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A LAW OF THE ITERATED LOGARITHM FOR RANDOM GEOMETRIC SERIES

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We consider the random variables $\xi(\beta) = \sum_{n=0}^{\infty} \beta^n \varepsilon_n$ for $\beta < 1$. We prove that if the ε_n are i.i.d. random variables with mean zero and variance 1, then a law of the iterated logarithm holds in the sense that the cluster set of

$$\frac{\sqrt{1-\beta^2}}{2\log\log(1/(1-\beta^2))}\xi(\beta),$$

when β converges to one, is the interval $[-1, 1]$.

1. Introduction. In a recent paper ([2]) we studied the random variables $\xi(\beta) \equiv \sum_{k=0}^{\infty} \beta^k \varepsilon_k$, where ε_n are independent, identically distributed symmetric Bernoulli random variables. Our main result was a variant of the law of the iterated logarithm (LIL) concerning the limit as β tends to one, that is, we proved that

$$\limsup_{\beta \uparrow 1} \frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log(1/(1-\beta^2))}}\xi(\beta) = 1 \quad \text{a.s.} \tag{1.1}$$

Our investigation of this random variable was motivated by the fact that there is a long-standing history surrounding it. As early as 1935, Wintner and coworkers [7, 8, 14, 15] studied the properties of the probability distribution of the random variable $\xi(\beta)$, proving in particular that it is always continuous for $0 < \beta < 1$ and moreover pure, that is, either absolutely continuous or purely singular continuous with respect to Lebesgue measure. In the case $\beta < 1/2$, they showed that the distribution is singular continuous, but for $\beta \geq 1/2$, which of the two cases is realized depends on the arithmetic properties of the number β, and a complete answer cannot be given even today. For a survey of this problem, the reader may consult an article by Garsia [5] or a recent paper by one of the authors [1].

From the point of view of limit theorems, it is natural to ask the question under what more general conditions on the random variables ε_n a LIL of the type (1.1) can be proven. In the case of the standard LIL, the first proof was given in 1924 by Khintchine [9] for Bernoulli random variables; it took 20 years until Hartman and Wintner [6] showed that in the i.i.d. case only finite second moments were required. It is an all too natural conjecture that under

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these same conditions on ε_n, (1.1) should hold. It is the purpose of the present article to prove this. Indeed, we will prove the following theorem:

THEOREM 1.1. *Let ε_i be i.i.d. random variables such that $\mathbb{E}\varepsilon_i = 0$, $\mathbb{E}\varepsilon_i^2 = 1$. Put*

$$\bar{\xi}(\beta) \equiv \frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log(1/(1-\beta^2))}} \sum_{k=0}^{\infty} \beta^k \varepsilon_k. \tag{1.2}$$

Then (i) $\mathbb{P}(\lim_{\beta\uparrow 1} \operatorname{dist}(\bar{\xi}(\beta), [-1,1]) = 0) = 1$; (ii) $\mathbb{P}(\mathscr{C}(\{\bar{\xi}(\beta)\}) = [-1,1]) = 1$, *where $\mathscr{C}(\{\bar{\xi}(\beta)\})$ denotes the cluster set (set of all limit points) of $\bar{\xi}(\beta)$ as β tends to one and* $\operatorname{dist}(a,b)$ *stands for the distance between a and b.*

REMARK. Theorem 1.1 is a LIL in Strassen's formulation [11]. It implies in particular the classical version: $\limsup_{\beta\uparrow 1} \bar{\xi} = 1$ a.s. and $\liminf_{\beta\uparrow 1} \bar{\xi} = -1$ a.s.

As pointed out in [2], the techniques used there can easily accommodate unbounded ε_n, provided the moment generating function $\mathbb{E}(e^{t\varepsilon})$ exists in a neighborhood of the origin and satisfies a bound of the form $\mathbb{E}(e^{t\varepsilon}) \leq \exp(t^2/2 + O(t^4))$. However, if only finite variance is assumed, the proof requires substantial modifications. In particular, we will need to make use of a remarkable result from Strassen's theory [13] on fluctuations of sums of random variables. On the technical level, our proof is otherwise inspired by the beautiful proof of the Hartman–Wintner LIL by De Acosta [3].

A LIL does not come without a central limit theorem (CLT), which will in fact be necessary for our proof. We will require the following version of it:

THEOREM 1.2 (Central-Limit Theorem). *Let ε_n be i.i.d. r.v.'s with mean zero and variance one. Let $g(\beta)$ be an increasing function from* $[0,1]$ *to $N \cup \{\infty\}$ such that*

$$\lim_{\beta\uparrow 1} g(\beta) = +\infty.$$

Put $f(\beta) \equiv \sum_{n=0}^{g(\beta)} \beta^{2n}$ and define

$$Z(\beta) \equiv \frac{1}{\sqrt{f(\beta)}} \sum_{n=0}^{g(\beta)} \beta^n \varepsilon_n. \tag{1.3}$$

Then $Z(\beta)$ converges in law to a Gaussian r.v. with mean zero and variance one as $\beta \uparrow 1$.

The proof of this theorem follows largely the standard proof of the CLT and will be given in Section 3, where this theorem will actually be used.

Let us give a brief outline of the proof of Theorem 1.1. Section 2 is devoted to proving (i). We first establish that, for suitable $N_2(\beta)$, the tails

$$(1.4)\qquad \bar{\xi}_2(\beta) \equiv \frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log(1/(1-\beta^2))}} \sum_{k=N_2(\beta)}^{\infty} \beta^k \varepsilon_k$$

are almost surely asymptotically negligible. For the remaining piece $\bar{\xi}_1(\beta)$, we prove an upper bound by showing that for a suitably chosen sequence $\beta_n \uparrow 1$,

$$(1.5)\qquad \limsup_{n\to\infty} |\bar{\xi}_1(\beta_n)| \leq 1 + \varepsilon \quad \text{a.s.},$$

while

$$(1.6)\qquad \lim_{n\to\infty} \sup_{\beta_n \leq \beta \leq \beta_{n+1}} |\bar{\xi}(\beta_n) - \bar{\xi}(\beta)| = 0 \quad \text{a.s.}$$

In reference [2], we proved the analogue of (1.6) in an elementary way, using renormalization group techniques. Under the weak hypothesis of the present theorem, we will have to take recourse to Strassen's results [13] in order to prove (1.6). [In our opinion, Stout's remark in [12] applies at this point: "One would labour long (and possibly in vain) to produce those results by classical methods."]

