that a pointwise limit of martingale functions is again a martingale function, by the dominated convergence theorem.

LEMMA 2.6. *Let $H(x)$ be a martingale function. If $H(x) = 0$ for some $x \in \mathbb{R}$, then $H(x) = 0, \forall\, x \in \mathbb{R}$, and if $H(x) = 1$ for some $x \in \mathbb{R}$, then $H(x) = 1$, $\forall\, x \in \mathbb{R}$.*

PROOF. If $H(x) = 0$, then, for any $t > 0$, $H(X_1(t)) = 0$ a.s. (P^x), because conditional on $X_1(t)$ there is positive P^x-probability that $N(t) = 1$. Now under any P^x, $x \in \mathbb{R}$, the distribution of $X_1(t)$ is equivalent to the speed measure $\mu(dy)$. Consequently, the distributions of $X_1(t)$ under the measures P^x, $x \in \mathbb{R}$, are all equivalent to each other, so that, $\forall\, x \in \mathbb{R}$, $P^x\{H(X_1(t)) = 0\} = 1$. This clearly implies that $H(x) = 0$, $\forall\, x \in \mathbb{R}$, in view of (2.11).

A similar argument proves the second statement of the lemma. □

3. Existence of a travelling wave. Recall that $R_t = \max(X_1(t), \dots, X_{N(t)}(t))$. Define $u(t, x, y)$ to be the distribution function of R_t under P^x, that is,

$$u(t,x,y) = P^x\{R_t \le y\} = E^x \prod_{j=1}^{N(t)} 1\{X_j(t) \le y\}. \tag{3.1}$$

For each $t > 0$ and $x \in \mathbb{R}$, $u(t, x, y)$ is a strictly increasing function of y, because, for any $a < b$, $P^x\{N(t) = 1 \text{ and } a < X_1(t) < b\} > 0$ ([1], Section 4.11, Problem 5 again). Thus, for $0 < \alpha < 1$, the αth quantile $\gamma(\alpha, t)$ of $u(t, 0, \cdot\,)$ is the unique real number such that $u(t, 0, \gamma(\alpha, t)) = \alpha$, for $0 < \alpha < 1$, and $\gamma(\alpha, t)$ is strictly increasing in α. For $\alpha = 0$ and $\alpha = 1$, we set $\gamma(0, t) = -\infty$ and $\gamma(1, t) = \infty$.

Let $-\infty < s_1 < s_2 < \infty$ and $-\infty < x_1 < x_2 < \infty$, and let $K = [s_1, s_2] \times [x_1, x_2] \subset \mathbb{R}^2$.

LEMMA 3.1. *For any $\delta > 0$, the set $\{u(t + s, x, y)\colon t \ge \delta - s_1,\ y \in \mathbb{R}\}$ is a uniformly equicontinuous family of functions of $(x, s) \in K$.*

PROOF. This follows from a simple coupling argument almost identical to that used in the proof of Lemma 2.1. □

Equation (2.2) implies that, for any $t > 0$, $s \ge 0$,

$$u(t+s,x,y) = E^x \prod_{j=1}^{N(s)} u\big(t, X_j(s), y\big). \tag{3.2}$$

Also, the sign-change lemma implies that if $t_1 < t_2$, then, for any $y_1, y_2 \in \mathbb{R}$, $u(t_2, x, y_2) - u(t_1, x, y_1)$ has at most one sign change in x ($-$ to $+$), that is,

$$u(t_2,x_1,y_2) > u(t_1,x_1,y_1) \quad \Rightarrow \quad u(t_2,x,y_2) > u(t_1,x,y_1), \qquad \forall\, x \ge x_1,$$
$$u(t_2,x_2,y_2) < u(t_1,x_2,y_1) \quad \Rightarrow \quad u(t_2,x,y_2) < u(t_1,x,y_1), \qquad \forall\, x \le x_2.$$

THEOREM 3.2. *For each $\alpha \in [0, 1]$, $s \in \mathbb{R}$ and $x \in \mathbb{R}$,*

$$\begin{aligned} g(\alpha,s,x) &= \lim_{t\to\infty} u(t-s,x,\gamma(\alpha,t)) \\ &= \lim_{t\to\infty} P^x\{R_{t-s} \le \gamma(\alpha,t)\} \end{aligned} \tag{3.3}$$

exists, and the convergence is uniform for (s, x) in any compact subset K of $\mathbb{R} \times \mathbb{R}$.

NOTE. $g(\alpha, 0, 0) = \alpha$, $\forall\, \alpha \in (0, 1)$.

PROOF OF THEOREM 3.2. It follows from Lemma 3.1 and the Arzela–Ascoli theorem that any sequence $t_n \to \infty$ has a subsequence $t_k \to \infty$ such that $u(t_k - s, x, \gamma(\alpha, t_k))$ converges uniformly for $(s, x) \in K$. Suppose that $t_1 <$

$t_1' < t_2 < t_2' < \cdots \to \infty$ are such that

$$u(t_k - s, x, \gamma(\alpha, t_k)) \to g(\alpha, s, x),$$
$$u(t_k' - s, x, \gamma(\alpha, t_k')) \to g'(\alpha, s, x).$$

We will show that $g \equiv g'$.

Since $t_k' > t_k$, $u(t_k' - s, x, \gamma(\alpha, t_k')) - u(t_k - s, x, \gamma(\alpha, t_k))$ has at most one sign change in x, from $-$ to $+$; consequently, $g'(\alpha, s, x) - g(\alpha, s, x)$ has at most one sign change, from $-$ to $+$. On the other hand, since $t_{k+1} > t_k'$ the same argument shows that $g(\alpha, s, x) - g'(\alpha, s, x)$ has at most one sign change, also from $-$ to $+$. Thus,

$$g(\alpha, s, x) > g'(\alpha, s, x) \text{ for some } x \quad \Rightarrow \quad g(\alpha, s, x) \geq g'(\alpha, s, x)\ \forall\, x;$$
$$g(\alpha, s, x) < g'(\alpha, s, x) \text{ for some } x \quad \Rightarrow \quad g(\alpha, s, x) \leq g'(\alpha, s, x)\ \forall\, x.$$

Suppose that for some $s > 0$, $g(\alpha, s, x) \geq g'(\alpha, s, x)\ \forall\, x \in \mathbb{R}$ or $g(\alpha, s, x) \leq g'(\alpha, s, x)\ \forall\, x \in \mathbb{R}$. By (3.2),

$$\alpha = u(t_k, 0, \gamma(\alpha, t_k)) = E^0 \prod_{j=1}^{N(s)} u(t_k - s, X_j(s), \gamma(\alpha, t_k)),$$

$$\alpha = u(t_k', 0, \gamma(\alpha, t_k')) = E^0 \prod_{j=1}^{N(s)} u(t_k' - s, X_j(s), \gamma(\alpha, t_k')),$$

so by the dominated convergence theorem

$$\alpha = E^0 \prod_{j=1}^{N(s)} g(\alpha, s, X_j(s)) = E^0 \prod_{j=1}^{N(s)} g'(\alpha, s, X_j(s)).$$

It now follows that $g(\alpha, s, X_1(s)) = g'(\alpha, s, X_1(s))$ P^0-a.s. Since $g(\alpha, s, x)$ and $g'(\alpha, s, x)$ are continuous in x it follows that $g(\alpha, s, x) = g'(\alpha, s, x)$, $\forall\, x \in \mathbb{R}$. Together with the results of the preceding paragraph this shows that $g(\alpha, s, x) = g'(\alpha, s, x)$, for all $x \in \mathbb{R}$ and $s > 0$, and $g(\alpha, 0, x) = g'(\alpha, 0, x)$ follows by continuity.

Now let $s < 0$; by (3.2) again,

$$u(t_k - s, x, \gamma(\alpha, t_k)) = E^x \prod_{j=1}^{N(-s)} u(t_k, X_j(-s), \gamma(\alpha, t_k))$$

and

$$u(t_k' - s, x, \gamma(\alpha, t_k')) = E^x \prod_{j=1}^{N(-s)} u(t_k', X_j(-s), \gamma(\alpha, t_k')),$$

$$\Rightarrow \quad g(\alpha, s, x) = E^x \prod_{j=1}^{N(-s)} g(\alpha, 0, X_j(-s))$$

and

$$g'(\alpha, s, x) = E^x \prod_{j=1}^{N(-s)} g'(\alpha, 0, X_j(-s)).$$

But $g(\alpha, 0, x) = g'(\alpha, 0, x)$, $\forall\, x \in \mathbb{R}$, so it follows that $g(\alpha, s, x) = g'(\alpha, s, x)$. This completes the proof that the limit in (3.3) exists. The local uniformity in (s, x) is an easy consequence of Lemma 3.1. □

PROPOSITION 3.3. *The function* $g(\alpha, s, x)$ *has the following properties*:

$$g(\alpha, s, x) = E^x \prod_{j=1}^{N(s')} g(\alpha, s + s', X_j(s')), \qquad \forall\, s' \geq 0; \tag{3.4}$$

$$g(\alpha, s, x) \text{ is strictly increasing in } \alpha; \tag{3.5}$$

$$g(\alpha, s, x) \text{ is jointly continuous in } \alpha, s, x; \tag{3.6}$$

$$\begin{aligned} g(\alpha, s, x) = g(\alpha', s', x) \text{ for some } x \Rightarrow g(\alpha, s + r, x) \\ = g(\alpha', s' + r, x), \forall\, x, r \in \mathbb{R}; \end{aligned} \tag{3.7}$$

(3.8) $\{g(\alpha, s + r, x)\colon 0 \leq \alpha \leq 1,\ s \in \mathbb{R}\}$ *is a uniformly equicontinuous family of functions of* $(r, x) \in K$, *for any compact* $K \subset \mathbb{R} \times \mathbb{R}$.

PROOF. (3.4) follows immediately from (3.2) and (3.3) by the dominated convergence theorem.

By Lemma 3.1, the family $\{u(t + s, x, y)\colon t \geq \delta - s_1,\ y \in \mathbb{R}\}$ is a uniformly equicontinuous family of functions of $(s, x) \in [s_1, s_2] \times [x_1, x_2]$. Since each $g(\alpha, s + r, x)$ is a uniform limit (locally) of functions in this collection, (3.8) follows.

Observe that, $\forall\, \alpha, \alpha' \in (0, 1)$, $\forall\, s, s' \in \mathbb{R}$, the function $g(\alpha, s, x) - g(\alpha', s', x)$ has no sign change in x. (This may be proved by an argument similar to that used in the proof of Theorem 3.2.) Thus, either $g(\alpha, s, x) \geq g(\alpha', s', x)\ \forall\, x \in \mathbb{R}$ or $g(\alpha, s, x) \leq g(\alpha', s', x)\ \forall\, x \in \mathbb{R}$.

