The Annals of Probability
1992, Vol. 20, No. 4, 1779–1804

NECESSARY AND SUFFICIENT CONDITIONS FOR ASYMPTOTIC NORMALITY OF L-STATISTICS

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It is now classical that the sample mean $\bar{Y}$ is known to be asymptotically normal with $\sqrt{n}$ norming if and only if $0 < \mathrm{Var}[Y] < \infty$ and with arbitrary norming if and only if the df of Y is in the domain of attraction of the normal df. Now let $T_n = n^{-1}\sum c_{ni} h(X_{n:i})$ for order statistics $X_{n:i}$ from a df F denote a general L-statistic subject to a bit of regularity; the key condition introduced into this problem in this paper is the regular variation of the score function J defining the c_{ni}'s. We now define a rv Y by $Y = K(\xi)$, where ξ is uniform $(0, 1)$ and where $dK = J\,dh(F^{-1})$. Then T_n is shown to be asymptotically normal with $\sqrt{n}$ norming if and only if $0 < \mathrm{Var}[Y] < \infty$ and with arbitrary norming if and only if the df of Y is in the domain of attraction of the normal df. As it completely parallels the classical theorem, this theorem gives the *right* conclusion for L-statistics. In order to establish the necessity above, we also obtain a nice necessary and sufficient condition for the stochastic compactness of T_n and give a representation formula for all possible subsequential limit laws.

1. Asymptotic normality. Let $X_{n:1} \le \cdots \le X_{n:n}$ denote the order statistics of an iid sample from the df F. Consider the L-statistics

$$T_n \equiv \sum_{i=k+1}^{n-m} c_{ni} h(X_{n:i}),$$

where $k, m \ge 1$ are fixed integers, c_{ni} are known constants and h is a known function, all of which are specified by the statistician. If ξ has a uniform $(0, 1)$ distribution, then $F^{-1}(\xi)$ has df F. Thus we may assume that

$$T_n \equiv \frac{1}{n} \sum_{i=k+1}^{n-m} c_{ni} g(\xi_{n:i}), \tag{1.1}$$

where $g \equiv h(F^{-1})$ and $0 \le \xi_{n:1} \le \cdots \le \xi_{n:n} \le 1$ are the order statistics of a sample from the uniform $(0, 1)$ distribution. [Division by $n - m - k$ instead of n in (1.1) requires only a trivial adjustment.] Our point of view is to suppose that the statistician chooses rather smooth scores c_{ni} and then wants to know for what functions g (or df's F) asymptotic normality will hold. Asymptotic normality is characterized in the present section. Analogous characterizations of stochastic compactness with a representation of all possible subsequential limits appear in Section 2.

Received February 1989; revised December 1990.
[1]Research supported in part by NSF Grant DMS-88-03209.
[2]Research supported in part by NSF Grant DMS-88-01083.
AMS 1980 *subject classifications*. Primary 60F05; secondary 62G30, 62F05, 62E20, 62E10.
Key words and phrases. L-statistics, regularly varying, stochastic compactness.

The first general theorem on the asymptotic normality of L-statistics was proven by Chernoff, Gastwirth and Johns (1967). They were concerned with this problem in connection with optimal estimators of location and scale parameters in parametric families of distributions based on L-statistics. A large number of authors since then have studied the asymptotic normality of L-statistics. A partial list consists of Bickel (1967), Shorack (1969, 1972), Stigler (1969, 1974), Ruymgaart and van Zuijlen (1977), Sen (1978), Boos (1979), Mason (1981), Singh (1981) and Helmers and Ruymgaart (1988). Consequently, many sets of sufficient conditions now exist in the literature which ensure asymptotic normality of L-statistics. If and only if conditions for asymptotic normality of L-statistics have been obtained only for the highly specialized cases of uniform rv's by Hecker (1976) and van Zwet (unpublished 1974 Oberwolfach lecture) and exponential random variables by Eicker and Puri (1976).

It is our aim to consider the purely probabilistic problem of finding necessary and sufficient conditions for the asymptotic normality of statistics of the form (1.1). Since the weights c_{ni}, $i = 1, \dots, n$, appearing in (1.1) can be arbitrarily chosen at each stage n, to determine the complete solution to this problem is without a doubt hopeless. Thus to make the problem tractable, it is appropriate to place some mild restrictions on how the weights are formed. To begin, we will suppose:

CONDITION G.

(1.2) $\quad g$ is a nondecreasing ($\nearrow$) left-continuous function on $(0, 1)$.

Assume that:

CONDITION C.

$$c_{ni}/n = \int_{(i-1)/n}^{i/n} J(t)\, dt \quad \text{for } 1 \le i \le n \tag{1.3}$$

for some

$$J \ge 0 \quad \text{with } J \text{ continuous on } (0, 1). \tag{1.4}$$

The reader should note from our proofs that it is a trivial matter to replace g by $g_1 - g_2$ and J by $J_1 - J_2$ in the *sufficiency* statements below with g_i and J_i that satisfy our assumptions.

The natural centering constant to associate with T_n is

$$\mu_n \equiv \int_{k/n}^{1-m/n} J(t) g(t)\, dt. \tag{1.5}$$

We then define the statistics

$$S_n \equiv \sqrt{n}\,(T_n - \mu_n)/\sigma_n \quad \text{and} \quad S_n^* \equiv \sqrt{n}\,(T_n - \mu_n)/\sigma_n^* \tag{1.6}$$

for the σ_n and σ_n^* of (1.14) below.

We use the convention that $\int_a^b = \int_{[a,b)}$ when integrated with respect to a left-continuous integrator like dg, while $\int_a^b = \int_{(a,b]}$ when integrated with respect to a right-continuous integrator like dF. Also, in either case, $\int_b^a \equiv -\int_a^b$ if $a < b$. We use f_+ and f_- for the right- and left-continuous versions of a function f and let $\Delta f = f_+ - f_-$.

We now define the key function K on (0, 1), with $0 < c < 1$ being a fixed continuity point of g, by

$$K(t) \equiv \int_c^t J\,dg \quad \text{so that } dK = J\,dg \text{ with } K(c) = 0 \text{ and } K \nearrow \text{ and left-continuous.} \tag{1.7}$$

In the sequel, for notational convenience we will *always* assume $c = \frac{1}{2}$. We also assume that K is not the trivial zero function and thus K is the left-continuous inverse of some nondegenerate df H. Hence if ξ denotes a generic uniform (0, 1) random variable, then

$$K(\xi) \quad \text{has the nondegenerate df } H. \tag{1.8}$$

This is the fundamental random variable of this paper.

For each $0 < a < b < 1$, we agree that

$$K_{ab}(t) \quad \text{equals } K(a), K(t), K(b) \text{ corresponding to } t < a, a \le t < b, b \le t, \tag{1.9}$$

and we then define

$$Y_{[a,b)}(\xi) \equiv K_{ab}(\xi) - EK_{ab}(\xi) = -\int_a^b (1_{[\xi \le t]} - t)\,dK(t). \tag{1.10}$$

[It is easy to show that $Y_{[a,b)}(\xi) \to_{\text{a.s}} Y - EY$ as $a \to 0$ and $b \to 1$, provided $E|Y| < \infty$.] Note that the statement

$$Y_{[a,b)}(\xi) \quad \text{has mean 0 and variance } \sigma^2_{[a,b)} \tag{1.11}$$

always holds, where

$$\sigma^2[a,b) \equiv \int_0^1 K_{ab}^2(t)\,dt - \left(\int_0^1 K_{ab}(t)\,dt\right)^2 \tag{1.12}$$

$$= \int_a^b \int_a^b (s \wedge t - st)\,dK(s)\,dK(t). \tag{1.13}$$

Since $s \wedge t - st \ge 0$, we see that $\sigma^2[a,b) \nearrow$ as $a \searrow 0$ and $b \nearrow 1$. We now define

$$\sigma_n \equiv \sigma[k/n, 1 - m/n), \qquad \sigma(a) \equiv \sigma[a, 1-a) \quad \text{and} \quad \sigma_n^* \equiv \sigma(1/n). \tag{1.14}$$

Let $\mathbb{B}$ denote a Brownian bridge and note from (1.13) [contrast this with (1.10)] that

$$\int_a^b \mathbb{B}\, dK \cong N(0, \sigma^2[a, b)). \tag{1.15}$$

We also let $\mathbb{B}_n$, $n \geq 1$, signify a sequence of Brownian bridges.

We seek a very broad class of useful J functions that includes all the standard continuous J functions. Roughly speaking, if J is regularly varying at 0 and 1 and fairly smooth, then our theorems apply. By allowing trimming in (1.1), we are able to require J to be smooth and still include all the usual examples. Specifically, we assume:

CONDITION J.

(1.16a) $J \geq 0$ is Lipschitz on any $[\delta, 1-\delta]$ with $0 < \delta \leq \frac{1}{2}$;

(1.16b) there exist $-\infty < \rho_0, \rho_1 < \infty$ such that $J(t) = t^{\rho_0} l_0(t)$ and $J(1-t) = t^{\rho_1} l_1(t)$ on some $(0, \delta]$ for l_i's slowly varying at 0;

and

(1.16c) $l_i'(t) = t^{-1} l_i(t) \varepsilon_i(t)$ with ε_i continuous and $\varepsilon_i(t) \to 0$ as $t \to 0, i = 0, 1$.

From the Karamata representation theorem [cf. de Haan (1970)], we expect any common regularly varying J to satisfy (1.16c). Note that (1.16b, c) *is* satisfied by any $t^a(1-t)^b$, any $(\log_i 1/t)^a(\log_j 1/(1-t))^b$, any product of such functions and by the inverse normal df Φ^{-1}.

CONDITION K.

$$k, m \geq 1 \text{ are fixed.} \tag{1.17}$$

(In Remark 3.2 it is explained how the case when $k = 0$ and/or $m = 0$ can be treated.)