Equation (1.5) is proved by applying a truncation technique similar to that of De Acosta [3]. That is we split $\bar{\xi}_1(\beta_n)$ into

$$(1.7)\qquad \bar{\xi}_1(\beta_n) = \bar{\xi}_1^q(\beta_n) + \bar{\xi}_1^t(\beta_n),$$

where

$$(1.8)\qquad \bar{\xi}_1^t(\beta_n) = \sum_{k=0}^{N_2(\beta_n)} \beta_n^k \varepsilon_k \mathbb{1}_{\left\{|\varepsilon_k| < \tau\beta_n^{-k}/\sqrt{2(1-\beta_n^2)\log\log(1/(1-\beta_n^2))}\right\}}$$

and

$$(1.9)\qquad \bar{\xi}_1^q(\beta_n) = \sum_{k=0}^{N_2(\beta_n)} \beta_n^k \varepsilon_k \mathbb{1}_{\left\{|\varepsilon_k| \geq \tau\beta_n^{-k}/\sqrt{2(1-\beta_n^2)\log\log(1/(1-\beta_n^2))}\right\}}.$$

Here $\mathbb{1}_A$ denotes the indicator function of the event A. $\bar{\xi}_1^q(\beta_n)$ will be shown to converge to zero almost surely and for $\bar{\xi}_1^t(\beta_n)$ the bound (1.5) will be proven in a rather standard fashion.

Then, in Section 3, to prove that the cluster set of $\bar{\xi}(b)$ is $[-1, 1]$, we combine the decomposition of $\xi(\beta)$ used in [2] with a moderate deviation estimate of De Acosta and Kuelbs [4]. This will require the use of our central limit theorem, Theorem 2.

2. The upper bound. Let us begin by introducing some notation. Given $0 < \beta < 1$, let

$$N(\beta) \equiv \left[\frac{1}{1-\beta^2}\right], \tag{2.1}$$

$$N_2(\beta) = \left[N(\beta)\log N(\beta)\right]. \tag{2.2}$$

Also, in the present section, we set $\beta_n = e^{-\theta_n^n}$, where $\theta_n < 1$ may be chosen as $\theta_n = 1 - (1/\ln n)^3$. The following asymptotic relations will be useful in the sequel:

(i) $$\frac{\beta^{N_2(\beta)}}{\sqrt{1-\beta^2}} \to 1 \quad \text{as } \beta \uparrow 1.$$

(ii) $$\beta^{N(\beta)} \sim \exp\left[\frac{\ln \beta}{(1-\beta^2)}\right] \to e^{1/2} \quad \text{as } \beta \uparrow 1.$$

(iii) $$\theta_n^n \sim \exp\left[-\frac{n}{(\ln n)^3}\right].$$

(iv) $$\left(\frac{\theta_{n+1}}{\theta_n}\right)^n \sim \exp\left[\frac{3}{(\ln n)^4}\frac{n}{n+1}\right] \to 1 \quad \text{as } \beta \uparrow 1.$$

(v) $$\frac{1-\beta_n}{1-\beta_{n+1}} \sim \left(\frac{\theta_n}{\theta_{n+1}}\right)^n \frac{1}{\theta_{n+1}} \to 1 \quad \text{as } \beta \uparrow 1.$$

Our first lemma concerns the tails of ξ:

LEMMA 2.1. *Let (ε_n) be i.i.d. r.v.'s with mean zero and variance one. Then*

$$\lim_{\beta \uparrow 1} \frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log(1/(1-\beta^2))}} \sum_{k=N_2(\beta)}^{\infty} \beta^k \varepsilon_k = 0 \quad a.s. \tag{2.3}$$

PROOF. By the Borel–Cantelli lemma, (2.3) will follow if we can show that for any $\varepsilon > 0$,

$$\sum_{n=n_0}^{\infty} \mathbb{P}\left(\sup_{\beta_n < \beta < \beta_{n+1}} \left|\sum_{k=N_2(\beta)}^{\infty} \beta^k \varepsilon_k\right| \geq \varepsilon\sqrt{\frac{2\log\log(1/(1-\beta_n^2))}{1-\beta_n^2}}\right) < \infty. \tag{2.4}$$

Now

$$\begin{aligned} &\sup_{\beta_n < \beta < \beta_{n+1}} \left|\sum_{k=N_2(\beta)}^{\infty} \beta^k \varepsilon_k\right| \\ &\quad \leq \sum_{k=N_2(\beta_n)}^{\infty} \beta_{n+1}^k |\varepsilon_k| \\ &\quad = \sum_{k=N_2(\beta_n)}^{\infty} \beta_{n+1}^k \left[|\varepsilon_k| - \mathbb{E}(|\varepsilon_k|)\right] + \sum_{k=N_2(\beta_n)}^{\infty} \beta_{n+1}^k \mathbb{E}(|\varepsilon_k|). \end{aligned} \tag{2.5}$$

For the second term we have

$$\sqrt{1-\beta^2}\sum_{k=N_2(\beta_n)}^{\infty}\beta_{n+1}^k\mathbb{E}(|\varepsilon_k|) \le \sqrt{1-\beta^2}\,\frac{\beta_{n+1}^{N_2(\beta_n)}}{1-\beta_{n+1}} \le \frac{2}{\theta_n}, \tag{2.6}$$

where use was made of the asymptotic property (v). For the first term we use Chebyshev's inequality to get

$$\begin{aligned}\mathbb{P}&\left(\sum_{k=N_2(\beta_n)}^{\infty}\beta_{n+1}^k\big[|\varepsilon_k|-\mathbb{E}(|\varepsilon_k|)\big] \ge \varepsilon\sqrt{\frac{2\log\log\big(1/(1-\beta_n^2)\big)}{1-\beta_n^2}}\right)\\ &\le \frac{1-\beta_n^2}{2\varepsilon^2\log\log\big(1/(1-\beta_n^2)\big)}\sum_{k=N_2(\beta_n)}^{\infty}\beta_{n+1}^{2k}\\ &\le \frac{1}{2\varepsilon^2\log\log\big(1/(1-\beta_n^2)\big)}\,\frac{\beta_{n+1}^{2N_2(\beta_n)}}{\theta_n}.\end{aligned} \tag{2.7}$$

