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STOCHASTIC MONOTONICITY AND SLEPIAN-TYPE INEQUALITIES FOR INFINITELY DIVISIBLE AND STABLE RANDOM VECTORS[1]

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We study the relation between stochastic domination of an infinitely divisible random vector $\mathbf{X}$ by another infinitely divisible random vector $\mathbf{Y}$ and their corresponding Lévy measures. The results are used to derive a Slepian-type inequality for a general class of symmetric infinitely divisible random vectors.

1. Introduction. The Slepian inequality (Slepian [19]) and its modifications are an essential ingredient in the proofs of many results concerning sample path properties of Gaussian processes. The inequality compares the behavior of the suprema of two Gaussian processes and is based on properties of the covariances. One version, due to Fernique [5], Corollaire 2.1.3, is as follows.

THEOREM 1.1. *Let* $\mathbf{X} = (X_1, \dots, X_d)$ *and* $\mathbf{Y} = (Y_1, \dots, Y_d)$ *be zero-mean Gaussian random vectors, such that for every* $i, j = 1, \dots, d$,

$$E(X_i - X_j)^2 \geq E(Y_i - Y_j)^2. \tag{1.1}$$

Then

$$E \max_{1 \leq i \leq d} X_i \geq E \max_{1 \leq i \leq d} Y_i.$$

(We have stated Theorem 1.1 for finite-dimensional Gaussian vectors and not for infinite-dimensional Gaussian processes in order to avoid considering suprema of random variables taken over an uncountable set. Slepian's inequality deals with qualitative properties of Gaussian processes, and thus there is no loss of generality in restricting our discussion to the finite-dimensional case. We shall do this throughout the paper.)

Note that condition (1.1) can be written succinctly as

$$ET\mathbf{X} \geq ET\mathbf{Y},$$

where $T\colon R^d \to R^{d(d-1)/2}$ is defined by

$$T(x_1, \dots, x_d) = \big((x_1 - x_2)^2, (x_1 - x_3)^2, \dots, \times (x_1 - x_d)^2, (x_2 - x_3)^2, \dots, (x_{d-1} - x_d)^2\big)$$

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and where an inequality between two vectors means inequality between each component. The transformation T is used many times in the sequel.

Theorem 1.1 can be explained roughly as follows: If, on average, the components of the vector $\mathbf{X}$ are more different from each other than the components of the vector $\mathbf{Y}$, then they are likely to fluctuate more and hence the maximal component of $\mathbf{X}$ is likely to be greater than the maximal component of $\mathbf{Y}$.

There is a different version of the Slepian inequality [19]: Assume (1.1) and $EX_i^2 = EY_i^2$, $i = 1, \dots, d$. Then $EX_iX_j \le EY_iY_j$, $i, j = 1, \dots, d$, and for every $-\infty < \lambda < \infty$, $P\{\max_{i=1,\dots,d} X_i \ge \lambda\} \ge P\{\max_{i=1,\dots,d} Y_i \ge \lambda\}$.

Theorem 1.1 compares $E \max_{1 \le i \le d} X_i$ with $E \max_{1 \le i \le d} Y_i$. The following corollary, which compares $E \max_{1 \le i \le d}|X_i|$ with $E \max_{1 \le i \le d}|Y_i|$, is obtained by applying Theorem 1.1 to the $(d+1)$-dimensional vectors $(\mathbf{X}, 0)$ and $(\mathbf{Y}, 0)$.

COROLLARY 1.1. *Let X and Y be zero-mean Gaussian random vectors in R^d. If*

$$E(X_i - X_j)^2 \ge E(Y_i - Y_j)^2, \qquad i, j = 1, \dots, d,$$

and

$$EX_i^2 \ge EY_i^2, \qquad i = 1, \dots, d,$$

then

$$E \max_{1 \le i \le d} |X_i| \ge \tfrac{1}{2} E \max_{1 \le i \le d} |Y_i|.$$

Being stated in terms of covariance matrices, Slepian's inequality is, in the preceding forms, a very specific property of Gaussian random vectors. Nevertheless, the inequality is so important that attempts have been made to generalize it to other random vectors and processes. What was striking to the authors was the absence of any positive results for symmetric α-stable processes, which are, in a sense, very close relatives of Gaussian processes.

A random vector $\mathbf{X} = (X_1, \dots, X_d)$ is called α-stable, $0 < \alpha < 2$, if for any $A > 0$, $B > 0$ there is a $\mathbf{D} \in R^d$ such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} =_d (A^\alpha + B^\alpha)^{1/\alpha}\mathbf{X} + \mathbf{D}, \tag{1.2}$$

where $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ are independent copies of X. An α-stable random vector is called strictly α-stable if $\mathbf{D} = \mathbf{0}$ for every A and B. An α-stable random vector $\mathbf{X}$ satisfying $\mathbf{X} =_d -\mathbf{X}$ is called symmetric α-stable (SαS). Clearly, an S2S vector is zero-mean Gaussian. Of course, one cannot use covariances or the L_2 distance in the SαS case, because $E|\mathbf{X}|^p < \infty$ only when $p < \alpha$, but one can try to mimic (1.1) by regarding it as a comparison between the *scale parameters* of $X_i - X_j$ and $Y_i - Y_j$, respectively. Scale parameters do make sense in the stable case too, and the scale parameter of an SαS random variable X is equal to $c(p, \alpha)(EX^p)^{1/p}$, $0 < p < \alpha$, where $c(p, \alpha)$ is a finite positive constant.

Therefore one might guess that the condition

$$E|X_i - X_j|^p \geq E|Y_i - Y_j|^p$$

for all $i, j = 1, \ldots, d$ implies the conclusion of Theorem 1.1 in the stable case. Unfortunately, this is not likely to be true, because in the SαS case, the scale parameters of the preceding differences give very little information on the actual distribution of the vectors $\mathbf{X}$ and $\mathbf{Y}$. Linde [13] (the title of his paper notwithstanding) demonstrates that a version of Slepian's inequality different from that of Theorem 1.1 fails in the stable case under these circumstances.

In fact, in contrast to the Gaussian case, a multivariate stable law cannot be specified in general by a finite number of numerical parameters, because every SαS random vector $\mathbf{X}$ has characteristic function of the form

$$(1.3) \quad \phi_{\mathbf{X}}(\boldsymbol{\theta}) = E \exp i\left(\sum_{j=1}^{d} \theta_j X_j\right) = \exp\left\{-\int_{S_d} |\theta_1 s_1 + \cdots + \theta_d s_d|^{\alpha} \Gamma(ds)\right\},$$

where Γ is a finite symmetric measure on Borel subsets of the unit sphere S_d in R^d, called the *spectral measures* of $\mathbf{X}$. Relation (1.3), moreover, defines a one-to-one correspondence between SαS laws in R^d and finite symmetric measures Γ on S_d. See Kuelbs [9] for details. Since it is the spectral measure Γ that specifies the SαS law, it is in terms of the spectral measures that one should try to compare two different stable random vectors.

Another argument supports this point. Joag-dev, Perlman and Pitt [6] have discovered a close relation between Slepian's inequality and *association* in the Gaussian case. We remind the reader that random variables $Z_1, \ldots, Z_d$ are associated if $\text{cov}(f(Z_1, \ldots, Z_d), g(Z_1, \ldots, Z_d)) \geq 0$ for all pairs of functions f, g: $R^d \to R^1$ which are nondecreasing in each argument and for which the covariance exists. It has been proved by Pitt [16] that jointly Gaussian random variables are associated if and only if all covariances are nonnegative. Now, the question of association of (not necessarily symmetric) stable random variables has been completely solved in Lee, Rachev and Samorodnitsky [11], and the criterion involves the spectral measure. This motivated the present research.

Technically, our approach is based on two key ideas. First, we derive a relation between Slepian's inequality and stochastic ordering in the context of positive random vectors. Then we exploit the fact that any SαS random vector is a mixture of zero-mean Gaussian vectors. This fact was brought to the consciousness of the mainstream of probability theory by LePage [12] and Marcus and Pisier [15]. Second, we derive a relation between Slepian's inequality and stochastic ordering, in the context of positive stable random vectors.