We now define [recall (1.8)]

$$\overline{T}_n \equiv (1/n) \sum_{i=k+1}^{n-m} K(\xi_{n:i}) \quad \text{and} \quad \overline{\mu}_n \equiv \int_{k/n}^{1-m/n} K(t)\, dt, \tag{1.18}$$

$$\overline{S}_n \equiv \sqrt{n}\left(\overline{T}_n - \overline{\mu}_n\right)/\sigma_n \tag{1.19}$$

and

$$\overline{S}_n^* \equiv \sqrt{n}\left(\overline{T}_n - \overline{\mu}_n\right)/\sigma_n^*. \tag{1.20}$$

THEOREM 1.1. *Suppose the basic regularity Conditions* G, J, C *and* K *hold.*

(i) (*Standard scaling, or domain of normal attraction of the normal*). *We have*

$$0 < \sigma^2(0) < \infty \text{ if and only if } \sqrt{n}\,(T_n - B_n) \to_d N(0, \tau^2) \text{ for some } 0 < \tau < \infty \text{ and some } B_n. \tag{1.21}$$

In such cases, $\tau = \sigma(0)$ *and, on a suitable space,*

$$\begin{aligned}
&\sqrt{n}\,(T_n - \mu_n)/\sigma(0) \\
&\quad = \sqrt{n}\left(\overline{T}_n - \overline{\mu}_n\right)/\sigma(0) + o_p(1) \\
(1.22)\qquad &\quad = -\int_0^1 \mathbb{B}_n \, dK/\sigma(0) + o_p(1) \\
&\quad = (1/\sqrt{n}) \sum_{i=1}^{n} \left[K(\xi_i) - \int_0^1 K(t)\, dt \right] / \sigma(0) + o_p(1) \\
&\quad \to_d N(0,1),
\end{aligned}$$

where the summation in (1.22) *extends from* 1 *to* n *no matter what* $k, m \geq 1$.

(ii) (*Domain of attraction of the normal*). *We have*

$$(1.23)\qquad \sqrt{n}\,(T_n - B_n)/A_n \to_d N(0,1)$$

for some $A_n > 0$ *and* B_n *if and only if the df* H (*recall* $K = H^{-1}$) *satisfies*:

(1.24) *H is in the domain of attraction of the normal distribution.*

Condition (1.24) *also implies* $\sigma_n/\sigma^* \to 1$, $\sigma_n/A_n \to 1$ *and*

$$(1.25)\qquad S_n = \sqrt{n}\,(T_n - \mu_n)/\sigma_n = \sqrt{n}\left(\overline{T}_n - \overline{\mu}_n\right)/\sigma_n + o_p(1) \to_d N(0,1).$$

From part (ii) of Theorem 1.1 it can be easily inferred that (1.24) holding is equivalent to (1.23) holding for some $k \geq 1$, $m \geq 1$ and is also equivalent to (1.23) being true for all $k \geq 1$, $m \geq 1$. A similar statement also holds for part (i).

There are many conditions known to be equivalent to (1.24), in particular:

$$(1.26)\qquad t\left[K^2(\lambda t) + K_+^2(1-\lambda t)\right]/\sigma^2(t) \to 0 \quad \text{as } t \to 0 \text{ for all } 0 < \lambda \leq 1.$$

$$(1.27)\qquad \sigma(\lambda_t)/\sigma(t) \to 1 \quad \text{as } t \to 0 \text{ for all } 0 < \lambda \leq 1.$$

$$(1.28)\qquad \limsup_n |\Phi_{in}(c)| = 0 \quad \text{for all } c > 0 \text{ and } i = 0, 1, \text{ with } \Phi_{in} \text{ as defined in (2.1).}$$

$$(1.29)\qquad t\left[K^2(t) + K_+^2(1-t)\right] \Big/ \int_t^{1-t} K^2(s)\, ds \to 0 \quad \text{as } t \to 0.$$

$$(1.30)\qquad x^2 \int_{|y|>x} dH(y) \Big/ \int_{|y|<x} y^2\, dH(y) \to 0 \quad \text{as } x \to 0.$$

The equivalence of (1.24), (1.26), (1.27) and (1.29) was first stated and proved in S. Csörgő, Haeusler and Mason (1988a) (from now on denoted CsHM). That (1.24) and (1.30) are equivalent is shown in Gnedenko and Kolmogorov (1954). Equivalence of (1.26) and (1.28) is trivial.

Mason and Shorack (1990a) establish versions of Theorem 1.1 and Theorem 2.1 below in case k and $n - m$ are replaced by sequences k_n and

m_n, where either $k_n \wedge (n - m_n) \to \infty$ and $(k_n \vee (n - m_n))/n \to 0$ or else $\sqrt{n}(k_n/n - a) \to 0$ and $\sqrt{n}(m_n/n - b) \to 0$ with $0 < a < b < 1$.

Theorem 1.1 will be derived from the much more general results presented in the next section.

2. General limit laws. We now obtain probabilistic representations of all possible limit laws of S_n and S_n^*.

Fix any $a_n \to 0$ with $0 < a_n < n$, and let

$$\Phi_{0n}(c) = \begin{cases} K(c/n)/(\sqrt{n}\,\sigma_n^*), & \text{if } 0 < c \le n - a_n, \\ K(1 - a_n/n)/(\sqrt{n}\,\sigma_n^*), & \text{if } n - a_n < c; \end{cases} \tag{2.1}$$

the second case pedantically ensures Φ_{0n} is well defined. Define Φ_{1n} analogously to equal $-K_+(1 - c/n)/(\sqrt{n}\,\sigma_n^*)$ or $-K_+(a_n/n)/(\sqrt{n}\,\sigma_n^*)$ according as $0 < c \le n - a_n$ or $n - a_n < c$. Note that the Φ_{in} are $\nearrow$ left-continuous functions on $(0, \infty)$ that equal 0 at $c = n/2$. From (3.15), we have immediately that $\limsup_n |\Phi_{in}(c)| \le 1/\sqrt{c}$ (which $\to 0$ as $c \to \infty$) for $c \ge 1$ and $i = 1, 2$. The cutoff at $c = 1$ comes from our definition of σ_n^*. For $i = 0, 1$, we will use Φ_i to denote a nonpositive, $\nearrow$ left-continuous function on $(0, \infty)$. We will write $\Phi_{in} \to_D \Phi_i$ if $\Phi_{in}(x) \to \Phi_i(x)$ at all continuity points x of Φ_i.

THEOREM 2.1. *Suppose Conditions* G, J, C *and* K *hold. Then*

$$\limsup_n |\Phi_{in}(c)| < \infty \quad \text{for all } c > 0 \text{ and } i = 0, 1 \tag{2.2}$$

is necessary and sufficient for:

$$S_n^* \quad \text{is stochastically compact.} \tag{2.3}$$

Moreover, any possible subsequential limit random variable is of the form

$$V_{0k} + \tau Z + V_{1m}, \tag{2.4}$$

where for all $x, y > 0$ and $-\infty < \rho < \infty$,

$$\begin{aligned} h_\rho(x, y) &\equiv \int_x^y u^\rho \, du / y^\rho \\ &= \begin{cases} (y^{\rho+1} - x^{\rho+1})/((\rho + 1) y^\rho), & \text{if } \rho \ne -1, \\ y \log(y/x), & \text{if } \rho = -1, \end{cases} \end{aligned} \tag{2.5}$$

and

$$V_{0k} \equiv \int_{S_k^0}^\infty h_{\rho_0}(\mathbb{N}^0(x), x)\, d\Phi_0(x) + \int_k^{S_k^0} h_{\rho_0}(k, x)\, d\Phi_0(x), \tag{2.6}$$

$$V_{1m} \equiv -\int_{S_m^1}^\infty h_{\rho_1}(\mathbb{N}^1(x), x)\, d\Phi_1(x) - \int_m^{S_m^1} h_{\rho_1}(m, x)\, d\Phi_1(x), \tag{2.7}$$

$$Z =_d N(0, 1) \quad \text{and} \quad 0 \le \tau \le 1, \tag{2.8}$$

and $\mathbb{N}^0$ *and* $\mathbb{N}^1$ *are independent Poisson processes that are independent of* Z *with* S_k^i *denoting the event times of* $\mathbb{N}^i$ *for* $i = 0, 1$.

Note Remarks 3.1–3.3 at the end of Section 3 below.

The asymptotic normality result of Section 1 is useful for statistics. Proving sufficiency is relatively straightforward. In order to show that Theorem 1.1 is the best possible, we need Theorem 2.1. In fact, the proof of necessity in Theorem 2.1 requires results slightly more general; see Propositions 3.1 and 3.2 below. We also require the following technical result.

PROPOSITION 2.1. *Suppose*

$$\int_\varepsilon^\infty \Phi_i^2(x)\,dx < \infty \quad \text{for all } \varepsilon > 0 \text{ for } i = 0, 1, \tag{2.9}$$

holds for a nonconstant, $\nearrow$ *, less than or equal to* 0 *and left-continuous* Φ. *As in* (2.6), *let*

$$V_k \equiv \int_{S_k}^\infty h_\rho(\mathbb{N}(x), x)\,d\Phi(x) + \int_k^{S_k} h_\rho(k, x)\,d\Phi(x) \tag{2.10}$$

for $-\infty < \rho < \infty$. *Then* V_k *is never a nondegenerate normal random variable for any choice of* $-\infty < \rho < \infty$ *and integer* $k \ge 1$.

The proof of Proposition 2.1 is very technical and long and is not included. For the proof, see Mason and Shorack (1990b).

REMARK 2.1. Proposition 2.1 in combination with Cramér's theorem implies that the random variable in (2.4) is nondegenerate normal if and only if $\Phi_0 = \Phi_1 = 0$.

REMARK 2.2. In Theorem 3 of CsHM (1988a) it is shown that any infinitely divisible random variable W can be uniquely represented as

$$W =_d V_{01} + \Phi_0(S_1^0) + \tau Z + V_{11} - \Phi_1(S_1^1) + d, \tag{2.11}$$

where Z, V_{01}, V_{11}, Φ_0 and Φ_1 are as in (2.6), (2.7) and (2.8) with $\rho_0 = \rho_1 = 0$, $0 \le \tau < \infty$ and $-\infty < d < \infty$. The class of laws belonging to random variables of the structure given in (2.11), where ρ_0 and ρ_1 can be arbitrary real numbers, obviously forms a class containing the infinitely divisible laws. One question that naturally arises concerning these laws is that of the uniqueness of their representation in terms of the quantities Φ_0, Φ_1, ρ_0, ρ_1, τ and d. Since (except in the known case when $\rho_0 = \rho_1 = 0$) it appears to be extremely difficult to write the characteristic function of these laws, to determine the answer to this open question will undoubtedly be a formidable task.

