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RADEMACHER'S THEOREM FOR WIENER FUNCTIONALS

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Given an $\mathbb{R}$-valued, Borel measurable function F on an abstract Wiener space (E, H, μ), we show that F is uniformly Lipschitz continuous in the directions of H if and only if it has one derivative in the sense of Malliavin and that derivative is an element of $L^\infty(\mu; H)$.

1. Formulation of the main result. Let H be a separable, real, Hilbert space with scalar product $(\cdot, \cdot)_H$, $\|\cdot\|$ a measurable norm on H (cf. [1]), and (E, H, μ) the corresponding (in the sense of Gross) abstract Wiener space. Next, let $\langle \ell, \omega\rangle$ denote the pairing between an element ω of E and an element ℓ of the dual space E^*, denote by $\mathscr{P}$ the algebra of functions $\phi\colon E \to \mathbb{R}$ of the form

$$\phi(\omega) = p(\langle \ell_1, \omega\rangle, \dots, \langle \ell_n, \omega\rangle), \qquad \omega \in E, \tag{1.1}$$

for some $n \in \mathbb{Z}^+$, polynomial $p\colon \mathbb{R}^n \to \mathbb{R}$, and $(\ell_1, \dots, \ell_n) \in (E^*)^n$, and take $\mathfrak{P}$ to be the vector space of maps $\Phi\colon E \to H$ which can be expressed in the form

$$\Phi(\omega) = \sum_{i=1}^{n} \phi_i(\omega)\ell_i \tag{1.2}$$

for some choice of $n \in \mathbb{Z}^+$, $(\phi_1, \dots, \phi_n) \in \mathscr{P}^n$ and $(\ell_1, \dots, \ell_n) \in (E^*)^n$. For $\ell \in E^*$, we define the *lowering operator* $\partial_\ell\colon \mathscr{P} \mapsto \mathscr{P}$ and the *raising operator* $\partial_\ell^\dagger\colon \mathscr{P} \mapsto \mathscr{P}$, respectively, by

$$[\partial_\ell \phi](\omega) = \frac{d}{dt}\phi(\omega + t\ell)\bigg|_{t=0},$$

$$\left[\partial_\ell^\dagger \phi\right](\omega) = \phi(\omega)\langle \ell, \omega\rangle - [\partial_\ell\phi](\omega),$$

and recall the integration by parts formula

$$\frac{d}{dt}\mathbb{E}_\mu\{F(\omega + t\ell)\phi(\omega)\}\bigg|_{t=0} = \mathbb{E}_\mu\left\{F(\omega)\left[\partial_\ell^\dagger\phi\right](\omega)\right\},$$

which holds for every $\ell \in E^*$, $\phi \in \mathscr{P}$ and $F \in L^2(E, \mu)$ (note that there is no smoothness requirement of any kind for F). Finally, we define the operators

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$\mathscr{D}$: $\mathscr{P} \mapsto \mathfrak{P}$ and $\mathscr{D}^\dagger$: $\mathfrak{P} \mapsto \mathscr{P}$ by

$$[\mathscr{D}\phi](\omega) = \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(\langle \ell_1, \omega\rangle, \dots, \langle \ell_n, \omega\rangle)\ell_i,$$

$$[\mathscr{D}^\dagger\Phi](\omega) = \sum_{i=1}^{n} \left[\partial_{\ell_i}^\dagger \phi_i\right](\omega),$$

when ϕ and Φ are given by (1.1) and (1.2), respectively. Clearly, the integration by parts formula can be expressed in terms of $\mathscr{D}$ and $\mathscr{D}^\dagger$ as

$$\mathbb{E}_\mu\{([\mathscr{D}\phi](\omega), \Phi(\omega))_H\} = \mathbb{E}_\mu\{\phi(\omega)[\mathscr{D}^\dagger\Phi](\omega)\}, \qquad \phi \in \mathscr{P} \text{ and } \Phi \in \mathfrak{P},$$

and using this relation one can show (see [2] or [3]) that the adjoints of $\mathscr{D}$ and $\mathscr{D}^\dagger$ coincide, respectively, with the minimal closures of $\mathscr{D}^\dagger$ and $\mathscr{D}$ as operators on $L^2(\mu; H) \mapsto L^2(\mu; \mathbb{R})$ and $L^2(\mu; \mathbb{R}) \mapsto L^2(\mu; H)$, respectively. Thus, if $W_1^2(\mathbb{R})$ is the completion of $\mathscr{P}$ with respect to the norm

