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ON THE SPECTRAL SLLN AND POINTWISE ERGODIC THEOREM IN L^{α}[1]

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We obtain criteria for the SLLN to hold for processes which are Fourier transforms of random measures. With this spectral approach, we also give criteria for the pointwise ergodic theorem to hold, for some classes of operators between L^{α}-spaces, $1 \le \alpha < +\infty$. These results apply in particular to contractions on L^2. Some random fields extensions are also studied.

1. Introduction. The criterion, obtained by Gaposhkin (1977a), for a (weakly) stationary process to satisfy the strong law of large numbers (SLLN) has had various extensions, in particular to second-order nonstationary harmonizable processes [Gaposhkin (1977b), Rousseau-Egelé (1979) and Dehay (1987)]. Outside of the L^2-framework, it has also been studied for Fourier transforms of independently scattered symmetric α-stable (SαS) measures in Cambanis, Hardin and Weron (1987). It is shown here that with a spectral approach, neither the L^2-requirement nor any distributional assumption are indispensable in establishing the SLLN. Only the harmonic representation with respect to a bounded (in a sense to be made precise) random measure is crucial. This is illustrated in the present work, where we obtain criteria for the SLLN to hold for processes which are Fourier transforms of random measures (with or without αth-moment conditions, $0 < \alpha \le 2$).

It is well known that stationary processes and unitary groups of operators are intimately related, and so are the corresponding strong law and pointwise ergodic theorem. This type of duality between operators and processes carries over to our framework, although, in general, the operators are not shifts. It is thus also the purpose of our work to obtain the pointwise ergodic theorem for some new classes of operators between L^{α}-spaces, $1 \le \alpha < +\infty$.

We now give a brief description of the contents of this paper. In the next section we set the stage. We introduce the processes under study and also illustrate the scope of our approach with various examples. Section 3 is the core of the paper, and ergodic properties of processes are developed. These

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results recover some classical strong laws and present new ones. In Section 4 we adapt our framework to operators to give a criterion for the pointwise ergodic theorem to hold for some new classes of operators between L^α-spaces. This also recovers some classical results. In the last section we discuss some random fields generalizations.

2. Preliminaries. Let $(\Omega, \mathscr{B}, \mathscr{P})$ be a probability space. For $0 \le \alpha < +\infty$, let $L^\alpha(\Omega, \mathscr{B}, \mathscr{P})$ [$L^\alpha(\mathscr{P})$ for short] be the corresponding space of complex-valued random variables equipped, for $0 < \alpha < +\infty$, with the (quasi-) norm $(\mathscr{E}|\cdot|^\alpha)^{1/\alpha} = \|\cdot\|_\alpha$, while on $L^0(\mathscr{P})$ the topology is the one induced by convergence in probability. Let $\mathscr{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$ and let $\mathscr{B}_0(\mathbb{R})$ be the δ-ring of Borel bounded sets. Let μ denote a positive σ-finite measure on $\mathbb{R}$ and let as above $L^\beta(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu)$ [$L^\beta(\mu)$ for short], $0 \le \beta < +\infty$, be the corresponding spaces of β-integrable functions. Finally, let $C_0(\mathbb{R})$ be the continuous functions vanishing at ∞ with the sup norm $\|\ \|_\infty$ and let, throughout, K denote a generic absolute constant whose value might change from one expression to another.

We now introduce some terminology and results which either generalize concepts in Houdré (1989, 1990a, b) or can be obtained from slight modifications of the results there. First, for $0 \le \alpha \le 2$ and $2 \le \beta < +\infty$ (resp. for $\beta = +\infty$), let Z be a continuous linear operator from $L^\beta(\mu)$ [resp. $C_0(\mathbb{R})$] to $L^\alpha(\mathscr{P})$. For such Z, using Lemma 2.1 below, we define $\int_{\mathbb{R}} f\,dZ$ via $\int_{\mathbb{R}} f\,dZ = \Lambda P \int_{\mathbb{R}} f\,dZ_0$. [This last integral makes sense for any $f \in L^\beta(\mu)$, when $\beta < +\infty$, and f Borel bounded when $\beta = +\infty$. More generally, it exists for any $f \in L^2(\nu)$, where ν is as in Lemma 2.1.] As defined, $\int_{\mathbb{R}} f\,dZ$ enjoys familiar properties, in particular linearity, with furthermore $\Lambda P \int_{\mathbb{R}} f\,dZ_0 = \Lambda \int_{\mathbb{R}} f\,dPZ_0 = \int_{\mathbb{R}} f\,d\Lambda PZ_0$. For $\beta < +\infty$, a process X: $\mathbb{R} \to L^\alpha(\mathscr{P})$ is said to be *(α, β)-bounded (with respect to μ)*, if

$$X_t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi}\,dZ(\zeta) \quad \text{in } L^\alpha(\mathscr{P}),$$

uniformly on compact subsets of $\mathbb{R}$. For $\beta = +\infty$, $L^\beta(\mu)$ is replaced by $C_0(\mathbb{R})$ and X is (α, ∞)-bounded whenever $X_t = \int_{\mathbb{R}} e^{it\xi}\,dZ(\xi)$. The finitely additive random measure Z: $\mathscr{B}_0(\mathbb{R}) \to L^0(\mathscr{P})$ is then said to have bounded (α, β)-μ-variation, $0 \le \alpha \le 2$, $2 \le \beta \le +\infty$. Moreover, for $\beta = +\infty$, Z is σ-additive on $\mathscr{B}(\mathbb{R})$ [this can be checked by using, e.g., Lemma 2.1 below as well as the results of Houdré (1990b)]. Finally, it is clear that a (α, β)-bounded process is L^α-continuous (continuous in probability when $\alpha = 0$) and that for $\beta = +\infty$, we also have L^α-boundedness. Furthermore, as in Houdré (1990b), but with the additional requirement $\int_{\{|u|>1\}} |u|^{-\beta}\,d\mu < +\infty$, equivalent characterizations of L^α-continuous (α, β)-bounded processes can be given when $\alpha \ge 1$.

To illustrate the scope and the applicability of our results, we now present some typical and some less typical examples (we do consider the case $\alpha > 2$ for future considerations on operators, although for probabilistic purposes the case $\alpha \le 2$ is the most interesting and furthermore contains $\alpha > 2$.) For

$\alpha = 2$, (α, ∞)-bounded processes are also known as *(weakly) harmonizable* or *V-bounded* and when $\mathscr{E}Z(\cdot)Z(\cdot)$: $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}) \to \mathbb{C}$ extends to a measure on $\mathscr{B}(\mathbb{R}^2)$ they are *(Loève or strongly) harmonizable*. For Z *orthogonally scattered*, that is, $\mathscr{E}Z(A)\overline{Z(B)} = 0$ whenever $A \cap B = \varnothing$, $A, B \in \mathscr{B}(\mathbb{R})$, $(2, \infty)$-boundedness is just weak stationarity. For $0 < \alpha < 2$, a typical example of random measure Z is an independently scattered (isotropic) SαS with finite control measure, in which case Z induces a continuous linear operator from $C_0(\mathbb{R})$ to $L^p(\mathscr{P})$, $p < \alpha$. We note that in the discrete-time case, μ is finite [see Houdré (1990a) and so (α, β)-boundedness, $\beta < +\infty$, is contained in (α, ∞)-boundedness. A discrete-time orthogonal process $X = \{X_n\}$ with $\mathscr{E}|X_n|^2 \le K$ is $(2, 2)$-bounded with respect to Lebesgue measure [see Houdré (1990a). By taking independent zero-mean random variables which are L^α-bounded, we can get, via Rosenthal's inequality, $(\alpha, 2)$-boundedness, $2 \le \alpha < +\infty$ (w.r.t. Lebesgue measure). For examples of random variables exhibiting more dependence, let $\alpha \ge 2$ and let X: $\mathbb{Z} \to L^\alpha(\mathscr{P})$ be an L^α-bounded martingale difference process. Then, by Burkholder's and Minkowski's inequality, we have

$$\begin{aligned} \left\| \sum_{i=1}^{N} p_i X_{n_i} \right\|_\alpha^\alpha &\le K \left\| \sum_{i=1}^{N} |p_i|^2 |X_{n_i}^2| \right\|_{\alpha/2}^{\alpha/2} \\ &\le K \left\{ \sum_{i=1}^{N} |p_i|^2 \| X_{n_i}^2 \|_{\alpha/2} \right\}^{\alpha/2} \\ &\le K \sup \| X_{n_i} \|_\alpha^\alpha \left\{ \sum_{i=1}^{N} |p_i|^2 \right\}^{\alpha/2}. \end{aligned}$$