Now,

$$\beta_{n+1}^{2N_2(\beta_n)} \sim \left(2\theta_n^n\right)^{\theta_{n+1}(\theta_{n+1}/\theta_n)^n},$$

and therefore, using the properties (iii) and (iv) from above, the left-hand side of (2.7) is the general term of a summable series and the lemma is proved. □

Our next lemma establishes that $\bar{\xi}(\beta)$ for β between β_n and β_{n+1} are strongly correlated.

LEMMA 2.2. *Let $\theta_n = 1 - 1/(\log n)^3$ and ε_n be as in Lemma 2.1. Then*

$$\lim_{\beta\uparrow 1}\ \sup_{\beta_n<\beta<\beta_{n+1}} |\bar{\xi}_1(\beta)-\bar{\xi}_1(\beta_n)| = 0 \quad a.s. \tag{2.8}$$

PROOF. We put $S_n \equiv \sum_{k=0}^n \varepsilon_k$. Define also

$$a(\beta) \equiv \frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log\big(1/(1-\beta^2)\big)}}.$$

This allows us to write

$$\begin{aligned}&\sum_{k=0}^{N_2(\beta_{n+1})}\varepsilon_k\big(\beta^k a(\beta)-\beta_n^k a(\beta_n)\big)\\ &\quad= \sum_{k=1}^{N_2(\beta_{n+1})} S_k\beta^k a(\beta)(1-\beta) - S_k\beta_n^k a(\beta_n)(1-\beta_n)\\ &\qquad+\big(\beta^{N_2+1}a(\beta)-\beta_n^{N_2+1}a(\beta_n)\big)S_{N_2}.\end{aligned} \tag{2.9}$$

The last term is easily seen to be insignificant: Notice simply that

$$\sup_{\beta_n<\beta<\beta_{n+1}} |\beta^{N_2+1}a(\beta) - \beta_n^{N_2+1}a(\beta_n)||S_{N_2}| \tag{2.10}$$
$$\leq \frac{2}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \frac{1}{\sqrt{N(\beta_n)}} |S_{N(\beta_{n+1})\log N(\beta_{n+1})}|,$$

where the right-hand side goes to zero a.s. by the Marcinkiewicz–Zygmund theorem [8] or the Hartman–Wintner LIL.

The difficult part is to control the first terms in (2.9). Here we use the following (and truly remarkable) result of Strassen [13] (see also Stout [12], page 295):

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{1}{\sqrt{2n^3\log\log n}} \sum_{k=1}^{n} |S_k| = \frac{1}{\sqrt{3}} \right) = 1. \tag{2.11}$$

Let us first write

$$\left| \sum_{k=1}^{N_2(\beta_{n+1})} S_k\beta^k a(\beta)(1-\beta) - S_k\beta_n^k a(\beta_n)(1-\beta_n) \right|$$
$$\leq (1-\beta_n)a(\beta_n) \sum_{k=1}^{N_2(\beta_{n+1})} |S_k|\left(\beta_{n+1}^k - \beta_n^k\right)$$
$$+ |a(\beta_{n+1})(1-\beta_{n+1}) - a(\beta_n)(1-\beta_n)| \sum_{k=1}^{N_2(\beta_{n+1})} |S_k|\beta_n^k \tag{2.12}$$
$$\leq (1-\beta_n)(\beta_{n+1}-\beta_n)a(\beta_n) \sum_{k=1}^{N_2(\beta_{n+1})} k|S_k|\beta_{n+1}^{k-1}$$
$$+ |a(\beta_{n+1})(1-\beta_{n+1}) - a(\beta_n)(1-\beta_n)| \sum_{k=1}^{N_2(\beta_{n+1})} |S_k|\beta_n^k.$$

Note that in deriving (2.12) we use the fact that for β between β_n and β_{n+1}, $a(\beta)(1-\beta)\beta^k \geq a(\beta_{n+1})(1-\beta_{n+1})\beta_{n+1}^k$. Let us consider the second term in (2.12) first. Notice that

$$|a(\beta_{n+1})(1-\beta_{n+1}) - a(\beta_n)(1-\beta_n)|$$
$$= a(\beta_n)(\beta_{n+1}-\beta_n) + (1-\beta_{n+1})\left(\dot{a}(\beta_n) - a(\beta_{n+1})\right) \tag{2.13}$$
$$\leq c\frac{1-\theta_n}{N(\beta_n)} \frac{1}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}},$$

where $c \approx 1.5$ for n large enough.

We split the sum into the piece where k runs from 1 to $3N(\beta_{n+1})\log\log N(\beta_{n+1})$ and the rest. Recall that

$$N_2(\beta_{n+1}) = N(\beta_{n+1})\log N(\beta_{n+1}).$$

For the second piece we have

$$\begin{aligned}
&|a(\beta_{n+1})(1-\beta_{n+1}) - a(\beta_n)(1-\beta_n)| \sum_{k=3N(\beta_{n+1})\log\log N(\beta_{n+1})}^{N_2(\beta_{n+1})} |S_k|\beta_{n+1}^k \\
&\quad\le |a(\beta_{n+1})(1-\beta_{n+1}) - a(\beta_n)(1-\beta_n)|\beta_{n+1}^{3N(\beta_{n+1})\log\log N(\beta_{n+1})} \sum_{k=1}^{N_2(\beta_{n+1})} |S_k| \\
&\quad\le \frac{c(1-\theta_n)}{\sqrt{2N(\beta_{n+1})^3\log\log N(\beta_{n+1})}} \\
&\qquad\times\left(1-\frac{1}{N(\beta_{n+1})}\right)^{(3/2)N(\beta_{n+1})\log\log N(\beta_{n+1})} \sum_{k=1}^{N_2(\beta_{n+1})} |S_k| \\
&\quad\le \frac{c(1-\theta_n)}{\sqrt{2N(\beta_{n+1})^3\log^3 N(\beta_{n+1})\log\log N(\beta_{n+1})}} \sum_{k=1}^{N_2(\beta_{n+1})} |S_k|.
\end{aligned} \tag{2.14}$$

In the last two lines we made use of the fact that $N(\beta_n)/N(\beta_{n+1}) \to 1$ as $n\uparrow\infty$. Since $1-\theta_n$ converges to zero as $n\to\infty$, using (2.11) we see that (2.14) converges to zero almost surely.