$$\phi \mapsto \left(\mathbb{E}_\mu\{|\phi(\omega)|^2\}\right)^{1/2} + \left(\mathbb{E}_\mu\{\|[\mathscr{D}\phi](\omega)\|_H^2\}\right)^{1/2},$$

then $W_1^2(\mathbb{R})$ can be identified as a subspace of $L^2(\mu; \mathbb{R})$ and $F \in L^2(\mu; \mathbb{R})$ is an element of $W_1^2(\mathbb{R})$ if and only if

$$\left|\mathbb{E}_\mu\{F[\mathscr{D}^\dagger\Phi]\}\right| \le C\|\Phi\|_{L^2(\mu; H)}, \qquad \Phi \in \mathfrak{P},$$

for some $C \in [0, \infty)$, in which case $\|\mathscr{D}F\|_{L^2(\mu; H)} \le C$. From now on we will use $\mathscr{D}$ and $\mathscr{D}^\dagger$ to denote their own closed extensions.

The mapping F: $E \mapsto \mathbb{R}$ will be said to be *H-Lipschitz continuous* with constant $C \in [0, \infty)$ if it is Borel measurable and

$$|f(\omega + h) - f(\omega)| \le C\|h\|_H, \qquad \omega \in E \text{ and } h \in H.$$

We will say that F is μ-a.e. *H-Lipschitz continuous*, if there is a μ-a.e. modification $\tilde{F}$ of F (i.e., $\tilde{F}$ coincides with F outside some μ-negligible set), which is H-Lipschitz. In the latter case we define

$$\mathrm{Lip}_H[F] = \operatorname*{ess\,sup}_{\omega \in E} \sup_{h \in H\setminus\{0\}} \left(\frac{1}{\|h\|_H}\left|\tilde{F}(\omega + h) - \tilde{F}(\omega)\right|\right).$$

Notice that, as a consequence of the quasitranslation invariance of μ in the directions of H, the right-hand side above does not depend on the particular μ-a.e., H-Lipschitz continuous modification $\tilde{F}$. Also, as our result will show, one can always find an μ-a.e. modification $\tilde{F}$, which is H-Lipschitz continuous with constant $C = \mathrm{Lip}_H[F]$. Finally, we say that F: $E \to \mathbb{R}$ is *locally H-differentiable* at $\omega \in E$ if there is $F'(\omega) \in H$ such that, for each $h \in H$,

$$F(\omega + \varepsilon h) - F(\omega) - \varepsilon(F'(\omega), h)_H = o_h(\varepsilon),$$

where the error term $o_h(\varepsilon)$ is such that $(1/\varepsilon)o_h(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly on compact subsets of H.

Our main result is the following:

THEOREM. *Let $F \in L^2(\mu, \mathbb{R})$. Then the following three conditions are equivalent*:

(i) *F is μ-a.e. H-Lipschitz continuous*;

(ii) *F has a μ-a.e. modification $\tilde{F}$ which is locally H-differentiable at μ-a.e. $\omega \in \Omega$ and $\tilde{F}' \in L^\infty(\mu; H)$*;

(iii) *$F \in W_1^2(E)$ and $\mathscr{D}F \in L^\infty(\mu; H)$.*

In fact, (i) *implies* (ii) *with* $\|\tilde{F}'\|_{L^\infty(\mu; H)} = \mathrm{Lip}_H[F]$, (ii) *implies* (iii) *with* $\mathscr{D}F = \tilde{F}'$ *μ-a.e., and* (iii) *implies that F admits a μ-a.e. modification $\tilde{F}$ which is H-Lipschitz with constant* $\|\mathscr{D}F\|_{L^\infty(\mu; H)}$.

REMARK. Our interest in the preceding result stems from its applications to the analysis of anticipating S.D.E.'s. A paper containing such applications is in preparation. In the meantime, we will content ourselves with the application given at the end of this article, and we take this opportunity to thank R. Leandre for leading us to this application.