3. Proofs. We begin by introducing some preparatory material. The actual proofs begin following (3.18). For independent uniform (0, 1) random variables $\xi_1, \dots, \xi_n$, the *empirical* df $\mathbb{G}_n$ and the *uniform empirical process* $\mathbb{U}_n$

are defined by

$$\mathbb{G}_n(t) = n^{-1} \sum_{i=1}^{n} 1_{[\xi_i \le t]} \quad \text{and} \quad \mathbb{U}_n(t) = \sqrt{n}\,[\mathbb{G}_n(t) - t] \quad \text{for } 0 \le t \le 1.$$

M. Csörgő, S. Csörgő, Horváth and Mason (1986) (from now on denoted CsCsHM) showed that on some probability space a sequence of pairs $(\mathbb{U}_n, \mathbb{B}_n)$, $n \ge 1$, can be constructed in such a way that

$$\left\|(\mathbb{U}_n - \mathbb{B}_n)/(I(1-I))^{(1/2)-\nu}\right\|_{1/n}^{1-1/n} = O_p(n^{-\nu}) \quad \text{as } n \to \infty \tag{3.1}$$

for any fixed $0 \le \nu < 1/4$. Here $\|h\|_a^b \equiv \sup\{|h(t)|\colon\ a \le t \le b\}$. In fact, $\xi_{n1}, \dots, \xi_{nn}$, $n \ge 1$, will denote a triangular array of row independent random variables and the $\mathbb{U}_n$ of (3.1) is actually the uniform empirical process of the random variables $\xi_{n1}, \dots, \xi_{nn}$ in the nth row. Now the random variables $\xi_{n:i}$ of (3.1) are actually of the form

$$\xi_{n:i} = (\eta_1^0 + \cdots + \eta_i^0)/(\eta_1^0 + \cdots + \eta_{n+1}^0) \equiv S_i^0/S_{n+1}^0 \tag{3.2}$$

for iid exponential (1) random variables η_j^0. Thus

$$\begin{aligned} n\mathbb{G}_n(t/n) &= \sum_{i=1}^{n} 1_{[\xi_{n:i} \le t/n]} = \sum_{i=1}^{n} 1_{[S_i^0 \le tS_{n+1}^0/n]} \\ &= \mathbb{N}^0\big(tS_{n+1}^0/n\big) \quad \text{for } 0 \le t \le n, \end{aligned} \tag{3.3}$$

where $\mathbb{N}^0$ is a Poisson process with interarrival times $\eta_1^0, \eta_2^0, \dots$ and arrival times S_i^0. Moreover, the construction (3.1) uses an analogous method to construct the random variables $1 - \xi_{n:n}$, $1 - \xi_{n:n-1}, \dots$ starting at the right-hand end. This leads to an *independent* Poisson process $\mathbb{N}^1$ associated with the right tail; its arrival times are denoted by S_i^1. Refer to CsCsHM (1986) for a more precise description of this construction.

CsCsHM (1986) also pointed out how the above construction can be modified to obtain a single sequence of iid uniform $(0, 1)$ random variables $\xi_1, \dots, \xi_n, \dots$ on which (3.1) holds with $\mathbb{U}_n$ formed from $\xi_1, \dots, \xi_n$. However, on this modified probability space, (3.2) is no longer valid.

In the proofs that follow, when we are dealing with L-statistics which we claim have a Poisson component in their limiting distribution, we will assume without comment that we are on the former probability space so that we do not have to deal with a sequence of Poisson processes, and, on the other hand, when considering L-statistics which we claim to be asymptotically normal, we will assume without comment that we are on the modified probability space.

It is well known [see Shorack and Wellner (1986), page 419, for example] that given $\varepsilon > 0$ one can choose $0 < \lambda \equiv \lambda\varepsilon < 1$ so small that

$$P\left(\begin{array}{l} \mathbb{G}_n(t) \le t/\lambda, \\ \quad \text{for all } 0 \le t \le 1 \text{ and } \mathbb{G}_n(t) \ge \lambda t \\ \quad \text{for all } \xi_{n:1} \le t \le 1; \text{ and} \\ \mathbb{G}_n(t) \ge 1 - (1-t)/\lambda, \\ \quad \text{for all } 0 \le t \le 1 \text{ and } \mathbb{G}_n(t) \le 1 - \lambda(1-t) \\ \quad \text{for all } 0 \le t < \xi_{n:n} \end{array}\right) > 1 - \varepsilon/6 \tag{3.4}$$

for all $n \geq 1$. Let $A_{n\varepsilon}$ denote the set in (3.4) and let $1_{n\varepsilon}$ denote its indicator function.

We also record the easily verified fact that

$$(3.5) \qquad P(a_n/n \leq \xi_{n:1} \leq 1/(na_n)) \to 1 \quad \text{if and only if } a_n \to 0.$$

Let A''_n denote the event in (3.5) and let $1''_n$ denote its indicator function. Since $\xi_{n:i} =_d (\eta_1 + \cdots + \eta_i)/(\eta_1 + \cdots + \eta_{n+1})$ for iid exponential (1) random variables η_i, we see trivially that $M \equiv M_\varepsilon$ can be chosen so large that

$$(3.6) \qquad P(1/(Mn) \leq \xi_{n:1} \leq \cdots \leq \xi_{n:k} \leq M/n) > 1 - \varepsilon/6$$

for all n. Let $A''_{n\varepsilon}$ denote the event in (3.6) and let $1''_{n\varepsilon}$ denote its indicator function.

Using (1.16) we have for $0 < t \leq \delta$ that

$$(3.7) \qquad \begin{aligned} J'(t) &= \rho_0 t^{\rho_0 - 1} l_0(t) + t^{\rho_0} l_0'(t) = t^{-1} J(t)[\rho_0 + \varepsilon_0(t)] \\ &= t^{\rho_0 - 1} l_0(t)[\rho_0 + \varepsilon_0(t)]. \end{aligned}$$

From de Haan [(1970), page 21] we know that any function l on $(0, \delta]$ that is slowly varying at 0 satisfies

$$(3.8) \qquad \sup_{a \leq r \leq b} \left| \frac{l(rt)}{l(t)} - 1 \right| \to 0 \quad \text{as } t \to 0$$

for all $0 < a < b$. Thus for any $0 < \lambda < 1$ and $\delta \equiv \delta_\lambda > 0$ small enough, we have

$$(3.9) \qquad \sup\{|l(rt)/l(t)| : \lambda \leq r \leq 1/\lambda \text{ and } 0 < t \leq \delta\} \leq M_\lambda$$

for some M_λ and hence

$$(3.10) \qquad \begin{aligned} &|J'(t^*)| \leq M_\lambda J(t)/t \\ &\text{for } 0 < t \leq \delta \text{ provided } \lambda t < t^* < t/\lambda \text{ for } 0 < t < \delta. \end{aligned}$$

We will write our proofs as though $\delta = 1/2$ in (1.16). This will allow us to use the bound $|J'(t^*)| \leq M_\lambda J(t)/[t(1-t)]$ over the *entire* interval on frequent occasions below. We thus save considerable notation. The case of the Lipschitz condition on the middle of the interval is trivial by comparison. Note (1.16a).

It is shown in CsHM [(1988b), Lemma 2.1] that any quantile function K satisfies

$$(3.11) \qquad \limsup_{a \to 0,\, b \to 1} \frac{aK^2(a) + (1-b)K^2(b)}{\sigma^2[a, b)} \leq 1$$

and in CsHM [(1988a), Lemma 3.2] that

$$(3.12) \qquad \begin{aligned} &\limsup_{a \to 0,\, b \to 1} \int_a^b K^2(t)\, dt \Big/ \sigma^2[a, b) < \infty \\ &[1 \text{ is an appropriate upper bound if } \sigma^2(0,1) = \infty]. \end{aligned}$$

Contrast (3.11) with (1.26). CsHM (1988b) also point out a Gnedenko and

Kolmogorov (1954) result [see also Tucker (1971)] that

$$\int_a^b K(t)\,dt \Big/ \left(\int_a^b K^2(t)\,dt \right)^{1/2} \to 0 \tag{3.13}$$

as $a \to 0$, $b \to 1$ provided $\int_0^1 K^2(t)\,dt = \infty$.

Note that (3.11) gives for $c \geq 1$ that

$$\limsup_n |\Phi_{0n}(c)| = \limsup_n c^{-1/2} \sqrt{c/n}\, K(c/n)/\sigma_n^* \leq c^{-1/2}, \tag{3.14}$$

which $\to 0$ as $c \to \infty$.

Thus if $\Phi_{0n'} \to_D \Phi_0$ on some subsequence n', then

$$|\Phi_0(c)| \leq c^{-1/2} \quad \text{for all } c > 1. \tag{3.15}$$

Analogous results hold for Φ_{1n} and Φ_1. The choice of 1 in the definition of σ_n^* controls the cutoff point $c = 1$ in (3.14).