If $r < 0$, then

$$g(\alpha, s + r, x) = E^x \prod_{j=1}^{N(-r)} g(\alpha, s, X_j(-r)),$$

$$g(\alpha', s' + r, x) = E^x \prod_{j=1}^{N(-r)} g(\alpha', s', X_j(-r)).$$

Thus, if $g(\alpha, s, x) = g(\alpha', s', x)\ \forall\ x \in \mathbb{R}$, it follows that $g(\alpha, s + r, x) = g(\alpha', s' + r, x)\ \forall\ r \leq 0,\ x \in \mathbb{R}$. On the other hand, if $g(\alpha, s, x) \geq g(\alpha', s', x)\ \forall\ x \in \mathbb{R}$ with strict inequality for some x, then strict inequality must hold for all x in some open interval J, since $g(\alpha, s, x)$ and $g(\alpha', s', x)$ are continuous functions of x. But then the integral representations above imply that $g(\alpha, s + r, x) > g(\alpha', s' + r, x)\ \forall\ r < 0,\ x \in \mathbb{R}$, because there is positive P^x-probability that $X_1(-r) \in J$ and $N(-r) = 1$. Thus, $g(\alpha, s, x) = g(\alpha', s', x)$ for some x implies $g(\alpha, s + r, x) = g(\alpha', s + r, x)\ \forall\ r > 0,\ x \in \mathbb{R}$, which in turn implies equality $\forall\ r \leq 0,\ x \in \mathbb{R}$. This proves (3.7).

Let $\alpha < \alpha'$. Then $g(\alpha, 0, 0) = \alpha < \alpha' = g(\alpha', 0, 0)$; hence by (3.7), $g(\alpha, s, x) < g(\alpha', s, x)\ \forall\ s, x \in \mathbb{R}$. This proves (3.5).

To prove (3.6) it suffices to show that $\lim_{\alpha \to \alpha_*} g(\alpha, s, x) = g(\alpha_*, s, x)$, for each (s, x), in view of (3.8). First consider the case $0 < \alpha_* < 1$. By (3.5), $\lim_{\alpha \uparrow \alpha_*} g(\alpha, s, x) = g'(\alpha_*, s, x)$ and $\lim_{\alpha \downarrow \alpha_*} g(\alpha, s, x) = g''(\alpha_*, s, x)$ exist, and $g' \le g \le g''$. By (3.4),

$$g'(\alpha_*, s, x) = E^x \prod_{j=1}^{N(s')} g'(\alpha_*, s + s', X_j(s'))$$

and

$$g''(\alpha_*, s, x) = E^x \prod_{j=1}^{N(s')} g''(\alpha_*, s + s', X_j(s'));$$

consequently, by the same argument used to prove (3.7), either $g'(\alpha_*, s, x) < g''(\alpha_*, s, x)\ \forall\ s, x \in \mathbb{R}$ or $g'(\alpha_*, s, x) = g''(\alpha_*, s, x)\ \forall\ s, x \in \mathbb{R}$. But $g(\alpha, 0, 0) = \alpha\ \forall\ \alpha \in (0,1)$, so $\alpha_* = g'(\alpha_*, 0, 0) = g''(\alpha_*, 0, 0)\ \forall\ \alpha_* \in (0,1)$. Hence $g' \equiv g''$. This proves that $\lim_{\alpha \to \alpha_*} g(\alpha, s, x) = g(\alpha_*, s, x)\ \forall\ \alpha_* \in (0,1)$, $\forall\ s, x \in \mathbb{R}$. The argument for $\alpha_* = 0$ and $\alpha_* = 1$ is essentially the same. □

Corollary 3.4. $g(\alpha, s, x) = g(g(\alpha, s, 0), 0, x)$.

Proof. Since $g(g(\alpha, s, 0), 0, 0) = g(\alpha, s, 0)$, this follows from (3.7). □

Proposition 3.5. *The function $g(\alpha, s, x)$ is strictly increasing or strictly decreasing, or is constant in s. Furthermore, for each α the same case holds simultaneously $\forall\ x \in \mathbb{R}$.*

Proof. By (3.7), for any $s > 0$ one of the following is true:

(i) $g(\alpha, r, x) < g(\alpha, s + r, x),\ \forall\ x, r \in \mathbb{R}$;
(ii) $g(\alpha, r, x) > g(\alpha, s + r, x),\ \forall\ x, r \in \mathbb{R}$;
(iii) $g(\alpha, r, x) = g(\alpha, s + r, x),\ \forall\ x, r \in \mathbb{R}$.

Setting $r = ks$, $k \in \mathbb{Z}$, we get that exactly one of the following is true:

(i′) $g(\alpha, ks, x) < g(\alpha, (k+1)s, x),\ \forall\ x \in \mathbb{R},\ k \in \mathbb{Z}$;
(ii′) $g(\alpha, ks, x) > g(\alpha, (k+1)s, x),\ \forall\ x \in \mathbb{R},\ k \in \mathbb{Z}$;
(iii′) $g(\alpha, ks, x) = g(\alpha, (k+1)s, x),\ \forall\ x \in \mathbb{R},\ k \in \mathbb{Z}$.

If we consider s of the form 2^{-n}, $n = 0, 1, 2, \dots$, it follows that $g(\alpha, s', x)$ is either strictly increasing in $s'\ \forall\ x$, strictly decreasing in $s'\ \forall\ x$, or constant in $s'\ \forall\ x$ when s' is restricted to integer multiples of 2^{-n}, $n = 0, 1, 2, \dots$. The proposition now follows from the continuity of g. □

Theorem 3.6. *Assume that*

$$\lim_{s \to \infty} g(1/2, s, x) = 1 \tag{3.9}$$

and

$$\lim_{s\to-\infty} g(1/2, s, x) = 0, \qquad \forall\, x \in \mathbb{R}. \tag{3.10}$$

Then there exists an increasing homeomorphism f: $\mathbb{R} \to \mathbb{R}$ such that the rescaled branching diffusion process $(\tilde{X}_j(t))_{1\le j\le \tilde{N}(t)}$ defined by $\tilde{N}(t) = N(t)$ and $\tilde{X}_j(t) = f(X_j(t))$ satisfies

$$\lim_{t\to\infty} P^x\{\tilde{R}_t \le t + y\} = g(1/2, y, x), \qquad \forall\, x, y \in \mathbb{R}. \tag{3.11}$$

NOTE. (i) If the underlying diffusion process $X_1(t)$ is recurrent, then (3.9) and (3.10) *must* hold. See Proposition 5.4 for a more general sufficient condition.

(ii) The hypotheses (3.9) and (3.10) imply that for each x the function $g(1/2, s, x)$ is strictly increasing in s, by Proposition 3.5, and by (3.7) the same is true for $g(\alpha, s, x)$, all $\alpha \in (0, 1)$. Furthermore, $g(1/2, y, x)$ is jointly continuous in x and y, by (3.6). Thus, (3.11) implies that under each P^x, $x \in \mathbb{R}$, the random variables $\tilde{R}_t - t$ converge in distribution to a proper, continuous, strictly increasing distribution function, as stated in the abstract.

PROOF OF THEOREM 3.6. First we will construct a suitable homeomorphism f: $\mathbb{R} \to \mathbb{R}$. Choose any $s > 0$; since $g(1/2, s, x) \uparrow$ in s and

$$\lim_{t\to\infty} P^x\{R_t \le \gamma(1/2, t + s)\} = g(1/2, s, x),$$
$$\lim_{t\to\infty} P^x\{R_t \le \gamma(1/2, t)\} = g(1/2, 0, x),$$

it follows that $\gamma(1/2, t) < \gamma(1/2, t + s)$ for all sufficiently large t. Furthermore, by (3.9), $\gamma(1/2, t) \to \infty$, as $t \to \infty$. For $n = 1, 2, \dots,$ define $t_n = \min\{k/2^n\colon k \ge 0 \text{ and } \gamma(1/2, j/2^n) < \gamma(1/2, (j + 1)/2^n),\ \forall\, j \ge k\}$ and note that $t_1 \le t_2 \le t_3 \le \cdots$. Now define

$$f(\gamma(1/2, j/2^n)) = j/2^n, \qquad \forall\, j/2^n \ge t_m;$$

f can be extended to an increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$.

If $\tilde{X}_j(t) = f(X_j(t))$, for $1 \le j \le N(t) = \tilde{N}(t)$, then clearly $\tilde{R}_t = f(R_t)$. Consequently, for any j, k such that $(j + k)/2^n \ge t_n$,

$$P^x\left\{\tilde{R}_{j/2^n} \le j/2^n + k/2^n\right\} = P^x\left\{R_{j/2^n} \le \gamma(1/2, (j + k)/2^n)\right\}.$$

It therefore follows from (3.3) and the continuity of $g(1/2, y, x)$ in y that as $t \to \infty$ through any of the discrete sets $D_n = [k/2^n\colon k \in \mathbb{Z}\}$,

$$P^x\left\{\tilde{R}_t \le t + y\right\} \to g(1/2, y, x).$$

The result (3.11) now follows from Lemma 3.1 [applied to the rescaled branching diffusion process $(\tilde{X}_j(t))$]. □

4. A family of martingales and a coupling argument.

For $\alpha \in (0, 1)$, $s \in \mathbb{R}$ and $t \ge 0$, define

$$Y_{\alpha, s}(t) = \prod_{j=1}^{N(t)} g(\alpha, s + t, X_j(t)). \tag{4.1}$$

Recall [cf. (3.7) and Proposition 3.5] that $g(\alpha, s, x)$ is increasing in α and monotone in s; consequently, for each $t \geq 0$, so is $Y_{\alpha, s}(t)$.

PROPOSITION 4.1. *$Y_{\alpha, s}(t)$ is a martingale relative to any admissible filtration $(\mathscr{F}_t)_{t \geq 0}$, under any P^x, $x \in \mathbb{R}$.*

PROOF. This follows from (3.4) by the same argument as in the proof of Lemma 2.2. □

We now examine more closely the martingales $Y_{\alpha, s}(t)$ defined by (4.1) and their limits. Recall that $\gamma(\alpha, t)$ is the αth quantile of R_t under P^0. By the Markov property, if $(\mathscr{F}_t)_{t \geq 0}$ is any admissible filtration for the branching diffusion process $(X_j(t))$, then

$$\begin{aligned} P^x\{R_{t-s} \leq \gamma(\alpha, t) | \mathscr{F}_r\} &= \prod_{i=1}^{N(r)} P^{X_j(r)}\{R_{t-s-r} \leq \gamma(\alpha, t)\} \\ &= \prod_{j=1}^{N(r)} u\big(t - s - r, X_j(r), \gamma(\alpha, t)\big), \end{aligned}$$

for $t - s \geq r$. Letting $t \to \infty$ and appealing to Theorem 3.2, we obtain

$$\begin{aligned} \lim_{t \to \infty} P^x\{R_{t-s} \leq \gamma(\alpha, t) | \mathscr{F}_r\} &= \prod_{j=1}^{N(r)} g\big(\alpha, s + r, X_j(r)\big) \\ &= Y_{\alpha, s}(r). \end{aligned}$$

Since $Y_{\alpha, s}(r)$ is a bounded martingale, it has a limit as $r \to \infty$; thus, we may define

$$Y_{\alpha, s} = \lim_{r \to \infty} Y_{\alpha, s}(r) = \lim_{r \to \infty} \lim_{t \to \infty} P^x\{R_{t-s} \leq \gamma(\alpha, t) | \mathscr{F}_r\}. \tag{4.2}$$

Observe that $Y_{\alpha, s}(r)$ and $Y_{\alpha, s}$ do not depend on the filtration $(\mathscr{F}_r)_{r \geq 0}$. Furthermore, with P^x-probability 1, $Y_{\alpha, s}$ is nondecreasing in α and monotone in s in the same direction as is $g(\alpha, s, x)$, because for each r the same properties obtain for $Y_{\alpha, s}(r)$. Also, since $Y_{\alpha, s}(r)$ is a bounded martingale under any P^x,

$$E^x Y_{\alpha, s} = E^x Y_{\alpha, s}(0) = g(\alpha, s, x). \tag{4.3}$$

Since $g(\alpha, s, x)$ is continuous and monotone, there is a version of the process $Y_{\alpha, s}$ which is right-continuous in s.