Again, X is $(\alpha, 2)$-bounded, for $\alpha \ge 2$; hence, $X_n = \int_{-\pi}^{\pi} e^{in\xi}\, dZ(\xi)$, $n \in \mathbb{Z}$, where again Z is "dominated" by Lebesgue measure. To further illustrate the scope of our framework, recall [see Cambanis, Hardin and Weron (1987)] that a continuous in probability SαS process $(1 < \alpha < 2)$ can be represented as $X_t = \int_0^1 f(t, \tau)\, dM(\tau)$, $t \in \mathbb{R}$, where M is α-stable Lévy motion. Hence, when $\{\int_{\mathbb{R}} (\int_0^1 |f(t, \tau)|^\alpha\, d\tau)^{2/\alpha}\, dt\}^{1/2} < +\infty$, we have (by Minkowski, Cauchy–Schwarz and Plancherel) and for any $1 \le p < \alpha$,

$$\begin{aligned} \left\{ \mathscr{E} \left| \int_{\mathbb{R}} g(t) X_t\, dt \right|^p \right\}^{1/p} &= K \left\{ \int_0^1 \left| \int_{\mathbb{R}} g(t) f(t, \tau)\, dt \right|^\alpha d\tau \right\}^{1/\alpha} \\ &\le K \int_{\mathbb{R}} |g(t)| \left\{ \int_0^1 | f(t, \tau) |^\alpha\, d\tau \right\}^{1/\alpha} dt \\ &\le K \left\{ \int_{\mathbb{R}} |g(t)|^2 \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left(\int_0^1 | f(t, \tau) |^\alpha\, d\tau \right)^{2/\alpha} dt \right\}^{1/2} \\ &\le K \left\{ \int_{\mathbb{R}} |\hat{g}(t)|^2 \right\}^{1/2}, \end{aligned}$$

and X is $(p, 2)$-bounded with respect to Lebesgue measure. It is clear that whenever f is a function of the difference of t and τ, the above condition is not verified. In fact, for such f no boundedness condition can be satisfied [see Cambanis and Houdré (1990)]. However, since the corresponding process is strictly stationary, the ergodic averages do converge. In ways similar to the ones developed above, conditions for (p, β)-boundedness can also be obtained for SαS random measure Z. Again, recall that a generic representation for such Z is given by $Z(A) = \int_{\mathbb{R}} N(A, \tau)\, dM(\tau)$, $A \in \mathscr{B}_0(\mathbb{R})$, $N(A, \tau) \in L^\alpha(\mathbb{R})$. In particular, $Z\colon \mathscr{B}(\mathbb{R}) \to L^p(\mathscr{P})$, $1 \le p < \alpha$, is σ-additive if and only if $N(\cdot, \tau)\colon \mathscr{B}(\mathbb{R}) \to L^\alpha(\mathbb{R})$ is σ-additive. Furthermore, whenever $N(dt, \tau) = N(t, \tau)\, dt$ with $\int_{\mathbb{R}}(\int_{\mathbb{R}} |N(t, \tau)|^\alpha\, d\tau)^{\gamma/\alpha}\, dt < +\infty$, Z is (p, β)-bounded, $1 \le p < \alpha$, $1/\beta + 1/\gamma = 1$ (the proof of this claim is essentially as above). In all these examples, X and Z can be recovered from one another by the usual inversion formulas.

Since a process X which is (α, β)-bounded is L^α-continuous, it has a (t, ω)-measurable modification with almost surely locally integrable sample paths and the averages $\sigma_T X(\omega) = (1/2T)\int_{-T}^{T} X(t, \omega)\, dt$, $T > 0$, $\omega \in \Omega$, are well defined. We then say that X satisfies the SLLN whenever $\lim_{T \to \infty} \sigma_T X(\omega) = 0$ [$\lim_{N \to \infty}(1/(2N + 1))\sum_{n=-N}^{N} X_n(\omega) = 0$ in the discrete-time case], for almost all ω [we will usually omit the reference to ω, e.g., write $\sigma_T X$ for $\sigma_T X(\omega)$]. Of course, considering one-sided averages gives results essentially identical to the ones presented below.

We now state a decomposition lemma which for $\alpha = 2$ and $\beta = +\infty$ is just one of the various forms of Grothendieck's inequality. More details can be found for $1 \le \alpha \le 2$ in Houdré (1990b) [the extension to $0 \le \alpha < 1$ can be obtained by combining the factorization results and techniques of Maurey (1974) with the results and techniques in Houdré (1990b)].

LEMMA 2.1. *Let the process X be (α, β)-bounded, $0 \le \alpha \le 2 \le \beta \le +\infty$, with associated random measure Z_X. Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\mathscr{P}})$ with $L^2(\mathscr{P}) \subset L^2(\tilde{\mathscr{P}})$, a $(2, \beta)$-bounded process Y with orthogonally scattered random measure Z_Y defined on $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\mathscr{P}})$ and a random variable $\Lambda \in L^{2\alpha/(2-\alpha)}(\mathscr{P})$ such that $X_t = \Lambda P Y_t$, $t \in \mathbb{R}$, where P is the orthogonal projection from $L^2(\tilde{\mathscr{P}})$ to $L^2(\mathscr{P})$.*

In Lemma 2.1, Λ can be chosen constant (a.s. $\mathscr{P}$) when $\alpha = 2$. Furthermore, since Z_Y is orthogonally scattered and also has finite $(2, \beta)$-variation, there exists [see Houdré (1990b)] a positive measure ν [finite when $\beta = +\infty$ while given by $d\nu = g\, d\mu$, $g \in L^{\beta/(\beta-2)}(\mu)$ when $\beta < +\infty$] such that

$$\mathscr{E}\left|\int_{\mathbb{R}} f\, dZ_Y\right|^2 = \int_{\mathbb{R}} |f|^2\, d\nu \tag{2.1}$$

for all $f \in L^2(\nu)$. The above ν is a *dominating measure* ($\mathscr{E}|\int_{\mathbb{R}} f\, dP Z_Y|^2 \le \|P\|^2 \int_{\mathbb{R}} |f|^2\, d\nu$).

Whenever X is (α, β)-bounded, $0 < \beta < +\infty$, Fubini's theorem and the defining boundedness property, as well as the condition $\int_{\{|u| > 1\}} |u|^{-\beta}\, d\mu < +\infty$, give $\lim_{T\to\infty} \sigma_T X = Z_X(0)$, in $L^\alpha(\mathscr{P})$ (for $2 \le \beta$, Lemma 2.1 and $\int_{\{|u|>1\}} |u|^{-\beta}\, d\mu < +\infty$ will also do it, while for $\beta = +\infty$, the integral condition is not needed). Hence, under these conditions the weak law of large numbers is verified if and only if $Z_X(\{0\}) = 0$. However, it is well known that, even in the stationary case, $Z_X(\{0\}) = 0$ a.s. is not a sufficient condition for the SLLN to hold. Similarly, it is not because the dominating ν in (2.1), is, say, the spectral measure of a stationary process satisfying the SLLN, that the dominated X satisfies the SLLN. After all, (2.1) is just a norm estimate. However, this norm estimate is a strong ingredient in obtaining a criterion for the almost sure convergence of the ergodic averages.

To conclude this section, we state another domination lemma, the proof of which is in Rousseau-Egelé (1979) and which goes back to Gàl and Koksma (1950). First, we need some more notation: Given any integer $p \ge 0$, an integer $n \ge 2$ such that $2^p < n \le 2^{p+1}$ has a unique binary decomposition $n = 2^p + 1 + \sum_{j=1}^{p} \varepsilon_j 2^{p-j}$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \in \{0,1\}^p$. Hence, to any such n, that is, to any sequence $\varepsilon \in \{0,1\}^p$, we can associate the (finite) sequence

$$a_k(\varepsilon, p) = \begin{cases} 2^p + 1 + \sum_{j=1}^{k} \varepsilon_j 2^{p-j}, & k = 1, 2, \dots, p, \\ 2^p, & k = 0. \end{cases}$$

With this notation and if a_k is short for $a_k(\varepsilon, p)$, we have the following lemma.

LEMMA 2.2. *Let $\{z_j\}$ be a sequence of complex numbers and let $\{t_j\}$ be a sequence of positive numbers. Then, for any $p \ge 1$,*

$$\max_{2^p < n \le 2^{p+1}} \left| \sum_{j=2^p+1}^{n} z_j \right|^2 \le \left(\sum_{k=1}^{p} t_k^{-1} \right) \left(\sum_{k=1}^{p} t_k \sum_{(\varepsilon_1, \dots, \varepsilon_k) \in \{0,1\}^k} \left| \sum_{j=a_{k-1}+1}^{a_k} z_j \right|^2 \right).$$

3. The spectral SLLN. With the results of the previous section, our approach in proving the SLLN follows classical paths, the first of which is another lemma showing that we can reduce the problem to the dyadic subsequences. We prove our results only for the most interesting situation of continuous-time processes; the discrete results are obtainable in an identical fashion. *Finally, throughout this section, and unless otherwise stated, (α, β)-bounded is short for (α, β)-bounded with respect to the σ-finite measure μ, $0 \le \alpha \le 2 \le \beta \le +\infty$, with, furthermore, $\int_{\{|u|>1\}} |u|^{-\beta}\, d\mu < +\infty$, whenever $\beta < +\infty$.*

LEMMA 3.1. *If X is (α, β)-bounded, then*

$$\lim_{p\to+\infty} \max_{2^p < n \le 2^{p+1}} |\sigma_n X - \sigma_{2^p} X| = 0 \quad (a.s.\ \mathscr{P}).$$

PROOF. Since, by Lemma 2.1, $X = \Lambda PY$, and since $\sigma X = \Lambda \sigma PY$, it is enough to show (as for stationary processes) that

$$\lim_{p \to +\infty} \max_{2^p < n \le 2^{p+1}} |\sigma_n PY - \sigma_{2^p} PY| = 0 \quad (\text{a.s. } \mathscr{P}).$$

As in the harmonizable case [strong or weak; see Rousseau-Egelé (1979) or Dehay (1987)], and since

$$\sigma_n PY - \sigma_{2^p} PY = \sum_{j=2^p+1}^{n} (\sigma_j PY - \sigma_{j-1} PY),$$

applying Lemma 2.2 with its notation, we get

$$\begin{aligned} &\mathscr{E} \max_{2^p < n \le 2^{p+1}} |\sigma_n PY - \sigma_{2^p} PY|^2 \\ (3.1) \qquad &\le \left(\sum_{k=1}^{p} t_k^{-1} \right) \left(\sum_{k=1}^{p} t_k 2^k \max_{(\varepsilon_1, \dots, \varepsilon_k) \in \{0,1\}^k} \mathscr{E} |\sigma_{a_k} PY - \sigma_{a_{k-1}} PY|^2 \right). \end{aligned}$$