Although our original motivation did come from the stable case, it turns out that a similar approach works for a more general class of infinitely divisible random vectors, the so-called type G vectors, introduced by Marcus [14], using a series representation of these vectors given by Rosinski [18].

Our main results on Slepian's inequality in the stable and, more generally, in the infinitely divisible case seem to be among the very few positive results available so far in this context. The only other positive result we are aware of is due to Brown and Rinott [2], who consider infinitely divisible random vectors of a very special "linear" form, which makes their structure close to the structure of Gaussian random vectors.

A random vector $\mathbf{X} = (X_1, \dots, X_d)$ is infinitely divisible (without Gaussian component), or simply i.d., if there exists a σ-finite measure ν on the Borel subsets of $R^d - \{0\}$, called the *Lévy measure* of $\mathbf{X}$, satisfying $\int_{R^d}(1 \wedge |\mathbf{x}|^2)\nu(d\mathbf{x}) < \infty$, and a vector $\mathbf{b}$ in R^d such that the characteristic function of $\mathbf{X}$ has the form

$$(1.4)\quad \theta_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\left\{ \int_{R^d - \{0\}} \left(e^{i(\boldsymbol{\theta}, \mathbf{x})} - 1 - \frac{i(\boldsymbol{\theta}, \mathbf{x})}{1 + |\mathbf{x}|^2} \right) \nu(dx) + i(\boldsymbol{\theta}, \mathbf{b}) \right\}.$$

Stable random vectors constitute a subclass of i.d. random vectors, with Lévy measures of a special form, which we will describe later.

In the next section we discuss stochastic ordering of stable and, more generally, i.d. random vectors. In Section 3 we apply the results of Section 2 to derive our version of Slepian's inequality.

2. Stochastic ordering of stable and infinitely divisible random vectors. Let $\mathbf{X} = (X_1, \dots, X_d)$ and $\mathbf{Y} = (Y_1, \dots, Y_d)$ be random vectors in R^d. We say that $\mathbf{X}$ *dominates stochastically* $\mathbf{Y}$ (denoted $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$) if there is a random vector $\mathbf{Z} = (Z_1, \dots, Z_{2d})$ on R^{2d} such that

$$(2.1)\qquad (Z_1, \dots, Z_d) =_d \mathbf{X},$$

$$(2.2)\qquad (Z_{d+1}, \dots, Z_{2d}) =_d \mathbf{Y},$$

$$(2.3)\qquad Z_i \geq Z_{i+d} \quad \text{a.s., } i = 1, \dots, d.$$

A set A in R^d is called *increasing* if $x \in A$ and $y \geq x$ (componentwise) implies $y \in A$. The following is fundamental.

THEOREM 2.1. *The following are all equivalent*:

(i) $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$.

(ii) *For any function* f: $R^d \to R_+$, *nondecreasing in each argument*, $E(f(\mathbf{X})) \geq E(f(\mathbf{Y}))$.

(iii) *For any increasing Borel set* A *in* R^d, $P\{\mathbf{X} \in A\} \geq P\{\mathbf{Y} \in A\}$.

PROOF. The implications (i) → (ii) and (ii) → (iii) are trivial. For the implication (iii) → (i), see Strassen [20], Theorem 11. (In connection with Theorem 2.1, see also [7] and [8].) □

The third equivalent condition of Theorem 2.1 is a natural test for $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$ because it involves explicitly the probability laws of $\mathbf{X}$ and $\mathbf{Y}$. However, it is, at best, practical only when the distribution functions of $\mathbf{X}$ and $\mathbf{Y}$ are available in an explicit form. When these distributions do not have a nice form, one has to obtain an alternative test which uses the information available on the random vectors $\mathbf{X}$ and $\mathbf{Y}$. For example, in the case of an infinitely divisible random vector the information commonly available is the Lévy measure of the vector (the distribution functions are unfortunately computable only in very few cases, even when $d = 1$).

We will investigate in this section whether one can derive a test for stochastic domination of one i.d. random vector by another i.d. random vector in terms of their corresponding Lévy measures. We will see that this is possible in some cases, and the characterization is complete when the random vectors are strictly α-stable with $\alpha < 1$.

Consider an i.d. random vector $\mathbf{X}$ whose Lévy measure ν satisfies

$$\int_{R^d} (1 \wedge |\mathbf{x}|)\nu(d\mathbf{x}) < \infty. \tag{2.4}$$

In this case the characteristic function of $\mathbf{X}$ can be written in the form

$$\phi_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\left\{\int_{R^d - \{0\}} (e^{i(\boldsymbol{\theta}, \mathbf{x})} - 1)\nu(d\mathbf{x}) + i(\boldsymbol{\theta}, \mathbf{b})\right\} \tag{2.5}$$

after changing the shift vector $\mathbf{b}$ in (1.4). In particular, all positive i.d. vectors are of this form, as well as all α-stable random vectors with $\alpha < 1$. In the latter case, strict stability occurs if and only if $\mathbf{b} = 0$ in (2.5) (see Feller [4] and Kuelbs [9]).

The following theorem gives criteria for $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$.

THEOREM 2.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two i.d. random vectors satisfying* (2.4), *and let* $(\nu_{\mathbf{X}}, \mathbf{b}_{\mathbf{X}})$ *and* $(\nu_{\mathbf{Y}}, \mathbf{b}_{\mathbf{Y}})$ *be the corresponding parameters of their characteristic functions given in the form* (2.5). *Assume that* $\nu_{\mathbf{X}}$ *and* $\nu_{\mathbf{Y}}$ *are concentrated on* $R_+^d = \{\mathbf{x} \in R^d\colon x_i \geq 0,\ i = 1, \dots, d\}$. *If*

$$\nu_{\mathbf{X}}(A) \geq \nu_{\mathbf{Y}}(A) \quad \textit{for every increasing Borel set } A \textit{ in } R^d, \tag{2.6}$$

and

$$\mathbf{b}_{\mathbf{X}} \geq \mathbf{b}_{\mathbf{Y}} \quad \textit{componentwise}, \tag{2.7}$$

then

$$\mathbf{X} \geq_{\text{st}} \mathbf{Y}.$$

PROOF. For a $\delta > 0$ let

$$A_\delta = \{\mathbf{x} \in R^d\colon x_i \geq \delta \text{ for some } i = 1, \dots, d\}. \tag{2.8}$$

Observe that A_δ is an increasing set and $A_\delta \to R_+^d \setminus \{\mathbf{0}\}$ as $\sigma \to 0$.

Let $\nu_{\mathbf{X}}^{(\delta)}$ and $\nu_{\mathbf{Y}}^{(\delta)}$ be the respective restrictions of $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ to A_δ. Let $\mathbf{X}^{(\delta)}$ and $\mathbf{Y}^{(\delta)}$ be i.d. random vectors in R_+^d with Lévy measures $\nu_{\mathbf{X}}^{(\delta)}$ and $\nu_{\mathbf{Y}}^{(\delta)}$ and shift vectors $\mathbf{b}_{\mathbf{X}}$ and $\mathbf{b}_{\mathbf{Y}}$, respectively. By (2.6), for every increasing set $A \subseteq R^d$,

$$\nu_{\mathbf{X}}^{(\delta)}(A) = \nu_{\mathbf{X}}(A \cap A_\delta) \geq \nu_{\mathbf{Y}}(A \cap A_\delta) = \nu_{\mathbf{Y}}^{(\delta)}(A). \tag{2.9}$$

In particular,

$$\Delta := \nu_{\mathbf{X}}^{(\delta)}(A_\delta) - \nu_{\mathbf{Y}}^{(\delta)}(A_\delta) \geq 0.$$

Both $\mathbf{X}^{(\delta)}$ and $\mathbf{Y}^{(\delta)}$ are compound Poisson in the following sense. Let

$$c := \nu_{\mathbf{X}}^{(\delta)}(A_\delta) = \nu_{\mathbf{Y}}^{(\delta)}(A_\delta) + \Delta \tag{2.10}$$

and

$$F^{(\delta)} := c^{-1}\nu_{\mathbf{X}}^{(\delta)},$$

$$G^{(\delta)} := c^{-1}\big(\nu_{\mathbf{Y}}^{(\delta)} + \Delta\delta_{\mathbf{0}}\big).$$

These are probability measures on R_+^δ because, by (2.10), $F^{(\delta)}(R_+^d) = c^{-1}\nu_{\mathbf{X}}^{(\delta)}(R_+^d) = c^{-1}\nu_{\mathbf{X}}^{(\delta)}(A_\delta) = 1$ and $G^{(\delta)}(R_+^d) = c^{-1}(\nu_{\mathbf{Y}}^{(\delta)}(A_\delta) + \Delta) = 1$. By (2.9),

$$F^{(\delta)}(A) \geq G^{(\delta)}(A) \tag{2.11}$$

for every increasing Borel set A in R_+^d. This is clearly true if $\mathbf{0} \notin A$. If $\mathbf{0} \in A$, then $A = R_+^d$ because A is increasing, and so (2.11) still holds because $F^{(\delta)}(R_+^d) = 1 = G^{(\delta)}(R_+^d)$.