PROPOSITION 4.2. *For any $\alpha, \alpha_* \in (0,1)$ and $s_*, x \in \mathbb{R}$, either*

$$Y_{\alpha, s} \geq Y_{\alpha_*, s+s_*}, \quad \forall\, s \in \mathbb{R} \text{ with } P^x\text{-probability } 1, \tag{4.4}$$

or

$$Y_{\alpha, s} \leq Y_{\alpha_*, s+s_*}, \quad \forall\, s \in \mathbb{R} \text{ with } P^x\text{-probability } 1. \tag{4.5}$$

PROOF. By (3.17), either $g(\alpha, s + r, x) \geq g(\alpha_*, s + s_* + r, x)\ \forall\, x, r \in \mathbb{R}$ or $g(\alpha, s + r, x) < g(\alpha_*, s + s_* + r, x)\ \forall\, x, r \in \mathbb{R}$. By (4.1), $Y_{\alpha, s}(r) \geq Y_{\alpha_*, s+s_*}(r)$

$\forall\, r > 0$ in the first case and $Y_{\alpha,s}(r) < Y_{\alpha_*, s+s_*}(r)\ \forall\, r > 0$ in the second case. □

The next result is similar, but requires a more sophisticated argument; it does not seem to follow from the comparison methods of Section 2.

PROPOSITION 4.3. *For any* $\alpha \in (0,1)$, $s_*, x \in \mathbb{R}$ *and positive integers* k_1, k_2, *with* P^x-*probability* 1, *one of the following holds*:

$$Y^{k_1}_{\alpha,s} \geq Y^{k_2}_{\alpha,s+s_*}, \qquad \forall\, s \in \mathbb{R}, \tag{4.6}$$

or

$$Y^{k_1}_{\alpha,s} \leq Y^{k_2}_{\alpha,s+s_*}, \qquad \forall\, s \in \mathbb{R}. \tag{4.7}$$

REMARK. According to Proposition 4.3, it may be random whether (4.6) holds or (4.7) holds. However, it will follow from results in Section 5 that either (4.6) holds a.s. (P^x) or (4.7) holds a.s. (P^x).

PROOF OF PROPOSITION 4.3. Since $Y_{\alpha,s}$ is a monotone right-continuous function of s a.s. (P^x), it suffices to prove that for any $s_1, s_2, x \in \mathbb{R}$ and any $\varepsilon > 0$,

$$P^x\left\{Y^{k_1}_{\alpha,s_1} > Y^{k_2}_{\alpha,s_1+s_*} + \varepsilon \text{ and } Y^{k_1}_{\alpha,s_2} < Y^{k_2}_{\alpha,s_2+s_*} - \varepsilon\right\} = 0. \tag{4.8}$$

We will use a coupling argument, which will involve the use of auxiliary branching diffusion processes $(\tilde{X}^i_j(t))$, $i = 1,2,\dots,k_1$, and $(\hat{X}^i_j(t))$, $i = 1,2,\dots,k_2$. Let $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ be the filtration generated by the processes $(\tilde{X}^i_j(t)$, $i = 1,2,\dots,k_1)$ *and* by the original process $(X_j(t))$. Let $(\hat{\mathscr{F}}_t)_{t\geq 0}$ be the filtration generated by the processes $(\hat{X}^i_j(t)$, $i = 1,2,\dots,k_2)$. The processes will be constructed in such a way that $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ is an admissible filtration for its generating processes and likewise for $(\hat{\mathscr{F}}_t)_{t\geq 0}$. Random variables defined in terms of $(\tilde{X}^i_j(t))$ will be denoted by a superscript i and a tilde, for example, $\tilde{N}^i(t)$ = number of particles in $(\tilde{X}^i_j(t))$ at time t, $\tilde{R}^i_t = \max(\tilde{X}^i_1(t),\dots,\tilde{X}^i_{\tilde{N}^i(t)}$, and so on; similarly for $(\hat{X}^i_j(t))$.

We may assume without loss of generality that $s_* < 0$.

STEP 1. The auxiliary processes $(\tilde{X}^i_j(t))$ and $(\hat{X}^i_j(t))$ will coincide with the original process $(X_j(t))$ up to a certain time. Specifically, for a certain $r_* < \infty$,

$$\tilde{N}^i(t) = N(t) \quad \text{and} \quad \tilde{X}^i_j(t) = X_j(t), \qquad \forall\, t \leq r_*,\ j = 1,\dots,N(t), \tag{4.9}$$

$$\hat{N}^i(t) = N(t) \quad \text{and} \quad \hat{X}^i_j(t) = X_j(t), \qquad \forall\, t \leq r_* - s_*,\ j = 1,\dots,N(t). \tag{4.10}$$

Fix $\delta > 0$ (small); $r_* < \infty$ should be chosen so that

$$P^x\left\{\sup_{r\geq r_*} |Y_{\alpha,s} - Y_{\alpha,s}(r)| \geq \varepsilon/4(k_1+k_2)\right\} < \delta$$

for $s = s_1, s_2, s_1 + s_*, s_2 + s_*$. It follows from (4.2) that such an r_* exists. Now each of the processes $(\tilde{X}_j^i(t))$ and $(\hat{X}_j^i(t))$ has the same law as $(X_j(t))$, so the preceding inequality is valid also for $\tilde{Y}_{\alpha,s}^i$, $\tilde{Y}_{\alpha,s}^i(r)$ and $\hat{Y}_{\alpha,s}^i$, $\hat{Y}_{\alpha,s}^i(r)$. But $Y_{\alpha,s}(r_*) = \tilde{Y}_{\alpha,s}^i(r_*) = \hat{Y}_{\alpha,s}^i(r_*)$, since all the processes coincide up to time r_*; consequently,

$$P^x\big\{|Y_{\alpha,s} - \tilde{Y}_{\alpha,s}^i| \geq \varepsilon/2(k_1 + k_2)\big\} < 2\delta, \qquad 1 \leq i \leq k_1,$$

and

$$P^x\big\{|Y_{\alpha,s} - \hat{Y}_{\alpha,s}^i| \geq \varepsilon/2(k_1 + k_2)\big\} < 2\delta, \qquad 1 \leq i \leq k_2,$$

for $s = s_1, s_2, s_1 + s_*, s_2 + s_*$. Since all of the random variables $Y_{\alpha,s}$, $\tilde{Y}_{\alpha,s}^i$, $\hat{Y}_{\alpha,s}^i$ take values in [0, 1], it follows that for $s = s_1, s_2, s_1 + s_*, s_2 + s_*$,

$$P^x\left\{\left|\prod_{i=1}^{k_1} \tilde{Y}_{\alpha,s}^i - Y_{\alpha,s}^{k_1}\right| \geq \varepsilon/2\right\} < 2k_1\delta$$

and

$$P^x\left\{\left|\prod_{i=1}^{k_2} \hat{Y}_{\alpha,s}^i - Y_{\alpha,s}^{k_2}\right| \geq \varepsilon/2\right\} < 2k_2\delta.$$

The auxiliary processes are constructed in such a way that

$$P^x\left\{\prod_{i=1}^{k_1} \tilde{Y}_{\alpha,s_1}^i > \prod_{i=1}^{k_2} \hat{Y}_{\alpha,s_1+s_*}^i \text{ and } \prod_{i=1}^{k_1} \tilde{Y}_{\alpha,s_2}^i < \prod_{i=1}^{k_2} \hat{Y}_{\alpha,s_2+s_*}^i\right\} = 0; \tag{4.11}$$

this therefore implies that

$$P^x\big\{Y_{\alpha,s_1}^{k_1} > Y_{\alpha,s_1+s_*}^{k_2} + \varepsilon \text{ and } Y_{\alpha,s_2}^{k_1} < Y_{\alpha,s_2+s_*}^{k_2} - \varepsilon\big\} < 4(k_1 + k_2)\delta.$$

Since $\delta > 0$ is arbitrary, (4.8) then follows.

STEP 2. The processes $(\tilde{X}_j^i(t))$, $i = 1, 2, \ldots, k_1$, are constructed as follows. Run the original branching diffusion process $(X_j(t))$ up to time $t = r_*$; by (4.9), this determines the evolution of each $(\tilde{X}_j^i(t))$ up to $t = r_*$. At time $t = r_*$, each of the processes $(X_j(t))$ and $(\tilde{X}_j^i(t))$, $i = 1, \ldots, k_1$, has a single particle at each of the locations $X_1(r_*), X_2(r_*), \ldots, X_{N(r_*)}(r_*)$. Conditional on $\tilde{\mathscr{F}}_{r_*}$, let each of these $(1 + k_1)N(r_*)$ particles begin its own branching diffusion process forward in time, independent of all the others; the paths of these particles and their progeny constitute the futures (after time r_*) of the processes $(X_j(t)), (\tilde{X}_j^1(t)), \ldots, (\tilde{X}_j^{k_1}(t))$. Note that conditional on $\tilde{\mathscr{F}}_{r_*}$ these processes are mutually independent and have the same law.

Let all of the particles in the processes $(\tilde{X}_j^i(t))$, $i = 1, \ldots, k_1$, be colored white, and let each white particle be shadowed in spacetime by a red particle $-s_*$ time units in the past. Thus, for each path $\tilde{X}_j^i(t)$ of a white particle, $t \geq \tilde{t}_j^i \geq r_*$, define

$$\overline{X}_j^i(t) = \tilde{X}_j^i(t + s_*), \qquad t \geq \tilde{t}_j^i - s_*,$$

and let each $\bar{X}_j^i(t)$ be the path of a red particle. [Here, $\tilde{t}_j^i$ is of course the birth time of the white particle following path $\tilde{X}_j^i(t)$.] Observe that if a white particle fissions at time t, then the corresponding red particle fissions at time $t - s_*$, and the offspring red is the shadow particle of the offspring white. Conditional on $\tilde{\mathscr{F}}_{r_*}$, the processes $(\bar{X}_j^i(t))$ are independent branching diffusion processes, each initiated by particles at $(r_* - s_*, X_1(r_*)), \dots, (r_* - s_*, X_{N(r_*)}(r_*))$ in spacetime.