To prove the result, it is enough to show that $\mathscr{E} \max_{2^p < n \le 2^{p+1}} |\sigma_n PY - \sigma_{2^p} PY|^2$ is the general term of a convergent series. Since X is (α, β)-bounded, for any $T > 0$, we have, using $\int_{\{|u|>1\}} |u|^{-\beta} \, d\mu < +\infty$,

$$\sigma_T PY = P \int_{\mathbb{R}} \left(\frac{\sin T\xi}{T\xi} \right) dZ_Y(\xi).$$

Hence, by (3.1) and (2.1), it is in turn enough to show that

$$(3.2) \quad \sum_{p=1}^{\infty} \left(\sum_{k=1}^{p} t_k^{-1} \right) \left(\sum_{k=1}^{p} t_k 2^k \max_{\varepsilon \in \{0,1\}^k} \left\{ \int_{\mathbb{R}} \left| \frac{\sin a_k \xi}{a_k \xi} - \frac{\sin a_{k-1} \xi}{a_{k-1} \xi} \right|^2 d\nu(\xi) \right\} \right) < +\infty.$$

To do so, and as for stationary or harmonizable processes, we divide $\mathbb{R}$ into four pieces, $\{|\xi| < 2^{-p-1}\}, \{2^{-p-1} < |\xi| \le 2^{-p+k}\}, \{2^{-p+k} < |\xi| \le 1\}, \{|\xi| > 1\}$. We then use the triangle inequality and proceed to estimate each one of the resulting four sums. The estimates over the four different regions are similar and so we just give the details for, say, the second and fourth regions, since in this latter case the nonfiniteness of ν makes the estimate different.

For the second region, since $2^{-p-1} < |\xi| \le 2^{-p+k}$ we have

$$\begin{aligned} \left| \frac{\sin a_k \xi}{a_k \xi} - \frac{\sin a_{k-1} \xi}{a_{k-1} \xi} \right|^2 &\le K |a_k - a_{k-1}|^2 / |a_{k-1}|^2 \\ &\le K 2^{(p-k)2} / 2^{2p} = K 2^{-2k}. \end{aligned}$$

Using this estimate and taking $t_k = t^k$, $1 < t < 2$, (3.2) with $\mathbb{R}$ replaced by

$2^{-p-1} < |\xi| \le 2^{-p+k}$ is dominated by

$$K \sum_{p=1}^{\infty} \sum_{k=1}^{p} t^k 2^k 2^{-2k} \left\{ \int_{\{2^{-p-1} < |\xi| \le 2^{-p+k}\}} d\nu(\xi) \right\}$$
$$= K \sum_{p=1}^{\infty} \sum_{k=1}^{p} t^k 2^{-k} \sum_{j=p-k}^{p} \nu\{2^{-j-1} < |\xi| \le 2^{-j}\}$$
$$= K \sum_{j=0}^{\infty} \nu\{2^{-j-1} < |\xi| \le 2^{-j}\} \sum_{k=1}^{\infty} \sum_{p=\max(j,k)}^{j+k} t^k 2^{-k}$$
$$\le K\nu\{0 < |\xi| \le 1\} \sum_{k=1}^{\infty} (k+1) t^k 2^{-k} < +\infty,$$

since ν is finite or σ-finite.

For the first region, using

$$\left| \frac{\sin a_k \xi}{a_k \xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi} \right|^2 \le K|\xi|^2 |a_k - a_{k-1}|^2 \le K|\xi|^2 2^{2(p-k)}$$

and proceeding similarly, the corresponding series is finite. The third series can be estimated using

$$\left| \frac{\sin a_k \xi}{a_k \xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi} \right|^2 \le K|a_k - a_{k-1}|/|a_{k-1}|\, |\xi a_{k-1}| \le K 2^{(p-k)} 2^{-2p}/|\xi|.$$

To estimate the last sum, we use

$$\left| \frac{\sin a_k \xi}{a_k \xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi} \right|^2 \le K/|a_{k-1}\xi|^2 \le K 2^{-2p}/|\xi|^2,$$

and proceeding as above, (3.2) with $\mathbb{R}$ replaced by $\{|\xi| > 1\}$ is majorized by

$$K \int_{\{|u|>1\}} |u|^{-2}\, d\nu \sum_{p=1}^{\infty} \sum_{k=1}^{p} t^k 2^{k-2p}$$
$$\le K \left(\int_{\{|u|>1} |u|^{-\beta}\, d\mu \right)^{2/\beta} \sum_{p=1}^{\infty} t^k 2^{-k} < +\infty,$$

since $d\nu = g\, d\mu$, $g \in L^{\beta/(\beta-2)}(\mu)$ and using Hölder's inequality when $\beta < +\infty$, while ν is finite for $\beta = +\infty$. Thus, (3.2) is finite, and the result follows. □

We now state the main result of this section, a criterion for the a.s. convergence of the ergodic averages.

THEOREM 3.2. *Let X be (α, β)-bounded with random measure Z_X, and let either X be L^α-bounded, $\alpha > 1$, or let PY be L^2-bounded. The following* *conditions are equivalent*:

(i) *For a.a.* ω, $\lim_{T\to+\infty}\sigma_T X(\omega)$ *exists*.
(ii) *For a.a.* ω, $\lim_{p\to+\infty} Z_X\{|\xi|<2^{-p}\}(\omega)$ *exists*.

Under either condition, and for a.a. ω, $\lim_{T\to\infty}\sigma_T X = \lim_{p\to+\infty} Z_X\{|\xi|<2^{-p}\} = Z(\{0\})$.

PROOF. Again, as in the L^2-case (stationary or harmonizable),

$$\sigma_T X = (\sigma_T X - \sigma_n X) + (\sigma_n X - \sigma_{2^p} X) + (\sigma_{2^p} X - Z_X\{|\xi|<2^{-p}\}) + Z_X\{|\xi|<2^{-p}\}.$$

The middle parentheses are taken care of by the previous lemma. To show that the third term tends to 0 a.s. as $p\to\infty$, we show that

$$\sum_{p=1}^{\infty}\mathscr{E}|\sigma_{2^p}PY - PZ_Y\{|\xi|<2^{-p}\}|^2 < +\infty.$$

But the triangle inequality, as well as (2.1), gives

$$\sum_{p=1}^{\infty}\mathscr{E}|\sigma_{2^p}PY - Z_Y\{|\xi|<2^{-p}\}|^2$$

$$\le K\sum_{p=1}^{\infty}\left\{\int_{\{|\xi|<2^{-p}\}}\left|\frac{\sin 2^p\xi}{2^p\xi}-1\right|^2 d\nu(\xi) + \int_{\{|\xi|\ge 2^{-p}\}}\left|\frac{\sin 2^p\xi}{2^p\xi}\right|^2 d\nu(\xi)\right\}.$$

To prove the result, it is thus again enough to show that both series converge. We provide the details only for the first integral. Since

$$\left|\frac{\sin 2^p\xi}{2^p\xi}-1\right|^2 \le K2^{2p}|\xi|^2,$$

the first sum is dominated by

$$K\sum_{p=1}^{\infty}2^{2p}\left\{\sum_{k=p}^{\infty}\int_{\{2^{-k-1}\le|\xi|<2^{-k}\}}|\xi|^2\,d\nu(\xi)\right\}$$

$$\le K\sum_{p=1}^{\infty}2^{2p}\left(\sum_{k=p}^{\infty}2^{-2k}\nu\{2^{-k-1}\le|\xi|<2^{-k}\}\right)$$

$$\le K\sum_{k=1}^{\infty}2^{-2k}\nu\{2^{-k-1}\le|\xi|<2^{-k}\}\sum_{p=1}^{k}2^{2p}$$

$$\le K\nu\{0<|\xi|<2^{-1}\}\left(1+\sum_{k=1}^{\infty}2^{-2k}\right) < +\infty.$$

Using

$$\left|\frac{\sin 2^p\xi}{2^p\xi}\right|^2 \le K/2|2^p\xi|^2,$$

the second sum can be estimated in a similar way. Hence,

$$\lim_{p\to\infty}\left(\sigma_{2^p}X - Z_X\{|\xi| < 2^{-p}\}\right) = 0 \quad (\text{a.s. } \mathscr{P}).$$

For the first parentheses,

$$|\sigma_T X - \sigma_n X| \le |1/2T - 1/2n|\left|\int_{-n}^{n} X_t\,dt\right| + 1/2T\left|\int_n^T X_t\,dt\right| + 1/2T\left|\int_{-T}^{-n} X_t\,dt\right|.$$

Hence, whenever X is L^α-bounded, we get

$$\begin{aligned}\mathscr{E}\sup_{n<T\le n+1}|\sigma_T X - \sigma_n X|^\alpha &\le Kn^{-2\alpha}\left\{\int_{-n}^{n} dt\right\}^\alpha + Kn^{-\alpha}\left\{\int_n^{n+1} dt\right\}^\alpha + Kn^{-\alpha}\left\{\int_{-n-1}^{-n} dt\right\}^\alpha\\ &\le Kn^{-\alpha}, \qquad \alpha > 1.\end{aligned}$$

It thus follows that

$$\lim_{n\to\infty}\sup_{n<T\le n+1}|\sigma_T X - \sigma_n X| = 0 \quad \text{a.s. } (\mathscr{P}).$$

In a similar way, whenever PY is L^2-bounded,

$$\mathscr{E}\sup_{n<T\le n+1}|\sigma_T PY - \sigma_n PY|^2 \le Kn^{-2}$$

and

$$\lim_{n\to\infty}\sup_{n<T\le n+1}|\sigma_T PY - \sigma_n PY| = 0 \quad \text{a.s. } (\mathscr{P}).$$

Finally, (α, β)-boundedness via Lemma 2.1 gives $\lim_{p\to\infty} Z_X\{|\xi| < 2^{-p}\} = Z_X(\{0\})$ in $L^\alpha(\mathscr{P})$, and the result follows. □

REMARK 3.3. The extra L^α or L^2-bounded hypotheses are particular to the case $\beta < +\infty$, and in particular for discrete-time processes they are always satisfied. For $\beta = +\infty$, the exponentials are Z_Y-integrable; hence, $PY_t = P\int_{\mathbb{R}} e^{it\lambda}\,dZ_Y(\lambda)$ and $\mathscr{E}|PY_t|^2 \le K\nu(\mathbb{R}) < +\infty$. Theorem 3.2 can be restated as follows: Let $Z_k = Z_k\{2^{-k-1} \le |\xi| < 2^{-k}\}$; then by (α, β)-boundedness, the series $\sum_{k=p}^{\infty} Z_k$, $p \ge 1$, converges in $L^\alpha(\mathscr{P})$ and its sum is $Z_X\{|\xi| < 2^{-p}\} - Z_X(\{0\})$. Hence, X satisfies the SLLN if and only if for a.a. ω, $\lim_{p\to+\infty}\sum_{k=1}^{p} Z_k(\omega)$ exists, and $Z_X(\{0\}) = 0$.