Define $\mathbb{G}_n^*$ by

$$\mathbb{G}_n^*(t) \text{ denotes } k/n,\ \mathbb{G}_n(t),\ 1 - m/n \quad \text{as } t \leq \xi_{n:k},\ \xi_{n:k} \leq t \leq \xi_{n:n-m},\ \xi_{n:n-m} \leq t \tag{3.16}$$

for fixed $k, m \geq 1$. Note that

$$|\mathbb{G}_n^*(t) - t| \leq |\mathbb{G}_n(t) - t| \quad \text{provided } k/n \leq t \leq 1 - m/n \tag{3.17}$$

and that

$$|\mathbb{G}_n^*(t) - \mathbb{G}_n(t)| \leq (k \vee m)/n \quad \text{for } 0 \leq t \leq 1. \tag{3.18}$$

Define

$$\Gamma(t) \equiv \int_{1/2}^t J(s)\,ds \quad \text{for } 0 < t < 1. \tag{3.19}$$

Fix integers $k \leq l < r \leq n - m$. Now

$$T_n(l, r] \equiv n^{-1} \sum_{l+1}^{r} c_{ni} g(\xi_{n:i}) = \int_{(\xi_{n:l},\ \xi_{n:r}]} g\, d\Gamma(\mathbb{G}_n^*) \tag{3.20}$$

$$= g(\xi_{n:r})\Gamma(r/n) - g(\xi_{n:l})\Gamma(l/n) - \int_{[\xi_{n:l},\ \xi_{n:r})} \int_{1/2}^{\mathbb{G}_n^*(t)} J(s)\,ds\,dg(t); \tag{3.21}$$

as soon as one notes that $\mathbb{G}_n$ can replace $\mathbb{G}_n^*$ in this identity since $\xi_{n:k} \leq \xi_{n:l} \leq t \leq \xi_{n:r} \leq \xi_{n:n-m}$, this is just integration by parts. Also

$$\begin{aligned} -\mu_n(l/n, r/n) &\equiv -\int_{l/n}^{r/n} g(t)\,d\Gamma(t) \\ &= -g(r/n)\Gamma(r/n) + g(l/n)\Gamma(l/n) \\ &\quad + \int_{l/n}^{r/n} \int_{1/2}^{t} J(s)\,ds\,dg(t). \end{aligned} \tag{3.22}$$

Scaling the difference by some norming constant A_n gives, after some rearrangement, that

$$
\begin{aligned}
S_n(l,r] &\equiv \sqrt{n}\{T_n(l,r] - \mu_n(l/n, r/n)\}/A_n \\
&= -\int_{l/n}^{r/n}\sqrt{n}\int_t^{\mathbb{G}_n^*(t)} J(s)\,ds\,dg(t)/A_n \\
&\quad -\int_{\xi_{n:l}}^{l/n}\sqrt{n}\int_{l/n}^{\mathbb{G}_n^*(t)} J(s)\,ds\,dg(t)/A_n \qquad (3.23)\\
&\quad +\int_{\xi_{n:r}}^{r/n}\sqrt{n}\int_{r/n}^{\mathbb{G}_n^*(t)} J(s)\,ds\,dg(t)/A_n \\
&\equiv \theta_n[l,r) + \alpha_n(l) - \alpha_n(r).
\end{aligned}
$$

[Note that having $\mathbb{G}_n^*$ in place of $\mathbb{G}_n$ in (3.23) is *important*, since otherwise the integral could be infinite in some cases.]

Most of the proofs of Theorems 1.1 and 2.1 will be contained in those of the following two propositions.

PROPOSITION 3.1. *Assume Conditions* G, J, C *and* K *hold. Suppose constants* $A_n > 0$ *and a subsequence* n' *of* n *exist for which*

$$(\sigma_{n'}^*/A_{n'})\Phi_{in'} \to_D \ (\text{some } \Phi_i) \quad \text{as } n' \to \infty \text{ for } i = 0,1, \qquad (3.24)$$

and for some $0 \le a < \infty$, *we have*

$$\sigma_{n'}^*/A_{n'} \to a \quad \text{as } n' \to \infty. \qquad (3.25)$$

(i) *We have*

$$\int_\varepsilon^\infty \Phi_i^2(x)\,dx < \infty \quad \text{for all } \varepsilon > 0 \text{ for } i = 0,1; \qquad (3.26)$$

and this implies that V_{0k} *and* V_{1m} *of Theorem* 2.1 *are well-defined random variables for all* $k, m \ge 1$.

(ii) *There exist sequences* $l_{n'} \to \infty$ *and* $r_{n'} \to \infty$ *for which*

$$l_{n'}/n' \to 0, \qquad r_{n'}/n' \to 0, \qquad l_{n'}/r_{n'} \to 0 \quad \text{as } n' \to \infty \qquad (3.27)$$

and

$$
\begin{aligned}
&\alpha_{n'}(k) + \theta_{n'}[k, l_{n'}) \to_d V_{0k}, \\
&-\alpha_{n'}(n'-m) + \theta_n[n' - l_{n'}, n'-m) \to_d V_{1m},
\end{aligned} \qquad (3.28)
$$

$$\theta_{n'}[l_{n'}, r_{n'}) \to_p 0, \qquad \theta_{n'}[n' - r_{n'}, n' - l_{n'}) \to_p 0 \qquad (3.29)$$

as $n' \to \infty$ *with* V_{0k} *and* V_{1m} *as in Theorem* 2.1.

(iii) *If* n'' *is a further subsequence on which, for some* $0 \le \tau < \infty$, *we have*

$$\sigma(r_{n''}/n'')/A_{n''} \to \tau, \qquad (3.30)$$

then, in addition to (3.28) *and* (3.29), *we have*

$$\theta_{n''}[r_{n''}, n'' - r_{n''}) = -\int_{(r_{n''}/n'')}^{1-(r_{n''}/n'')} \mathbb{B}_n\,dK + o_p(1) \to_d \tau Z \quad \text{as } n'' \to \infty \qquad (3.31)$$

with Z independent of V_{0k} *and* V_{1m}; *and hence*

$$S_{n''}(k, n'' - m] \to_d V_{0k} + \tau Z + V_{1m} \quad \text{as } n'' \to \infty. \tag{3.32}$$

(iv) *If* $\tau > 0$, *then the sequences specified above can be chosen so that*

$$\sigma(l_{n''}/n'')/\sigma(r_{n''}/n'') \to 1 \quad \text{as } n'' \to \infty. \tag{3.33}$$

(v) *Also*

$$a = 0 \text{ in } (3.25) \text{ implies } \tau = 0 \text{ and } \Phi_i(x) = 0 \quad \text{for all } x > 1 \text{ and } i = 0, 1. \tag{3.34}$$

(vi) *Finally*,

$$\begin{aligned} &\text{if } \Phi_0 = \Phi_1 = 0 \text{ and } a \in (0, \infty),\\ &\text{then } \tau = a \text{ and we may suppose that } n'' = n'. \end{aligned} \tag{3.35}$$

PROPOSITION 3.2. *Assume Conditions* G, J, C *and* K *hold*. (*Recall that* $k, m \geq 1$.) *Suppose constants* $A_n > 0$ *and* B_n *and a subsequence* n' *of* n *exist for which*, *for some nondegenerate random variable* W,

$$\sqrt{n'}\,(T_{n'} - B_{n'})/A_{n'} \to_d W. \tag{3.36}$$

Then (3.24) *and* (3.25) *hold along some further subsequence* n''. *Furthermore*

$$W =_d d' + d(V_0 + \tau Z + V_1) \quad \text{with } \Phi_0, \Phi_1 \text{ satisfying } (3.26) \tag{3.37}$$

for some $d > 0$ *and* d' *and with* τ, V_0, Z, V_1 *as in Proposition* 3.1. *Moreover*, *if* $\tau = 0$, *then at least one of* Φ_0 *and* Φ_1 *is not identically a constant. Also*, *with* $c_0 > 0$,

$$\begin{aligned} &\text{if } \lim_{n'} \Phi_{in'}(c_0) = -\infty \text{ for either } i = 0 \text{ or } 1 \text{ and } (3.36) \text{ holds},\\ &\text{then } \tau = 0. \end{aligned} \tag{3.38}$$

Toward establishing these propositions, let us first analyze the behavior of $\theta_n[r, n - r)$ for r *large*, assuming that (3.25) holds on some subsequence n'. Using a Taylor series expansion and letting $\mathbb{U}_n^*(t) \equiv \sqrt{n}\,[\mathbb{G}_n^*(t) - t]$ gives

$$\begin{aligned} \theta_n[r, n - r) &= -\int_{r/n}^{1-r/n} \mathbb{U}_n^* J\, dg/A_n - (1/2)n^{-1/2}\\ &\qquad \times \int_{r/n}^{1-r/n} \mathbb{U}_n^*(t)^2 J'(t_n^*)\, dg(t)/A_n\\ &= -\int_{r/n}^{1-r/n} \mathbb{B}_n\, dK/A_n - \int_{r/n}^{1-r/n} (\mathbb{U}_n - \mathbb{B}_n)\, dK/A_n\\ &\qquad -\int_{r/n}^{1-r/n} (\mathbb{U}_n^* - \mathbb{U}_n)\, dK/A_n\\ &\qquad -(1/2)n^{-1/2} \int_{r/n}^{1-r/n} \mathbb{U}_n^*(t)^2 J'(t_n^*)\, dg(t)/A_n\\ &\equiv Z_n(r) + \gamma_{n1}(r) + Y_{n2}(r) + \gamma_{n3}(r); \end{aligned} \tag{3.39}$$

here t_n^* lies between $\mathbb{G}_n^*(t)$ and t.

We next turn to consideration of the $\gamma_{nj}(r)$, culminating in (3.45)–(3.47). Consider $\gamma_{n1}(r)$. Now

$$
\begin{aligned}
|\gamma_{n1}(r)| &\le \left\|(\mathbb{U}_n - \mathbb{B}_n)/(I(1-I))^{(1/2)-\nu}\right\|_{1/n}^{1-1/n} \\
&\quad \times \int_{r/n}^{1-r/n} (t(1-t))^{(1/2)-\nu}\, dK(t)/A_n \\
&\le O_p(n^{-\nu})\biggl\{(r/n)^{(1/2)-\nu}K(1-r/n) - (r/n)^{(1/2)-\nu}K(r/n) \\
&\qquad + \int_{r/n}^{1-r/n} K(t)(t(1-t))^{-(1/2)-\nu}\, dt\biggr\}/A_n \\
&\le O_p(r^{-\nu})\bigl\{\sqrt{r/n}\,(K(1-r/n) - K(r/n))/\sigma_n^*\bigr\}(\sigma_n^*/A_n) \\
&\quad + O_p(n^{-\nu})\left\{\int_{r/n}^{1-r/n} K^2(t)\, dt \int_{r/n}^{1-r/n}(t(1-t))^{-1-2\nu}\, dt\right\}^{1/2}/A_n \qquad (3.40) \\
&= \left[O_p(r^{-\nu}) + O_p(n^{-\nu})\left\{\int_{r/n}^{1-r/n} K^2(t)\, dt/(\sigma_n^*)^2\right\}^{1/2} O((r/n)^{-\nu})\right] \\
&\quad \times(\sigma_n^*/A_n) \\
&= O_p(r^{-\nu})\left[1 + \left\{\int_{r/n}^{1-r/n} K^2(t)\, dt/(\sigma_n^*)^2\right\}^{1/2}\right](\sigma_n^*/A_n) \\
&= O_p(r^{-\nu}) \quad \text{on the subsequence } n' \text{ of } (3.25),
\end{aligned}
$$

using (3.1) in the second step, Cauchy–Schwarz in the third step, (3.11) and (1.13) in the fourth step and (3.12), (1.13) and (3.25) in the sixth step.