STEP 3. The construction of the processes $(\hat{X}_j^i(t))$, $i = 1, \dots, k_2$, will involve the paths of the red particles. The evolution of $(\hat{X}_j^i(t))$, for $t \le r_* - s_*$, is determined by (4.10). At time $t = r_* - s_*$ each of the processes $(\hat{X}_j^i(t))$ has a particle at each of the positions $X_1(r_* - s_*), \dots, X_{N(r_* - s_*)}(r_* - s_*)$; let all of these particles and their offspring be colored blue. At time $t = r_* - s_*$ there are also red particles at positions $X_1(r_*), \dots, X_{N(r_*)}(r_*)$, each of which will initiate a branching diffusion process of red particles.

Conditional on $\hat{\mathscr{F}}_{r_* - s_*}$, let each of the blue particles begin its own branching diffusion process, independent of all the other blue and red particles. However, whenever a free (uncoupled) blue particle meets a free red particle the two particles are coupled, and thereafter both particles follow the path of the red (including fissions—whenever the red particle fissions, so does the blue, and the offspring red and blue are coupled). The paths of the blue particles for $t \ge t_* - s_*$ constitute the processes $(\hat{X}_j^i(t))$, $i = 1, \dots, k_2$. Observe that conditional on $\hat{\mathscr{F}}_{r_* - s_*}$ the processes $(\hat{X}_j^i(t))$, $i = 1, \dots, k_2$, are mutually independent but are *not* independent of the processes $(\tilde{X}_j^i(t))$, $i = 1, \dots, k_1$.

For $t \ge r_* - s_*$, let $\mathscr{F}_t^* = \tilde{\mathscr{F}}_{t + s_*} \vee \hat{\mathscr{F}}_t$. Observe that, for $t \ge r_* - s_*$, $(\mathscr{F}_t^*)_{t \ge r_* - s_*}$ is an admissible filtration for both the red processes $(\bar{X}_j^i(t))$ and for the blue processes $(\hat{X}_j^i(t))$.

STEP 4. Consider the aggregation of all white particles, that is, the collection of all particles in the process $(\tilde{X}_j^1(t)), \dots, (\tilde{X}_j^{k_1}(t))$. Let $R_t^W = \max_{1 \le i \le k_1} \tilde{R}_t^i$ be the position of the rightmost white particle at time t. Conditional on $\tilde{\mathscr{F}}_{r_*}$ the random variables $\tilde{R}_t^1, \dots, \tilde{R}_t^{k_1}$ are independent; also, for each i,

$$\tilde{Y}_{\alpha, s}^i = \lim_{r \to \infty} \lim_{t \to \infty} P^x\{\tilde{R}_{t-s}^i \le \gamma(\alpha, t) | \tilde{\mathscr{F}}_r\},$$

by (4.2). It follows that

$$\prod_{i=1}^{k_1} \tilde{Y}_{\alpha, s}^i = \lim_{r \to \infty} \lim_{t \to \infty} P^x\{R_{t-s}^W \le \gamma(\alpha, t) | \tilde{\mathscr{F}}_r\}.$$

Recall that each white particle is shadowed by a red particle $-s_*$ time units in the past. Hence, if R_t^R is the position of the rightmost red particle at time t, then $R^R_{t-s_*} = R^W_t$, so

$$(4.12)\qquad \begin{aligned}\prod_{i=1}^{k_1} \tilde{Y}^i_{\alpha,s} &= \lim_{r\to\infty}\lim_{t\to\infty} P^x\Big\{R^R_{t-s-s_*} \le \gamma(\alpha,t) | \tilde{\mathscr{F}}_r\Big\}\\ &= \lim_{r\to\infty}\lim_{t\to\infty} P^x\big\{R^R_{t-s-s_*} \le \gamma(\alpha,t) | \mathscr{F}^*_r\big\}.\end{aligned}$$

Similarly, if $R^B_t = \max_{1\le i\le k_2} \hat{R}^i_t$ is the position of the rightmost blue particle at time t, then

$$(4.13)\qquad \begin{aligned}\prod_{i=1}^{k_2} \hat{Y}^i_{\alpha,s} &= \lim_{r\to\infty}\lim_{t\to\infty} P^x\{R^B_{t-s} \le \gamma(\alpha,t) | \hat{\mathscr{F}}_r\}\\ &= \lim_{r\to\infty}\lim_{t\to\infty} P^x\{R^B_{t-s} \le \gamma(\alpha,t) | \mathscr{F}^*_r\}.\end{aligned}$$

Consider next the relationship between R^B_t and R^R_t. If the rightmost particle at time t (among all particles, red or blue) is a free red, then $R^R_t > R^B_t$; if it is a free blue, then $R^B_t > R^R_t$; if it is a coupled blue–red, then $R^B_t = R^R_t$. (*Note*: The P^x-probability that two uncoupled particles are at the same location at time $t > 0$ is zero.) Call a set of free blue particles a *blue cluster* if no two have a free red between them, and call a set of free red particles a *red cluster* if no two have a free blue between them. If $t_1 < t_2$ and $R^B_{t_1} > R^R_{t_1}$ but $R^B_{t_2} < R^R_{t_2}$, then between times t_1 and t_2 the rightmost blue cluster disappears. (Recall that whenever a free red and a free blue meet they couple.) Similarly, if $t_1 < t_2$ and $R^R_{t_1} > R^B_{t_1}$ but $R^R_{t_2} < R^B_{t_2}$, then between times t_1 and t_2 the rightmost red cluster disappears. Now after time $r_* - s_*$ no new red clusters or blue clusters arise, because a free red particle cannot cross a free blue particle without coupling. Since there are only finitely many red and blue clusters at time $r_* - s_*$, it follows that only finitely many clusters can disappear after time $r_* - s_*$; therefore,

$$P^x\big\{R^B_t \le R^R_t \text{ eventually or } R^B_t \ge R^R_t \text{ eventually}\big\} = 1.$$

Define events $F = \{R^R_t \le R^B_t \text{ eventually}\}$ and $G = \{R^B_t \le R^R_t \text{ eventually}\}$. We will show that, $\forall\, s \in \mathbb{R}$,

$$(4.14)\qquad \prod_{i=1}^{k_1} \tilde{Y}^i_{\alpha,s} \ge \prod_{i=1}^{k_2} \hat{Y}^i_{\alpha,s+s_*} \quad \text{on } F \text{ a.s. } (P^x)$$

and

$$(4.15)\qquad \prod_{i=1}^{k_1} \tilde{Y}^i_{\alpha,s} \le \prod_{i=1}^{k_2} \hat{Y}^i_{\alpha,s+s_*} \quad \text{on } G \text{ a.s. } (P^x).$$

Since by the preceding paragraph $P^x(F \cup G) = 1$, this will imply (4.11) and therefore complete the proof of Proposition 4.2. Fix $s \in \mathbb{R}$, and define events

$$H_t = \big\{R^R_{t-s-s_*} \le \gamma(\alpha,t)\big\} \quad\text{and}\quad K_t = \big\{R^B_{t-s-s_*} \le \gamma(\alpha,t)\big\};$$

clearly

$$\lim_{t\to\infty} 1_F 1_{K_t\cap H_t^c} = 0 \quad \text{a.s.}\ (P^x) \quad \text{and} \quad \lim_{t\to\infty} 1_G 1_{H_t\cap K_t^c} = 0 \quad \text{a.s.}\ (P^x).$$

By the martingale convergence theorem,

$$\lim_{r\to\infty} P^x(F|\mathscr{F}_r^*) = 1_F \quad \text{a.s.}\ (P^x) \quad \text{and} \quad \lim_{r\to\infty} P^x(G|\mathscr{F}_r^*) = 1_G \quad \text{a.s.}\ (P^x).$$

Choose $\delta > 0$ (small); then there exist $r_\delta < \infty$ and $t_\delta < \infty$ such that for all $r \ge r_\delta$ and $t \ge t_\delta$,

$$\begin{aligned}
&E^x|1_F - P^x(F|\mathscr{F}_r^*)| < \delta,\\
&E^x\big(1_F 1_{K_t\cap H_t^c}\big) < \delta,\\
\Rightarrow\ &E^x\big(P^x(F|\mathscr{F}_r^*)1_{K_t\cap H_t^c}\big) < 2\delta,\\
\Rightarrow\ &E^x\big(P^x(F|\mathscr{F}_r^*)P^x(K_t\cap H_t^c|\mathscr{F}_r^*)\big) < 2\delta,\\
\Rightarrow\ &E^x\big(1_F P^x(K_t\cap H_t^c|\mathscr{F}_r^*)\big) < 2\delta,\\
\Rightarrow\ &P^x\big\{F\cap\big\{P^x(K_t\cap H_t^c|\mathscr{F}_r^*) > \sqrt{2\delta}\big\}\big\} < \sqrt{2\delta},\\
\Rightarrow\ &P^x\big\{F\cap\big\{P^x(K_t|\mathscr{F}_r^*) - P^x(H_t|\mathscr{F}_r^*) > \sqrt{2\delta}\big\}\big\} < \sqrt{2\delta}.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, it follows that

$$1_F\Big(\lim_{r\to\infty}\lim_{t\to\infty} P^x(K_t|\mathscr{F}_r^*)\Big) \le 1_F\Big(\lim_{r\to\infty}\lim_{t\to\infty} P^x(H_t|\mathscr{F}_r^*)\Big) \quad \text{a.s.}\ (P^x).$$

By (4.12) and (4.13), this proves (4.14); (4.15) follows by a similar argument. □

COROLLARY 4.4. *For any $\alpha \in (0,1)$, $s_*, x \in \mathbb{R}$ and $\beta \in (0,\infty)$, with P^x-probability 1 either*

$$Y_{\alpha,s} \ge Y^\beta_{\alpha,s+s_*}, \qquad \forall\, s \in \mathbb{R}, \tag{4.16}$$

or

$$Y_{\alpha,s} \le Y^\beta_{\alpha,s+s_*}, \qquad \forall\, s \in \mathbb{R}. \tag{4.17}$$

PROOF. For rational β this follows immediately from Proposition 4.3. Any irrational β is the limit of an increasing sequence of rationals. □

Let $(X_j(t))$ and $(\tilde{X}_j(t))$ be independent branching diffusion processes started from single particles at x and $\tilde{x}$, respectively, under $P = P^{x,\tilde{x}}$. Let random variables defined in terms of $(\tilde{X}_j(t))$ be denoted by a tilde; for example, $\tilde{N}(t)$ is the number of particles in $(\tilde{X}_j(t))$ at time t.