COROLLARY 3.4. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be an L^2-bounded martingale difference process. Then $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} X_n = 0$ with probability* 1.

PROOF. From the estimates of Section 2, X is $(2, 2)$-bounded with respect to Lebesgue measure. Hence,

$$\sum_{n=1}^{\infty}\mathscr{E}\left|Z_X\{|\xi| < 2^{-n}\}\right|^2 \le K\sum_{n=1}^{\infty} 1/2^{(n-1)} < +\infty.$$

The result now follows because $Z_X(\{0\}) = 0$. □

COROLLARY 3.5. *Let X satisfy the hypotheses of Theorem 3.2 and let its random measure Z be independently scattered. Then X satisfy the SLLN if and only if $Z(\{0\}) = 0$.*

PROOF. The Z_k are independent and the convergence of the series in Remark 3.3 already holds in $L^\alpha(\mathscr{P})$. □

In our framework, and if log denotes the logarithm of base 2, one of the sufficient conditions given in Gaposhkin (1977a, b) or Dehay (1987) for the strong law to hold in the stationary or in the weakly harmonizable, case, becomes:

THEOREM 3.6. *Let X be (α,β)-bounded, $\alpha > 1$, with random measure Z such that $Z(\{0\}) = 0$ and let $\sup_{t\in\mathbb{R}} \mathscr{E}|X_t|^\alpha < +\infty$. If there exists a positive measure ν on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$ such that:*

(i) $$\mathscr{E}|Z(A)|^\alpha \le \nu(A\times A), \qquad A \in \mathscr{B}_0(\mathbb{R}),$$

(ii) $$\iint_{\{0<|\xi|,\,|\eta|<\delta\}} \left(\log\log\frac{1}{|\xi|}\right)^{\alpha/2}\left(\log\log\frac{1}{|\eta|}\right)^{\alpha/2} d\nu(\xi,\eta) < +\infty$$

for some $\delta > 0$,

then X satisfies the SLLN.

PROOF. To prove the assertion, it is enough to show that the sequence $Z\{|\xi| < 2^{-p}\}$ converges to $Z(\{0\}) = 0$ with probability 1, and to do so, we again show that $\mathscr{E}|Z\{0 < |\xi| < 2^{-p}\}|^\alpha$ is the general term of a convergent series. Our proof is only sketched. First, for $2^q < p \le 2^{q+1}$, we have

$$Z\{|\xi| < 2^{-p}\} = Z(\{0\}) + Z\{0 < |\xi| < 2^{-2^q}\} - Z\{2^{-p} \le |\xi| < 2^{-2^q}\}.$$

But, by (i) and (ii) above with q_0 any integer such that $2^{-2^{q_0}} < \delta$, we have, if $A_q = \{0 < |\xi| < 2^{-2^q}\}$,

$$\sum_{q=q_0}^{\infty} \mathscr{E}|Z(A_q)|^\alpha$$
$$\le \sum_{q=q_0}^{\infty} \nu(A_q\times A_q)$$
$$\le \sum_{q=q_0}^{\infty} q^{-\alpha}\iint_{A_q\times A_q}\left(\log\log\frac{1}{|\xi|}\right)^{\alpha/2}\left(\log\log\frac{1}{|\eta|}\right)^{\alpha/2} d\nu(\xi,\eta)$$
$$\le \iint_{\{0<|\xi|,\,|\eta|<\delta\}}\left(\log\log\frac{1}{|\xi|}\right)^{\alpha/2}\left(\log\log\frac{1}{|\eta|}\right)^{\alpha/2} d\nu(\xi,\eta)\sum_{q=q_0}^{\infty} q^{-\alpha} < +\infty.$$

Hence, $\lim_{q\to+\infty} Z\{0 < |\xi| < 2^{-2^q}\} = 0$ a.s. For $Z\{2^{-p} \le |\xi| < 2^{-2^q}\}$, let $B_k =$

$\{2^{-a_k} \le |\xi| < 2^{-a_{k-1}}\}$, where a_k is defined as before, and let $C_q = \{2^{-2^{q+1}} \le |\xi| < 2^{-2^q}\}$. Applying Lemma 2.2 with $t_k = 1$ [actually what is needed is a slight extension of the lemma to cover the case $1 < \alpha < 2$; this can be obtained by replacing the Cauchy–Schwarz inequality by Hölder's inequality in the proof given for $\alpha = 2$ in Rousseau-Egelé (1979)], as well as (i) and (ii), we get

$$\begin{aligned}
&\sum_{q=q_0}^{\infty} \mathscr{E}\left(\max_{2^q < p \le 2^{q+1}} \left| Z\{2^{-p} \le |\xi| < 2^{-2^q}\} \right|^{\alpha} \right) \\
&\quad \le \sum_{q=q_0}^{\infty} q^{\alpha-1} \left(\sum_{k=1}^{q} \sum_{(\varepsilon_1, \dots, \varepsilon_k) \in \{0,1\}^k} \mathscr{E}|Z(B_k)|^{\alpha} \right) \\
&\quad \le \sum_{q=q_0}^{\infty} q^{\alpha-1} \left(\sum_{k=1}^{q} \nu(C_k \times C_k) \right) \le \sum_{q=q_0}^{\infty} q^{\alpha} \nu(C_q \times C_q) \\
&\quad \le \iint_{\{0 < |\xi|, |\eta| < \delta\}} \left(\log\log \frac{1}{|\zeta|} \right)^{\alpha/2} \left(\log\log \frac{1}{|\eta|} \right)^{\alpha/2} d\nu(\xi, \eta) < +\infty.
\end{aligned}$$

Hence, with probability 1,

$$\lim_{q \to +\infty} \max_{2^q < p \le 2^{q+1}} Z\{2^{-p} \le |\xi| < 2^{-2^q}\} = 0. \qquad \square$$

REMARK 3.7. Starting with a discrete-time stationary process X such that $\sigma_{2^p} X$ diverges on an arbitrary set of positive measure [Menchov's counterexample ensures that X exists; see Alexits (1961), Chapter 2], the process $Y = \Lambda X$, $\Lambda \in L^{2\alpha/(2-\alpha)}(\mathscr{P})$, $\Lambda > 0$, is a discrete-time (α, ∞)-bounded process such that $\sigma_{2^p} Y$ diverges on the same arbitrary set of positive measure. In fact, such Y can be chosen with absolutely continuous dominating measure $d\nu(t) = g(t)\,dt$, $g \in L^1(]-\pi, \pi])$ by also adapting the arguments in Gaposhkin (1977a).

From the above proofs, it is clear that the integrability condition $\int_{\{|u|>1\}} |u|^{-\beta}\, d\mu < +\infty$ is not minimal. It is also clear that conditions (i) and (ii) can be replaced by $\mathscr{E}|Z(A)|^{\alpha} \le \nu(A)$, $A \in \mathscr{B}_0(\mathbb{R})$ and $\sum_{p=2}^{\infty} (\log p)^{\alpha} \nu\{2^{-p-1} \le |\xi| < 2^{-p}\} < +\infty$. When $\alpha = \beta = 2$, for example, for L^2-bounded martingale difference or orthogonal processes, our results are also not optimal, since the rate of convergence can be improved. By techniques similar to the ones in Theorem 3.4(ii) with an appropriate extension of the classical Rademacher–Menchov theory, it can be shown that with probability 1,

$$\lim_{N \to \infty} \frac{1}{\sqrt{(2N+1) \log^{3+\varepsilon}(2N+1)}} \sum_{n=-N}^{N} X_n(\omega) = 0, \qquad \varepsilon > 0.$$

For $\beta = +\infty$ and $\alpha > 1$, it can also be shown with the above techniques and as in Gaposhkin (1977a, b), Rousseau-Egelé (1979) and Dehay (1987) for $\alpha = 2$, that with probability 1, $\lim_{T \to \infty} (\log\log 2T)^{-\alpha/2} \sigma_T X = 0$, the log log speed being the best possible. The various necessary or sufficient conditions given in the works mentioned above also admit α-counterparts.

REMARK 3.8. By now, the reader might have wondered what happens in the cases $0 < \beta < 2$. Using, for example, the factorization results of Maurey (1974), some partial results can be obtained: Replacing 2 by $\alpha > 1$ in Lemma 2.1, whenever a dominating measure ν satisfies

$$\sum_{k=0}^{\infty} \nu\{2^{-k-1} < |\xi| \le 2^{-k}\}^{\alpha/\gamma} < +\infty, \qquad 1 < \alpha \le \gamma < \beta,$$

then Theorem 3.2 and its proof carry over (replacing $\mathscr{E}|\cdot|^2$ by $\mathscr{E}|\cdot|^\alpha$ in the proof).