For the first piece of (2.12) we get similarly the bound

$$\frac{c(1-\theta_n)}{\sqrt{2N(\beta_{n+1})^3\log\log N(\beta_{n+1})}} \sum_{k=1}^{3N(\beta_n)\log\log N(\beta_n)} |S_k| \tag{2.15}$$

and using that

$$1-\theta_n = \frac{1}{\log^3 n} \le \frac{c}{(\log\log N(\beta_{n+1}))^3},$$

(2.15) goes to zero a.s. for the same reason as before.

The term

$$(1-\beta_n)(\beta_{n+1}-\beta_n)a(\beta_n)\sum_{k=1}^{N_2(\beta_{n+1})} k|S_k|\beta_{n+1}^{k-1}$$

is treated along the same lines, but splitting the sum this time at $5N(\beta_{n+1})\log\log N(\beta_{n+1})$. This concludes the proof of Lemma 2.2. □

REMARK. We have chosen $\theta_n = 1-(1/(\log n)^3)$ in order to simplify the proof of the previous lemma; in principle, more refined estimates obtained by splitting the sums into m pieces at the points $c_j N \log_j^+ N$ will allow us to prove this lemma for sequences θ_n approaching one in an arbitrarily slow way.

We can now concentrate on the $\bar{\xi}(\beta_n)$. Following the ideas of De Acosta, we will introduce a (β and k dependent) truncation of the variables ε_k, that is, we write

$$\varepsilon_k = \varepsilon_k^t(\beta) + \varepsilon_k^q(\beta), \tag{2.16}$$

where

$$\varepsilon_k^t(\beta) \equiv \varepsilon_k \mathbb{1}_{\left\{|\varepsilon_k| < \tau\beta^{-k}/\sqrt{2(1-\beta^2)\log\log(1/(1-\beta^2))}\right\}}, \tag{2.17}$$

and

$$\varepsilon_k^q(\beta) \equiv \varepsilon_k \mathbb{1}_{\left\{|\varepsilon_k| \geq \tau\beta^{-k}/\sqrt{2(1-\beta^2)\log\log(1/(1-\beta^2))}\right\}}, \tag{2.18}$$

where τ is a positive real number to be chosen later.

The following lemma establishes that the tails of the truncation are negligible:

Lemma 2.3. *Let ε_k be i.i.d. r.v.'s with mean zero and variance one. Choose β_n and θ_n as in Lemma* 2.2 *and define*

$$\bar{\xi}_1^q(\beta_n) \equiv \frac{\sqrt{1-\beta_n^2}}{\sqrt{2\log\log\left(1/(1-\beta_n^2)\right)}} \sum_{k=0}^{N_2(\beta_n)} \beta_n^k \varepsilon_k^q(\beta_n). \tag{2.19}$$

Then

$$\lim_{n\to\infty} \bar{\xi}_1^q(\beta_n) = 0 \quad a.s. \tag{2.20}$$

Proof. Let us first observe that our truncation is related to that of De Acosta [3]: Since [for $k \leq N_2(\beta_n)$]

$$\frac{\beta^{-k}}{\sqrt{2(1-\beta^2)\log\log\left(1/(1-\beta^2)\right)}} \geq \sqrt{\frac{k}{2\log\log k}}, \tag{2.21}$$

$$\mathbb{1}_{\left\{|\varepsilon_k| \geq \tau\beta^{-k}/\sqrt{2(1-\beta^2)\log\log(1/(1-\beta^2))}\right\}} \leq \mathbb{1}_{\left\{|\varepsilon_k| \geq \tau\sqrt{(k/2\log\log k)}\right\}}.$$

We can easily adopt De Acosta's proof of his Lemma 2.3 of [3] to show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \sum_{k=0}^{N(\beta_n)} \beta_n^k \varepsilon_k^q(\beta_n) = 0 \quad \text{a.s.} \tag{2.22}$$

Just notice that by (2.21),

$$\begin{aligned} &\left| \frac{1}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \sum_{k=0}^{N(\beta_n)} \beta_n^k \varepsilon_k^q(\beta_n) \right| \\ &\quad \leq \frac{1}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \sum_{k=0}^{N(\beta_n)} |\varepsilon_k| \mathbb{1}_{\left\{|\varepsilon_k| \geq \tau\sqrt{(k/2\log\log k)}\right\}}. \end{aligned} \tag{2.23}$$

The right-hand side of (2.23) has been shown to converge to zero in [3]. We repeat the argument for completeness: Kronecker's lemma implies that (2.23) converges to zero as $N(\beta_n)$ goes to infinity if

$$\sum_{k=5}^{\infty} \frac{1}{\sqrt{2k \log\log k}} |\varepsilon_k| \mathbb{1}_{\{|\varepsilon_k| \geq \tau\sqrt{(k/2\log\log k)}\}} < \infty \quad \text{a.s.,} \tag{2.24}$$

which in turn is true if the expectation of the sum is finite. But

$$\sum_{k=5}^{\infty} \frac{1}{\sqrt{2k \log\log k}} \mathbb{E}\Big(|\varepsilon_k| \mathbb{1}_{\{|\varepsilon_k| \geq \tau\sqrt{(k/2\log\log k)}\}}\Big)$$