2. Proof of the main result. Much of what we do will turn on the following simple lemma.

LEMMA . *Let $\mathfrak{L}$ be a finite dimensional subspace of E^* and denote by $\mathfrak{L}^\perp$ the orthogonal complement of $\mathfrak{L}$ as a subspace of H. Then the orothogonal projection map $P_\mathfrak{L}$ of H onto $\mathfrak{L}$ admits a unique continuous extension as a map from E onto $\mathfrak{L}$. Furthermore, if $P_\mathfrak{L}^\perp \equiv \mathbf{I} - P_\mathfrak{L}$, $E_{\mathfrak{L}^\perp} \equiv P_\mathfrak{L}^\perp(E)$, and $\mu_{\mathfrak{L}^\perp} \equiv \mu \circ P_\mathfrak{L}^{\perp\,-1}$, then $E_{\mathfrak{L}^\perp}$ is the completion of $\mathfrak{L}^\perp$ with respect to $\|\cdot\| \restriction \mathfrak{L}^\perp$ and $(E_{\mathfrak{L}^\perp}, \mathfrak{L}^\perp, \mu_{\mathfrak{L}^\perp})$ is an abstract Wiener space. Finally, if $\mu_\mathfrak{L} \equiv \mu \circ P_\mathfrak{L}^{-1}$, then the mapping*

$$\omega \in E \mapsto \Psi_\mathfrak{L}(\omega) \equiv \left(P_\mathfrak{L}^\perp \omega, P_\mathfrak{L}\omega\right) \in E_{\mathfrak{L}^\perp} \times \mathfrak{L}$$

is a homeomorphism which takes the measure μ onto the measure $\mu_{\mathfrak{L}^\perp} \times \mu_\mathfrak{L}$. In particular, if F: $E \to [0, \infty)$ is Borel measurable and $F_\mathfrak{L}$: $\mathfrak{L} \to [0, \infty]$ is the Borel measurable function given by

$$F_\mathfrak{L}(\ell) = \int_{E_{\mathfrak{L}^\perp}} F \circ \Psi_\mathfrak{L}^{-1}(\omega_{\mathfrak{L}^\perp}, \ell)\mu_{\mathfrak{L}^\perp}(d\omega_{\mathfrak{L}^\perp}),$$

then

$$\mathbb{E}_\mu[F | \mathscr{F}_\mathfrak{L}] = F_\mathfrak{L} \circ P_\mathfrak{L} \qquad \mu\text{-}a.e.$$

PROOF. To see that $P_\mathfrak{L}$ admits a continuous extension to E, choose a basis $(\ell_1, \dots, \ell_n)$ for $\mathfrak{L}$ so that $(\ell_i, \ell_j)_H = \delta_{i,j}$, and define $P_\mathfrak{L}\omega = \sum_1^n \langle \ell_i, \omega \rangle \ell_i$ for $\omega \in E$. Next, note that, because of continuity, $E_{\mathfrak{L}^\perp}$ is closed in E and can therefore be identified as the completion of $\mathfrak{L}^{\perp}$ with respect to $\|\cdot\| \upharpoonright \mathfrak{L}^{\perp}$; and obviously $\Psi_{\mathfrak{L}}$ is a homeomorphism. In addition, use of characteristic functions shows that $(E_{\mathfrak{L}^{\perp}}, \mathfrak{L}^{\perp}, \mu_{\mathfrak{L}^{\perp}})$ is an abstract Wiener space and that $\mu = (\mu_{\mathfrak{L}^{\perp}} \times \mu_{\mathfrak{L}}) \circ \Psi_{\mathfrak{L}}$. Finally, because F is $\mathscr{F}_{\mathfrak{L}}$-measurable if and only if $F = F \circ P_{\mathfrak{L}}$, the last assertion is an immediate consequence of the preceding ones. □

2.1. (i) *implies* (ii) *and* (ii) *implies* (iii). Suppose that $\tilde{F} \in L^2(\mu; \mathbb{R})$ is H-Lipschitz continuous with some constant $C \in [0, \infty)$. Given $\ell \in E^* \setminus \{0\}$, let $\mathfrak{L}$ be the linear span of $\{\ell\}$ and set

$$\tilde{F}(\omega_{\mathfrak{L}^{\perp}}, t) = \tilde{F}(\omega_{\mathfrak{L}^{\perp}} + t\ell) \quad \text{for } \omega_{\mathfrak{L}^{\perp}} \in E_{\mathfrak{L}^{\perp}} \text{ and } t \in \mathbb{R}.$$