It is easy to check that

$$\mathbf{X}^{(\delta)} =_d \mathbf{V}_1 + \cdots + \mathbf{V}_M,$$

$$\mathbf{Y}^{(\delta)} =_d \mathbf{W}_1 + \cdots \mathbf{W}_M,$$

where $\mathbf{V}_1, \mathbf{V}_2, \dots$ and $\mathbf{W}_1, \mathbf{W}_2, \dots$ are sequences of i.i.d. random vectors in R_+^d with common laws $F^{(\delta)}$ and $G^{(\delta)}$, respectively. M is a Poisson random variable with parameter c, independent of the vectors $\mathbf{V}_1, \mathbf{V}_2, \dots$ and $\mathbf{W}_1, \mathbf{W}_2, \dots$. It follows from Theorem 2.1 that $\mathbf{V}_1 \geq_{\text{st}} \mathbf{W}_1$, and from the definition of stochastic dominance that $\mathbf{X}^{(\delta)} \geq_{\text{st}} \mathbf{Y}^{(\delta)}$ for every $\delta > 0$. Since $A_\delta \to R_+^d \setminus \{\mathbf{0}\}$ as $\delta \to 0$, we have $\mathbf{X}^{(\delta)} \Rightarrow \mathbf{X}$ and $\mathbf{Y}^{(\delta)} \Rightarrow \mathbf{Y}$, establishing the theorem. □

REMARK 1. Theorem 2.2 has the following extension to the case where $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ are concentrated on $R_+^d \cup R_-^d$. (A set A in R^d is called decreasing if $-A$ is increasing.)

If

$$\nu_{\mathbf{X}}(A) \geq \nu_{\mathbf{Y}}(A) \quad \text{for every increasing set } A \text{ in } R_+^d,$$

$$\nu_{\mathbf{X}}(A) \leq \nu_{\mathbf{Y}}(A) \quad \text{for every increasing set } A \text{ in } R_-^d,$$

$$b_{\mathbf{X}} \geq b_{\mathbf{Y}} \quad \text{componentwise},$$

then

$$\mathbf{X} \geq_{\text{st}} \mathbf{Y}.$$

Suppose indeed that $\mathbf{X}$ and $\mathbf{Y}$ are concentrated on R_-^d. Since $\nu_{-\mathbf{X}}(-A) = \nu_{\mathbf{X}}(A) \leq \nu_{\mathbf{Y}}(A) = \nu_{-\mathbf{Y}}(-A)$ for any increasing set $-A$, we have $-\mathbf{X} \leq_{\text{st}} -\mathbf{Y}$ by Theorem 2.2; that is, $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$. This observation and Theorem 2.2 imply the general case since we can write $\mathbf{X} = \mathbf{X}_+ + \mathbf{X}_-$ and $\mathbf{Y} = \mathbf{Y}_+ + \mathbf{Y}_-$, where $\mathbf{X}_+$ and $\mathbf{Y}_+$ are concentrated on R_+^d, $\mathbf{X}_-$ and $\mathbf{Y}_-$ are concentrated on $R_+^d \setminus \{\mathbf{0}\}$, $\mathbf{X}_+$ is independent of $\mathbf{X}_-$ and $\mathbf{Y}_+$ is independent of $\mathbf{Y}_-$.

REMARK 2. Theorem 2.2 fails in the case of i.d. random vectors whose Lévy measure does not satisfy (2.4). In fact, even a much weaker (and, at first glance, plausible) statement is false. Namely, it is not true, in general, that if the Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ of i.d. random vectors $\mathbf{X}$ and $\mathbf{Y}$ satisfy (2.6) and are concentrated on R_+^d, then there is a vector $\mathbf{a} \in R^d$ big enough so that $\mathbf{X} + \mathbf{a} \geq_{\text{st}} \mathbf{Y}$. For a simple counterexample (with $d = 1$), let $1 < \alpha < 2$, and let X and Y be totally skewed to the right α-stable random variables with scale parameters σ_1 and σ_2 correspondingly. That is, the (one-dimensional) Lévy measures ν_X and ν_Y are concentrated on $(0, \infty)$ and

$$\nu_X(dx) = c\sigma_1^\alpha x^{-(1+\alpha)}\,dx, \qquad \nu_Y(dx) = c\sigma_2^\alpha x^{-(1+\alpha)}\,dx, \qquad x > 0,$$

for some positive constant c. Now suppose

$$\sigma_1 > \sigma_2 > 0.$$

Then condition (2.6) of Theorem 2.2 is satisfied. We claim that there is no real a such that $X + a \geq_{\text{st}} Y$. Indeed, as $x \to \infty$,

$$-\log P(X \leq -x) \sim kx^{\alpha/(\alpha-1)}/\sigma_1^{\alpha/(\alpha-1)} \sim -\log P(X + a \leq -x),$$

$$-\log P(Y \leq -x) \sim kx^{\alpha(\alpha-1)}/\sigma_2^{\alpha/(\alpha-1)},$$

for some positive constant k. Since $\sigma_1 > \sigma_2 > 0$, no matter what a is,

$$P(Y \leq -x) = o(P(X + a \leq -x)), \qquad x \to \infty,$$

so that $X + a \geq_{\text{st}} Y$ is impossible. [In this example, the Lévy measures are concentrated on $(0, \infty)$ but the distributions have support on R^1.]

REMARK 3. Since the structure of i.d. random vectors considered in Brown and Rinott [2] is very simple, one can explicitly compute the Lévy measure of these vectors and then apply Theorem 2.2 to obtain conditions for these vectors to be stochastically ordered. We will not state these conditions here.

The following is a partial converse to Theorem 2.2.

THEOREM 2.3. *Suppose* $\mathbf{X}$ *and* $\mathbf{Y}$ *are* α*-stable random vectors with* $\alpha < 1$. *Then* $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$ *implies*

(2.12) $\qquad \nu_{\mathbf{X}}(A) \geq \nu_{\mathbf{Y}}(A)$ *for every increasing set* A *in* R^d,

(2.13) $\qquad \nu_{\mathbf{X}}(A) \leq \nu_{\mathbf{Y}}(A)$ *for every decreasing set* A *in* R^d.