Consider $\gamma_{n2}(r)$. Now by (3.18) and then (3.11), (1.13) and (3.25), we have

$$
\begin{aligned}
|\gamma_{n2}(r)| &\le \left[\int_{r/n}^{1-r/n} (k \vee m) n^{-1/2}\, dK/\sigma_n^*\right](\sigma_n^*/A_n) \qquad (3.41) \\
&= O(r^{-1/2}) \quad \text{on the subsequence } n' \text{ of } (3.25).
\end{aligned}
$$

Consider $\gamma_{n3}(r)$. Set

$$
\gamma_{n3}^{(1)}(r) = -(1/2)n^{-1/2}\int_{r/n}^{1/2} \mathbb{U}_n^*(t)^2 J'(t_n^*)\, dg(t)/A_n \quad \text{and}
$$

$$
\gamma_{n3}^{(2)}(r) = \gamma_{n3}(r) - \gamma_{n3}^{(1)}(r).
$$

Recall (3.6) with $r = M \equiv M_\varepsilon$ so that $r/n \ge \xi_{n:k}$ on $A_{n\varepsilon}'''$ and recall (3.4). Now (3.39) and (3.10) give

$$
\begin{aligned}
\gamma_{n\varepsilon}^{(1)}(r) &\equiv 1_{n\varepsilon}1_{n\varepsilon}''|\gamma_{n3}^{(1)}(r)| \qquad (3.42) \\
&\le n^{-1/2}\int_{r/n}^{1/2} \mathbb{U}_n^2(t) M_{\lambda_\varepsilon} J(t)(t(1-t))^{-1}\, dg(t)/A_n.
\end{aligned}
$$

Using Fubini's theorem gives

$$e_{n\varepsilon}(r) \equiv E\gamma_{n\varepsilon}^{(1)} \le M_{\lambda_\varepsilon} n^{-1/2} \int_{r/n}^{1/2} dK(t)/A_n$$

$$= O(r^{-1/2}) \quad \text{on the subsequence } n' \text{ of } (3.25)$$

as in the previous paragraph. Given $\varepsilon > 0$ and $r = M_\varepsilon$ as in (3.6), we find by Markov's inequality that on the n' of (3.25),

$$P\big(|\gamma_{n3}^{(1)}(r)| \ge \varepsilon\big) \le P(A_{n\varepsilon}^c) + P(A_{n\varepsilon}''^c) + \varepsilon^{-1} e_{n\varepsilon}(r) \le \varepsilon \tag{3.43}$$
$$\text{for all } n \ge \text{ some } n_\varepsilon \text{ provided } r \ge r_\varepsilon.$$

Since by symmetry the same statement holds for $\gamma_{n3}^{(2)}(r)$, we get

$$\lim_{r\to\infty} \overline{\lim_{n'\to\infty}} \sup P\big(|\gamma_{n'3}(r)| \ge \varepsilon\big) = 0 \quad \text{for all } \varepsilon > 0. \tag{3.44}$$

Combine (3.40), (3.41) and (3.44) to get

$$\lim_{r\to\infty} \overline{\lim_{n'\to\infty}} \sup P\big(|\gamma_{n'}(r)| \ge \varepsilon\big) = 0 \tag{3.45}$$
$$\text{for all } \varepsilon > 0, \text{ where } \gamma_n(r) \equiv \sum_{j=1}^{3} \gamma_{nj}(r).$$

Now from (3.39) and (3.45) we get the representation

$$\theta_n[r, n-r) = Z_n(r) + \gamma_n(r) \tag{3.46}$$

with $\gamma_n(r)$ satisfying (3.45) and

$$Z_n(r) = -\int_{r/n}^{1-r/n} \mathbb{B}_n \, dK/A_n =_d \tau_n(r) Z, \tag{3.47}$$

where $\tau_n(r) = \sigma(r/n)/A_n$ and $Z =_d N(0,1)$.

LEMMA 3.1. *Suppose $J(t) = t^\rho L(t)$, where $J > 0$, $-\infty < \rho < \infty$, and L is slowly varying at* 0. *Let*

$$\Delta_n(x,y) = \frac{\int_x^y J(u/n)\,du}{J(y/n)}. \tag{3.48}$$

Then for all $0 < a < b < \infty$, we have

$$\sup_{a \le x, y \le b} \big|\Delta_n(x,y) - h_\rho(x,y)\big| \to 0 \quad \text{as } n \to \infty, \tag{3.49}$$

where $h_\rho(x,y)$ is as in (2.5).

PROOF OF LEMMA 3.1. Now the sup in (3.49) is bounded by

$$\sup_{a\le x,y\le b} \frac{|\int_x^y\{L(u/n)-L(y/n)\}u^\rho\,du|}{L(y/n)y^\rho}$$

$$\le \max(a^{-\rho},b^{-\rho})\int_a^b u^\rho\,du \sup_{a\le u,y\le b}\frac{|L(u/n)-L(y/n)|}{L(y/n)}$$

$$\to 0 \quad \text{as } n\to\infty.$$

The final convergence to 0 follows from (3.8). □

PROOF OF PROPOSITION 3.1. Recall (3.23) and (3.45)–(3.47). We now fix integers $1\le k$, $m<l<r<n/2$ and use (3.33) to write

$$S_n \equiv \sqrt{n}\,(T_n-\mu_n)/A_n = \sqrt{n}\,\{T_n(k,n-m]-\mu_n(k/n,1-m/n)\}/A_n \tag{3.50}$$

$$\begin{aligned} &= S_n(k,l]+S_n(l,r]+S_n(r,n-r]\\ &\quad+S_n(n-r,n-l]+S_n(n-1,n-m]\\ &= \{\alpha_n(k)+\theta_n[k,l)\}+\theta_n[l,r)+\theta_n[r,n-r)+\theta_n[n-r,n-l)\\ &\quad+\{\theta_n[n-l,n-m)-\alpha_n(n-m)\}.\end{aligned} \tag{3.51}$$

In the spirit of (3.24), note also (2.1), we define

$$\Phi_{in}^*(c) \equiv (\sigma_n^*/A_n)\Phi_{in}(c) \quad \text{for } c>0 \text{ and } i=0,1. \tag{3.52}$$

Consider the $\theta_n[l,r)$ of (3.23) and (3.51). Recall (3.4) for $1_{n\varepsilon}$. Then for n' as in (3.24) we have from (3.23) and a change of variables that

$$1_{n\varepsilon}\theta_n[l,r) = -1_{n\varepsilon}\int_{l/n}^{r/n}\left[\frac{n\int_t^{\mathbb{G}_n^*(t)}J(s)\,ds}{J(t)}\right]J(t)\frac{dg(t)}{\sqrt{n}\,A_n}$$

$$= -1_{n\varepsilon}\int_l^r\left[\frac{n\int_{t/n}^{\mathbb{G}_n^*(t/n)}J(s)\,ds}{J(t/n)}\right]d\Phi_{0n}^*(t)$$

$$= -1_{n\varepsilon}\int_l^r\frac{\int_t^{n\mathbb{G}_n^*(t/n)}J(u/n)\,du}{J(t/n)}\,d\Phi_{0n}^*(t) \tag{3.53}$$

$$= 1_{n\varepsilon}\int_l^r h_{\rho_0}(n\mathbb{G}_n^*(t/n),t)\,d\Phi_{0n}^*(t)+o(1) \quad \text{on } n' \tag{3.54}$$

$$=_{\text{a.s.}} 1_{n\varepsilon}\int_l^r h_{\rho_0}\big(\mathbb{N}^0(tS_{n+1}^0/n)\vee k,t\big)\,d\Phi_{0n}^*(t)+o(1) \quad \text{on } n' \tag{3.55}$$

$$\begin{aligned} &= 1_{n\varepsilon}\int_l^r\big[h_{\rho_0}\big(\mathbb{N}^0(tS_{n+1}^0/n)\vee k,t\big)-h_{\rho_0}(k,t)\big]\,d\Phi_{0n}^*(t)\\ &\quad+1_{n\varepsilon}\int_l^r h_{\rho_0}(k,t)\,d\Phi_{0n}^*(t)+o(1) \quad \text{on } n',\end{aligned} \tag{3.56}$$

using Lemma 3.1 in step (3.54) (this lemma is applicable because the $1_{n\varepsilon}$ multiplier leads to bounds a and b for this lemma), and using (3.3) in (3.55).

Next consider the $\alpha_n(k)$ of (3.23) and (3.51). Recall (3.4) and (3.5). As in the previous paragraph,

$$
\begin{aligned}
1_{n\varepsilon}1''_{n\varepsilon}\alpha_n(k) &= -1_{n\varepsilon}1''_{n\varepsilon}\int_{\xi_{n:k}}^{k/n}\left[\frac{n\int_{k/n}^{\mathbb{G}_n^*(t)}J(s)\,ds}{J(t)}\right]J(t)\frac{dg(t)}{\sqrt{n}\,A_n}\\
&= -1_{n\varepsilon}1''_{n\varepsilon}\int_{n\xi_{n:k}}^{k}\left[\frac{n\int_{k/n}^{\mathbb{G}_n^*(t/n)}J(s)\,ds}{J(t/n)}\right]d\Phi_{0n}^*(t)\\
&= -1_{n\varepsilon}1''_{n\varepsilon}\int_{nS_k^0/S_{n+1}^0}^{k}\frac{\int_k^{n\mathbb{G}_n^*(t/n)}J(u/n)\,du}{J(t/n)}\,d\Phi_{0n}^*(t) \qquad (3.57)\\
&= 1_{n\varepsilon}1''_{n\varepsilon}\int_{nS_k^0/S_{n+1}^0}^{k}\left[h_{\rho_0}(\mathbb{N}^0(tS_{n+1}^0/n)\vee k,t)\right.\\
&\qquad\left.-h_{\rho_0}(k,t)\right]d\Phi_{0n}^*(t)+o(1)\quad\text{on } n'.
\end{aligned}
$$