PROPOSITION 4.5. *For any $\alpha \in (0,1)$, $s_* \in \mathbb{R}$ and $x, \tilde{x} \in \mathbb{R}$, with $P^{x,\tilde{x}}$-probability 1 either*

$$Y_{\alpha,s} \ge \tilde{Y}_{\alpha,s+s_*}, \qquad \forall\, s \in \mathbb{R}, \tag{4.18}$$

or

$$Y_{\alpha,s} \le \tilde{Y}_{\alpha,s+s_*} \qquad \forall\, s \in \mathbb{R}. \tag{4.19}$$

PROOF. This is virtually the same as that of Proposition 4.3. Define a third branching diffusion process $(\hat{X}_j(t))$ as follows. Let $(\hat{X}_j(t))$ coincide with $(\tilde{X}_j(t))$ up to time r. After time r, whenever a particle from $(\hat{X}_j(t))$ meets a particle from $(X_j(t+s_*))$ the two particles are coupled and thereafter follow the path of the particle from $(X_j(t+s_*))$. As in the proof of Proposition 4.3, for any s_1, $s \in \mathbb{R}$,

$$P^{x,\tilde{x}}\Big\{Y_{\alpha,s_1} > \hat{Y}_{\alpha,s_1+s_*} \text{ and } Y_{\alpha,s_2} < \hat{Y}_{\alpha,s_2+s_*}\Big\} = 0.$$

Furthermore, if r is chosen sufficiently large then

$$P^{x,\tilde{x}}\Big\{|\hat{Y}_{\alpha,s+s_*} - \tilde{Y}_{\alpha,s+s_*}| \ge \varepsilon/2\Big\} < \varepsilon,\ s = s_1, s_2.$$

Letting $\varepsilon \to 0$, one obtains the desired result. □

5. The extreme value law.

THEOREM 5.1. *For each $\alpha \in (0,1)$, with P^x-probability* 1 (*for any $x \in \mathbb{R}$*) *the random functions $Y_{\alpha,s}$ must assume one of the following forms*:

$$Y_{\alpha,s} = \begin{cases} 1, & \text{if } s > U_\alpha, \\ 0, & \text{if } s < U_\alpha; \end{cases} \tag{5.1}$$

$$Y_{\alpha,s} = \begin{cases} 1, & \text{if } s < U_\alpha, \\ 0, & \text{if } s > U_\alpha; \end{cases} \tag{5.2}$$

$$Y_{\alpha,s} = \exp\{-Z_\alpha e^{-C_\alpha s}\}, \qquad \forall\, s \in \mathbb{R}, \tag{5.3}$$

where $C_\alpha \in \mathbb{R}$ is a constant independent of x and the random variables U_α and Z_α satisfy $-\infty \le U_\alpha \le \infty$ and $0 \le Z_\alpha \le \infty$. For a given α, only one of the three types (5.1)–(5.3) *may occur with positive P^x-probability* (*excluding the trivial overlaps where $U_\alpha = \pm\infty$ and $Z_\alpha = 0$ or ∞*). *Moreover, with P^x-probability* 1 *all of the functions $Y_{\alpha,s}$, $\alpha \in (0,1)$, are of the same type.*

In Section 7 we give examples showing all three types (5.1)–(5.3) are possible. Note that types (5.1) and (5.2) may be regarded as limiting cases of (5.3) in which $C_\alpha = \pm\infty$. In the event that the functions $Y_{\alpha,s}$ are of type (5.1) or (5.2), the behavior of R_t is ultimately predictable (see Proposition 5.2).

PROOF OF THEOREM 5.1. Recall that $Y_{\alpha,s}$ is monotone in s and nondecreasing in α. Suppose that $0 < Y_{\alpha,s} < 1$ and $0 < Y_{\alpha,s+s_*} < 1$ for some $s \in \mathbb{R}$, $s_* > 0$. Then $0 < Y_{\alpha,s+r} < 1$ for all $r \in [0, s_*]$. By Corollary 4.4, for each $r \in [0, s_*]$ there exists $b(r) \in \mathbb{R}$ such that

$$\frac{\log Y_{\alpha,s+r}}{\log Y_{\alpha,s}} = b(r), \qquad \forall\, s \in \mathbb{R};$$

a routine argument using the monotonicity in s of $Y_{\alpha,s}$ now shows that (5.3) holds. This proves that with P^x-probability 1 for each α the function $Y_{\alpha,s}$ must assume one of the forms (5.1)–(5.3).

Suppose that for some α case $(5.i)$ obtains ($i = 1, 2$ or 3) with $-\infty < U_\alpha < \infty$ (if $i = 1$ or 2) and $0 < Z_\alpha < \infty$ (if $i = 3$). Then by Proposition 4.2 the *same* case $(5.i)$ must obtain for *all* $\alpha \in (0,1)$ (although perhaps with $U_\alpha = \pm\infty$ or $Z_\alpha = 0$ or ∞). Thus the probability space $(\Omega, \mathscr{F}, P^x)$ is partitioned into four disjoint events F_1, F_2, F_3, F_4; on F_i ($i = 1, 2$ or 3) all $Y_{\alpha,s}$ are of type $(5.i)$ with $-\infty < U_\alpha < \infty$ or $0 < Z_\alpha < \infty$ for some α, and on F_4 all $Y_{\alpha,s}$ are either identically 1 or identically 0.

Suppose now that for some $\alpha = \alpha_*$ case $(5.i)$ (with $i = 1$ or 2) holds and that $-\infty < U_{\alpha_*} < \infty$ with positive P^x-probability. Then by Proposition 4.5 the same case $(5.i)$ must hold with P^x-probability 1 for $\alpha = \alpha_*$ (although U_α may assume the values $\pm\infty$ with positive probability). Similarly, if for some $\alpha = \alpha_*$ case (5.3) holds and $0 < Z_{\alpha_*} < \infty$ with positive P^x-probability, then by Proposition 4.5 case (5.3) must hold with $P^{\tilde{x}}$-probability 1, for $\alpha = \alpha_*$, and C_{α_*} must be a constant r.v., the same for all initial points $\tilde{x}$. □

Define $Q_t = \inf\{\alpha: R_t \le \gamma(\alpha, t)\}$, that is, Q_t is the observed quantile of R_t at time t. As in the proof of Theorem 5.1, define F_i ($i = 1, 2$ or 3) to be the event on which all $Y_{\alpha,s}$, $\alpha \in (0,1)$, are of type i and not all of another type i', and define F_4 to be the event on which all $Y_{\alpha,s}$, $\alpha \in (0,1)$, are identically 0 or 1.

PROPOSITION 5.2. *There exists a random variable Q such that Q_t converges to Q in probability on $F_1 \cup F_2 \cup F_4$, that is, $\forall\, \varepsilon > 0$, $\forall\, x \in \mathbb{R}$,*

$$\lim_{t\to\infty} P^x\left\{|Q_t 1_{F_1\cup F_2\cup F_4} - Q 1_{F_1\cup F_2\cup F_4}| > \varepsilon\right\} = 0. \tag{5.4}$$

PROOF. This is a routine consequence of (4.2) with $s = 0$, because on $F_1 \cup F_2 \cup F_4$,

$$Y_{\alpha,0} = \begin{cases} 1, & \text{if } \alpha > Q, \\ 0, & \text{if } \alpha < Q, \end{cases}$$

for a suitable Q valued in $[0,1]$. □

PROPOSITION 5.3. *Assume $P^x(F_3) = 1$ for some $x \in \mathbb{R}$. Then*

$$g(\alpha, s, x) = E^x \exp\left\{-Z_\alpha e^{-C_\alpha s}\right\}. \tag{5.5}$$

For any $\alpha \in (0,1)$ the constant $C_\alpha = 0$ iff $g(\alpha, s, 0)$ is constant in s, $C_\alpha > 0$ iff $g(\alpha, s, 0)$ is increasing in s and $C_\alpha < 0$ iff $g(\alpha, s, 0)$ is decreasing in s. For any $\alpha < \alpha'$, if $P^x\{0 < Z_\alpha < \infty$ and $0 < Z_{\alpha'} < \infty\} > 0$, then $C_\alpha = C_{\alpha''}$ $\forall\, \alpha'' \in [\alpha, \alpha']$. If, for some $\alpha \in (0,1)$ and $x \in \mathbb{R}$, $P^x\{0 < Z_\alpha < \infty\} = 1$, then $C_\alpha = C$ is the *same* $\forall\, \alpha \in (0,1)$. *Finally, if* $C_\alpha \neq 0$, *then*

$$(5.6)\qquad P^x\{Z_\alpha = 0\} = \begin{cases} \lim_{s\to\infty} g(\alpha,s,x), & \text{if } C_\alpha < 0,\\ \lim_{s\to-\infty} g(\alpha,s,x), & \text{if } C_\alpha > 0,\end{cases}$$

$$(5.7)\qquad P^x\{Z_\alpha = \infty\} = \begin{cases} \lim_{s\to\infty} (1-g(\alpha,s,x)), & \text{if } C_\alpha > 0,\\ \lim_{s\to-\infty} (1-g(\alpha,s,x)), & \text{if } C_\alpha < 0.\end{cases}$$

PROOF. Since $Y_{\alpha,s}$ is the limit as $r \to \infty$ of a bounded martingale $Y_{\alpha,s}(r)$ with $Y_{\alpha,s}(0) = g(\alpha,s,x)$, (5.5) follows directly from (5.3), and (5.6) and (5.7) are immediate consequences of (5.5). The necessary and sufficient conditions for $C_\alpha = 0$, $C_\alpha < 0$ and $C_\alpha > 0$ also follow immediately from (5.5).

Suppose that $P^x\{0 < Z_\alpha < \infty \text{ and } 0 < Z_{\alpha'} < \infty\} > 0$; then with positive P^x-probability

$$Y_{\alpha,s} = \exp\{-Z_\alpha e^{-C_\alpha s}\}, \qquad \forall\, s,\ 0 < Z_\alpha < \infty,$$
$$Y_{\alpha',s} = \exp\{-Z_{\alpha'} e^{-C_{\alpha'} s}\}, \qquad \forall\, s,\ 0 < Z_{\alpha'} < \infty,$$
$$Y_{\alpha'',s} = \exp\{-Z_{\alpha''} e^{-C_{\alpha''} s}\}, \qquad \forall\, s,\ 0 < Z_{\alpha''} < \infty$$

(that $0 < Z_{\alpha''} < \infty$ when $0 < Z_\alpha, Z_{\alpha'} < \infty$ follows from the monotonicity in α of $Y_{\alpha,s}$). By Proposition 4.2, $C_\alpha = C_{\alpha'} = C_{\alpha''}$.

Finally, suppose $P^x\{0 < Z_\alpha < \infty\} = 1$. Then for any $\alpha' \neq \alpha$, either $P^x\{0 < Z_{\alpha'} < \infty\} = 0$, in which case the choice of $C_{\alpha'}$ is irrelevant, or $P^x\{0 < Z_\alpha, Z_{\alpha'} < \infty\} > 0$, in which case $C_\alpha = C_{\alpha'}$. □

Define

$$(5.8)\qquad G^+(\alpha,x) = \lim_{s\to\infty} g(\alpha,s,x),$$

$$(5.9)\qquad G^-(\alpha,x) = \lim_{s\to-\infty} g(\alpha,s,x).$$

Let $T = \inf\{t \geq 0: N(t) \geq 2\}$ and for $x \in \mathbb{R}$ let $\tau_x = \inf\{t \geq T: \text{some } X_j(t) = x\}$.