$$\leq \sum_{k=5}^{\infty} \frac{1}{\sqrt{2k \log\log k}} \sum_{j=k}^{\infty} \mathbb{E}\Big(|\varepsilon_k| \mathbb{1}_{\{\tau\sqrt{(j/2\log\log j)} \leq |\varepsilon_k| \leq \tau\sqrt{((j+1)/(2\log\log(j+1)))}\}}\Big)$$

$$\begin{aligned} &\leq \sum_{j=5}^{\infty} \tau\sqrt{\frac{j+1}{\log\log(j+1)}}\, \mathbb{P}\left(\tau\sqrt{\frac{j}{2\log\log j}} \leq |\varepsilon| \leq \tau\sqrt{\frac{j+1}{2\log\log(j+1)}}\right) \\ &\quad \times \sum_{k=5}^{j} \frac{1}{\sqrt{2k\log\log k}} \end{aligned} \tag{2.25}$$

$$\leq c \sum_{j=5}^{\infty} \tau \frac{j+1}{\log\log(j+1)} \mathbb{P}\left(\tau\sqrt{\frac{j}{2\log\log j}} \leq |\varepsilon| \leq \tau\sqrt{\frac{j+1}{2\log\log(j+1)}}\right)$$

$$\leq c'\mathbb{E}\big(|\varepsilon|^2\big) = c',$$

and we are done.

Notice that the same argument shows that with

$$S_k^q \equiv \sum_{j=0}^{k} \varepsilon_j^q(\beta_n), \tag{2.26}$$

$$\lim_{k\to\infty} \frac{1}{\sqrt{2k \log\log k}} S_k^q = 0 \quad \text{a.s.} \tag{2.27}$$

By (2.22) we are left with showing that

$$\frac{1}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \sum_{k=N(\beta_n)}^{N(\beta_n)\log N(\beta_n)} \beta_n^k \varepsilon_k^q(\beta_n) \to 0 \quad \text{a.s.} \tag{2.28}$$

Now notice that

$$\begin{aligned} \sum_{k=N(\beta_n)}^{N(\beta_n)\log N(\beta_n)} \beta_n^k \varepsilon_k^q(\beta_n) &= \sum_{k=N(\beta_n)}^{N(\beta_n)\log N(\beta_n)} \beta_n^k (S_k^q - S_{k-1}^q) \\ &= \sum_{k=N(\beta_n)}^{N(\beta_n)\log N(\beta_n)} \beta_n^k S_k^q (1-\beta_n) \\ &\quad + \beta^{N\ln N+1} S_{N\ln N}^q - \beta^N S_{N-1}^q. \end{aligned} \tag{2.29}$$

Now $\beta_n^{N(\beta_n)} \sim e^{-1/2}$, and by the remark (2.27),

$$\lim_{n\to\infty} \frac{1}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \beta_n^{N(\beta_n)} S^q_{N(\beta_n)-1} = 0 \quad \text{a.s.} \tag{2.30}$$

Also, since

$$\beta_n^{N(\beta_n)\log N(\beta_n)} \sim \frac{1}{\sqrt{N(\beta_n)}},$$

$$\frac{\beta_n^{N(\beta_n)\log N(\beta_n)+1} S^q_{N(\beta_n)\log N(\beta_n)}}{\sqrt{2N(\beta_n)\log\log N(\beta_n)}} \sim \frac{S^q_{N(\beta_n)\log N(\beta_n)}}{N(\beta_n)\sqrt{2\log\log N(\beta_n)}},$$

which is easily seen to converge to zero by the Marcinkiewicz–Zygmund theorem [10] (see also [12, 4]).

Thus, the last two terms in (2.29) converge to zero a.s., and we just have to consider the sum. Since

$$1-\beta_n \sim \frac{1}{2N(\beta_n)} \quad \text{and for } N \le k \le N\log N, \quad \beta_n^k \sim e^{-(k/2N)},$$

what we have to show is that

$$Z_N \equiv \frac{1}{N^{3/2}\sqrt{2\log\log N}} \sum_{k=N}^{N\log N} e^{-(k/2N)} |S_k^q| \to 0 \quad \text{a.s.} \tag{2.31}$$

Putting $X_k \equiv |S_k^q|/\sqrt{2k\log\log k}$, we have

$$\begin{aligned} Z_N &= \frac{1}{N^{3/2}\sqrt{2\log\log N}} \sum_{k=N}^{N\log N} e^{-(k/2N)}\sqrt{2k\log\log k}\, X_k \\ &\le \max_{N\le k\le N\log N} X_k \frac{1}{N^{3/2}\sqrt{2\log\log N}} \sum_{k=N}^{N\log N} e^{-(k/2N)}\sqrt{2k\log\log k}. \end{aligned} \tag{2.32}$$

Now

$$\begin{aligned} &\frac{1}{N^{3/2}\sqrt{2\log\log N}} \sum_{k=N}^{N\log N} e^{-(k/2N)}\sqrt{2k\log\log k} \\ &\le 2\frac{1}{N}\sum_{k=N}^{N\log N} e^{-(k/2N)}\sqrt{\frac{k}{N}} \\ &\le c\int_1^\infty e^{-x/2}\sqrt{x}\,dx \equiv \gamma < \infty \end{aligned} \tag{2.33}$$

and thus

$$Z_{N(\beta_n)} \le \gamma \max_{k\ge N(\beta_n)} X_k. \tag{2.34}$$

But since $X_k \to 0$ a.s., $\max_{k\ge N} X_k$ converges to zero a.s., too. Using (2.34) together with the fact that Z_N is positive, this implies that $Z_{N(\beta_n)}$ converges to zero a.s. as n goes to infinity, and the lemma is proved. □

It remains now to consider the variables

$$\bar{\xi}_1^t(\beta_n) \equiv \frac{\sqrt{1-\beta_n^2}}{\sqrt{2\log\log(1/(1-\beta_n^2))}} \sum_{k=0}^{N_2(\beta_n)} \beta_n^k \varepsilon_k^t(\beta_n). \tag{2.35}$$

LEMMA 2.4. *Under the same conditions as in Lemma* 2.3,

$$\begin{aligned} &\limsup_{n\to\infty} \bar{\xi}_1^t(\beta_n) \le 1, \quad a.s.,\\ &\liminf_{n\to\infty} \bar{\xi}_1^t(\beta_n) \ge -1, \quad a.s. \end{aligned} \tag{2.36}$$