Obviously, for each $\omega_{\mathfrak{L}^{\perp}}$, $t \in \mathbb{R} \mapsto \tilde{F}(\omega_{\mathfrak{L}^{\perp}}, t) \in \mathbb{R}$ is an absolutely continuous function and, as such, has the property that the set $\mathscr{T}(\omega_{\mathfrak{L}^{\perp}}, \ell)$ of $t \in \mathbb{R}$ for which the derivative

$$\lim_{\varepsilon \to 0} \frac{\tilde{F}(\omega_{\mathfrak{L}^{\perp}}, t + \varepsilon) - \tilde{F}(\omega_{\mathfrak{L}^{\perp}}, t)}{\varepsilon}$$

exists has full Lebesgue measure. Hence, if $\mathscr{A}(\ell)$ denotes that set of $\omega \in E$ for which the limit

$$G(\omega, \ell) \equiv \lim_{\varepsilon \to 0} \frac{\tilde{F}(\omega + \varepsilon\ell) - \tilde{F}(\omega)}{\varepsilon}$$

exists, then, by our lemma and Fubini's theorem, we know that $\mu(A(\ell)^{\mathrm{C}}) = 0$ for every $\ell \in E^*$. We extend $G(\cdot, \ell)$ to the whole of E by making it vanish off of $A(\ell)$. Obviously, $|G(\omega, \ell)| \leq C\|\ell\|_H$ for all $(\omega, \ell) \in E \times E^*$.

Next, select an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for H consisting of elements from E^*, and set

$$\mathscr{A} \equiv \bigcup_{n=1}^{\infty} \mathscr{A}_n,$$

where

$$\mathscr{A}_n \equiv \left\{ \sum_{k=1}^{n} a_k e_k : \{a_k\}_{k=1}^{n} \subset \mathbb{Q} \text{ with } \sum_{k=1}^{n} a_k^2 = 1 \right\}$$

and $\mathbb{Q}$ stands for the set of rational numbers in $\mathbb{R}$. For every $\ell = \sum_{k=1}^{n} a_k e_k \in \mathscr{A}_n$ and every $\phi \in \mathscr{P}$, we have

$$\begin{aligned}
\mathbb{E}_{\mu}\{G(\omega, \ell)\phi(\omega)\} &= \lim_{\varepsilon \to 0} \mathbb{E}_{\mu}\left\{ \frac{\tilde{F}(\omega + \varepsilon\ell) - \tilde{F}(\omega)}{\varepsilon} \phi(\omega) \right\} \\
&= \mathbb{E}_{\mu}\left\{ \tilde{F}(\omega)\left[\partial_{\ell}^{\dagger}\phi\right](\omega) \right\} = \sum_{k=1}^{n} a_k \mathbb{E}_{\mu}\left\{ \tilde{F}(\omega)\left[\partial_{e_k}^{\dagger}\phi\right](\omega) \right\} \\
&= \sum_{k=1}^{n} a_k \lim_{\varepsilon \to 0} \mathbb{E}_{\mu}\left\{ \frac{\tilde{F}(\omega + \varepsilon e_k) - \tilde{F}(\omega)}{\varepsilon} \phi(\omega) \right\} \\
&= \mathbb{E}_{\mu}\left\{ \sum_{k=1}^{n} a_k G(\omega, e_k)\phi(\omega) \right\}
\end{aligned}$$

and so we now know that, for every $\ell \in \mathscr{A}$,

$$G(\omega, \ell) = \sum_{k=1}^{\infty} (\ell, e_k)_H G(\omega, e_k) \quad \text{for } \mu\text{-a.e. } \omega \in E,$$

and therefore that the set

$$B \equiv \left\{ \omega \in \bigcap_{k=1}^{\infty} A(e_k) : G(\omega, \ell) = \sum_{k=1}^{\infty} (\ell, e_k)_H G(\omega, e_k) \text{ for all } \ell \in \mathscr{A} \right\}$$

has full μ measure. In particular, for each $\omega \in B$ and all $\ell \in \mathscr{A}$,

$$\left| \sum_{k=1}^{\infty} (\ell, e_k)_H G(\omega, e_k) \right| = |G(\omega, \ell)| \leq C \|\ell\|_H,$$

from which it is clear that

$$\sum_{k=1}^{\infty} G(\omega, e_k)^2 \leq C^2 < \infty \quad \text{for all } \omega \in B.$$

Finally, define

$$G(\omega) \equiv \begin{cases} \sum_{k=1}^{\infty} G(\omega, e_k) e_k \in H, & \text{for } \omega \in B, \\ 0, & \text{for } \omega \notin B. \end{cases}$$