PROPOSITION 5.4. *Suppose that either* (5.10) *or* (5.11) *holds*:

$$(5.10)\qquad G^+(\alpha,x) = 1 \quad \text{and} \quad G^-(\alpha,x) = 0, \qquad \forall\, \alpha, x.$$

$$(5.11)\qquad G^+(\alpha,x) = 0 \quad \text{and} \quad G^-(\alpha,x) = 1, \qquad \forall\, \alpha, x.$$

Then, for some $i = 1$, 2 *or* 3,

$$(5.12)\qquad P^x(F_i) = 1, \qquad \forall\, x \in \mathbb{R}.$$

If $P^x(F_1) = 1$ *or* $P^x(F_2) = 1$, *then* $-\infty < U_\alpha < \infty$ *a.s.* (P^x). *If* $P^x(F_3) = 1$, *then* $C_\alpha = C$ *is independent of* α, $C \neq 0$ *and* $0 < Z_\alpha < \infty$ *a.s.* (P^x).

If, for some $x \in \mathbb{R}$, $P^x\{\tau_x < \infty\} = 1$, *then* (5.10) *holds.*

NOTE. If $X_1(t)$ is recurrent and $\beta(x) > 0$ somewhere, then $P^x\{\tau_x < \infty\} = 1$, $\forall\, x \in \mathbb{R}$.

PROOF OF PROPOSITION 5.4. Recall from (4.3) that $g(\alpha, s, x) = E^x Y_{\alpha, s}$. Consequently, if either (5.10) or (5.11) holds, then, for each $\alpha \in (0,1)$ and each $x \in \mathbb{R}$,

$$P^x\{Y_{\alpha, s} = 0\ \forall\, s \in \mathbb{R} \text{ or } Y_{\alpha, s} = 1\ \forall\, s \in \mathbb{R}\} = 0.$$

Thus, $Y_{\alpha, s}$ must be of the form (5.1) or (5.2) with $-\infty < U_\alpha < \infty$ or of the form (5.3) with $0 < Z_\alpha < \infty$ and $C_\alpha \neq 0$. But now Proposition 4.5 implies that $P^x(F_1) = 1\ \forall\, x$ or $P^x(F_2) = 1\ \forall\, x$ or $P^x(F_3) = 1\ \forall\, x$. If $P^x(F_3) = 1$, then $C_\alpha = C$ is independent of α, by Proposition 5.3.

Note that $G^+(\alpha, x)$ and $G^-(\alpha, x)$ are continuous functions of x, by (3.8). Also, $\forall\, \alpha$, $\prod_{j=1}^{N(t)} G^+(\alpha, X_j(t))$ and $\prod_{j=1}^{N(t)} G^-(\alpha, X_j(t))$ are martingales relative to any admissible filtration $(\mathscr{F}_t)$, under any P^x, $x \in \mathbb{R}$, by Proposition 4.1 and the dominated convergence theorem for conditional expectations. Thus, $\forall\, \alpha$, the functions $G^+(\alpha, x)$ and $G^-(\alpha, x)$ are martingale functions of x, so if $G^\pm(\alpha, x) < 1$ for some $x \in \mathbb{R}$, then $G^\pm(\alpha, x) < 1\ \forall\, x \in \mathbb{R}$, by Lemma 2.6.

Assume now that $P^x(\tau_x < \infty\} = 1$ for some x. The martingale property implies that

$$G^\pm(\alpha, x) = E^x \prod_{j=1}^{N(\tau_x)} G^\pm(\alpha, X_j(\tau_x)),$$

so by the result of the previous paragraph, if $G^+(\alpha, x) < 1$ for some $x \in \mathbb{R}$, then $G^+(\alpha, x) = 0$ for all $x \in \mathbb{R}$, and similarly for G^-. For each $\alpha \in (0,1)$, $g(\alpha, 0, 0) = \alpha$, so by Proposition 3.5 the function $g(\alpha, s, x)$ is either strictly increasing or strictly decreasing in s. Furthermore, it follows from $g(\alpha, 0, x) = E^x(Y_{\alpha,0}(\tau_x))$ and

$$Y_{\alpha,0}(\tau_x) = g(\alpha, \tau_x, x) \prod_{X_j(\tau_x) \neq x} g(\alpha, \tau_x, X_j(\tau_x))$$

that $g(\alpha, s, x)$ is increasing rather than decreasing in s. Consequently (5.10) must hold. □

6. Travelling waves.

For certain branching diffusion processes the distribution of $R_t - \gamma(1/2, t)$ has a weak limit as $t \to \infty$ (cf. Theorem 3.6). In this section we investigate some of the consequences of our previous results for such processes. We assume throughout this section that

$$\lim_{t \to \infty} P^0\{R_t \leq \gamma(1/2, t) + y\} = w_0(y), \qquad \forall\, y \in \mathbb{R}, \tag{6.1}$$

where $w_0(y)$ is a *proper*, continuous c.d.f. [thus $w_0(y) \to 1$ as $y \to \infty$ and $w_0(y) \to 0$ as $y \to -\infty$). Let $\gamma(\alpha) = \inf\{y: w_0(y) \geq \alpha\}$ be the αth quantile of $w_0(y)$. Note that γ is continuous at α iff w_0 is strictly increasing at $\gamma(\alpha)$ [i.e., $\forall\, \varepsilon > 0$, $w_0(\gamma(\alpha) + \varepsilon) - w_0(\gamma(\alpha)) > 0$ and $w_0(\gamma(\alpha)) - w_0(\gamma(\alpha) - \varepsilon) > 0$].

LEMMA 6.1. *Assume* (6.1). *Then, for any* $\alpha \in (0,1)$ *and* $s, x \in \mathbb{R}$,

$$\lim_{t\to\infty} P^x\{R_{t-s} \le \gamma(1/2,t) + \gamma(\alpha)\} = g(\alpha, s, x). \tag{6.2}$$

PROOF. By Theorem 3.2, for every $\alpha' \in (0,1)$,

$$\lim_{t\to\infty} P^x\{R_{t-s} \le \gamma(\alpha', t)\} = g(\alpha', s, x), \tag{6.3}$$

and by Proposition 3.3, $g(\alpha', s, x)$ is continuous in α'. Consequently, to prove (6.2) it suffices to show that $\forall\, \varepsilon > 0\ \exists\ t_\varepsilon$ such that for $t \ge t_\varepsilon$,

$$\gamma(\alpha - \varepsilon, t) \le \gamma(1/2, t) + \gamma(\alpha) \le \gamma(\alpha + \varepsilon, t).$$

But this follows from (6.1) and (6.3) with $s = x = 0$ and $\alpha' = \alpha \pm \varepsilon$, since $g(\alpha', 0, 0) = \alpha'$ and $w_0(\gamma(\alpha)) = \alpha$. □

PROPOSITION 6.2. *Assume* (6.1). *Then there exists a constant* $v \in \mathbb{R}$ *such that*, $\forall\, \alpha \in (0,1)$ *and* $s \in \mathbb{R}$,

$$\lim_{t\to\infty} (\gamma(\alpha, t) - \gamma(\alpha, t - s)) = vs, \tag{6.4}$$

$$\lim_{t\to\infty} \gamma(\alpha, t)/t = v. \tag{6.5}$$

Moreover, $v = 0$ *iff* $g(\alpha, s, x)$ *is constant in* s *for all* $\alpha \in (0,1)$ *and* $x \in \mathbb{R}$. *If* $v \ne 0$, *then*

$$\lim_{t\to\infty} (\gamma(1/2, t) - \gamma(\alpha, t)) = -\gamma(\alpha), \qquad \forall\, \alpha \in (0,1), \tag{6.6}$$

$$w_0(y) = g(1/2, y/v, 0), \qquad \forall\, y \in \mathbb{R}, \tag{6.7}$$

$$g(\alpha, s, 0) = g(1/2, s + \gamma(\alpha)/v, 0), \qquad \forall\, \alpha \in (0,1),\ s \in \mathbb{R}; \tag{6.8}$$

thus w_0 *is strictly increasing and* γ *is continuous.*

PROOF. Recall (Proposition 3.5) that $\forall\, \alpha \in (0,1)$ the function $g(\alpha, s, x)$ is strictly increasing, strictly increasing or constant in s, and that the same case obtains $\forall\, x \in \mathbb{R}$. Suppose that, for some $\alpha' \in (0,1)$, $g(\alpha', s, 0)$ is not constant in s; choose $\alpha'' \ne \alpha'$ such that $g(\alpha', s, 0) = \alpha''$ for some s [recall that $g(\alpha', 0, 0) = \alpha'$]. Since $\gamma(\alpha)$ has at most countably many discontinuities and $g(\alpha, s, x)$ is continuous and monotone in s, α'' may be chosen so that $\gamma(\alpha)$ is continuous at $\alpha = \alpha''$, and thus w_0 is strictly increasing at $\gamma(\alpha'')$.

It follows from (6.1) and (6.2) that if γ is continuous at α, then $(\gamma(1/2, t) - \gamma(\alpha, t)) \to -\gamma(\alpha)$ as $t \to \infty$. But by Lemma 6.1,

$$\lim_{t\to\infty} P^0\{R_{t-s} \le \gamma(1/2, t) + \gamma(\alpha')\} = g(\alpha', s, 0) = \alpha'',$$

$$\lim_{t\to\infty} P^0\{R_{t-s} \le \gamma(1/2, t - s) + \gamma(\alpha'')\} = g(\alpha'', 0, 0) = \alpha'',$$

so

$$\lim_{t\to\infty} (\gamma(1/2, t) - \gamma(1/2, t - s)) = \gamma(\alpha'') - \gamma(\alpha') \ne 0.$$

[*Note*: $\gamma(\alpha') \neq \gamma(\alpha'')$ because $\alpha' \neq \alpha''$ and w_0 is continuous.] Now (6.4) follows for each α such that $\gamma(\alpha)$ is continuous at α, since $(\gamma(1/2, t) - \gamma(\alpha, t)) \to -\gamma(\alpha)$; but $\gamma(\alpha)$ has at most countably many discontinuities and $\gamma(\alpha, t)$ is monotone in α, so (6.4) must hold $\forall\ \alpha \in (0,1)$. The relation (6.5) follows from (6.4) by a routine but tedious argument which we shall omit.

We have shown that if $g(\alpha, s, 0)$ is not constant in s for some α, then (6.4) holds with $v \neq 0$. It follows, by (6.1) and (6.2), that

$$\begin{aligned} w_0(y) &= \lim_{t\to\infty} P^0\{R_t \le \gamma(1/2,t) + y\} \\ &= \lim_{t\to\infty} P^0\{R_t \le \gamma(1/2, t + y/v) + o(1)\} \\ &= g(1/2, y/v, 0), \end{aligned}$$

provided w_0 is strictly increasing at y. But by the continuity and monotonicity in y of w_0 and in s of $g(\alpha, s, 0)$, it follows that (6.7) holds $\forall\ y \in \mathbb{R}$. Similarly, by (6.1), (6.2) and (6.4),

$$\begin{aligned} g(\alpha, s, 0) &= \lim_{t\to\infty} P^0\{R_{t-s} \le \gamma(1/2,t) + \gamma(\alpha)\} \\ &= \lim_{t\to\infty} P^0\{R_{t-s} \le \gamma(1/2, t + \gamma(\alpha)/v) + o(1)\} \\ &= g(1/2, s + \gamma(\alpha)/v, 0), \end{aligned}$$

proving (6.8). Now if (6.7) holds then by Proposition 3.5, $w_0(y)$ is strictly increasing, so, by (6.1) and (6.2), (6.6) must hold $\forall\ \alpha \in (0,1)$.