PROOF. Let $\mathbf{X}$ and $\mathbf{Y}$ be α-stable, $0 < \alpha < 1$, with $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$ and let $\mathbf{Z}$ be a random vector on R^{2d} satisfying (2.1), (2.2) and (2.3). Note that $\mathbf{Z} =_{\text{d}} (\mathbf{X}, \mathbf{Y})$ is not necessarily α-stable. Let $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \dots$ be i.i.d. copies of $\mathbf{Z}$. For $n \geq 1$ let

$$\mathbf{S}^{(n)} = n^{-1/\alpha} \sum_{j=1}^{n} \mathbf{Z}^{(j)}$$

and write $\mathbf{S}^{(n)} = (S_1^{(n)}, \dots, S_{2d}^{(n)})$. Two things are clear about the sequence

$\{\mathbf{S}^{(n)}\}_{n=1,2,\ldots}^{\infty}$. First, for every $n \geq 1$,

$$\left(S_1^{(n)},\ldots,S_d^{(n)}\right) =_d \mathbf{X} + (n^{1-1/\alpha} - 1)\mathbf{b}_{\mathbf{X}},$$
$$\left(S_{d+1}^{(n)},\ldots,S_{2d}^{(n)}\right) =_d \mathbf{Y} + (n^{1-1/\alpha} - 1)\mathbf{b}_{Y},$$
$$S_i^{(n)} \geq S_{i+d}^{(n)} \quad \text{a.s. for every } i = 1,\ldots,d.$$

Second, the sequence $\{\mathbf{S}^{(n)}\}_{n=1,2,\ldots}^{\infty}$ is tight because $0 < \alpha < 1$ implies that the d-dimensional marginals of $\mathbf{S}^{(n)}$ converge weakly. Therefore, there is a subsequence $\{\mathbf{S}^{(n_k)}\}_{k=1,2,\ldots}$ converging weakly to a probability law H on R^{2d}. Let $\mathbf{U} = (U_1,\ldots,U_{2d})$ be a random vector in R^{2d} with the law H. Clearly, $(U_1,\ldots,U_d) =_d \mathbf{X}$ and $(U_{d+1},\ldots,U_{2d}) =_d \mathbf{Y}$. Moreover, the set $C = \{\mathbf{x} \in R^{2d}: x_i \geq x_{i+d},\ i = 1,\ldots,d\}$ is closed, and so by Theorem 29.1 of Billingsley [1] we have $H(C) = 1$, and hence $U_i \geq U_{i+d}$ a.s. for every $i = 1,\ldots,d$.

It also follows that the original vector $\mathbf{Z}$ belongs to the domain of partial attraction of the distribution H, so that H must be infinitely divisible. Let ν be the corresponding Lévy measure.

By the uniqueness of the Lévy measure,

$$\nu_{\mathbf{X}} = \nu \circ T_{\mathbf{X}}^{-1}, \qquad \nu_{\mathbf{Y}} = \nu \circ T_{\mathbf{Y}}^{-1}, \tag{2.14}$$

where $T_{\mathbf{X}}$ and $T_{\mathbf{Y}}$ are the projections of R^{2d} on the first d and the last d coordinates respectively. The Lévy measure ν satisfies (2.4) because $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ do and

$$\int_{R^{2d}}(1 \wedge |\mathbf{z}|)\nu(d\mathbf{z}) \leq \int_{R^d}(1 \wedge |\mathbf{x}|)\nu_{\mathbf{X}}(d\mathbf{x}) + \int_{R^d}(1 \wedge |\mathbf{y}|)\nu_{\mathbf{Y}}(d\mathbf{y}).$$

Therefore, the characteristic function of H can be written in the form (2.5). Let $\mathbf{c}$ be the corresponding shift vector.

We obtain

$$\begin{aligned} &E \exp i\left\{\sum_{k=1}^{d} \theta_k(U_k - U_{k+d})\right\} \\ &\quad = \exp\left\{\int_{R^{2d}-\mathbf{0}}\left(\exp\left(i\sum_{k=1}^{d}\theta_k(x_k - x_{k+d})\right) - 1\right)\nu(d\mathbf{x}) \right. \\ &\qquad\qquad \left. + i\sum_{k=1}^{d}\theta_k(c_k - c_{k+d})\right\}. \end{aligned} \tag{2.15}$$

Therefore,

$$\begin{aligned} &E \exp i\left\{\sum_{k=1}^{d} \theta_k(U_k - U_{k+d})\right\} \\ &\quad = \exp\left\{\int_{R^d-\mathbf{0}}\left(e^{i(\boldsymbol{\theta},\mathbf{y})} - 1 - \frac{i(\theta,\mathbf{y})}{1+|\mathbf{y}|^2}\right)\eta(d\mathbf{y}) + i(\boldsymbol{\theta},\boldsymbol{\gamma})\right\} \end{aligned} \tag{2.16}$$

for every $\boldsymbol{\theta} \in R^d$ for some $\boldsymbol{\gamma} \in R^d$, where the Lévy measure η is given by

$\eta = \nu \circ T^{-1}$, and where T: $R^{2d} \to R^d$ is given by

$$T(x_1, \dots, x_d, x_{d+1}, \dots, x_{2d}) = (x_1 - x_{d+1}, x_2 - x_{d+2}, \dots, x_d - x_{2d}).$$

Since the i.d. random vector $(U_1 - U_{d+1}, U_2 - U_{d+2}, \dots, U_d - U_{2d})$ is concentrated on R_+^d, we conclude that $\eta(R^d - R_+^d) = 0$, which is, of course, equivalent to

$$\nu\{\mathbf{x} \in R^{2d}: x_i < x_{i+d} \text{ for some } i = 1, 2, \dots, d\} = 0. \tag{2.17}$$

Let A be an increasing Borel set in R^d. Then by (2.17) and (2.14) we get

$$\nu_{\mathbf{Y}}(A) = \nu(R^d \times A) = \nu((R^d \times A) \cap C)$$
$$= \nu((A \times A) \cap C) = \nu(A \times A) \le \nu(A \times R^d) = \nu_{\mathbf{X}}(A),$$

where $C = \{\mathbf{x} \in R^{2d}: x_i \ge x_{i+d} \text{ for every } i = 1, 2, \dots, d\}$, thus proving (2.12). The proof of (2.13) is similar. □

REMARK 1. Theorem 2.3 is false, in general, in the case of nonstable i.d. random vectors, even those satisfying (2.4). For a simple counterexample (again, with $d = 1$), let Y be a standard (i.e., mean 1) Poisson random variable and X be a nonnegative i.d. random variable with Lévy measure given by

$$\nu_X(dx) = \begin{cases} n\delta_{1/2}(dx), & 0 < x < 1, \\ \frac{1}{4}x^{-3/2}\,dx, & x \ge 1, \end{cases} \tag{2.18}$$

where n is some positive integer to be specified later. Note that

$$X =_d X_1 + X_2,$$

where X_1 and X_2 are independent i.d. random variables with Lévy measures

$$\nu_{X_1}(dx) = n\delta_{1/2}(dx),$$
$$\nu_{X_2}(dx) = \begin{cases} 0, & 0 < x < 1, \\ \frac{1}{4}x^{-3/2}\,dx, & x \ge 1. \end{cases} \tag{2.19}$$

X_1 and X_2 are both positive random variables because their Lévy measures are supported on $[0, \infty)$ and satisfy (2.4). It is simple to check that the Laplace transform of X_2 satisfies

$$1 - L_{X_2}(\theta) \sim \text{const.}\,\theta^{1/2} \quad \text{as } \theta \to 0.$$

Therefore, by (5.22) of Feller ([4], page 447), we have

$$P(X_2 > \lambda) \sim \text{const.}\,\lambda^{-1/2} \quad \text{as } \lambda \to \infty.$$

Hence, there is a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$,

$$P(X > \lambda) \ge P(X_2 > \lambda) \ge P(Y > \lambda). \tag{2.20}$$

On the other hand, note that

$$X_1 =_d \tfrac{1}{2}(U_1 + U_2 + \cdots + U_n),$$

where U_i's are i.i.d. Poisson random variables with mean 1. Choose now n so big that

$$[P(U_1 \le 2\lambda_0)]^n \le P(Y = 0).$$

Then for every $0 < \lambda < \lambda_0$ we have

$$\begin{aligned} P(X > \lambda) &\ge P(X_1 > \lambda) = 1 - P(X_1 \le \lambda) \\ &\ge 1 - P(U_1 \le 2\lambda, \dots, U_n \le 2\lambda) = 1 - P(U_1 \le 2\lambda)^n \\ &\ge 1 - P(Y = 0) = P(Y > 0) \ge P(Y > \lambda). \end{aligned}$$