Clearly $\|G(\omega)\|_H \leq C$ for all $\omega \in E$. In order to complete the proof that (i) implies (ii), let $\omega \in B$ be chosen and fixed, and define the mapping $\Psi: \mathscr{A} \to C([-1,1], \mathbb{R})$ by

$$[\Psi(\ell)](\varepsilon) = \begin{cases} \dfrac{\tilde{F}(\omega + \varepsilon l) - \tilde{F}(\omega)}{\varepsilon} - (G(\omega), \ell)_H, & \text{for } \varepsilon \in [-1,1] \setminus \{0\}, \\ 0, & \text{for } \varepsilon = 0. \end{cases}$$

After observing that

$$|[\Psi(\ell')](\varepsilon) - [\Psi(\ell)](\varepsilon)| \leq 2C|\ell' - \ell|_H \quad \text{for } \ell, \ell' \in \mathscr{A} \text{ and } \varepsilon \in [-1,1],$$

we conclude that Ψ admits a unique continuous extension to the whole of $\mathbf{S}(H) = \{h \in H: \|h\|_H = 1\}$. We again use Ψ to denote this extension and remark that, for every $h \in \mathbf{S}(H)$, we have, on the one hand, $[\Psi(h)](0) = 0$ and, on the other hand,

$$\frac{\tilde{F}(\omega + \varepsilon h) - \tilde{F}(\omega)}{\varepsilon} - (G(\omega), h)_H = [\Psi(h)](\varepsilon) \quad \text{for } \varepsilon \in [-1,1] \setminus \{0\}.$$

Thus, (ii), with $\tilde{F}' = G$, follows immediately from the fact that Ψ is uniformly continuous on compacts. Furthermore, because there is one μ-negligible set off of which

$$(G(\omega), h)_H = \lim_{\varepsilon \to 0} \frac{\tilde{F}(\omega + \varepsilon h) - \tilde{F}(\omega)}{\varepsilon} \quad \text{for all } h \in H,$$

it is clear that

$$\|G\|_{L^\infty(\mu;H)} \le \operatorname*{ess\,sup}_{\omega\in E} \sup_{h\in H\setminus\{0\}} \left(\frac{1}{\|h\|_H}\left|\tilde{F}(\omega+h)-\tilde{F}(\omega)\right|\right).$$

Turning to the implication (ii) implies (iii), assume (ii) and observe that, just as before, we have that

$$\mathbb{E}_\mu\left\{F(\omega)\left[\partial_\ell^\dagger\phi\right](\omega)\right\} = \mathbb{E}_\mu\left\{\left(\tilde{F}'(\omega),\ell\right)_H\phi(\omega)\right\} \quad \text{for all } \ell\in E^* \text{ and } \phi\in\mathscr{P},$$

or, equivalently, that

$$\mathbb{E}_\mu\{F(\omega)[\mathscr{D}^\dagger\Phi](\omega)\} = \mathbb{E}_\mu\left\{\left(\tilde{F}'(\omega),\Phi(\omega)\right)_H\right\} \quad \text{for all } \Phi\in\mathfrak{P}.$$

Hence, because $\tilde{F}'\in L^\infty(\mu;H)\subseteq L^2(\mu;H)$, we have now proved both that $F\in W_1^2(\mathbb{R})$ and that $DF=\tilde{F}'\in L^\infty(\mu;H)$.

2.2. (iii) *implies* (i). Choose $\{e_n\}_{n=1}^\infty\subset E^*$ to be an orthonormal basis for H, let $\mathfrak{L}_n$ be the linear span of $\{e_1,\dots,e_n\}$ in E^*, and use $E_n^\perp$, Ψ_n, μ_n, and $\mu_n^\perp$ to denote, respectively, the associated quantities $E_{\mathfrak{L}_n^\perp}$, $\Psi_{\mathfrak{L}_n}$, $\mu_{\mathfrak{L}_n}$, and $\mu_{\mathfrak{L}_n^\perp}$ described in our lemma. Then the set B_n of $\ell_n\in\mathfrak{L}_n$ for which

$$\omega_n^\perp\in E_n^\perp\mapsto F\circ\Psi_n^{-1}\left(\omega_n^\perp,\ell_n\right)\in\mathbb{R}$$

is $\mu_n^\perp$-square integrable has full μ_n measure; and if $f_n\colon\mathbb{R}^n\to\mathbb{R}$ is defined by