Finally, suppose that, for all $\alpha \in (0,1)$, $g(\alpha, s, 0)$ is constant in s. Choose α' so that $\gamma(\alpha)$ is continuous at $\alpha = \alpha'$; then, by (6.2),

$$\lim_{t\to\infty} P^0\{R_{t-s} \le \gamma(1/2,t) + \gamma(\alpha')\} = g(\alpha', s, 0) = g(\alpha', 0, 0) = \alpha',$$

$$\lim_{t\to\infty} P^0\{R_{t-s} \le \gamma(1/2,t-s) + \gamma(\alpha')\} = g(\alpha', 0, 0) = \alpha'.$$

Since w_0 is strictly increasing at $\gamma(\alpha')$, it follows that $(\gamma(1/2, t) - \gamma(1/2, t-s)) \to 0$ as $t \to \infty$. Routine arguments based on (6.1) now show that $(\gamma(\alpha, t) - \gamma(\alpha, t-s)) \to 0$ as $t \to \infty$, $\forall\ \alpha \in (0,1)$, and (6.5) follows from (6.4). □

Recall (Section 5) that F_4 is the event that each of the functions $Y_{\alpha,s}$, $\alpha \in (0,1)$, is either identically 1 or identically 0 in s and that F_i, $i = 1, 2, 3$, is the event that (5.i) holds $\forall\ \alpha$ but not all of the $Y_{\alpha,s}$ are identically 1 or identically 0.

PROPOSITION 6.3. *Assume* (6.1). *If* $v = 0$ *in* (6.4), *then* $P^x(F_3 \cup F_4) = 1$ $\forall\ x \in \mathbb{R}$, *and* $C_\alpha = 0\ \forall\ \alpha \in (0,1)$. *If* $v \neq 0$, *then either* (5.10) *or* (5.11) *holds, and therefore* (5.12) *also holds. If* $v > 0$, *then* $P^x(F_2) = 0\ \forall\ x$, *and if* $v < 0$, *then* $P^x(F_1) = 0\ \forall\ x$. *If* $P^x(F_1) = 1$ *or* $P^x(F_2) = 1$ *and* $v \neq 0$, *then* $-\infty < U_{1/2} < \infty$ *a.s.* (P^x) *and*

$$U_\alpha = U_{1/2} + \gamma(\alpha)/v \quad a.s.\ (P^x), \tag{6.9}$$

$\forall\ \alpha \in (0,1)$ *and* $x \in \mathbb{R}$. *If* $P^x(F_3) = 1$ *and* $v \neq 0$, *then* $C_\alpha = C$ *is independent of* α, $C > 0$ *if* $v > 0$, $C < 0$ *if* $v < 0$, $0 < Z_{1/2} < \infty$ *a.s.* (P^x), *and*

$$Z_\alpha = Z_{1/2} e^{-C\gamma(\alpha)/v} \quad a.s.\ (P^x), \tag{6.10}$$

$\forall\ \alpha \in (0,1)$ *and* $x \in \mathbb{R}$.

PROOF. Suppose first that $v = 0$. Then by Proposition 6.2, $g(\alpha, s, x)$ is constant in s for all α, x. But $g(\alpha, s, x) = E^x Y_{\alpha, s}$, by (4.3), and $\forall\ \alpha$ the function $Y_{\alpha, s}$ is either nondecreasing in s a.s. (P^x) or nonincreasing in s a.s. (P^x), by Theorem 5.1. Consequently, $Y_{\alpha, s}$ is constant in s a.s. (P^x), so $P^x(F_3 \cup F_4) = 1$.

Now suppose that $v \neq 0$. Since $w_0(y)$ is a proper c.d.f., (6.7) and (6.8) imply that $g(\alpha, s, 0) \to 0$ and 1 as $s \to \pm\infty$ or $\mp\infty$. Thus, either

$$G^+(\alpha, 0) = 1 \quad \text{and} \quad G^-(\alpha, 0) = 0, \qquad \forall\ \alpha \in (0,1),$$

or

$$G^+(\alpha, 0) = 0 \quad \text{and} \quad G^-(\alpha, 0) = 1, \qquad \forall\ \alpha \in (0,1).$$

Recall from the proof of Proposition 5.4 that $\forall\ \alpha$, $G^+(\alpha, x)$ and $G^-(\alpha, x)$ are martingale functions of x; therefore, by Lemma 2.6, either (5.10) or (5.11) holds. Proposition 5.4 implies that (5.12) also holds, and in addition implies that $-\infty < U_\alpha < \infty$ a.s. (P^x) if $P^x(F_i) = 1$ for $i = 1$ or 2 and that $C_\alpha = C$ is independent of α, that $C \neq 0$ and that $0 < Z_\alpha < \infty$ a.s. (P^x) if $P^x(F_3) = 1$. If $v > 0$ then (6.7) and (6.8) imply that $g(\alpha, s, 0)$ is increasing in s, so $P^x(F_2) = 0$ by (4.3) and (5.12); similarly, if $v < 0$, then $P^x(F_1) = 0$. If $v > 0$ and $P^x(F_3) = 1$, then, by (4.3), $C > 0$; if $v < 0$ and $P^x(F_3) = 1$, then $C < 0$.

If $v \neq 0$, then, by Proposition 6.2, $w_0(y)$ is strictly increasing and $\gamma(\alpha)$ is continuous. Recall from (4.2) that

$$Y_{\alpha, s} = \lim_{r \to \infty} \lim_{t \to \infty} P^x\{R_{t-s} \leq \gamma(\alpha, t) | \mathscr{F}_r\};$$

by (6.4) and (6.6),

$$\begin{aligned}\lim_{t \to \infty} P^x\{R_{t-s} \leq \gamma(\alpha, t) | \mathscr{F}_r\} &= \lim_{t \to \infty} P^x\{R_{t-s} \leq \gamma(1/2, t) + \gamma(\alpha) | \mathscr{F}_r\} \\ &= \lim_{t \to \infty} P^x\{R_{t-s} \leq \gamma(1/2, t + \gamma(\alpha)/v) | \mathscr{F}_r\}\end{aligned}$$

so $Y_{\alpha, s} = Y_{1/2, s + \gamma(\alpha)/v}$ a.s. (P^x). The relations (6.9) and (6.1) follow directly. □

PROPOSITION 6.4. *Assume* (6.1); *define* $w_x(y) = g(w_0(y), 0, x)$. *Then*, $\forall\ x \in \mathbb{R}$, $w_x(y)$ *is a proper*, *continuous c.d.f. and*

$$\lim_{t \to \infty} P^x\{R_t \leq \gamma(1/2, t) + y\} = w_x(y). \tag{6.11}$$

PROOF. Lemma 6.1 with $s = 0$ implies (6.11) for $y = \gamma(\alpha)$, $\alpha \in (0,1)$. Since g and w_0 are continuous, it follows that (6.11) holds $\forall\ y \in \mathbb{R}$ and that $w_x(y)$ is jointly continuous in $x, y \in \mathbb{R}$. It remains to prove that $w_x(y)$ is a proper c.d.f, that is, that $w_x(y) \to 0$ as $y \to -\infty$ and $w_x(y) \to 1$ as $y \to \infty$.

Suppose first that $g(\alpha, s, x)$ is constant in s for all α, x (cf. Proposition 3.5). Then by Proposition 4.1, for each $\alpha \in (0,1)$, $g(\alpha, 0, x)$ is a martingale function (of x). Define

$$H^+(x) = \lim_{y\to\infty} w_x(y) = \lim_{\alpha\to 1} g(\alpha, 0, x)$$

and

$$H^-(x) = \lim_{y\to -\infty} w_x(y) = \lim_{\alpha\to 0} g(\alpha, 0, x).$$

Then H^+ and H^- are martingale functions. Since $g(\alpha, 0, 0) \equiv \alpha$, $H^+(0) = 1$ and $H^-(0) = 0$, so, by Lemma 2.6, $H^+(x) = 1$ and $H^-(x) = 0$, $\forall\, x \in \mathbb{R}$. Thus w_x is a proper c.d.f. $\forall\, x \in \mathbb{R}$.

Suppose now that for some α, $g(\alpha, s, x)$ is not constant in s. Consider the functions $G^{\pm}(1/2, x)$ defined by (5.8) and (5.9); by Proposition 6.2, these are identically 1 or 0, so, by (5.8) and (5.9),

$$\lim_{s\to\infty} g(1/2, s, x) = 1 \text{ or } 0, \qquad \forall\, x \in \mathbb{R},$$

and

$$\lim_{s\to -\infty} g(1/2, s, x) = 0 \text{ or } 1, \qquad \forall\, x \in \mathbb{R}.$$

Now (6.11) and Corollary 3.4 imply that

$$\begin{aligned} w_x(y) &= g(w_0(y), 0, x) \\ &= g(g(1/2, y/v, 0), 0, x) \\ &= g(1/2, y/v, x), \end{aligned} \tag{6.12}$$

so it follows that $w_x(y) \to 1$ as $y \to \infty$ and $w_x(y) \to 0$ as $y \to -\infty$. □

When (6.1) holds it is often fairly easy to ascertain which of the cases (5.1)–(5.3) holds in Theorem 5.1. We shall state a simple sufficient condition for (5.3). Assume for simplicity that the underlying diffusion $X_1(t)$ satisfies a stochastic differential equation

$$dX_1(t) = \alpha(X_1(t))\, dt + \sigma(X_1(t))\, dW(t),$$

where $dW(t)$ is white noise and the local drift and diffusion coefficients $\alpha(x)$ and $\sigma(x)$ are continuous.

PROPOSITION 6.5. *Assume* (6.1). *If* $v > 0$, $|\alpha(x)|$ *is bounded as* $x \to \infty$, *and* $1/\sigma(x)$ *is bounded as* $x \to \infty$, *then* $P^x(F_3) = 1$. *If* $v < 0$, $|\alpha(x)|$ *is bounded as* $x \to -\infty$, *and* $1/\sigma(x)$ *is bounded as* $x \to -\infty$, *then* $P^x(F_3) = 1$. *In either case* $0 < Z_\alpha < 0$ *a.s.* (P^x) *and*

$$w_x(y) = E^x \exp\{-Z_{1/2} e^{-Cy/v}\}. \tag{6.13}$$

PROOF. If $v \neq 0$, then, by Proposition 6.2, $P^x(F_i) = 1$, $\forall\, x \in \mathbb{R}$, for some $i = 1, 2$ or 3. Suppose that $P^x(F_i) = 1$ for $i = 1$ or 2; then by Proposition 5.2

the observed quantile Q_t converges in probability to a random variable Q. We will show that if $v \neq 0$ and the functions $\alpha(x)$, $\sigma(x)$ satisfy the hypothesis stated, then this cannot occur. This will prove that $P^x(F_3) = 1$.