Under appropriate distributional assumptions, Corollary 3.5 can be obtained by different methods. A case at hand is the class of stable (or infinitely divisible) harmonizable processes. To that effect, we now present such results using a beautiful argument due to Rosinski. It was shown to us at a time when we could neither prove nor disprove the result below by the methods presented above.

THEOREM 3.9. *Let X be a continuous in probability "harmonizable" $S\alpha S$ process with control measure m. Then X satisfies the SLLN if and only if $m(\{0\}) = 0$.*

PROOF. Since $X_t = \int_{\mathbb{R}} e^{itx}\, dZ(x)$, $t \in \mathbb{R}$, is harmonizable SαS, it has a LePage-type decomposition [see Rosinski (1990)], that is, there exists a process Y on $(\Omega', \mathscr{B}', \mathscr{P}')$ such that $X =_{\mathscr{L}} Y$, where $\mathscr{L}$ denotes equality in law; furthermore,

$$Y_t = \sum_{k=1}^{\infty} \varepsilon_k R(\tau_k, \xi_k) e^{it\xi_k} = \int_{\mathbb{R}} e^{itx}\, dZ_1(x), \qquad t \in \mathbb{R},$$

where the series converges a.s. $\mathscr{P}'$. The three series ξ, τ, ε are mutually independent, the ξ_k's form a sequence of i.i.d. random variables with law $m_1 = m/m(\mathbb{R})$, the τ_k's are arrival times in a Poisson process with rate 1, the ε_k's are Bernoulli random variables with $\mathscr{P}_\varepsilon\{\varepsilon_k = 1\} = \mathscr{P}_\varepsilon\{\varepsilon_k = -1\} = 1/2$ and finally $R\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is given by $R(u, s) = s/u^{1/\alpha}$. Now, if M_1 denotes Z_1 conditioned on the ξ_k's and τ_k's, then, as shown now, M_1 is a second-order independently scattered measure. First, $Z_1(\mathbb{R}) = \sum_{k=1}^{\infty} \varepsilon_k R(\tau_k, \xi_k)$, where this last sum converges a.s. $(\mathscr{P}')$; hence,

$$\mathscr{E}|M_1(\mathbb{R})|^2 = \sum_{k=1}^{\infty} |R(\tau_k, \xi_k)|^2 < +\infty.$$

Second,

$$\begin{aligned}\mathscr{E}\left(e^{i\sum t_n Z_1(B_n)} | (\tau_k, \xi_k)\right) &= \prod_k \cos R(\tau_k, \xi_k) \sum_n t_n \chi_{B_n}(\xi_k) \\ &= \prod_n \prod_k \cos R(\tau_k, \xi_k) t_n \chi_{B_n}(\xi_k),\end{aligned}$$

since the B_n's are disjoint (χ_{B_n} is the indicator function of the set B_n) and the ξ_k i.i.d. Hence, M_1 is a second-order independently scattered measure and by Gaposhkin's result, $\mathscr{P}_\varepsilon\{\sigma_T X_1 \to M_1(\{0\})\} = 1$, where $X_1(t) = \int_{\mathbb{R}} e^{itx}\, dM_1(x)$. Now, Fubini's theorem and independence give $\mathscr{P}'\{\sigma_T Y \to Z_1(\{0\})\} = \mathscr{P}_\tau \otimes \mathscr{P}_\xi \otimes \mathscr{P}_\varepsilon\{\sigma_T X_1 \to M_1(\{0\})\} = 1$ [note too that whenever $Z_1(\{0\}) = 0$, $M_1(\{0\}) = 0$]. Finally, $\mathscr{P}\{\sigma_T X \to Z(\{0\})\} = 1$, and the result follows. □

We end this section (see, however, the next section for related results on the existence of the ergodic Hilbert transform) with a result which, once more, has its stationary counterpart in Gaposhkin (1977b). This theorem related a.s. (C, r)-convergence with the corresponding a.s. $(C, 1)$-convergence, that is, the SLLN. It is stated only for the more familiar Cesàro sums; small adjustments will give the continuous case, while other classical summability methods can also be analyzed by the same approach. Recall that the Cesàro means of order $r > -1$ of a sequence $\{X_n\}$ are given by $\sigma_N^r X = (A_N^r)^{-1}\sum_{n=0}^N A_{N-n}^{r-1} X_n$, where $A_n^r = (r+1)(r+2)\cdots(r+n)/n!$.

THEOREM 3.10. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be (α,∞)-bounded, $0 \le \alpha \le 2$, and let $r > 1/2$. Then X satisfies the SLLN if and only if with probability* 1, $\lim_{N\to\infty} \sigma_N^r X = 0$.

PROOF. It is clear (via Lemma 2.1) that the $\sigma_N^r X$ converge to $Z(\{0\})$ in L^α. Thus, the theorem is equivalent to showing that $\{\sigma_N^r X\}$ converges to 0 if and only if $\lim_{p\to+\infty} Z(|\xi| < 2^{-p}) = 0$ a.s. This follows from the methods presented above combined with the estimates on the (C, r)-kernels obtained in Gaposhkin (1977b), Theorem 6. □

4. The pointwise ergodic theorem. In this section we now turn our attention to operators and introduce first some definitions which parallel the corresponding notions for processes. We replace our probability space $(\Omega, \mathscr{B}, \mathscr{P})$ by a σ-finite measure space [also denoted $(\Omega, \mathscr{B}, \mathscr{P})$], and we extend the range of α beyond 2 and assume that $1 \le \alpha < +\infty$, while throughout the section we also take $1 \le \beta \le +\infty$. Furthermore, μ is as in the previous section, and, in particular, $\int_{\{|u|>1\}} |u|^{-\beta}\, d\mu < +\infty$.

Let $B(L^\alpha)$ be the algebra of bounded linear operators on $L^\alpha(\mathscr{P})$ equipped with the strong operator topology and let $\|\cdot\|$ denote the usual norm on $B(L^\alpha)$. Throughout, let $T\colon \mathbb{R} \to B(L^\alpha)$ also be an operator function, that is, let T be bounded ($\sup_t \|T^t g\|_\alpha \le K$, $g \in L^\alpha(\mathscr{P})$) and measurable ($t \to T^t g$ is strongly measurable). Since T is bounded, $\int_{\mathbb{R}} f(t) T^t\, dt$ is a well-defined Lebesgue–Bochner integral for any $f \in L^1(\mathbb{R})$. Recalling that for $\beta < +\infty$ (resp. $\beta = +\infty$), $\|\cdot\|_\beta$ is the norm of $L^\beta(\mu)$ [resp. on $C_0(\mathbb{R})$], we have the following definition.

DEFINITION 4.1. An operator function T is (α, β)-*bounded* (with respect to μ) if there exists $K > 0$ such that

$$\left\| \int_{\mathbb{R}} f(t) T^t \, dt \right\| \leq K \| \hat{f} \|_\beta,$$

for all $f \in L^\beta(\mu)^v = \{ f \in L^1(\mathbb{R}) \colon \hat{f} \in L^\beta(\mu)\}$ [$f \in L^1(\mathbb{R})$ when $\beta = +\infty$].

A function $E\colon \mathscr{B}_0(\mathbb{R}) \to B(L^\alpha)$ is called an *operator measure* whenever it is finitely additive and a *spectral measure* if in addition it is multiplicative, that is, $E(A \cap B) = E(A)E(B)$, $A, B \in \mathscr{B}_0(\mathbb{R})$. The relation between operator and spectral measures is easy to draw: An operator measure is a spectral measure if and only if it is (commuting) projection valued [this can be proved as in Helson (1986)]. For any $g \in L^\alpha(\mathscr{P})$, E_g given by $E_g(A) = E(A)g$, $A \in \mathscr{B}_0(\mathbb{R})$, defines a finitely additive L^α-valued set function. So, we have the following definition.

DEFINITION 4.2. An operator measure E has finite (α, β)-*variation* if for every g in $L^\alpha(\mathscr{P})$, $||| E_g ||| = \sup\{ ||| E_g |||(A) \colon A \in \mathscr{B}_0(\mathbb{R})\} < +\infty$, where

$$||| E_g |||(A) = \sup \left\{ \left\| \sum_{i=1}^N a_i E_g(A_i) \right\|_\alpha \colon \{A_i\}_1^N \subset \mathscr{B}_0(\mathbb{R}) \text{ partition of } A,\ a_i \in \mathbb{C}, \left\| \sum_{i=1}^N a_i \chi_{A_i} \right\|_\beta \leq 1 \right\}.$$

The integral $\int_{\mathbb{R}} f(\xi)\, dE(\xi)$ of the scalar function f with respect to the operator measure E of bounded (α, β)-variation can now be defined as the element of $B(L^\alpha)$ for which $(\int_{\mathbb{R}} f(\xi)\, dE(\xi))g = \int_{\mathbb{R}} f(\xi)\, dE_g(\xi)$, $g \in L^\alpha(\mathscr{P})$, where this last integral (w.r.t. the random measure E_g) is defined as before. For E of bounded (α, β)-variation ($\beta < +\infty$), any f in $L^\beta(\mu)$ is integrable with respect to E, while for $\beta = +\infty$, the Borel bounded functions are also E-integrable. Moreover, $\| \int_{\mathbb{R}} f(\xi)\, dE(\xi) g \|_\alpha \leq ||| E_g |||\, \| f \|_\beta$, for $f \in L^\beta(\mu)$ and $g \in L^\alpha(\mathscr{P})$.

With the above definitions we can state our first result.