$$f_n(x) = \int_{E_n^\perp} F\circ\Psi_n^{-1}\left(\omega_n^\perp,\ell_n(x)\right)\mu_n^\perp\left(d\omega_n^\perp\right)$$

when $\ell_n(x)\equiv\sum_1^n x_ie_i\in B_n$ and $f_n(x)=0$ when $\ell_n(x)\notin B_n$, then

$$\mathbb{E}_\mu\{F|\mathscr{F}_n\} = f_n\circ\ell_n^{-1}\circ P_n, \qquad \mu\text{-a.e.},$$

where (cf. the lemma) $\mathscr{F}_n=\mathscr{F}_{\mathfrak{L}_n}$ and $P_n=P_{\mathfrak{L}_n}$. In particular, because the Borel field over E is contained in the μ-completion of $\bigvee_1^\infty\mathscr{F}_n$, we know that

$$F = \lim_{n\to\infty} f_n\circ\ell_n^{-1}\circ P_n, \qquad \mu\text{-almost surely}.$$

We next examine the functions f_n. For this purpose, let $\psi\colon\mathbb{R}^n\to\mathbb{R}^n$ be a smooth function with the property that

$$x\in\mathbb{R}^n\mapsto\phi(x)\equiv(2\pi)^{n/2}e^{|x|^2/2}\psi(x)$$

is a vector-valued function with polynomial coefficients. If $\operatorname{div}(\psi)$ denotes the Euclidean divergence of ψ, then an easy computation shows that

$$-\int_{\mathbb{R}^n} f_n(x)[\operatorname{div}(\psi)](x)\,dx = \int_E F(\omega)[\mathscr{D}^\dagger\Phi](\omega)\mu(d\omega),$$

where $\Phi\in\mathfrak{P}$ is given by

$$\Phi(\omega) = \sum_1^n\left(\phi_i\circ\ell_n^{-1}\circ P_n(\omega)\right)e_i.$$

Hence,

$$\left|\int_{\mathbb{R}^n} f_n(x)[\operatorname{div}(\psi)](x)\,dx\right| = \left|\int_E (\mathscr{D}F(\omega), \Phi(\omega))_H\,\mu(d\omega)\right|$$
$$\le \|\mathscr{D}F\|_{L^\infty(\mu;H)}\|\Phi\|_{L^1(\mu;H)} = \|\mathscr{D}F\|_{L^\infty(\mu;H)}\|\psi\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)},$$

from which it follows that the distributional gradient of f_n is given by a function $\nabla f \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\nabla f_n\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le \|\mathscr{D}F\|_{L^\infty(\mu;H)}$. In particular, f_n can be modified on a set of Lebesgue measure 0 to obtain to an $\tilde{f}_n \in C(\mathbb{R}^n;\mathbb{R})$ which is Lipschitz continuous with constant $\|\mathscr{D}F\|_{L^\infty(\mu;H)}$. Hence, if

$$B = \left\{\omega \in E : \tilde{F}(\omega) \equiv \limsup_{n\to\infty} \tilde{f}_n \circ \ell_n^{-1} \circ P_n(\omega) \in \mathbb{R}\right\},$$

then, for all $(\omega, h) \in E \times H$, $\omega \in B$ if and only if $\omega + h \in B$, and so the extension of $\tilde{F}$ obtained by taking $\tilde{F} = 0$ on $E \setminus B$ is H-Lipschitz continuous with constant $\|\mathscr{D}F\|_{L^\infty(\mu;H)}$. Thus, since $F = \tilde{F}$ μ-almost surely, we are done.

3. Concluding remarks and an application. The familiar Rademacher theorem, from which our title derives, translates into the language of this article as the statement that, when F is H-Lipschitz continuous, then there is an $F' \in L^\infty(\mu; H)$ with the property that

$$\mathbf{R(H)} \qquad \lim_{\|h\|_H\to 0} \frac{F(\omega+h) - F(\omega) - (h, F'(\omega))_H}{\|h\|_H} = 0 \quad \text{for } \mu\text{-a.e. } \omega \in E.$$

However, so far as we know, this theorem has only been proved when $\dim(H) < \infty$, in which case $\mathbf{S}(H)$ is compact and therefore the assertion follows immediately from our result. On the other hand, when $\dim(H) = \infty$, we have been unable to decide whether **R(H)** follows from Lipschitz continuity; although we tend to believe that it does not. Indeed, we suspect that the best that one can do in this direction is replace **R(H)** by

$$\mathbf{R(E^*)} \qquad \lim_{\|\ell\|_{E^*}\to 0} \frac{F(\omega+\ell) - F(\omega) - (\ell, F'(\omega))_H}{\|\ell\|_{E^*}} = 0 \quad \text{for } \mu\text{-a.e. } \omega \in E.$$

Because, when $\dim(H) < \infty$, $E^* = H = E$, **R(H)** and **R($\mathbf{E^*}$)** are equivalent in the finite dimensional setting. However, in infinite dimensions, **R($\mathbf{E^*}$)** is much weaker than **R(H)** and, in fact, by our result, does follow from the assumption that F is H-Lipschitz continuous.