Suppose that $v < 0$ and that $|\alpha(x)|$, $1/\sigma(x)$ are bounded as $x \to -\infty$ (the other case is similar). Then $R_t \to -\infty$ in P^0-probability, by (6.5). Take t_* so large that $P^0\{|Q_t - Q| > \varepsilon\} < \varepsilon$, for all $t \geq t_*$, where $\varepsilon > 0$ is small. Then at time t_*, R_{t_*} is near $\gamma(Q, t_*)$ with P^0 probability greater than or equal to $1 - \varepsilon$. But if $|\alpha(x)|$ and $1/\sigma(x)$ are bounded as $x \to -\infty$, then there is a good chance $\delta \gg \varepsilon$ that the lead particle will wander out to beyond $\gamma(Q + \delta, t_* + 1)$ at time $t_* + 1$, in view of (6.1). But this contradicts the statement $P^0\{|Q_t - Q| > \varepsilon\} < \varepsilon$, $\forall\, t \geq t_*$.

Thus, if $v \neq 0$, then $P^0(F_3) = 1$. But, by Proposition 6.3, if $v \neq 0$, then (5.12) holds, so it follows that $P^x(F_3) = 1$, $\forall\, x \in \mathbb{R}$. Since either (5.10) or (5.11) holds, $0 < Z_\alpha < \infty$ a.s. (P^x), $C_\alpha = C$ is independent of α, and $C \neq 0$ (see Proposition 6.3). Finally,

$$\begin{aligned} w_x(y) &= g(1/2, -y/v, x) \\ &= E^x \exp\left\{-Z_{1/2} e^{-Cy/v}\right\} \end{aligned}$$

by Proposition 5.3 and (6.12), proving (6.13). □

7. Examples. In this section we discuss briefly some particular examples which illustrate various facets of the theory developed in the previous sections. In each example we will specify the branching diffusion process by identifying (i) the branching rate function $\beta(x)$, and (ii) the diffusion law, which in most cases is determined by the local drift coefficient $\alpha(x)$ and the local diffusion coefficient $\sigma(x)$. We always use $p_1(x) \equiv 1$.

EXAMPLE 7.1 [$\alpha(x) \equiv \mu$, $\sigma(x) \equiv 1$, $\beta(x) \equiv 1$]. For $\mu = 0$ this process is branching Brownian motion. It is known [5] that when $\mu = 0$, (6.1) obtains and that $\gamma(1/2, t) \sim \sqrt{2}\, t$ as $t \to \infty$. For arbitrary μ, (6.1) must also obtain, with $\gamma(1/2, t) \sim (\sqrt{2} + \mu)t$; thus the velocity v of the travelling wave is positive, negative, or zero depending on whether μ is greater than, less than or equal to $-\sqrt{2}$. By Proposition 6.2, the function $g(\alpha, s, x)$ is constant in s iff $\mu = -\sqrt{2}$; by (6.7), $g(\alpha, s, x)$ is strictly increasing in s if $\mu < -\sqrt{2}$ and strictly decreasing if $\mu > -\sqrt{2}$. Proposition 6.5 implies that $P^x(F_3 \cup F_4) = 1$, that is, case (5.3) of Theorem 5.1 obtains a.s., and (6.10) implies that $C < 0$ if $\mu > -\sqrt{2}$ and $C > 0$ if $\mu < -\sqrt{2}$. Also, if $\mu \neq -\sqrt{2}$, then $0 < Z_\alpha < \infty$ a.s.

Now consider the case $\mu = -\sqrt{2}$, for which $v = 0$. The law of $(X_j(t))$ under P^x is the same as that of $(X_j(t) + x)$ under P^0, since $\alpha(x)$, $\sigma(x)$ and $\beta(x)$ are all constant in x. Consequently $P^x\{R_t \leq y\} = P^0\{R_t \leq y - x\}$, and (6.1), (6.2) and (6.4) imply that $g(\alpha, s, x) = g(\alpha, 0, x) = w_0(\gamma(\alpha) - x)$, so

$$Y_{\alpha, s} = \lim_{t \to \infty} \prod_{j=1}^{N(t)} w_0\big(\gamma(\alpha) - X_j(t)\big).$$

By the arguments of [2], Section 2, we have $0 < Y_{\alpha,s} < 1$ a.s. (P^x). Therefore, in the representation (5.1), $C = 0$ and $0 < Z_\alpha < \infty$ a.s. (P^x).

EXAMPLE 7.2 [$\alpha(x) = -x$, $\sigma(x) \equiv 1$, $\beta(x) \equiv 1$]. In this example the underlying diffusion $X_1(t)$ is the Ornstein–Uhlenbeck process. This diffusion is positive recurrent, with stationary probability density $(2\pi)^{-1/2}e^{-x^2/2}$; hence, by Proposition 5.4, the function $g(\alpha, s, x)$ is not constant in s. If the initial point of the first particle is chosen at random from the stationary distribution, then at any time t the expected number of particles in dx is $e^t(\pi)^{-1/2}e^{-x^2}\,dx$; consequently,

$$\lim_{t\to\infty} P\{R_t > \sqrt{t}\log t\} = 0.$$

Thus, (6.5) does not hold with $v \neq 0$. Since $g(\alpha, s, x)$ is not constant in s, Proposition 6.2 implies that (6.1) does not hold. In fact it is not difficult to show that, $\forall\, x \in \mathbb{R}$, $\forall\, \varepsilon > 0$,

$$P^x\{|R_t - \sqrt{t}| > \varepsilon\} \to 0;$$

thus, (6.1) holds but with a *discontinuous* limiting c.d.f. $w_0(\cdot)$. An argument like that of [3] seems to show that $\sqrt{t}(R_t - \sqrt{t})$ converges in distribution to a nontrivial limit.

EXAMPLE 7.3 [$\alpha(x) = -x/|x|$, $\sigma(x) \equiv 1$, $\beta(x) \equiv 1$]. In this example the underlying diffusion $X_1(t)$ is positive recurrent, with stationary probability density $(1/2)e^{-|x|}$. The distribution of R_t behaves differently in this example then in Example 7.2 because of the different tail behavior of the stationary distribution. Here (6.1) holds and the velocity of the wave is $v = 1$. This may be proved by an adaptation of the methods of [3]. By Proposition 6.5, $P^x(F_3) = 1\ \forall\, x$. Note that the wave velocity is different than that in Example 7.1 with $\alpha(x) \equiv -1$, even though the two processes behave the same way on $(0, \infty)$.

EXAMPLE 7.4 [$\alpha(x) = 2x/|x|$, $\sigma(x) \equiv 1$, $\beta(x) \equiv 1$]. Here the underlying diffusion $X_1(t)$ is transient, with either $X_1(t) \to \infty$ or $X_1(t) \to -\infty$ a.s. (P^x). Under any P^x, there is positive probability that eventually there are no particles to the right of the origin, because to the left of the origin particles behave as in Example 7.1 with $\mu = -2$. But there is also positive P^x-probability that $R_t/t \to 2 + \sqrt{2}$, because to the right of the origin particles behave as in Example 7.1 with $\mu = +2$. Using the results concerning the distribution of R_t in Example 7.1 with $\mu = \pm 2$, it is not difficult to show that here

$$P^x\{R_t \le \gamma(\alpha_+, t) + y\} \to w_x^+(y) \quad \text{and} \quad P^x\{R_t \le \gamma(\alpha_-, t) + y\} \to w_x^-(y),$$

where

$$\lim_{y\to\infty} w_x^+(y) = 1, \qquad \lim_{y\to-\infty} w_x^+(y) = \alpha_x,$$

$$\lim_{y\to\infty} w_x^-(y) = \alpha_x, \qquad \lim_{y\to-\infty} w_x^-(y) = 0,$$

$$\alpha_- < \alpha_0 < \alpha_+ \quad \text{and} \quad \alpha_x = P^x\{\text{no particles to the right of 0 as } t \to \infty\}.$$

Furthermore, $\gamma(\alpha_+, t)/t \to 2 + \sqrt{2}$ and $\gamma(\alpha_-, t)/t \to -2 + \sqrt{2}$ as $t \to \infty$. Thus in this example the distribution of R_t splits into two distinct travelling waves: one travelling at velocity $2 + \sqrt{2}$, the other at $-2 + \sqrt{2}$. Each of the two waves is a *defective* c.d.f., and each is continuous.

Now consider the function $g(\alpha, s, x)$; it satisfies

$$g(w_0^+(y), s, 0) = w_0^+(y + s(2 + \sqrt{2})), \qquad \forall\, y, s \in \mathbb{R},$$

$$g(w_0^-(y), s, 0) = w_0^-(y + s(-2 + \sqrt{2})), \qquad \forall\, y, s \in \mathbb{R}.$$

Consequently, $g(\alpha, s, x)$ is strictly increasing in s for $\alpha > \alpha_0$, strictly decreasing in s for $\alpha < \alpha_0$ and (by continuity) constant in s for $\alpha = \alpha_0$. (See Proposition 3.5—this shows that the same case need not obtain for all α.)

Since $|\alpha(x)|$ and $1/\sigma(x)$ are bounded as $x \to \pm\infty$, an argument similar to that in the proof of Proposition 6.5 shows that the observed quantile Q_t cannot stabilize as $t \to \infty$; thus, by Proposition 5.2, $P^x(F_3) = 1\ \forall\, x \in \mathbb{R}$. Let $A = \{$no particles to the right of 0 eventually$\}$. Then, by (4.2) and (5.3), for $\alpha > \alpha_0$,

$$1\{Z_\alpha = 0\} = 1_A \quad \text{a.s. } (P^x),$$

$$1\{Z_\alpha = \infty\} = 0 \quad \text{a.s. } (P^x),$$

but for $\alpha < \alpha_0$

$$1\{Z_\alpha = 0\} = 0 \qquad \text{a.s. } (P^x),$$

$$1\{Z_\alpha = \infty\} = 1 - 1_A \quad \text{a.s. } (P^x);$$

moreover, $C_\alpha = C_- < 0$, for $\alpha < \alpha_0$, $C_{\alpha_0} = 0$ and $C_\alpha = C_+ > 0$, for $\alpha > \alpha_0$. This shows that the constants C_α in (5.3) need not always be equal or even of the same sign.

EXAMPLE 7.5 [$\alpha(x) \equiv 1$, $\sigma(x) = \min(1, e^{1-x})$, $\beta(x) \equiv 0$]. In this example there are no fissions, only a single particle executing a diffusion R_t with coefficients $\alpha(x)$, $\sigma(x)$. Furthermore, $R_t - t$ is a martingale whose quadratic variation $V = \int_0^\infty \sigma^2(R_t)\,dt$ is finite a.s. (P^x). Consequently $R_t - t \to U$ a.s. (P^x) for a suitable random variable U. Under P^0, the random variable U has a nonatomic distribution, because under P^0, $U = -T + U'$, where T, U' are independent and $T = \inf\{t\colon R_t = 1\}$, which is known to have a density. Thus (6.1) holds, with $w_0(y)$ being a translate of the c.d.f. of U under P^0. The velocity v of the wave is 1.

Clearly, as $t \to \infty$ the position R_t of the particle becomes predictable, and thus $P^x(F_1) = 1$. This shows that (5.1) is possible.

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