PROOF. Let us first note that $\mathbb{E}(\bar{\xi}_1^t(\beta_n)) \to 0$. This follows from $\mathbb{E}(\bar{\xi}_1^t(\beta_n)) = -\mathbb{E}(\bar{\xi}_1^q(\beta_n))$, since $\mathbb{E}(\varepsilon_k) = 0$, and the proof of Lemma 2.3. Thus let $\tilde{\varepsilon}_k^t(\beta) \equiv \varepsilon_k^t(\beta) - \mathbb{E}(\varepsilon_k^t(\beta))$. Thus

$$\begin{aligned} \tilde{\bar{\xi}}_1^t(\beta_n) &\equiv \bar{\xi}_1^t(\beta_n) - \mathbb{E}(\bar{\xi}_1^t(\beta_n))\\ &= \frac{\sqrt{1-\beta_n^2}}{\sqrt{2\log\log(1/(1-\beta_n^2))}} \sum_{k=0}^{N_2(\beta_n)} \beta_n^k \tilde{\varepsilon}_k^t(\beta). \end{aligned} \tag{2.37}$$

Now, for any real s, we have

$$\begin{aligned} \mathbb{E}\big(e^{s\tilde{\bar{\xi}}_1^t(\beta_n)}\big) \le \prod_{k=0}^{N_2(\beta_n)} \Bigg(1 &+ \frac{s^2}{2}\frac{\beta_n^{2k}(1-\beta_n^2)}{2\log\log(1/(1-\beta_n^2))}\\ &\times \exp\left[|s|\sqrt{\frac{1-\beta_n^2}{2\log\log(1/(1-\beta_n^2))}}\,\|\tilde{\varepsilon}_k^t(\beta_n)\|_\infty \beta_n^k\right]\Bigg), \end{aligned} \tag{2.38}$$

where we have used that $e^x \le 1 + x + (x^2/2)e^{|x|}$ and $\mathbb{E}(\tilde{\varepsilon}_k^t(\beta_n)^2) \le \mathbb{E}(\varepsilon_k^2) = 1$.

Using the fact that $|\tilde{\varepsilon}_k^t(\beta_n)|$ is bounded by

$$\frac{2\tau\beta_n^{-k}}{\sqrt{2(1-\beta_n^2)\log\log(1/(1-\beta_n^2))}}$$

and that for positive a, $1 + a \le e^a$, we get

$$\begin{aligned} \mathbb{E}\big(e^{t\tilde{\bar{\xi}}_1^t(\beta_n)}\big) \le \exp\Bigg\{&\frac{s^2}{4\log\log(1/(1-\beta_n^2))}\\ &\times \exp\left[|s|\frac{2\tau}{2\log\log(1/(1-\beta_n^2))}\right]\Bigg\}. \end{aligned} \tag{2.39}$$

Using the exponential Markov inequality, we get for all $t > 0$,

$$\mathbb{P}\left(\bar{\xi}_1^t(\beta_n) \geq 1 + \gamma\right) \leq e^{-s(1+\gamma)}\mathbb{E}\left(e^{s\bar{\xi}_1^t(\beta_n)}\right). \tag{2.40}$$

Choosing $t = 2(1+\gamma)\log\log(1/(1-\beta_n^2))$, we obtain

$$\mathbb{P}\left(\bar{\xi}_1^t(\beta_n) \geq 1 + \gamma\right) \leq \exp\left\{-(1+\gamma)^2(2 - e^{(1+\gamma)2\tau})\log\log\frac{1}{1-\beta_n^2}\right\}. \tag{2.41}$$

For any positive γ we can now choose $\tau > 0$ small enough such that $(1+\gamma)^2(2 - e^{(1+\gamma)2\tau}) > 1$. But

$$\begin{aligned} \exp\{-\log\log(1/(1-\beta_n^2))\} &= \frac{1}{\log(1/(1-\beta_n^2))} \\ &\sim \frac{1}{-n\log\theta_n} \sim \frac{\log^3 n}{n}. \end{aligned} \tag{2.42}$$

Thus, the right-hand side of (2.41) is the general term of a summable series, and using the Borel–Cantelli lemma, we conclude the proof of the lemma for the lim sup. The proof for the lim inf is identical, and we are done. □

Collecting the results of this section, we have proven the following:

PROPOSITION 2.5. *Let ε_k be i.i.d. r.v.'s with mean zero and variance one. Then*

$$\limsup_{\beta\uparrow 1}\frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log(1/(1-\beta^2))}}\sum_{k=0}^{\infty}\beta^k\varepsilon_k \leq 1 \quad a.s., \tag{2.43}$$

the lim inf *being greater than or equal to* -1.

PROOF. Just notice that

$$\bar{\xi}(\beta) = \bar{\xi}_2(\beta) + \left(\bar{\xi}_1(\beta) - \bar{\xi}_1(\beta_n)\right) + \bar{\xi}_1^q(\beta_n) + \bar{\xi}_1^t(\beta_n), \tag{2.44}$$

and apply the four preceding lemmas to the four terms. □

With Proposition 2.5 we have proved the first part of Theorem 1.1, namely, that all possible limit points of $\xi(\beta)$ lie in the interval $[-1, 1]$. The second part, namely to show that all points in $[-1, 1]$ are limit points a.s., will be proved in the next section.