Thus [replacing l, r by k, l in (3.56)] adding (3.56) and (3.57) gives

$$
\begin{aligned}
&1_{n\varepsilon}1''_{n\varepsilon}\{\alpha_n(k)+\theta_n[(k,l)]\}\\
&\quad= 1_{n\varepsilon}1''_{n\varepsilon}\left\{\int_{nS_k^0/S_{n+1}^0}^{l}\left[h_{\rho_0}(\mathbb{N}^0(tS_{n+1}/n)\vee k,t)-h_\rho(k,t)\right]d\Phi_{0n}^*(t)\right.\\
&\qquad\qquad\left.+\int_k^l h_{\rho_0}(k,t)\,d\Phi_{0n}^*(t)\right\}+o(1)\quad\text{on } n' \qquad (3.58)\\
&\quad=_{\text{a.s.}} 1_{n\varepsilon}1''_{n\varepsilon}\left\{\int_{S_k}^{l}\left[h_{\rho_0}(\mathbb{N}^0(t)\vee k,t)-h_{\rho_0}(k,t)\right]d\Phi_0(t)\right.\\
&\qquad\qquad\left.+\int_k^l h_{\rho_0}(k,t)\,d\Phi_0(t)\right\}+o(1)\quad\text{on } n',
\end{aligned}
$$

by (3.24), under the (temporary) assumption that l is a continuity point of all Φ_{0n}^* and Φ_0, noting that the jumps of $\mathbb{N}^0$ a.s. miss the discontinuities of Φ_0 while $\Phi_{0n'}^*\to_D\Phi_0$. [In case the integers l and r in (3.51) are not continuity points of Φ_{0n}^* and Φ_0, we replace them by real numbers $\tilde{l}$ and $\tilde{r}$ that are and are within distance $\frac{1}{2}$. We will not mention this point again, but merely take l and r to be such continuity points.] Now that we have added $\alpha_n(k)$ and $\theta_n[k,l)$ and passed to the limit, we can break the limiting sum back apart to observe that

$$
\begin{aligned}
&1_{n\varepsilon}1''_{n\varepsilon}\{\alpha_n(k)+\theta_n[k,l)\}\\
&\quad= 1_{n\varepsilon}1''_{n\varepsilon}\left\{\int_{S_k}^{l}h_{\rho_0}(\mathbb{N}^0(t)\vee k,t)\,d\Phi_0(t)+\int_k^{S_k}h_p(k,t)\,d\Phi_0(t)\right\} \qquad (3.59)\\
&\qquad+o(1)\quad\text{on } n'.
\end{aligned}
$$

Similarly we obtain in (3.56) that

$$1_{n\varepsilon}\theta_n[l,k) = 1_{n\varepsilon}\int_l^r h_{\rho_0}(\mathbb{N}^0(t) \vee k, t)\, d\Phi_0(t) + o(1) \quad \text{on } n'. \tag{3.60}$$

Thus on the subsequence n' of (3.24) we have under (3.24) that

$$\begin{aligned} \alpha_{n'}(k) + \theta_{n'}[k,l) &\to_p \beta_0(k) + \delta_0(k) + \theta_0[k,l) \\ &\equiv \int_k^{S_k^0} h_{\rho_0}(k,t)(t)\, d\Phi_0(t) + \int_{S_k^0}^k h_{\rho_0}(\mathbb{N}^0(t) \vee k, t)\, d\Phi_0(t) \\ &\quad + \int_k^l h_{\rho_0}(\mathbb{N}^0(t) \vee k, t)\, d\Phi_0(t) \end{aligned} \tag{3.61}$$

and

$$\theta_{n'}[l,r) \to_p \theta_0[l,r) \equiv \int_l^r h_{\rho_0}(\mathbb{N}^0(t) \vee k, t)\, d\Phi_0(t). \tag{3.62}$$

Moreover, in case Φ_0 is constant, which by (3.15) must be zero, we have

$$\alpha_{n'}(k) + \theta_{n'}[k,l) \to_p 0 \quad \text{and} \quad \theta_{n'}[l,r) \to_p 0. \tag{3.63}$$

The obvious analog in the right tail is

$$\begin{aligned} \theta_{n'}[n'-l, n'-m) - \alpha_{n'}(n'-m) &\to_p \theta_1[l,m) + \delta_1(m) + \beta_1(m) \\ &\equiv -\int_m^l h_{\rho_1}(\mathbb{N}^1(t) \vee m, t)\, d\Phi_1(t) \\ &\quad - \int_{S_n^1}^m h_{\rho_1}(\mathbb{N}^1(t) \vee m, t)\, d\Phi_1(t) \\ &\quad - \int_m^{S_m^1} h_{\rho_1}(m,t)\, d\Phi_1(t) \end{aligned} \tag{3.64}$$

and

$$\theta_{n'}[n'-r, n'-l) \to_p \theta_1[r,l) \equiv -\int_l^r h_{\rho_1}(\mathbb{N}^1(t) \vee m, t)\, d\Phi_1(t). \tag{3.65}$$

It is also convenient to define

$$\Delta_0[l,r) \equiv \delta_0(l) + \theta_0[l,r) - \delta_0(r) = \int_{S_l^0}^{S_r^0} h_{\rho_0}(\mathbb{N}^0(t), t)\, d\Phi_0(t) \tag{3.66}$$

and

$$\Delta_1[l,r) = \delta_1(l) + \theta_1[l,r) - \delta_1(r) = -\int_{S_l^1}^{S_r^1} h_{\rho_1}(\mathbb{N}^1(t), t)\, d\Phi_1(t). \tag{3.67}$$

PROOF OF PART (i) OF PROPOSITION 3.1. Since our K is the inverse of a df, Lemma 2.5 of CsHM (1988a) shows directly that under (3.25),

$$\int_\varepsilon^\infty \Phi_i^2(x)\, dx < \infty \quad \text{for all } \varepsilon > 0 \text{ and } i = 1, 2. \tag{3.68}$$

We now note the expansion

$$h_\rho(x,t) = (t-x) - (\rho/2)(x-t)^2 t^{-1}(t^*/t)^{\rho-1} \tag{3.69}$$

for t^* between x and t.

We shall now show that

$$\Delta_0(k,\infty) \equiv \int_{S_k^0}^{\infty} h_{\rho_0}(\mathbb{N}^0(t),t)\, d\Phi_0(t) \text{ is a well-defined random variable.} \tag{3.70}$$

Now applying (3.69) to (3.66), intending to let $r \to \infty$ in the well-defined random variable, we see from

$$\begin{aligned} \Delta_0(k,\infty) &= -\int_{S_k^0}^{\infty} [\mathbb{N}^0(t) - t]\, d\Phi_0(t) \\ &\quad -(\rho_0/2)\int_{S_k^0}^{\infty} [\mathbb{N}^0(t) - t]^2 t^{-1}(t^*/t)^{\rho-1}\, d\Phi_0(t) \\ &\equiv D_1 - (\rho_0/2) D_2, \end{aligned} \tag{3.71}$$

that it is enough to show that D_1 and D_2 are well defined when ∞ is used for their upper limits. Choose $\lambda \equiv \lambda_\varepsilon$ so small that $P(A_\varepsilon) > 1 - \varepsilon$, where

$$A_\varepsilon \equiv [\mathbb{N}^0(t) \le t/\lambda_\varepsilon \text{ for all } t > 0 \text{ and } \mathbb{N}^0(t) \ge \lambda_\varepsilon t \text{ for all } t \ge S_1^0], \tag{3.72}$$

and let 1_ε denote the indicator function of A_ε. Now

$$1_\varepsilon 1_{[S_k^0 > \varepsilon]} D_2 \le \int_\varepsilon^\infty [\mathbb{N}^0(t) - t]^2 t^{-1} \lambda_\varepsilon^{(\rho-1)\wedge(-(\rho-1))}\, d\Phi_0(t);$$

and using Fubini's theorem we see that this random variable clearly has a finite mean and hence is a.s. finite. Letting $\varepsilon \to 0$ shows D_2 is well defined. That D_1 is a well-defined random variable follows from CsHM [(1986a), Theorem 3].

Proof of parts (ii)–(vi) of Proposition 3.1. All that is now needed is to show that our expansions (requiring the new device of $\mathbb{G}_n^*$) of the function Γ of (3.19) and, especially, our introduction and application of regularly varying functions J to obtain (3.61) and (3.62), have allowed us to reach a point where we can plug into the exactly analogous parts of the proof of the special case $J \equiv 1$ presented in CsHM (1988a). Let us now fix an integer $s \ge 2$. Note that $1 \le k, m < l < sl$, with fixed integers k, m, l, s. We now combine (3.23), (3.46), (3.62), (3.65), (3.61) and (3.64) into (3.51), using the definitions (3.66) and (3.67) with $r = sl$ to get, with appropriate remainders R,

$$\begin{aligned} S_n &= \{\alpha_n(k) + \theta_n[k,l)\} + \theta_n[l,sl) + \theta_n[sl,n-sl) + \theta_n[n-sl,n-l) \\ &\quad + \{\theta_n[n-l,n-m) - \alpha_n(n-m)\} \\ &= \{\beta_0(k) + \delta_0(k) + \theta_0[k,l) + R^0_{n,k,l}\} \\ &\quad + \{\theta_0[l,sl) + R^0_{n,l,sl}\} + \{Z_n(sl) + \gamma_n(sl)\} \\ &\quad + \{\theta_1[l,sl) + R^1_{n,l,sl}\} + \{\beta_1(m) + \delta_1(m) + \theta_1[m,l) + R^1_{n,m,l}\} \\ &= \beta_0(k) + [\delta_0(k) + \theta_0[k,l) - \delta_0(l)] \\ &\quad + [\delta_0(l) + \theta_0[l,sl) - \delta_0(sl)] + \delta_0(sl) \\ &\quad + R^0_{n,k,l} + R^0_{n,l,sl} + Z_n(sl) + \gamma_n(sl) + R^1_{n,l,sl} + R^1_{n,m,l} \\ &\quad + \beta_1(m) + [\delta_1(m) + \theta_1[m,l) - \delta_1(l)] \\ &\quad + [\delta_1(l) + \theta_1[l,sl) - \delta_1(sl)] + \delta_1(sl) \\ &= \beta_0(k) + \Delta_0[k,l) + \Delta_0[l,sl) + \delta_0(sl) \\ &\quad + R^0_{n,k,l} + R^0_{n,l,sl} + Z_n(sl) + \gamma_n(sl) \\ &\quad + \beta_1(m) + \Delta_1[m,l) + \Delta_1[l,sl) + \delta_1(sl) + R^1_{n,m,l} + R^1_{n,l,sl} \end{aligned} \tag{3.73}$$