Together with (2.20), this implies that $X \ge_{\mathrm{st}} Y$. But

$$\nu_X([1,\infty]) = \tfrac{1}{2} < 1 = \nu_Y([1,\infty]).$$

REMARK 2. Theorem 2.2 and 2.3 give a complete characterization of stochastic ordering for strictly α-stable random vectors with $\alpha < 1$ whose Lévy measures are concentrated on R_+^d (or, more generally, $R_+^d \cup R_-^d$). Namely, $\mathbf{X} \ge_{\mathrm{st}} \mathbf{Y}$ if and only if (2.12) and (2.13) hold. It is not true, however, that if $\mathbf{X}$ and $\mathbf{Y}$ are not necessarily strict α-stable random vectors with $\alpha < 1$, then $\mathbf{X} \ge_{\mathrm{st}} \mathbf{Y}$ implies (2.7) as well. For a counterexample (again with $d = 1$) take $0 < \alpha < 1$, $\sigma > 0$ and let X be a totally skewed to the right strictly α-stable random variable with scale parameter σ, and define Y as $Y_1 - Y_2 + 1$, where Y_1 and Y_2 are independent totally skewed to the right strictly α-stable random variables. (Of course, Y is not strictly stable.) Let Y_1 have scale parameter $\sigma/2$. Since totally skewed to the right strictly α-stable random variables with $\alpha < 1$ are supported by $(0,\infty)$, we have, for every $\lambda \le 0$, $0 = P(X \le \lambda) \le P(Y \le \lambda)$. Since $P(X > \lambda) \sim k\sigma^\alpha\lambda^{-\alpha}$ and $P(Y_1 > \lambda) \sim k(\sigma/2)^\alpha\lambda^{-\alpha}$ as $\lambda \to \infty$ for some positive constant k, there is a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$,

$$P(X > \lambda) > P(Y_1 + 1 > \lambda) \ge P(Y > \lambda).$$

Finally, it is a simple observation that we can always choose the scale parameter of Y_2 large enough so that for every $0 < \lambda < \lambda_0$,

$$P(X \le \lambda) \le P(X \le \lambda_0) \le P(Y \le 0) \le P(Y \le \lambda).$$

Therefore, $X \ge_{\mathrm{st}} Y$. Since $b_X = 0 < 1 = b_Y$, (2.7) does not hold in this case.

REMARK 3. Observe from the proof of Theorem 2.3 that if $\mathbf{X}$ and $\mathbf{Y}$ are strictly α-stable random vectors with $\alpha < 1$ and $\mathbf{X} \ge_{\mathrm{st}} \mathbf{Y}$, then we can choose the vector $\mathbf{Z}$ in (2.1)–(2.3) to be an i.d. vector. We do not yet know whether one can actually choose the vector $\mathbf{Z}$ to be α-stable as well.

3. Slepian inequality for type G infinitely divisible random vectors. In this section, we obtain a Slepian-type inequality for SαS random vectors with $1 < \alpha < 2$, and also for a more general class of i.d. random vectors, the so-called type G random vectors, which were introduced by Marcus [14]. As noted in Rosinski [18], the following are examples of one-dimensional distributions of type G: the SαS distributions, convolutions of stable distributions of different orders [whose characteristic function is $\phi(s) =$

$\exp\{-\int_{(0,2)}|s|^{\alpha}P(dx)\}$, where P is a measure on $(0,2)]$, the Laplace distribution, symmetrized gamma distributions (whose characteristic function is $\phi(s) = [\lambda^2/(\lambda^2+s^2)]^p$, $p, \lambda > 0$), the t distribution, a variance mixture of normal distributions $X = ZS$, where $Z \sim N(0,1)$ and where $S \geq 0$ has a completely monotone density on $(0,\infty)$. Complete monotonicity will be used extensively in the sequel. A function $(0,\infty)$ is said to be *completely monotone* if it is nonnegative and possesses derivatives of all orders that alternate in sign ([4]).

We shall establish the Slepian-type inequality in the context of type G i.d. random vectors and then state as a corollary the corresponding result for SαS random vectors. We start with a formal definition of type G i.d. random vectors and indicate several equivalent representations (see Rosinski [18] for details).

DEFINITION 3.1. A symmetric i.d. random vector $\mathbf{X} = (X_1, \dots, X_d)$ is said to be of type G if there is a function ψ with completely monotone derivative on $(0,\infty)$ satisfying $\psi(0) = 0$, and a σ-finite measure η on the Borel sets of R^d such that the characteristic function of $\mathbf{X}$ has the form

$$\phi_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\left\{-\int_{R^d}\psi\big(2^{-1}(\boldsymbol{\theta},\mathbf{x})^2\big)\eta(d\mathbf{x})\right\}. \tag{3.1}$$

The i.d. random vector $\mathbf{X}$ does not have a Gaussian component if and only if $\psi'(\infty) = 0$, and this is what we shall assume throughout our discussion. In this case one has the following representation of the function ψ:

$$\psi(s) = 2^{1/2}s^{-1/2}\int_0^{\infty}(1-\cos u)g\left(\frac{u^2}{2s}\right)du, \tag{3.2}$$

where $g\colon R^+ \to R^+$ is a completely monotone function such that

$$\int_0^{\infty}(1\wedge x^2)g(x^2)\,dx < \infty$$

(Rosinski [18], Theorem 1). It follows from the proof of Theorem 1 of Rosinski that the function ψ admits also the representation

$$\psi(s) = \int_0^{\infty}(1-e^{-su})\rho(du), \tag{3.3}$$

where ρ is a σ-finite measure on Borel subsets of $(0,\infty)$ satisfying $\int_0^{\infty}(1\wedge u)\rho(du) < \infty$. (This ρ is Rosinski's ρ_1.)

Finally, the Lévy measure ν of the i.d. vector $\mathbf{X}$ can be represented as

$$\nu(A) = \int_{R^d}\int_{-\infty}^{\infty}I_{(A-\{\mathbf{0}\})}(u\mathbf{x})g(u^2)\,du\,\eta(d\mathbf{x}) \tag{3.4}$$

([18], relation (15)).

To characterize the type G i.d. random vector $\mathbf{X}$, we shall use its Lévy measure ν and also the "parameters" η, ψ, g, ρ.

EXAMPLE 3.1. When $\mathbf{X}$ is SαS, one can write (3.1) as

$$\phi_{\mathbf{X}}(\theta) = \exp\left\{ -\int_{\mathbf{S}_d} |(\theta, x)|^{\alpha} \eta(d\mathbf{x}) \right\},$$

where η is a finite symmetric measure concentrated on the unit ball S_d of $\mathbf{R}^d$. In this case $\psi(x) = \text{const.}\ s^{\alpha/2}$, $g(u) = \text{const.}\ u^{-(\alpha+1)/2}$ and $\rho(du) = \text{const.}\ u^{-(\alpha+2)/2}$. The Lévy measure is given by

$$\nu(A) = \text{const.} \int_{S^d} \int_0^{\infty} I_{A-\{0\}}(u\mathbf{x}) u^{-(\alpha+1)}\, du\, \eta(d\mathbf{x})$$

and satisfies $\nu(tA) = t^{\alpha}\nu(A)$, $t > 0$.

To state the main result of this section, we need the following definition.

DEFINITION 3.2. For a type G i.d. random vector $\mathbf{X}$ as before, the *conjugate i.d. random vector* is a (symmetric) i.d. random vector $\mathbf{W}$ whose Lévy measure $\hat{\nu}$ is given by

$$\hat{\nu}(A) = \int_{R^d} \int_{-\infty}^{\infty} I_{A-\{0\}}(u\mathbf{v}) \rho(d(u^2)) \eta(d\mathbf{v}). \tag{3.5}$$

One can easily check that the measure $\hat{\nu}$ as defined in (3.5) is indeed a Lévy measure. $\mathbf{W}$, however, is not necessarily of type G.

EXAMPLE 3.2. If $\mathbf{X}$ is SαS, then the conjugate vector $\mathbf{W}$ is equal in distribution to const. $\mathbf{X}$.

EXAMPLE 3.3. If $g(x^2) = \text{const.} \exp\{-x^2/2\sigma^2\}$, then the conjugate vector has an especially simple form because in this case the measure ρ is simply a point mass.

We are now ready to state our result.