3. The cluster set. In this section we conclude the proof of Theorem 1.1 by showing that for any point $b \in [-1, 1]$, there exists some subsequence β_k such that

$$\lim_{k\to\infty}\bar{\xi}(\beta_k) = b \quad \text{a.s.} \tag{3.1}$$

That is, if we denote by $\mathscr{C}(\{\bar{\xi}(\beta)\})$ the cluster set of $\bar{\xi}(\beta)$,

$$\mathbb{P}\big(\mathscr{C}(\{\bar{\xi}(\beta)\}) = [-1,1]\big) = 1. \tag{3.2}$$

To prove this, we first introduce, following [2], for some $\delta > 0$, the sequence

$$\beta_k \equiv \exp\left\{-\frac{1}{k!(\log k)^2!(\log\log k)^\delta!}\right\}, \tag{3.3}$$

where $(\log k)^x! \equiv \prod_{j=2}^k(\log j)^x$, and so on. We also set

$$N_1(\beta_k) \equiv \frac{\log(1-(1/\log k))}{2\log\beta_k}. \tag{3.4}$$

Notice that

$$N(\beta_k) \sim \frac{1}{2}k!(\log k)^2!(\log\log k)^\delta!, \tag{3.5}$$

$$N_1(\beta_k) \sim \frac{N(\beta_k)}{2\log k} \tag{3.6}$$

and

$$N_2(\beta_k) = N(\beta_k)\log N(\beta_k) \sim N(\beta_k)k\log k. \tag{3.7}$$

We split $\xi(\beta_k)$ into three pieces,

$$\xi(\beta_k) = \sum_{i=1}^{3}\xi_i(\beta_k),$$

where

$$\xi_i(\beta_k) \equiv \sum_{n=N_{i-1}(\beta_k)}^{N_i(\beta_k)} \beta_k^n\varepsilon_n, \tag{3.8}$$

where we set $N_0 = 0$ and $N_3 = \infty$. For the remainder of this paper we set

$$\frac{1}{a_k} \equiv \sqrt{\frac{1-\beta_k^2}{2\log\log\big(1/(1-\beta_k^2)\big)}}. \tag{3.9}$$

The purpose of this new decomposition of $\xi(\beta)$ is the following. We want to extract a subsequence β_k in such a way that the $\xi(\beta_k)$ are essentially independent random variables. The idea is that first $\xi(\beta_k)$ will differ from $\xi_2(\beta_k)$ only by terms that divided by a_k converge to zero, and second that if $N_1(\beta_{k+1}) > N_2(\beta_k)$, for them $\xi_2(\beta_{k+1})$ and $\xi_2(\beta_k)$ are manifestly independent. This sets up a number of recursive conditions on $N_i(\beta_k)$ that will be seen to be satisfied for our choice of β_k.

First, notice that by Lemma 2.1,

$$\lim_{k\to\infty}\frac{1}{a_k}\xi_3(\beta_k) = 0 \quad \text{a.s.} \tag{3.10}$$

For $\xi_1(\beta_k)$, we get

$$(3.11)\quad \begin{aligned} \frac{1}{a_k}|\xi_1(\beta_k)| &= \frac{1}{\sqrt{2N(\beta_k)\log\log N(\beta_k)}}\left|\sum_{n=0}^{N_1(\beta_k)} \beta_k^n \varepsilon_n\right| \\ &\le \sqrt{\frac{N_1(\beta_k)}{N(\beta_k)}}\frac{1}{\sqrt{2N_1(\beta_k)\log\log N_1(\beta_k)}}\left|\sum_{n=0}^{N_1(\beta_k)} \beta_k^n \varepsilon_n\right|. \end{aligned}$$

Now

$$(3.12)\quad \limsup_{k\to\infty}\left|\frac{1}{\sqrt{2N_1(\beta_k)\log\log N_1(\beta_k)}}\left|\sum_{n=0}^{N_1(\beta_k)} \beta_k^n \varepsilon_n\right|\right| \le 1 \quad \text{a.s.}$$

Indeed, with β_k^n in the sum replaced by 1, this would be implied by the standard LIL. But on the other hand, for the range of summation, β_k^n is always close to one. One may thus simply repeat the proof of the LIL as given by De Acosta (or see our Section 2), taking into account that when estimating the tails of the truncation the bound $|\beta_k^n \varepsilon_n^q(\beta_k)| \le |\varepsilon_n^q(\beta_k)|$ can be used, and that again in the estimates of the truncated variables one may bound $\sum_{n=0}^{N_1(\beta_k)}\beta_k^{2n}$ by $N_1(\beta_k)$, which will give the result.

Since $N_1(\beta_k)/N(\beta_k) \sim 1/(\log k) \to 0$, it is immediate that the lim sup of (3.9) is zero almost surely.

Finally one may check, using (3.4), (3.5) and (3.6) (see also [2]) that

$$N_1(\beta_{k+1}) \ge N_2(\beta_k),$$

so that the random variables $\{\xi_2(\beta_k)\}_{k=1}^{\infty}$ are independent. In order to prove (3.1), it is enough to show that for any interval $[c, d]$ containing b,

$$(3.13)\quad \mathbb{P}\left(\frac{1}{a_k}\xi_2(\beta_k) \in [c, d], \text{i.o.}\right) = 1.$$

Using independence and the second Borel–Cantelli lemma, this in turn will follow from the following:

LEMMA 3.1. *For any $b \in [-1, 1]$ and any interval $[c, d]$ s.t. $b \in (c, d)$,*

$$(3.14)\quad \sum_{k=k_0}^{\infty} \mathbb{P}\left(\frac{1}{a_k}\xi_2(\beta_k) \in [c, d]\right) = +\infty.$$

The proof of this lemma will closely follow the ideas of De Acosta [3] and De Acosta and Kuelbs [4]. Since in this proof we make use of our (central limit) Theorem 1.2, let us restate this theorem and outline its proof:

THEOREM 1.2 (Central-Limit Theorem). *Let ε_n be i.i.d. r.v.'s with mean zero and variance one. Let $g(\beta)$ be an increasing function from* $[0, 1]$ *to* $\mathbb{N} \cup \infty$

such that

$$\lim_{\beta \to 1} g(\beta) = +\infty.$$

Put $f(\beta) \equiv \sum_{n=0}^{g(\beta)} \beta^{2n}$ *and define*

$$Z(\beta) \equiv \frac{1}{\sqrt{f(\beta)}} \sum_{n=0}^{g(\beta)} \beta^n \varepsilon_n.$$

Then $Z(\beta)$ *converges in law to a Gaussian r.v. with mean zero and variance one as* $\beta \to 1$.

PROOF. This theorem can be proved along the lines of the standard proofs of the central limit theorem; the one found in Shiryayev's text [11] is particularly convenient.