with the following relationships satisfied. First, we have

$$R^0_{n,k,l} \to_p 0 \quad \text{and} \quad R^1_{n,m,l} \to_p 0 \tag{3.74}$$

by (3.61) and (3.64). Second, we have

$$R^i_{n',l,ls} \to_p 0 \quad \text{as } n' \to \infty,\ i = 0,1, \tag{3.75}$$

by (3.62) with l, r equal to l, sl and by (3.65) with l, r equal to l, sl. Third, we have easily that

$$\delta_i(sl) \to_p 0 \quad \text{as } l \to \infty,\ i = 0,1, \tag{3.76}$$

as suggested by (3.70). Fourth, we have

$$\max_{0 \le h \le l} |\Delta_i(h,l] - \Delta_i(h,\infty)| = |\Delta_i(l,\infty)| \to_p 0 \quad \text{as } l \to \infty,\ i = 0,1, \tag{3.77}$$

by trivial observations based on (3.70). Fifth, we have

$$\Delta_i(l,sl) \to_p 0 \quad \text{as } l \to \infty,\ i = 0,1, \tag{3.78}$$

by an easy argument based on (3.70). Equations (3.74), (3.75), (3.76), (3.77), (3.78), (3.47), (3.45) of this paper correspond to equations in CsHM (1988a) bearing the numbers (2.24), (2.26), Lemma 2.6, (2.25), (2.27), first result in Lemma 2.10, second result in Lemma 2.10. Because we have reduced our problem to a complete analog of the CsHM (1988a) problem, we can now claim our conclusion from their conclusion. The conclusions (3.34) and (3.35) are a bit harder to locate in CsHM (1988a); they correspond to the last sentence of their Theorem 5 and their equation (1.4), respectively. [As a help to understanding, we merely mention that the introduction of the terms in (3.29) was essential to make the central term and the tail terms independent.] □

We note that

(3.79) not all CsHM (1988a) results carry over, since their limiting random variable only corresponds to ours when $\rho = 0$.

PROOF OF PROPOSITION 3.2. Let $W_n \equiv \sqrt{n}(T_n - B_n)/A_n$, so that $W_{n'} \to_d W$ by hypothesis.

CASE 1. $\limsup|\Phi_{in'}(c)| < \infty$ for all real c and $i = 0, 1$. Then on some further subsequence n'' we have by Helly–Bray that $\Phi_{in''} \to_D$ (some Φ_i) for $i = 0, 1$. Thus Proposition 3.1 implies that on a further n''' we have $S_n^* \to_d V_0 + \tau Z + V_1$ with Φ_0 and Φ_1 satisfying (3.26). But $W_n = (\sigma_n^*/A_n)S_n^* + \sqrt{n}(\mu_n - B_n)/A_n$. Thus the theorem on convergence of types implies that on n''' we have that σ_n^*/A_n converges to some positive d and $\sqrt{n}(\mu_n - B_n)/A_n$ converges to some real d'. Thus (3.37) holds. Also (3.24) and (3.25) hold on n'''. Suppose $\tau = 0$. If both Φ_0 and Φ_1 are constant functions (necessarily 0), then W would be degenerate. Since W is not degenerate, both Φ_0 and Φ_1 cannot be constant.

CASE 2. $\lim \Phi_{in''}(c_0) = -\infty$ for some n'' and c_0 with $i = 0$ or 1; suppose $i = 0$ for definiteness. Note that by (3.15), $0 < c_0 < 1$ since $\limsup|\Phi_{in}(1)| \le 1$ by (3.14) with $c = 1$. Using the integration by parts of (3.23), we now write

$$\begin{aligned} S_n^* &=_{\text{a.s.}} -\int_{k/n}^{1-m/n} \sqrt{n} \int_t^{\mathbb{G}_n^*(t)} J(s)\, ds\, dg(t)/\sigma_n^* \\ &\quad - \int_{\xi_{n:k}}^{k/n} \sqrt{n} \int_{k/n}^{\mathbb{G}_n^*(t)} J(s)\, ds\, dg(t)/\sigma_n^* \qquad (3.80)\\ &\quad + \int_{\xi_{n:n-m}}^{1-m/n} \sqrt{n} \int_{1-m/n}^{\mathbb{G}_n^*(t)} J(s)\, ds\, dg(t)/\sigma_n^* \\ &\equiv \theta_n[k, n-m] + \alpha_n(k) - \alpha_n(n-m). \qquad (3.81) \end{aligned}$$

Using (3.39)–(3.47) shows that

$$\theta_n[k, n-m] = O_p(1). \qquad (3.82)$$

We next prove that

$$\lim_{M\to\infty} \liminf_{n\to\infty} P(|\alpha_n(k)| \le M) > 0 \qquad (3.83)$$
$$\text{[and the same result for } \alpha_n(n-m)].$$

On the set $C_n \equiv [\xi_{n:k} > k/n]$ which has $\liminf P(C_n) = P(S_k > k) > 0$, we have

$$|\alpha_n(k)| = \int_{k/n}^{\xi_{n:k}} n \int_{k/n}^{k/n} J(s)\, ds\, dg(t)/(\sqrt{n}\,\sigma_n^*) = 0.$$

Thus (3.83) holds. [Our proof of a version of (3.83) trivialized when we changed to the present definition of C_n, as suggested to us by L. Vilharos.]

Lemmas 2.9–2.11 of CsHM (1988b) in fact establish, in the context of our notation, the following result: If $\theta_n[k, n-m) = O_p(1)$, the random variables $\alpha_n(k)$ and $\alpha_n(n-m)$ are asymptotically independent, the α_n of (3.81) both satisfy (3.83) and $W_n = (\theta_n[k, n-m) + \alpha_n(k) - \alpha_n(n-m) - B_n)/A_n = O_p(1)$ all on n'' for some constants $A_n > 0$ and B_n, then we may conclude that along n'', both

$$\sigma_n^* \alpha_n(k)/(\alpha_n^* \vee A_n) = O_p(1) \quad \text{and} \\ \alpha_n^* \alpha_n(n-m)/(\sigma_n^* \vee A_n) = O_p(1). \tag{3.84}$$

The required asymptotic independence of our $\alpha_n(k)$ and $\alpha_n(n-m)$ is a special case of Satz 4 of Rossberg (1967) and the other requirements are shown in (3.82), (3.83) or hypothesized; thus (3.84) holds from this CsHM result.

Let $D_{cn} \equiv [\xi_{n:2k} \le c/n]$ for $0 < c < c_0 < 1$ and note that $\liminf P(D_{cn}) > 0$. Now on the event D_{cn} and the subsequence n'' we have for all n large enough that

$$\begin{aligned} O_p(1) = \frac{\sigma_n^* |\alpha_n(k)|}{\sigma_n^* \vee A_n} &\ge \int_{c/n}^{k/n} n \int_{k/n}^{2k/n} J(s)\, ds \frac{dg(t)}{\sqrt{n}\,\sigma_n^*} \frac{\sigma_n^*}{\sigma_n^* \vee A_n} \\ &= \int_c^k \left\{ \int_k^{2k} \frac{J(u/n)\, du}{J(t/n)} \right\} \frac{dK(t/n)}{\sqrt{n}\,\sigma_n^*} \frac{\sigma_n^*}{\sigma_n^* \vee A_n} \\ &\ge \left(\frac{1}{2}\right) h_{\rho_0}(k, 2k) \int_c^k \left(\frac{2k}{t}\right)^{\rho_0} d\Phi_{0n}(t) \frac{\sigma_n^*}{(\sigma_n^* \vee A_n)} \\ &\ge (\text{some } \varepsilon) |\Phi_{0n}(c)| \frac{\sigma_n^*}{(\sigma_n^* \vee A_n)}. \end{aligned}$$

These inequalities obviously forces

$$\limsup \left(\frac{\sigma_n^*}{(\sigma_n^* \vee A_n)} \right) |\Phi_{0n}(c)| < \infty \text{ on } n'' \quad \text{for all } 0 < c < c_0 < 1 \\ \text{and hence for all } 0 < c < \infty, \tag{3.85}$$

which when coupled with $|\Phi_{0n}(c_0)| \to \infty$ on n'' gives

$$\frac{\sigma_n^*}{A_n} \to 0 \quad \text{on } n''; \tag{3.86}$$

this is just a statement that (3.25) holds on n'' with $a = 0$. Applying Helly selection we get that on a further n''', both (3.24) and (3.25) hold for $i = 0$. A repetition of the above argument provides a further n'''' on which $(\sigma_n^*/A_n)\Phi_{in} \to_D$ (some Φ_i) for both $i = 0, 1$. Thus Proposition 3.1 gives $\sqrt{n}(T_n - \mu_n)/A_n \to_d V_0 + V_1$ on a further n''''', with Φ_0 and Φ_1 satisfying (3.26) and with $\tau = 0$. Hence by the convergence of types of theorem, $\sqrt{n}(\mu_n - B_n)/A_n \to$ (some d') on n''''' and $W =_d V_0 + V_1 + d'$. Now this W would be degenerate if $\Phi_0 = \Phi_1 \equiv$ (Constant). Since W is not degenerate, at least one of Φ_0 and Φ_1 is not constant. □

PROOF OF THEOREM 2.1. Suppose (2.2) holds. Then every n' has a further n'' on which $\Phi_{in} \to_D \Phi_i$ for $i = 0, 1$. Thus (3.24) and (3.25) hold on n'' with $A_n = \sigma_n^*$ and $a = 1$. Thus on a further n''' on which (3.30) holds, we have (3.32) with $A_n = \sigma_n^*$. That is, $S_n^* \to_d V_{0k} + \tau Z + V_{1m}$ on n''' with $0 \le \tau \le 1$, necessarily. Thus (2.3) and (2.4) hold.