THEOREM 3.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be type* G *i.d. random vectors in* R^d *with Lévy measures* $\nu_{\mathbf{X}}$ *and* $\nu_{\mathbf{Y}}$, *and parameters* $\eta_{\mathbf{X}}, \psi_{\mathbf{X}}, g_{\mathbf{X}}, \rho_{\mathbf{X}}$ *and* $\eta_{\mathbf{Y}}, \psi_{\mathbf{Y}}, g_{\mathbf{Y}}, \rho_{\mathbf{Y}}$ [*defined by* (3.1), (3.2), (3.3) *and* (3.4)], *respectively. Let* $\mathbf{W}_{\mathbf{X}}$ *and* $\mathbf{W}_{\mathbf{Y}}$ *be the i.d. random vectors conjugate to* $\mathbf{X}$ *and* $\mathbf{Y}$, *whose Lévy measures are* $\hat{\nu}_{\mathbf{X}}$ *and* $\hat{\nu}_{\mathbf{Y}}$, *respectively. Assume* $E|\mathbf{X}| < \infty$ *and* $E|\mathbf{Y}| < \infty$.

If for every increasing Borel set A *in* $R_+^{d(d-1)/2}$,

$$\hat{\nu}_{\mathbf{X}}(\mathbf{x} \in R^d\colon T\mathbf{x} \in A) \geq \hat{\nu}_{\mathbf{Y}}(\mathbf{x} \in R^d\colon T\mathbf{x} \in A), \tag{3.6}$$

where $T\colon R^d \to R_+^{d(d-1)/2}$ *is defined by*

$$\begin{aligned} T(x_1, \ldots, x_d) = \big(&(x_1 - x_2)^2, (x_1 - x_3)^2, \ldots, \\ &(x_1 - x_d)^2, (x_2 - x_3)^2, \ldots, (x_{d-1} - x_d)^2 \big), \end{aligned} \tag{3.7}$$

then

$$E \max_{1\le i\le d} X_i \ge E \max_{1\le i\le d} Y_i.$$

PROOF. The main ingredient in our proof is a series representation of type G i.d. random vectors first presented in Marcus [14] which allows us to regard these vectors as a probabilistic mixture of Gaussian vectors, thus generalizing a well-known property of SαS vectors. The form we use is due to Rosinski [18]. Define

$$R_{\mathbf{X}}(x) := \inf\{u > 0: \rho_{\mathbf{X}}((u,\infty)) \le x\}, \qquad x > 0, \tag{3.8}$$

and let $R_{\mathbf{Y}}$ be defined similarly. Let $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$ be probability measures on R^d such that $\eta_{\mathbf{X}} \ll \lambda_{\mathbf{X}}$ and $\eta_{\mathbf{Y}} \ll \lambda_{\mathbf{Y}}$, and let

$$h_{\mathbf{X}} = \frac{d\eta_{\mathbf{X}}}{d\lambda_{\mathbf{X}}}, \qquad h_{\mathbf{Y}} = \frac{d\eta_{\mathbf{Y}}}{d\lambda_{\mathbf{Y}}}.$$

Then $\mathbf{X}$ admits the series representation

$$\mathbf{X} =_d \sum_{n=1}^{\infty} Z_n^{\mathbf{X}} R_{\mathbf{X}}\left(\frac{\Gamma_n^{\mathbf{X}}}{h_{\mathbf{X}}(\mathbf{S}_{n,\mathbf{X}})}\right)\mathbf{S}_{n,\mathbf{X}}, \tag{3.9}$$

where $\{Z_n^{\mathbf{X}}\}_{n=1}^{\infty}$, $\{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}$ and $\{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}$ are independent sequences. The sequence $\{Z_n^{\mathbf{X}}\}_{n=1}^{\infty}$ is a sequence of i.i.d. standard normal random variables; $\Gamma_n^{\mathbf{X}}$, $n = 1, 2, \dots,$ are the points of a unit rate Poisson process on $(0,\infty)$; and $\{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables in R^d with the common law $\lambda_{\mathbf{X}}$. Using a similar notation,

$$\mathbf{Y} =_d \sum_{n=1}^{\infty} Z_n^{\mathbf{Y}} R_{\mathbf{Y}}\left(\frac{\Gamma_n^{\mathbf{Y}}}{h_{\mathbf{Y}}(\mathbf{S}_{n,\mathbf{Y}})}\right)\mathbf{S}_{n,\mathbf{Y}}. \tag{3.10}$$

Let $\mathscr{F}_{\mathbf{X}}$ and $\mathscr{F}_{\mathbf{Y}}$ be the σ-algebras generated on the corresponding sample spaces by $\{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}$ and $\{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}$ and by $\{\Gamma_n^{\mathbf{Y}}\}_{n=1}^{\infty}$ and $\{\mathbf{S}_{n,\mathbf{Y}}\}_{n=1}^{\infty}$, respectively, and let $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ denote the right-hand sides of (3.9) and (3.10), respectively. Note that the regular conditional distributions of $\mathbf{X}$ given $\mathscr{F}_{\mathbf{X}}$ and $\mathbf{Y}$ given $\mathscr{F}_{\mathbf{Y}}$ are zero-mean Gaussian. Moreover, denoting by $E_{\mathscr{F}_{\mathbf{X}}}$ ($E_{\mathscr{F}_{\mathbf{Y}}}$) the conditional expectation given $\mathscr{F}_{\mathbf{X}}$ ($\mathscr{F}_{\mathbf{Y}}$), we obtain for $k_1 < k_2 < d$,

$$E_{\mathscr{F}_{\mathbf{X}}}\left(\tilde{X}_{k_1} - \tilde{X}_{k_2}\right)^2 = \sum_{n=1}^{\infty} R_{\mathbf{X}}^2\left(\frac{\Gamma_n^{\mathbf{X}}}{h_{\mathbf{X}}(\mathbf{S}_{n,\mathbf{X}})}\right)\left(S_{n,\mathbf{X}}^{k_1} - S_{n,\mathbf{X}}^{k_2}\right)^2, \tag{3.11}$$

$$E_{\mathscr{F}_{\mathbf{Y}}}\left(\tilde{Y}_{k_1} - \tilde{Y}_{k_2}\right)^2 = \sum_{n=1}^{\infty} R_{\mathbf{Y}}^2\left(\frac{\Gamma_n^{\mathbf{Y}}}{h_{\mathbf{Y}}(\mathbf{S}_{n,\mathbf{Y}})}\right)\left(S_{n,\mathbf{Y}}^{k_1} - S_{n,\mathbf{Y}}^{k_2}\right)^2, \tag{3.12}$$

where $\tilde{\mathbf{X}} = (\tilde{X}_1, \dots, \tilde{X}_d)$ and $\mathbf{S}_{n,\mathbf{X}} = (S_{n,\mathbf{X}}^1, \dots, S_{h,\mathbf{X}}^d)$.

Consider now the two nonnegative random vectors in $R^{d(d-1)/2}$, $\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$ whose components, labeled (k_1, k_2), $k_1 < k_2 < d$, are given by the right-hand sides of (3.11) and (3.12), respectively. We claim that (3.6) implies

$$\mathbf{\Delta}_{\mathbf{X}} \geq_{\mathrm{st}} \mathbf{\Delta}_{\mathbf{Y}}. \tag{3.13}$$

Before proving this, let us show that (3.13) implies the conclusion of the theorem.