THEOREM 4.3. *An operator function T is continuous and (α, β)-bounded if and only if there exists a (unique regular) operator measure E with finite (α, β)-variation such that $T^t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi}\, dE(\xi)$ [in $B(L^\alpha)$ with the strong operator topology], uniformly on compact subsets of $\mathbb{R}$.*

PROOF. Let T be continuous and (α, β)-bounded, then for any $g \in L^\alpha(\mathscr{P})$, $\{T^t g\}_{t \in \mathbb{R}}$ is an (α, β)-bounded L^α-continuous process. Thus [Houdré (1990b), Theorem 3.2; actually Theorem 3.2 there is stated in terms of the triangular kernel, but the proof carries over to the step kernel case], there exists a (unique regular) random measure Z_g with finite (α, β)-variation such that

$$T^t g = \lim_{\lambda \to -\infty} \int_{-\lambda}^{\lambda} e^{it\xi}\, dZ_g(\xi) \quad \text{in } L^\alpha(\mathscr{P}),$$

uniformly on compact subsets of $\mathbb{R}$. Moreover, $\||Z_g\|| \le K$ and $\|Z_g(A)\|_\alpha \le K\|g\|_\alpha$, for every $A \in \mathscr{B}_0(\mathbb{R})$. Hence, E: $\mathscr{B}_0(\mathbb{R}) \to B(L^\alpha)$ defined via $E(A)g = Z_g(A)$, $g \in L^\alpha(\mathscr{P})$, satisfies all the stated requirements. For the converse, let $T^t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi}\, dE(\xi)$ in the strong operator topology, uniformly on compact subsets of $\mathbb{R}$. Then, again [Houdré (1990b), Theorem 3.2] and for any $g \in L^\alpha(\mathscr{P})$, $\{T^t g\}_{t\in\mathbb{R}}$ is strongly continuous with, moreover, $\|\int_{\mathbb{R}} f(t) T^t g\, dt\|_\alpha \le \||E_g\|| \, \|\hat{f}\|_\beta$. □

Remark 4.4. Since $L^\alpha(\mathscr{P})$ is weakly complete, it has been possible to replace the relative weak compactness of the sets $\{\|\int_{\mathbb{R}} f(t) T^t g\, dt\|_\alpha : \|\hat{f}\|_\beta \le 1, f \in L^\beta(\mathbb{R})^v \ (f \in L^1(\mathbb{R}) \text{ when } \beta = +\infty)\}$, $g \in L^\alpha(\mathscr{P})$, by their boundedness. For $\beta = +\infty$, E can be defined on $\mathscr{B}(\mathbb{R})$ and is also σ-additive (in the strong operator topology). Hence, by dominated convergence and since the exponentials are E-integrable, we have $\lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi}\, dE(\xi) = \int_{\mathbb{R}} e^{it\xi}\, dE(\xi)$, $t \in \mathbb{R}$, and this recovers a result of Kluvánek (1967). For $\alpha = 2$ and $\beta = +\infty$, T will be called *harmonizable* and *strongly harmonizable* whenever $\mathscr{E} E_g(\cdot)\overline{E}_g(\cdot)$ can be extended to a complex measure on $\mathbb{R}^2$.

Corollary 4.5. *Let T be continuous and (α, β)-bounded with associated operator measure E. Then T is additive, that is, $T^{t+s} = T^t T^s$ for all $t, s \in \mathbb{R}$, if and only if E is multiplicative.*

Proof. Let E be multiplicative, then for any simple functions f_1 and f_2 with bounded support, $\int_{\mathbb{R}} f_1 f_2\, dE = \int_{\mathbb{R}} f_1\, dE \int_{\mathbb{R}} f_2\, dE$. By (α, β)-boundedness, this equality can be extended to compactly supported Borel bounded functions, since they can be uniformly approximated by simple functions. It thus follows that

$$\begin{aligned} T^{t+s} &= \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{i(t+s)\xi}\, dE(\xi) \\ &= \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi}\, dE(\xi) \int_{-\lambda}^{\lambda} e^{is\xi}\, dE(\xi) \\ &= \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi}\, dE(\xi) \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{is\xi}\, dE(\xi), \end{aligned}$$

since each individual limit exists. Hence, $T^{t+s} = T^t T^s$. For the converse, since $T^{t+s} = T^t T^s$ and since via Lemma 2.1, the (truncated) exponentials are dense in the corresponding spaces of ν-square integrable functions, we have for any smooth f_1 and f_2, $\int_{\mathbb{R}} f_1 f_2\, dE = \int_{\mathbb{R}} f_1\, dE \int_{\mathbb{R}} f_2\, dE$. Thus, the same result follows from any Borel bounded f_1 and f_2, and this proves the corollary. □

For $\alpha = 2$, when E is orthogonal projection valued, that is, when for every $A \in \mathscr{B}_0(\mathbb{R})$, $E(A)$ is Hermitian, T is not only additive but also *unitary*, namely, $T^t T^{t*} = T^{t*} T^t = I$ (I is the identity operator), while the martingale difference case corresponds to operator measures whose values are differences of increasing orthogonal projections.

In general, and in contrast to unitary operators, *(α, β)-bounded groups T (T is additive with $T^0 = I$) are not shifts*. This can be seen as follows: Let $X = \{X_n\}_{n \in \mathbb{Z}}$ be a discrete-time (α, ∞)-bounded process, $1 \le \alpha < +\infty$, $X_n = \int_{-\pi}^{\pi} e^{in\theta}\, dZ(\theta)$, and let Z be of bounded variation. Then, for any trigonometric polynomial P, $\|\int_{-\pi}^{\pi} P(\theta)\, dZ(\theta)\|_\alpha \le \int_{-\pi}^{\pi} |P(\theta)|\, d|Z|(\theta)$, where $|Z|$ is the total variation measure. It is then not difficult to see (as proved below) that X has a well-defined shift if and only if the following condition holds: If for some P, $\|\int_{-\pi}^{\pi} P(\theta)\, dZ(\theta)\|_\alpha = 0$, then $\int_{-\pi}^{\pi} |P(\theta)|\, d|Z|(\theta) = 0$. But for

$$dZ(\theta) = Z\{\chi_{]-\pi, 0]}(\theta) - \chi_{]0, \pi]}(\theta)\}\, d\theta,$$

where $Z \in L^\alpha(\mathscr{P})$, $Z \ne 0$, this cannot happen unless $P = 0$. To prove the above claim, that is, to verify that the shift is a well-defined operator, we need to show that the stated condition and Getoor's (1956) (C_1) condition are the same. Recall that X satisfies (C_1) [actually in Getoor (1956) the condition is given for $\alpha = 2$ and so in terms of the covariance] if for $n_1, n_2, \dots, n_N \in \mathbb{Z}$ and $p_1, p_2, \dots, p_N \in \mathbb{C}$ such that $\|\sum_{k=1}^N p_k X_{n_k}\|_\alpha = 0$, then $\|\sum_{k=1}^N p_k X_{n_k+m}\|_\alpha = 0$ for all $m \in \mathbb{C}$. Let (C_1) be satisfied and let P be a trigonometric polynomial such that $\|\int_{-\pi}^{\pi} P(\theta)\, dZ(\theta)\|_\alpha = 0$, then $\|\int_{-\pi}^{\pi} e^{im\theta} P(\theta)\, dZ(\theta)\|_\alpha = 0$ for all $m \in \mathbb{Z}$. Hence, by the uniqueness of the Fourier transform and since Z has bounded variation, $P\,dZ = 0 = |P|\, d|Z|$ and $\int_{-\pi}^{\pi} |P|\, d|Z| = 0$, which gives one-half of the equivalence. Now if $\|\int_{-\pi}^{\pi} P(\theta)\, dZ(\theta)\|_\alpha = 0$ implies

$$\int_{-\pi}^{\pi} |P(\theta)|\, d|Z|(\theta) = 0,$$

we finally have

$$\left\| \int_{-\pi}^{\pi} e^{im\theta} P(\theta)\, dZ(\theta) \right\|_\alpha \le \int_{-\pi}^{\pi} |e^{im\theta} P(\theta)|\, d|Z|(\theta) = 0,$$

and Getoor's (C_1) condition is verified.

After these preliminaries, we can now state the main result of this section. Again, we say that T satisfies the pointwise ergodic theorem whenever for any $g \in L^\alpha(\mathscr{P})$ the averages $\sigma_S Tg = (1/2S)\int_{-S}^{S} T^t g(\omega)\, dt$, $S > 0$, $\omega \in \Omega$, converge a.e. ($\mathscr{P}$) with, of course, in discrete-time the integral replaced by $(1/(2N+1))\sum_{n=-N}^{N} T^n g(\omega)$ or by $(1/N)\sum_{n=0}^{N-1} T^n g(\omega)$, when T^n is only defined for nonnegative powers, with in either case $T^0 = I$. The only discrepancy between the result below and Theorem 3.2 is in the case $\alpha > 2$, where for existence purposes [see Houdré (1990a)] we do require $\alpha \le \beta$.