One introduces $X_n(\beta) \equiv (1/\sqrt{f(\beta)})\beta^n \varepsilon_n$ and puts

$$F_{n,\beta}(x) \equiv \mathbb{P}(X_n(\beta) \le x)$$

and

$$\Phi_{n,\beta}(x) \equiv \Phi\left(x\beta^{-n}\sqrt{f(\beta)}\right),$$

where $\Phi(x)$ is the distribution function of the normal distribution.

The analogue of the Lindeberg condition [11] that will be required for the proof is that for all $\delta > 0$,

$$C(\beta) \equiv \sum_{n=0}^{g(\beta)} \int_{|x|>\delta} x^2 \, dF_{n,\beta}(x) \to 0 \quad \text{as } \beta \to 1. \tag{3.15}$$

With this condition and the notation above, Theorem 3.2 follows step by step as in [11]. We will leave it to the reader to check the details.

To prove that (3.15) holds, notice that

$$\sum_{n=0}^{g(\beta)} \int_{|x|>\delta} x^2 \, dF_{n,\beta}(x) = \sum_{n=0}^{g(\beta)} \int_{|x|>\delta} x^2 \, dF\left(\sqrt{f(\beta)}\,\beta^{-n}x\right), \tag{3.16}$$

where $F(x)$ denotes the common distribution function of the ε_n. Thus

$$\begin{aligned} C(\beta) &= \sum_{n=0}^{g(\beta)} \frac{1}{f(\beta)} \beta^{2n} \int_{|x|>\delta\beta^{-n}\sqrt{f(\beta)}} x^2 \, dF(x) \\ &\le \sum_{n=0}^{g(\beta)} \frac{1}{f(\beta)} \beta^{2n} \int_{|x|>\delta\sqrt{f(\beta)}} x^2 \, dF(x) \qquad (3.17) \\ &= \int_{|x|>\delta\sqrt{f(\beta)}} x^2 \, dF(x). \end{aligned}$$

The last expression converges to zero since $f(\beta) \to \infty$ and $\int x^2 \, dF(x) = 1$.

This concludes the proof of Theorem 3.2. □

PROOF OF LEMMA 3.1. The main idea of the proof of the lemma is now to represent $\xi_2(\beta_k)$ as an average of $q_k \equiv t^{-2} 2 \log\log N(\beta_k)$ variables, each of which converges to a Gaussian r.v. of variance one by the preceding central limit theorem. To do this, we introduce numbers p_{ik} satisfying

$$\beta_k^{2\sum_{i=0}^{l-1} p_{ik}}\left(1 - \beta_k^{2p_{lk}}\right) t^{-2} 2 \log\log \frac{1}{1-\beta_k^2} = 1, \tag{3.18}$$

and put

$$\begin{aligned} Z_l(\beta_k) \equiv \beta_k^{\sum_{i=0}^{l-1} p_{ik}} \sum_{n=0}^{p_{lk}-1} \beta_k^n \varepsilon_{n+\sum_{i=0}^{l-1} p_{ik} + N_1(\beta_k)} t^{-1} \\ \times \sqrt{(1-\beta_k^2) 2 \log\log \frac{1}{1-\beta_k^2}}. \end{aligned} \tag{3.19}$$

Then

$$\bar{\xi}_2(\beta_k) = \frac{t^{-1}}{q_k} \beta_k^{N_1(\beta_k)} \sum_{l=0}^{q_k} Z_l(\beta_k). \tag{3.20}$$

The p_{lk} are defined in such a way that $\mathbb{E}(Z_l(\beta_k)^2) = 1$, and that moreover p_{ik} converges to infinity with k uniformly in l. To see this, just note that $\sum_{i=0}^{l-1} P_{ik} \le N_2(\beta_k)$ for all l considered and thus

$$\left(1 - \beta_k^{2p_{lk}}\right) \sim -2 \log \beta_k p_{lk} \ge \frac{t^2}{2 \log\log\left(1/\left(1-\beta_k^2\right)\right)} \beta_k^{-2N_2(\beta_k)},$$

from which we get, using (3.3) and (3.7), that $p_{lk} \ge (\log^2 k)!$. Notice further that $\beta_k^{N_1(\beta_k)} \sim \exp[-(1/4 \log k)]$ converges to one in the limit $k \to \infty$ and may thus be ignored. Now,

$$\mathbb{P}\left(\bar{\xi}_2(\beta_k) \in [c,d]\right) \ge \prod_{l=0}^{q_k} \mathbb{P}\left(Z_l(\beta_k) \in t[c,d]\right). \tag{3.21}$$

By the remarks above and using Theorem 1.2, for any fixed $t < \infty$,

$$\mathbb{P}\left(Z_l(\beta_k) \in t[c,d]\right) \to \gamma(t[c,d]), \tag{3.22}$$

uniformly in l, where

$$\gamma(t[c,d]) \equiv \frac{1}{\sqrt{2\pi}} \int_{x \in t[c,d]} e^{-(x^2/2)}\, dx. \tag{3.23}$$

Now let $c = b - \delta$, $d = b + \delta$. We have (see De Acosta [3])

$$\gamma(t[c,d]) \ge \exp\left[-(t^2 b^2/2) + \log \gamma([-t\delta, t\delta])\right], \tag{3.24}$$

and thus, asymptotically

$$\begin{aligned} \mathbb{P}\left(\bar{\xi}_2(\beta_k) \in [c,d]\right) \ge \exp\Bigg[-\frac{1}{2} b^2 \log\log \frac{1}{1-\beta_k^2} \\ + \frac{1}{t^2} \log \gamma([-t\varepsilon, t\varepsilon]) \log\log \frac{1}{1-\beta_k^2} \Bigg]. \end{aligned} \tag{3.25}$$

Recalling that $\log(1/(1-\beta_k^2)) \sim k \log k$, we see that for any $b \in (-1, 1)$ and $\delta > 0$, if t is chosen large enough, the right-hand side of (3.25) is the general term of a divergent series. But this proves the lemma. □

Combined with the discussion in the beginning of this section, we see that we have actually proven (3.2) and thus the second assertion of Theorem 1.1. □

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