Suppose that (3.13) holds. The idea is to condition on $\mathscr{F}_{\mathbf{X}}$ (respectively, on $\mathscr{F}_{\mathbf{Y}}$), view $\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$ as the conditional variances of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ and then apply the Slepian inequality for Gaussian vectors. This will be done in a number of steps. We first generate $\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$ on the same probability space such that $\mathbf{\Delta}_{\mathbf{X}} \geq \mathbf{\Delta}_{\mathbf{Y}}$ a.s. We then construct the sequences $\{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}$, $\{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}$, $\{\mathbf{S}_{n,\mathbf{Y}}\}_{n=1}^{\infty}$ and $\{\Gamma_n^{\mathbf{Y}}\}_{n=1}^{\infty}$ on that space. This is possible because we can regard all four sequences as a random variable taking values in $(R^{\infty})^{2d+2}$, which is a Polish space, use Thereom 10.2.2 of Dudley [3] to conclude that there exist regular conditional probabilities of these random variables given $\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$, and then generate $\{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}$, $\{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}$, $\{\mathbf{S}_{n,\mathbf{Y}}\}_{n=1}^{\infty}$ and $\{\Gamma_n^{\mathbf{Y}}\}_{n=1}^{\infty}$ from these corresponding conditional probabilities. We now have $\mathbf{\Delta}_{\mathbf{X}}$, $\mathbf{\Delta}_{\mathbf{Y}}$, $\{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}$, $\{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}$, $\{\mathbf{S}_{n,\mathbf{Y}}\}_{n=1}^{\infty}$ and $\{\Gamma_n^{\mathbf{Y}}\}_{n=1}^{\infty}$ defined on the same probability space. Since $\mathbf{\Delta}_{\mathbf{X}} \geq \mathbf{\Delta}_{\mathbf{Y}}$ a.s., by Theorem 1.1, with probability 1,

$$\begin{aligned} &E\Big(\max_{i=1,\dots,d} \tilde{X}_i \big| \{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}, \{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}, \{\mathbf{S}_{n,\mathbf{Y}}\}_{n=1}^{\infty}, \{\Gamma_n^{\mathbf{Y}}\}_{n=1}^{\infty}\Big) \\ &\qquad \geq E\Big(\max_{i=1,\dots,d} \tilde{Y}_i \big| \{\mathbf{S}_{n,\mathbf{X}}\}_{n=1}^{\infty}, \{\Gamma_n^{\mathbf{X}}\}_{n=1}^{\infty}, \{\mathbf{S}_{n,\mathbf{Y}}\}_{n=1}^{\infty}, \{\Gamma_n^{\mathbf{Y}}\}_{n=1}^{\infty}\Big), \end{aligned}$$

so that

$$E \max_{1\leq i\leq d} X_i = E \max_{1\leq i\leq d} \tilde{X}_i \geq E \max_{1\leq i\leq d} \tilde{Y}_i = E \max_{1\leq i\leq d} Y_i.$$

It remains, therefore, to show that (3.13) holds.

Theorem 2.4 of Rosinski [17] (see also Corollary 4.3 of Rosinski [18]) ensures that the random vectors $\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$ are infinitely divisible. ($\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$ are not symmetric and hence not of type G.) Let $\nu_{\mathbf{\Delta}_{\mathbf{X}}}$ and $\nu_{\mathbf{\Delta}_{\mathbf{Y}}}$ be their respective Lévy measures. It follows from Theorem 2.2 that we only need to prove that for every increasing Borel set A in $R_+^{d(d-1)/2}$,

$$\nu_{\mathbf{\Delta}_{\mathbf{X}}}(A) \geq \nu_{\mathbf{\Delta}_{\mathbf{Y}}}(A). \tag{3.14}$$

We compute first the Lévy measures $\nu_{\mathbf{\Delta}_{\mathbf{X}}}$ and $\nu_{\mathbf{\Delta}_{\mathbf{Y}}}$. ($\mathbf{\Delta}_{\mathbf{X}}$ and $\mathbf{\Delta}_{\mathbf{Y}}$ are not symmetric and hence not of type G.) Define a probability measure $\tilde{\lambda}_{\mathbf{X}}$ on $R_+^{d(d-1)/2}$ by $\tilde{\lambda}_{\mathbf{X}} = \lambda_{\mathbf{X}} \circ T^{-1}$ and let $\tilde{\eta}_{\mathbf{X}} = \eta_{\mathbf{X}} \circ T^{-1}$. Clearly, $\tilde{\eta}_{\mathbf{X}} \ll \tilde{\lambda}_{\mathbf{X}}$. Let $\tilde{h}_{\mathbf{X}} := d\tilde{\eta}_{\mathbf{X}}/d\tilde{\lambda}_{\mathbf{X}}$. Then a version of $\tilde{h}_{\mathbf{X}}$ is given by

$$\tilde{h}_{\mathbf{X}}(y) = \int_{R^d} h_{\mathbf{X}}(\mathbf{x})\theta_{\mathbf{X}}^{(y)}(d\mathbf{x}), \tag{3.15}$$

where $\theta_{\mathbf{X}}^{(\cdot)}$ is the regular conditional distribution of the probability law $\lambda_{\mathbf{X}}$ given the σ-algebra $\mathscr{F}_T$ generated by the transformation T.

We now use Theorems 2.4 and 3.2 of Rosinski [17]. Let H: $(0,\infty) \times R^d \to R_+^{d(d-1)/2}$ be given by

$$H(u,\mathbf{v}) = \{H_{k_1,k_2}(u,\mathbf{v}),\, k_1 = 1,\ldots,d,\, k_2 = k_1+1,\ldots,d\},$$

where

$$H_{k_1,k_2}(u,\mathbf{v}) = R_{\mathbf{X}}^2\left(\frac{u}{h_{\mathbf{X}}(\mathbf{v})}\right)(v_{k_1} - v_{k_2})^2, \qquad u \ge 0, \mathbf{v} = (v_1,\ldots,v_d). \tag{3.16}$$

Define a measure on Borel sets in $R_+^{d(d-1)/2}$ by

$$F(A) = \int_0^\infty \int_{R^d} I_{A-\{\mathbf{0}\}}(H(u,\mathbf{v}))\, du\, \lambda_{\mathbf{X}}(d\mathbf{v}). \tag{3.17}$$

Let $Q_{\mathbf{v}}$: $(0,\infty) \to (0,\infty)$, $\mathbf{v} \in R^{d(d-1)/2}$, be given by

$$Q_{\mathbf{v}}(u) = R_{\mathbf{X}}^2\left(\frac{u}{h_{\mathbf{X}}(\mathbf{v})}\right), \qquad u > 0.$$

Let $m_{\mathbf{v}} := \mathrm{Leb} \circ Q_{\mathbf{v}}^{-1}$. Then, for every $a > 0$,

$$\begin{aligned} m_{\mathbf{v}}((a,\infty)) &= \mathrm{Leb}\left\{u: R_{\mathbf{X}}^2\left(\frac{u}{h_{\mathbf{X}}(\mathbf{v})}\right) > a\right\} \\ &= \mathrm{Leb}\left\{u: \rho_{\mathbf{X}}((a,\infty)) > \frac{u}{h_{\mathbf{X}}(\mathbf{v})}\right\} = h_{\mathbf{X}}(\mathbf{v})\rho_{\mathbf{X}}((a,\infty)). \end{aligned}$$

Therefore, $m_{\mathbf{v}}(du) = h_{\mathbf{X}}(\mathbf{v})\rho_{\mathbf{X}}(du)$, and thus

$$\begin{aligned} F(A) &= \int_{R^d}\left(\int_0^\infty I_{A-\{\mathbf{0}\}}\left(R_{\mathbf{X}}^2\left(\frac{u}{h_{\mathbf{X}}(\mathbf{v})}\right)T\mathbf{v}\right) du\right)\lambda_{\mathbf{X}}(d\mathbf{v}) \\ &= \int_{R^d}\left(\int_0^\infty I_{A-\{\mathbf{0}\}}(uT\mathbf{v})\, m_{\mathbf{v}}(du)\right)\lambda_{\mathbf{X}}(d\mathbf{v}) \\ &= \int_{R^d}\left(\int_0^\infty I_{A-\{\mathbf{0}\}}(uT\mathbf{v})\, h_{\mathbf{X}}(\mathbf{v})\rho_{\mathbf{X}}(du)\right)\lambda_{\mathbf{X}}(d\mathbf{v}) \\ &= \int_0^\infty\left(\int_{R^d} I_{A-\{\mathbf{0}\}}(uT\mathbf{v})\, h_{\mathbf{X}}(\mathbf{v})\lambda_{\mathbf{X}}(d\mathbf{v})\right)\rho_{\mathbf{X}}(du). \end{aligned} \tag{3.18}$$

Note that (3.15) implies that for every $u > 0$,

$$\int_{R^d} I_{A-\{\mathbf{0}\}}(uT\mathbf{v})\, h_{\mathbf{X}}(\mathbf{v})\lambda_{\mathbf{X}}(d\mathbf{v}) = \int_{R_+^{d(d-1)/2}} I_{A-\{\mathbf{0}\}}(u\mathbf{y})\, \tilde{h}_{\mathbf{X}}(\mathbf{y})\tilde{\lambda}_{\mathbf{X}}(d\mathbf{y}).$$