THEOREM 4.6. *Let T be (α, β)-bounded with either $1 < \alpha \le 2$ and $2 \le \beta \le +\infty$, or $2 < \alpha < +\infty$ and $\alpha \le \beta \le +\infty$, or $\alpha = 1$ and $\beta = +\infty$, and let $g \in L^\alpha(\mathscr{P})$. Then T satisfies the pointwise ergodic theorem if and only if $\lim_{n \to +\infty} E_g\{0 < |\xi| < 2^{-n}\} = 0$ a.e. $\mathscr{P}$, where E is the representing operator measure of T.*

PROOF. For $1 \le \alpha \le 2$, the proof is exactly as in Theorem 3.2 (Lemma 2.1 continues to hold for σ-finite measures). For $2 < \alpha < \infty$, since $\mathscr{P}$ is σ-finite, there exists $Y > 0$ (a.e. $\mathscr{P}$) such that $d\mathscr{P}_0 = Y d\mathscr{P}$ is a probability measure and $W \to Y^{-1/\alpha}W$ is an isometry from $L^\alpha(\mathscr{P})$ to $L^\alpha(\mathscr{P}_0)$. But since $L^\alpha(\mathscr{P}_0) \subset L^2(\mathscr{P}_0)$, the L^2-results of the previous section give

$$\lim_{S \to \infty} Y^{-1/\alpha} \frac{1}{2S} \int_{-S}^{S} T^t g\, dt = \lim_{n \to +\infty} Y^{-1/\alpha} E_g\{|\xi| < 2^{-n}\} \quad (\text{a.e. } \mathscr{P}_0).$$

This last equality gives the result. □

REMARK 4.7. For $\alpha = 1$ and $\beta < +\infty$, and in similarity to Theorem 3.2, an added L^2-boundedness assumption on the dilation part of T^t will also give conditions for the a.s. convergence of the ergodic averages. For $2 < \alpha < +\infty$, the very nice Lemma 2.1 does not hold. In fact, for $\beta = +\infty$, the dominating inequality (2.1) becomes

$$\left\| \int_{\mathbb{R}} f\, dE_g \right\|_\alpha \le \|g\|_\alpha \left(\int_{\mathbb{R}} |f|^{\alpha+\varepsilon}\, d\nu \right)^{1/\alpha+\varepsilon} \qquad \varepsilon > 0$$

[see Pisier (1986), page 69; the result there is actually not given for $\mathbb{R}$ but for a compact space and also not for operator measures; however, the version stated above can be easily obtained]. Hence, no direct (i.e., via studying $\sum \mathscr{E}|\cdot|^\alpha$) proof of the theorem seems likely. We remark too, that when μ is Lebesgue measure, and for $\beta < +\infty$, the pointwise ergodic theorem is always satisfied.

For operators between Hilbert spaces, Theorem 4.6 has the following interesting particular cases.

COROLLARY 4.8. *Let T be either a bounded invertible linear operator on $L^2(\mathscr{P})$ with $\sup_{n \in \mathbb{Z}} \|T^n\| < +\infty$ or let T be a contraction, that is, let $\|T\| \le 1$ and let $g \in L^2(\mathscr{P})$. Then there exists a σ-additive (on $\mathscr{B}(]-\pi, \pi])$) operator measure E such that $T^n = \int_{-\pi}^{\pi} e^{in\theta}\, dE(\theta)$, $n \in \mathbb{Z}$ or $n \ge 0$ when T is a contraction, and T satisfies the pointwise ergodic theorem if and only if $\lim_{n \to +\infty} E_g\{0 < |\xi| < 2^{-n}\} = 0$ a.e. $\mathscr{P}$. Moreover, the $(C,1)$ and (C,r), $r > 1/2$, almost sure convergence are equivalent.*

PROOF. For both cases, we use the results of Sz.-Nagy. When T is invertible with $\sup_{n \in \mathbb{Z}} \|T^n\| < +\infty$, T is similar to a unitary operator [see Sz.-Nagy (1947)], that is, there exists a unitary operator U and an invertible Hermitian operator Q such that $T = Q^{-1}UQ$. Hence, we have for any $g \in L^2(\mathscr{P})$,

$$\begin{aligned} \left\| \sum_{n=-N}^{N} a_n T^n g \right\|_2^2 &\le \|Q^{-1}\|^2 \|Q\|^2 \left\| \sum_{n=-N}^{N} a_n U^n g \right\|_2^2 \\ &= K \int_{-\pi}^{\pi} \left| \sum_{n=-N}^{N} a_n e^{in\theta} \right|^2 d\|Eg\|^2 \\ &\le K \|g\|_2^2 \sup_\theta \left| \sum_{n=-N}^{N} a_n e^{in\theta} \right|^2, \end{aligned}$$

and the group T is $(2,\infty)$-bounded. In other words, invertible T's such that $\sup_{n\in\mathbb{Z}}\|T^n\| < +\infty$ are exactly the Fourier transforms of the σ-additive spectral measures from $\mathscr{B}(]-\pi,\pi])$ to $B(L^2)$ [the $(2,\infty)$-bounded defining property trivially gives $\sup_{n\in\mathbb{Z}}\|T^n\| < +\infty$]. Of course, $E = Q^{-1}FQ$, where F is the spectral measure of the unitary group and E is projection valued. When T is a mere contraction, it has a unitary dilation [see Sz.-Nagy and Foias (1967) or use Lemma 2.1], that is, $T^n = PU^n$, where U is unitary on $L^2(\tilde{\mathscr{P}}) \supset L^2(\mathscr{P})$ and where P is the orthogonal projection from $L^2(\tilde{P})$ to $L^2(\mathscr{P})$. Hence, $T^n = P\int_{-\pi}^{\pi} e^{in\theta}\,dF(\theta) = \int_{-\pi}^{\pi} e^{in\theta}\,dPF(\theta)$, $n \geq 1$. Again it follows that $E = PF$, but this time E is not projection valued but rather positive definite operator valued. Combining these observations with the previous results, we get the corollary [for T a contraction, the sinc kernel has to be replaced by $(e^{iN\theta}-1)/N(e^{i\theta}-1)$, but estimates and methods completely analogous to the ones used in the bilateral case give the result]. To get the rest of the statement, it suffices to proceed as in the proof of Theorem 3.10. □

Our next result shows that on Hilbert space, when studying the ergodic averages of contractions or of invertible operators such that $\sup_{n\in\mathbb{Z}}\|T^n\| < +\infty$, only the unitary ones really matter.

Corollary 4.9. *Let T be either a contraction on $L^2(\mathscr{P})$ or an invertible operator such that $\sup_{n\in\mathbb{Z}}\|T^n\| < +\infty$ and let U be any associated unitary dilation or similarity. Then, for any $g \in L^2(\mathscr{P})$,*

$$\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} T^n g \text{ exists } (a.e.\ \mathscr{P}) \quad \text{if and only if} \quad \lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} U^n g$$

exists $(a.e.\ \mathscr{P})$.

Proof. For $g, h \in L^2(\mathscr{P})$,

$$\mathscr{E}(E_g]-2^{-n},2^{-n}[\bar{h}) = \mathscr{E}(PF_g]-2^{-n},2^{-n}[\bar{h}) = \mathscr{E}(F_g]-2^{-n},2^{-n}[P\bar{h})$$
$$\doteq \mathscr{E}(F_g]-2^{-n},2^{-n}[\bar{h}).$$

So, on $L^2(\mathscr{P})$, $E_g]-2^{-n},2^{-n}[$ and $F_g]-2^{-n},2^{-N}[$ coincide. □

Remark 4.10. We do not know how the above results relate to Akcoglu's (1975), de la Torre's (1976) or Stein's (1970), page 87, ergodic theorem, that is, we do not understand why for T positive ($Tf \geq 0$ whenever $f \geq 0$) the condition $\lim_{n\to+\infty} E_g\{0 < |\xi| < 2^{-n}\} = 0$ a.e. is always satisfied. A better understanding (a characterization?) of the effects of positivity on the spectral measure is certainly the key to this problem. Unfortunately, Theorem 4.6 does not give much information about, say, the ergodicity of the isometries in $L^\alpha(\mathscr{P})$, $\alpha \neq 2$. As already mentioned, it is shown in Cambanis and Houdré (1990) that the class of moving averages of Lévy motion (for which the shift exists and is an invertible isometry) and the (α,∞)-bounded class are disjoint [the results in Cambanis and Houdré (1990) continue to hold for shifts T such that $\sup_{n \in \mathbb{Z}} \|T^n\| < +\infty$]. Isometries and more generally power-bounded and invertible power-bounded operators on $L^\alpha(\mathscr{P})$ do admit another type of spectral representation [see Berkson and Gillespie (1987)] which for $\alpha = 2$ corresponds to $(2, \infty)$-boundedness. This spectral representation will certainly help to study the ergodicity of such operators on $L^\alpha(\mathscr{P})$, $\alpha \neq 2$. For processes whose unilateral (resp. bilateral) shift is a well-defined contraction (resp. such that $\sup_{n \in \mathbb{Z}} \|T^n\| < +\infty$), the above result strengthen weak laws obtained in Getoor (1956). Finally, if the finite positive measure $\mu_g(A) = \mathscr{E}(E_g(A)\bar{g}) = \mathscr{E}(F_g(A)\bar{g})$ is absolutely continuous with density in $L^{1+\varepsilon}(\Pi)$, then the ergodic averages converge.

It is clear that there are various potential extensions and generalizations of the above results. First, the results of Gaposhkin (1981) can be extended (via the methods used above) to give when T is, say, (α, ∞)-bounded, $\alpha \geq 1$, or a strongly continuous one-parameter contractive semigroup or a uniformly bounded group on L^2, the following: T satisfies the local ergodic theorem if and only if $\lim_{n \to \infty} E_g\{|\xi| > 2^n\} = 0$ a.e. $\mathscr{P}$. Other extensions include, for example, the pointwise ergodic theorem or the local ergodic theorem for "pseudo" Hermitian operators [see Kluvánek (1967)], more generally for operators for which some kind of spectral representation with respect to a nonorthogonally scattered operator measure holds. This is the case for the operators of the class C_ρ, $\rho > 0$ [see Sz.-Nagy and Foias (1967)], or more generally for operators which are similar to contractions. Except for random fields, to which our last section is devoted, we only state another result for which the passage from unitary to (α, ∞)-bounded, $\alpha \geq 1$, is rather safe.

For unitary operators Jajte (1987) gave a spectral criterion for the existence of the ergodic Hilbert transform. More generally, we have the following corollary [with the abuse of language of using (α, ∞)-bounded for the operator instead of the group].