Suppose (2.3) holds. Assume (2.2) fails; that is, $\lim_{n'} \Phi_{in'}(c_0) = -\infty$ for some $c_0 > 0$, with $i = 0$ or 1 and some n'. According to (2.3), we have (3.36) on a further n'', with $A_n = \sigma_n^*$, and hence (3.24) and (3.25) with $A_n = \sigma_n^*$ hold on n'''. Thus $c = 1$ in (3.25). Obviously now $\lim_{n'} \Phi_{in'}(c_0) = -\infty$ for $i = 0$ or 1 gives us a contradiction of (3.24) at c_0 on n''' and hence (2.2) must hold. □

PROOF OF THEOREM 1.1, PART (ii). Suppose (1.24) holds. Set $t = 1/(2n)$ and $\lambda = 2c$ in the equivalent (1.26) to obtain (1.28). Now set $A_n = \sigma_n^*$ in Proposition 3.1 and note that (3.24) and (3.25) hold on n with $\Phi_1 = \Phi_2 = 0$ and $a = 1$. Now every subsequence n' has a further subsequence n'' on which (3.30) holds, with $\tau = 1$ by (3.35). Thus (3.30) holds on the original n with $\tau = 1$. Thus $S_n^* \to_d Z$ by (3.32). Note also that by (3.28) and (3.29) on n,

$$
\begin{aligned}
S_n^* &= \theta_n(r_n, n - r_n) + o_p(1) \\
&= -\int_{r_n/n}^{1-r_n/n} \mathbb{B}_n \, dK/\sigma_n^* + o_p(1) \qquad [\text{by } (3.31)] \\
&= -\int_{r_n/n}^{1-r_n/n} \mathbb{U}_n \, dK/\sigma_n^* + o_p(1) \qquad [\text{by } (3.40) \text{ with } r_n \text{ for } r] \\
&= \left(1/\sqrt{n}\,\sigma_n^*\right) \sum_{i=1}^{n} Y_{[r_n/n,\,1-r_n/n)}(\xi_i) + o_p(1) \qquad [\text{by } (1.10)].
\end{aligned} \tag{3.87}
$$

This is also the limiting form of $\bar{S}_n^*$.

Suppose $\sqrt{n}(T_n - B_n)/A_n \to_d Z$ as in (1.23); that is, (3.36) holds. Thus for every n' we have (3.25) and (3.37) on some further n'', where $\Phi_0 = \Phi_1 = 0$ by Proposition 2.1. That is, every n' has a further n'' on which $\Phi_{in''} \to_D 0$ for $i = 0, 1$. Thus $\Phi_{in} \to_D 0$ for $i = 0, 1$. This is (1.28), which implies (1.24). □

PROOF OF THEOREM 1.1, PART (i). Suppose $0 < \sigma^2(0) < \infty$. Then $t[K^2(t) + K_+^2(1-t)] \to 0$ since $\int_0^1 K^2(t)\,dt < \infty$ with $K \nearrow$ and $K(\frac{1}{2}) = 0$. Thus with $\Phi_{0n}(c) \equiv K(c/n)/\sqrt{n} \to \Phi_0(c) \equiv 0$ for all $c > 0$ and $\Phi_{1n}(c) \equiv K_+(1 - c/n)/\sqrt{n} \to \Phi_1(c) \equiv 0$ for all $c > 0$. That is, (3.24) and (3.25) hold on n with $A_n = 1$, $\Phi_0 = \Phi_1 = 0$ and $a = \sigma(0)$. Thus every n' has a further n'' on which (3.32) holds, with $\Phi_1 = \Phi_0 = 0$, with $\tau = a = \sigma(0)$ by (3.35), and hence with limiting random variable $\sigma(0)Z$. Thus (3.32) holds on n with limiting random variable $\sigma(0)Z$.

Suppose $\sqrt{n}(S_n(k, m] - B_n) \to_d \tau Z$ with $0 < \tau < \infty$. That is, (3.36) holds on n with $A_n = 1$ and $W = \tau Z$. Thus every n' has a further n'' on which (3.37) and (3.25) hold, with $a = \sigma(0)$ the obvious limit of σ_n^* and with $\Phi_0 = \Phi_1 = 0$ by Remark 2.1. Thus (3.35) gives $\tau = \sigma(0)$.

In such cases we use (3.87) to write

$$\sqrt{n}\,(T_n - \mu_n)/\sigma(0) = -\int_{r_n/n}^{1-r_n/n} \mathbb{U}_n \, dK/\sigma(0) + o_p(1)$$

$$= -\int_{k/n}^{1-m/n} \mathbb{U}_n \, dK/\sigma(0) + o_p(1) \tag{3.88}$$

$$= -\int_0^1 \mathbb{U}_n \, dK/\sigma(0) + o_p(1)$$

$$= \sqrt{n}\left[(1/n)\sum_1^n Y(\xi_i) - EY\right] + o_p(1). \tag{3.89}$$

Now (3.88) holds since each of the two bits added has a variance that goes to zero because $\sigma(0) < \infty$. □

REMARK 3.1. We now simplify our expression for V_k. We see, most easily from (3.59), that some algebra yields

$$V_k = \int_{S_k}^{\infty} h_\rho(\mathbb{N}(x), x)\, d\Phi(x) + \int_k^{S_k} h_\rho(k, x)\, d\Phi(x) \equiv \Delta V_k + V_{k+1}. \tag{3.90}$$

Thus we obtain the formal series (it is rigorous with a finite upper limit)

$$V_k = \sum_{i=k}^{\infty} \Delta V_i, \tag{3.91}$$

so that including one more order statistic adds one more term to the series. Unfortunately, in this formula $k = 1$ corresponds to starting the sum in the T_n of (1.1) at $k + 1 = 2$. Moreover our proof only establishes this for $k + 1 \geq 2$. That is, we had to trim at least one order statistic from each tail in order to guarantee that the extreme terms were finite.

REMARK 3.2. Had we considered instead

$$T_n^{\#} \equiv \frac{1}{n} \sum_{i=k}^{n+1-k'} c_{ni} g(\xi_{n:i}), \qquad \mu_n^{\#} \equiv \int_{a_n}^{1-a'_n} J(t) g(t)\, dt \tag{3.92}$$

and

$$\sigma_n^{\#} \equiv \sigma[a_n, 1 - a'_n), \quad \text{where } a_n = k/(n+2) \text{ and } a'_n = k'/(n+2) \text{ for } k, k' \geq 1, \tag{3.93}$$

then the identity (3.23) would have been

$$S_n^{\#}(l, r] = -\int_{l/(n+2)}^{r/(n+2)} \sqrt{n} \int_t^{\hat{\mathbb{G}}_n(t)} J(s)\, ds\, dg(t)/A_n$$

$$- \int_{\xi_{n:l}}^{l/(n+2)} \sqrt{n} \int_{l/n}^{\hat{\mathbb{G}}_n(t)} J(s)\, ds\, dg(t)/A_n \tag{3.94}$$

$$+ \int_{\xi_{n:r}}^{r/n} \sqrt{n} \int_{r/n}^{\hat{\mathbb{G}}_n(t)} J(s)\, ds\, dg(t)/A_n,$$

where

(3.95) $\hat{\mathbb{G}}_n(t) \equiv (i+1)/(n+2)$ for $\xi_{n:i} \le t < \xi_{n:i+1}$ and $0 \le i \le n+1$.

This would have led to (leaving off the 0 − subscript)

$$\alpha_n(k) + \theta_n[k,r] \to V_k^{\#} \equiv \int_{S_k}^{r} \left[h_\rho(1+\mathbb{N}(t),t) - h_\rho(k,t)\right] d\Phi(t) + \int_k^r h_\rho(k,t)\, d\Phi(t) \tag{3.96}$$

along appropriate subsequences, as is easily seen from (3.54) and (3.57). Observe that

$$\begin{aligned} V_k^{\#} &= \int_{S_k}^{r} \int_{1+\mathbb{N}}^{k} u^\rho \, du\, t^{-\rho}\, d\Phi + \int_k^r \int_k^t u^\rho\, du\, t^{-\rho}\, d\Phi \\ &= \left(\int_k^{k+1} u^\rho\, du\right)\left(\int_k^{S_k} t^{-\rho}\, d\Phi(t)\right) \\ &\quad - \left(\int_k^{k+1}\int_t^{k+1} u^\rho\, du\, t^{-\rho}\, d\Phi(t)\right) + V_{k+1}^{\#} \\ &\equiv \Delta V_k^{\#} + V_{k+1}^{\#}. \end{aligned} \tag{3.97}$$

Again,

$$V_k = \sum_{i=k}^{\infty} \Delta V_k^{\#}, \quad \text{valid for } k \ge 1, \tag{3.98}$$

so that including one more order statistic adds one more term to the series. Happily, in this formula $k = 1$ corresponds to starting the sum in the T_n^* of (3.92) at $k = 1$. Furthermore, the corresponding proof for T_n^* establishes this for $k \ge 1$. Thus, no trimming is needed in this version. [We acknowledge that the uniqueness proof of Proposition 2.1 needs to be modified for this T_n^* case. However, the above format makes clear that it carries over. Moreover, the sufficiency for normality in the new case of $k \ge 1$ under the condition of this present paper is worked out in detail in Shorack (1972). We choose the formulation of Theorem 3.1 in order to readily apply the needed uniqueness result from Mason and Shorack (1990a).]

REMARK 3.3. We note that if K corresponds to a stable random variable in $D(\alpha)$ for $0 < \alpha < 2$, then in the representation

$$\begin{aligned} K(t) &= -[\delta_0 + o(1)]t^{-1/\alpha}L(t) \quad \text{and} \\ K(1-t) &= [\delta_1 + o(1)]t^{-1/\alpha}L(t) \quad \text{as } t \to 0, \end{aligned} \tag{3.99}$$

we have

$$\sigma(t) = \sqrt{\frac{2}{2-\alpha}}\, t^{(1/2)-1/\alpha} L(t)\left[\delta_0^2 + \delta_1^2 + o(1)\right]^{1/2}, \tag{3.100}$$

so that for $c > 0$, we have

$$\frac{\sigma_n^*}{\sigma_n}\Phi_{on}(c) \to -\sqrt{2 - \frac{\alpha}{2}}\,\frac{\delta_0}{\sqrt{\delta_0^2 + \delta_1^2}}c^{-1/\alpha} \quad \text{and}$$

(3.101)

$$\frac{\sigma_n^*}{\sigma_n}\Phi_{1n}(c) \to -\sqrt{\frac{2 - \alpha}{2}}\,\frac{\delta_1}{\sqrt{\delta_0^2 + \delta_1^2}}c^{-1/\alpha}.$$

Acknowledgment. We would like to thank the referee for a *very* thorough reading of the manuscript. His perceptive comments guided us to a more careful treatment of some subtleties.

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