It follows that

$$\begin{aligned} F(A) &= \int_0^\infty\left(\int_{R_+^{d(d-1)/2}} I_{A-\{\mathbf{0}\}}(u\mathbf{y})\, \tilde{h}_{\mathbf{X}}(\mathbf{y})\tilde{\lambda}_{\mathbf{X}}(d\mathbf{y})\right)\rho_{\mathbf{X}}(du) \\ &= \int_0^\infty \int_{R_+^{d(d-1)/2}} I_{A-\{\mathbf{0}\}}(u\mathbf{y})\, \tilde{\eta}_{\mathbf{X}}(d\mathbf{y})\rho_{\mathbf{X}}(du). \end{aligned} \tag{3.19}$$

It is straightforward to check that $\int_{R^{d(d-1)/2}}(I \wedge |\mathbf{x}|)F(d\mathbf{x}) < \infty$. Therefore, by Theorem 3.2 of Rosinski [17] we conclude that

$$\nu_{\Delta_{\mathbf{X}}}(A) = F(A) = \int_0^\infty \int_{R_+^{d(d-1)/2}} I_{A-\{\mathbf{0}\}}(u\mathbf{y})\tilde{\eta}_{\mathbf{X}}(d\mathbf{y})\rho_{\mathbf{X}}(du). \tag{3.20}$$

Similarly,

$$\nu_{\Delta_{\mathbf{Y}}}(A) = \int_0^\infty \int_{R_+^{d(d-1)/2}} I_{A-\{\mathbf{0}\}}(u\mathbf{y})\tilde{\eta}_{\mathbf{Y}}(d\mathbf{y})\rho_{\mathbf{Y}}(du). \tag{3.21}$$

We conclude by (3.20) and (3.18), $h_{\mathbf{X}} = d\eta_{\mathbf{X}}/d\lambda_{\mathbf{X}}$ and by (3.21) and (3.6) that for every increasing Borel set $A \in R_+^{d(d-1)/2}$,

$$\begin{aligned}
\nu_{\Delta_{\mathbf{X}}}(A) &= \int_0^\infty \int_{R_+^{d(d-1)/2}} I_{A-\{\mathbf{0}\}}(u\mathbf{y})\tilde{\eta}_{\mathbf{X}}(d\mathbf{y})\rho_{\mathbf{X}}(du) \\
&= \int_{R^d}\int_0^\infty I_{A-\{\mathbf{0}\}}(uT\mathbf{v})\rho_{\mathbf{X}}(du)\eta_{\mathbf{X}}(d\mathbf{v}) \\
&= \tfrac{1}{2}\int_{R^d}\int_{-\infty}^\infty I_{A-\{\mathbf{0}\}}(u^2T\mathbf{v})\rho_{\mathbf{X}}(d(u^2))\eta_{\mathbf{X}}(d\mathbf{v}) \\
&= \tfrac{1}{2}\int_{R^d}\int_{-\infty}^\infty I_{A-\{\mathbf{0}\}}(T(u\mathbf{v}))\rho_{\mathbf{X}}(d(u^2))\eta_{\mathbf{X}}(d\mathbf{v}) \\
&= \tfrac{1}{2}\int_{R^d}\int_{-\infty}^\infty I_{T^{-1}(A-\{\mathbf{0}\})}(u\mathbf{v})\rho_{\mathbf{X}}(d(u^2))\eta_{\mathbf{X}}(d\mathbf{v}) \\
&= \tfrac{1}{2}\hat{\nu}_{\mathbf{X}}(T^{-1}A) = \tfrac{1}{2}\hat{\nu}_{\mathbf{X}}(\mathbf{x} \in R^d\colon T\mathbf{x} \in A) \\
&\geq \tfrac{1}{2}\hat{\nu}_{\mathbf{Y}}(\mathbf{x} \in R^d\colon T\mathbf{x} \in A) = \tfrac{1}{2}\hat{\nu}_{\mathbf{Y}}(T^{-1}A) \\
&= \nu_{\Delta_{\mathbf{Y}}}(A),
\end{aligned}$$

thus proving (3.14). The proof of the theorem is now complete. □

REMARK. In the preceding proof, the random vector $\mathbf{\Delta_X}$ is i.d. and defined on $R_+^{d(d-1)/2}$. The i.d. random vector $\mathbf{W_X}$ which is conjugate to $\mathbf{X}$ is defined on $\mathbf{R}^d$, is symmetric and has a Lévy measure $\hat{\nu}_{\mathbf{X}}$ satisfying $\hat{\nu}_{\mathbf{X}}(T^{-1}A) = 2\nu_{\Delta_{\mathbf{X}}}(A)$, where A is any Borel set in $\mathbf{R}_+^{d(d-1)/2}$. As noted in Example 3.2, $\mathbf{W_X} =_d$ const. $\mathbf{X}$ if $\mathbf{X}$ is SαS.

The following is an immediate consequence of Theorem 3.1 and Example 3.2.

COROLLARY 3.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be* $S\alpha S$ *random vectors in* R^d, $1 < \alpha < 2$, *with Lévy measures* $\nu_{\mathbf{X}}$ *and* $\nu_{\mathbf{Y}}$, *respectively, and let* T *be defined as in* (3.7). *If for every increasing Borel set* A *in* $R_+^{d(d-1)/2}$,

$$\nu_{\mathbf{X}}(\mathbf{x} \in R^d\colon T\mathbf{x} \in A) \geq \nu_{\mathbf{Y}}(\mathbf{x} \in R^d\colon T\mathbf{x} \in A), \tag{3.22}$$

then

$$E \max_{1 \le i \le d} X_i \ge E \max_{1 \le i \le d} Y_i.$$

REMARK 1. Recall that condition (1.1) in the Gaussian Slepian inequality is

$$E(T\mathbf{X}) \ge E(T\mathbf{Y}).$$

REMARK 2. It is important to note that Theorem 3.1 allows us to compare two type G i.d. vectors of *different types*, that is, with different functions ψ (e.g., stable and nonstable random vectors). This is in sharp contrast to, say, the result of Brown and Rinott [2].

REMARK 3. Unfortunately, condition (3.6) is not easy to verify in practice. Much work remains to be done in order to derive a Slepian-type inequality, at least in the stable case, which is as useful as its Gaussian counterpart.

REMARK 4. Theorem 3.1 and Corollary 3.1 used the Gaussian Slepian inequality given in Theorem 1.1. If we use Corollary 1.1 instead, we obtain the following result: Let T: $R^d \to R_+^{d(d+1)/2}$ be given by

$$T(x_1, \dots, x_d) = \left\{ \left\{ (x_{k_1} - x_{k_2})^2 \right\}_{1 \le k_1, k_2 \le d}, \left\{ x_i^2 \right\}_{1 \le i \le d} \right\}.$$

Suppose that the conditions of either Theorem 3.1 or Corollary 3.1 hold for this new T. Then

$$E \max_{1 \le i \le d} |X_i| \ge \tfrac{1}{2} E \max_{1 \le i \le d} |Y_i|.$$

REMARK 5. Using Theorem 3.15 of Ledoux and Talagrand [10], we conclude that under the conditions of either Theorem 3.1 or Corollary 3.1, the following holds: For every nonnegative convex increasing function Φ on R_+,

$$E\Phi\Big(\max_{i,j=1,\dots,d} |X_i - X_j| \Big) \ge E\Phi\Big(\max_{i,j=1,\dots,d} |Y_i - Y_j| \Big).$$

Conclusion. Theorem 3.1 demonstrates that Slepian-type inequalities do exist for a wide variety of non-Gaussian infinitely divisible distributions. In the SαS case, inequalities relating the Lévy measures replace the covariance inequalities of the Gaussian case. In the more general case of type G i.d. vectors, the inequalities relate the Lévy measures of the *conjugate* i.d. vectors. These inequalities, moreover, involve only images of increasing sets.

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