Corollary 4.11. *Let the operator T be (α, ∞)-bounded, $\alpha \geq 1$ [resp. let T satisfy the hypotheses of Corollary 4.9 (when T is a contraction, T^{-n} has to be replaced by $T^{*n} = \int_{-\pi}^{\pi} e^{-in\theta}\, dE(\theta)$, $n > 1$)] and let $g \in L^\alpha(\mathscr{P})$ [resp. $g \in L^2(\mathscr{P})$]. Then $\lim_{N \to \infty} \sum_{0 < |n| \leq N} T^n g/n$ exists a.e. $\mathscr{P}$, if and only if $\lim_{p \to \infty} E_g\{-2^{-p} < \xi < 0\} - E_g\{0 < \xi < 2^{-p}\} = 0$ a.e. $\mathscr{P}$. Moreover, $\lim_{N \to \infty}(1/N)\sum_{n=0}^{N-1} T^n g$ exists for all $g \in L^\alpha(\mathscr{P})$ [resp. all $g \in L^2(\mathscr{P})$], a.s. $\mathscr{P}$, if and only if $\lim_{N \to \infty} \sum_{0 < |n| \leq N} T^n g/n$ exists for all $g \in L^\alpha(\mathscr{P})$ [resp. all $g \in L^2(\mathscr{P})$], a.s. $\mathscr{P}$.*

Proof. The first part of the statement follows by combining the methods presented above with the estimates on the corresponding exponential kernels obtained in Jajte (1987). The rest of the statement also follows by the methods devised in Jajte (1987). □

It is clear that the same method gives a necessary and sufficient condition for the existence of the stochastic Hilbert transform whenever X is (α, ∞)-

bounded, $0 \le \alpha \le 2$, namely $\lim_{N\to\infty} \sum_{0<|n|\le N} X_n/n$ exists a.s. $\mathscr{P}$ if and only if $\lim_{p\to\infty} Z\{-2^{-p} < \xi < 0\} - Z\{0 < \xi < 2^{-p}\} = 0$ a.s. $\mathscr{P}$. The continuous case can also be obtained in a similar way.

5. The spectral SLLN for random fields. We assume to the end of this paper that the standing assumptions on α, β are identical to the ones made in Section 3. However, μ is now defined on $\mathscr{B}(\mathbb{R}^m)$ with also $\int_{\{|\xi|>1\}} |\xi|^{-\beta}\, d\mu < +\infty$. It is easily seen that the univariate (α, β)-boundedness definition given in Houdré (1990b) carried over to the case of fields $X = \{X_t\}_{t\in\mathbb{R}^m}$, in fact, even to the LCA framework. Essentially, as in Houdré (1990b), X is (α, β)-bounded if and only if

$$X_t = \lim_{\lambda=(\lambda_1,\dots,\lambda_m)\to+\infty} \int_{-\lambda_1}^{\lambda_1} \cdots \int_{-\lambda_m}^{\lambda_m} \left(1 - \frac{|\xi_1|}{\lambda_1}\right) e^{it_1\xi_1} \cdots \left(1 - \frac{|\xi_m|}{\lambda_m}\right) e^{it_m\xi_m}\, dZ(\xi_1,\dots,\xi_m) \quad \text{in } L^\alpha(\mathscr{P}),$$

uniformly on the compacts, where $Z\colon \mathscr{B}_0(\mathbb{R}^m) \to L^0(\mathscr{P})$ has finite (α, β)-variation. The multidimensional version of Lemma 2.1 continues to hold.

LEMMA 5.1. *Let the field X be (α, β)-bounded, $0 \le \alpha \le 2 \le \beta \le +\infty$, with associated random measure Z_x. Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\mathscr{P}})$ with $L^2(\mathscr{P}) \subset {}^2(\tilde{\mathscr{P}})$, a $(2, \beta)$-bounded field Y with orthogonally scattered random measure Z_Y defined on $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\mathscr{P}})$ and a random variable $\Lambda \in L^{2\alpha/(2-\alpha)}(\mathscr{P})$ such that $X_t = \Lambda P Y_t$, $t \in \mathbb{R}$, where P is the orthogonal projection from $L^2(\tilde{\mathscr{P}})$ to $L^2(\mathscr{P})$.*

Again, since Y is a homogeneous field, there exists a positive measure ν (finite for $\beta = +\infty$) given by $d\nu = g\, d\mu$, $g \in L^{\beta/(\beta-2)}(\mu)$ on $\mathbb{R}^m$ such that

$$\left\| \int_{\mathbb{R}^m} f\, dZ_Y \right\|_2 = \left(\int_{\mathbb{R}^m} |f|^2\, d\nu \right)^{1/2} \tag{5.1}$$

for all $f \in L^2(\nu)$.

For fields, averaging is always more delicate than for processes. Throughout, we follow Gaposhkin (1977a), denote by $|A_\rho|$ the volume of A_ρ and study the averages,

$$\sigma_\rho X(\omega) = \frac{1}{|A_\rho|} \int_{A_\rho} X(t,\omega)\, dt, \qquad \rho > 0, \qquad \omega \in \Omega,$$

where $X(t,\omega) = X(t_1, t_2, \dots, t_m, \omega)$, $dt = dt_1\, dt_2, \dots, dt_m$, and where the A_ρ

satisfy the following three conditions:

(i) For each ρ, A_ρ is a bounded convex body containing the origin.

(ii) For $0 < \rho_0 \le \rho < \rho'$, $A_\rho \subset A_{\rho'}$ and $(|A_{\rho'}| - |A_\rho|)/|A_\rho| \le K(\rho' - \rho)/\rho$.

(iii) There exist two positive constants K_1 and K_2 such that the length $d(\rho)$ of any chord of A_ρ passing through the origin satisfies $K_1\rho \le d(\rho) \le K_2\rho$, $\rho \ge \rho_0 > 0$.

It is clear that n-dimensional spheres of radius ρ with center at the origin satisfy the above three conditions. This is also true of n-dimensional cubes centered at the origin with side of length ρ. Rectangles which do not flatten out also satisfy these conditions. We finally say that X satisfies the SLLN whenever $\lim_{\rho\to+\infty}\sigma_\rho X = 0$, with probability 1. For $\xi \in \mathbb{R}^m$, we see $|\xi| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_m^2)^{1/2}$ and then we have the following result.

THEOREM 5.2. *Let the random field X be (α, β)-bounded with random measure Z_X, and let either X be L^α-bounded, $\alpha > 1$, or let PY be L^2-bounded. Then X satisfies the SLLN if and only if for almost all ω, $\lim_{p\to+\infty} Z_X\{|\xi| < 2^{-p}\}(\omega) = 0$.*

PROOF. The proof requires only adjustments from the univariate results and so will only be sketched. Again,

$$\sigma_\rho X = \left(\sigma_\rho X - \sigma_n X\right) + \left(\sigma_n X - \sigma_{2^p} X\right) + \left(\sigma_{2^p} X - Z_X\{|\xi| < 2^{-p}\}\right) + Z_X\{|\xi| < 2^{-p}\}.$$

Using condition (ii) as well as arguments similar to the univariate ones, we easily see that $\sup_{n<\rho\le n+1} \mathscr{E}|\sigma_\rho X - \sigma_n X|^\alpha \le Kn^{-\alpha}$ or that

$$\sup_{n<\rho\le n+1} \mathscr{E}|\sigma_\rho PY - \sigma_n PY|^2 \le Kn^{-2}.$$

Hence, with probability 1, $\lim_{n\to\infty}\sup_{n<\rho\le n+1}|\sigma_\rho X - \sigma_n X| = 0$. For the third bracket, let

$$K_\rho(\xi) = \frac{1}{|A_\rho|}\int_{A_\rho} e^{it\cdot\xi}\,dt,$$

where $t\cdot\xi = \sum_{j=1}^m t_j\xi_j$, then clearly $\sigma_{2^p}X = \int_{\mathbb{R}^m} K_{2^p}(\xi)\,dZ_X(\xi)$. Since (ii) and (iii) give $|K_{2^p}(\xi) - 1| \le K2^p|\xi|$ whenever $|\xi| < 2^{-p}$, and $|K_{2^p}(\xi)| \le K/(2^p|\xi|)$ for $|\xi| \ge 2^{-p}$, breaking $\mathbb{R}^m$ into $\{|\xi| < 2^{-p}\}$ and $\{|\xi| \ge 2^{-p}\}$, we get, using Lemma 5.1, $\sum_{p=1}^\infty \mathscr{E}|\sigma_{2^p}PY - PZ_Y\{|\xi| < 2^{-p}\}|^2 < +\infty$. Hence, for almost all ω, $\lim_{p\to+\infty}\sigma_{2^p}X - Z_X\{|\xi| < 2^{-p}\} = 0$. For the middle bracket, we note that Lemma 2.2 (with its notation) continues to hold, we apply Lemma 5.1 and we also break $\mathbb{R}^m$ into four pieces: $\{|\xi| \le 2^{-p-1}\}, \{2^{-p-1} < |\xi| \le 2^{-p+k}\}, \{2^{-p+k} < |\xi| \le 1\}, \{|\xi| > 1\}$. Now, using estimates obtained in Gaposhkin (1977a), Theorem 6′, and proceeding as in the proof of Lemma 3.1 leads to four convergent series from which the result follows. □

REMARK 5.3. The requirements on the A_ρ's are just set to ensure that the kernels K_ρ do satisfy the right estimates and so, for any average for which such estimates hold, Theorem 5.2 continues to be true. It is also clear that Theorems 3.6 and Theorem 3.9 as well as the corollaries of Section 3 admit multidimensional versions and that as in Gaposhkin (1977a), the sequence $\{2^{-p}\}$ can be replaced by more general sequences

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