To see this latter statement, it suffices for us to show that the closed unit ball $\overline{B}(E^*) \equiv \{\ell : \|\ell\|_{E^*} \le 1\}$ is relatively compact in H. For this purpose, recall that $\overline{B}(E^*)$ is bounded in H and sequentially weak* compact in E^*. Thus, all that we have to do is show that if $\{\ell_m\}_1^\infty \subset \overline{B}(E^*)$ is weak* convergent to $\ell \in \overline{B}(E^*)$, then $\ell_m \to \ell$ in H. However, because

$$\mathbb{E}_\mu\{\langle \ell, \omega\rangle^4\} = 3\|\ell\|_H^4, \qquad \ell \in E^*,$$

we see that $\{\langle \ell_m, \omega\rangle^2\}_1^\infty$ is uniformly μ-integrable, and therefore that

$$\|\ell_m - \ell\|_H^2 = \mathbb{E}_\mu\Big\{\big(\langle \ell_m, \omega\rangle - \langle \ell, \omega\rangle\big)^2\Big\} \to 0$$

follows immediately from the fact that $\langle \ell_m, \omega\rangle \to \langle \ell, \omega\rangle$ for every $\omega \in E$.

Although we have been unable to settle the question raised above, the search for an answer led to discussion with Leandre in which we stumbled onto the following curious application to our version of Rademacher's theorem. Namely, let K denote a weak* compact subset of the dual space E^*, consider the function $F_K\colon \Omega \to \mathbb{R}$ given by

$$F_K(\omega) = \sup_{\ell \in K} \langle \ell, \omega\rangle.$$

Obviously F_K does more than satisfy condition (i) in our theorem, and therefore, we know that there exists a $A_K \in \mathscr{B}_\Omega$ of full μ measure and a $\mathscr{B}_\Omega$-measurable $F'_K\colon \Omega \to H$ such that

$$\lim_{\varepsilon \to 0} \frac{F_K(\omega + \varepsilon h) - F_K(\omega)}{\varepsilon} = (F'_K(\omega), h)_H \quad \text{for all } (\omega, h) \in A_K \times H.$$

On the other hand, a simple computation shows that, for all $(\omega, h) \in \Omega \times H$,

$$[D^+F(\omega)](h) \equiv \lim_{\varepsilon \searrow 0} \frac{F_K(\omega + \varepsilon h) - F(\omega)}{\varepsilon} = F_K(h, \omega),$$

where

$$F_K(h, \omega) \equiv \max\big\{\langle \ell, h\rangle : \ell \in K \text{ such that } \langle \ell, \omega\rangle = F_K(\omega)\big\}.$$

Hence, for $\omega \in A_K$,

$$h \in H \mapsto F_K(h, \omega) \quad \text{is linear},$$

which is possible only if: *For each $\omega \in A_K$, there is a unique $\ell_K(\omega) \in K$ with the property that* $\langle \ell_K(\omega), \omega\rangle = F_K(\omega)$. For example, when applied to the standard Brownian motion on the line, this result leads immediately to the observation that almost every Brownian path achieves its maximum value precisely once on each closed time interval. In fact, if K denotes any weak* compact set of signed Radon measures λ on $[0, \infty)$ and if

$$\sup_{\lambda \in K} \int_{[0,\infty)} (1 + t)|\lambda|(dt) < \infty,$$

then almost every Brownian path ω will have the property that there is precisely one $\lambda_K(\omega) \in K$ for which

$$\int_{[0,\infty)} \omega(t)(\lambda_K(\omega))(dt) = \max_{\lambda \in K} \int_{[0,\infty)} \omega(t)\lambda(dt) < \infty.$$

This is not the first time that such results have been discovered (cf. [4]), but it is probably the first time that they have been derived by this sort of analysis.

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