The Annals of Probability
1993, Vol. 21, No. 1, 1–13

THE WIENER SPHERE AND WIENER MEASURE

BY NIGEL CUTLAND AND SIU-AH NG[1]

University of Hull

The Loeb measure construction of nonstandard analysis is used to define uniform probability μ_L on the infinite-dimensional sphere of Poincaré, Wiener and Lévy, and we construct Wiener measure from it, thus giving rigorous sense to the informal discussion by McKean. From this follows an elementary proof of a weak convergence result. The relation to the infinite product of Gaussian measures is studied. We investigate transformations of the sphere induced by shifts and the associated transformations of μ_L. The Cameron–Martin density is derived as a Jacobian. We also prove a dichotomy theorem for the family of shifted measures.

0. Introduction. The connection between Wiener measure and the uniform probability measure on the infinite-dimensional sphere $S^\infty(\sqrt{\infty})$ has been known for a long time; it plays an important role in the understanding of Wiener measure and white noise and has motivated many important results. The intuitive idea behind it is clear, but so far it lacks a rigorous theory which is both natural and useful at the same time. With the advent of nonstandard analysis, this situation is easily remedied, and the sphere can once again occupy a more respectable niche.

The history of $S^\infty(\sqrt{\infty})$ in probability theory started with Poincaré [10] early in this century. His remarkable observation was the following. Fix an interval (a, b) on a coordinate axis of $\mathbb{R}^n$. Consider the uniform probability measure on $S^{n-1}(\sqrt{n})$. Then as n increases, the measure of the portion of $S^{n-1}(\sqrt{n})$ that projects onto (a, b) approaches that given by the Gaussian distribution $\mathscr{N}(0, 1)$ between a and b. In a different vein, Lévy [6] made use of this idea to study functional analysis in an infinite-dimensional space. These results eventually led to Wiener's construction of Brownian motion which appeared in [11].

In [11], $S^\infty(\sqrt{\infty})$ is called the "differential space." The justification for this term is as follows. Let w be a Brownian path. Intuitively, since $dw_t^2 = dt$, writing $\dot{w} = dw_t/dt$, we have $\int_0^1 \dot{w}_t^2\, dt = (dt)^{-1} = \infty$, so $\|\dot{w}\| = \sqrt{\infty}$. If we regard each dw_t as the coordinate of dw on the "t-axis," then the differential $\dot{w}$ lies on the "infinite-dimensional sphere in L^2 of radius $\sqrt{\infty}$." Wiener's construction was based on this heuristic picture. However, during the next several decades, the sphere faded from view and information from [11] was extracted without mentioning the sphere. Wiener measure was given many rigorous constructions, but the sphere was abandoned.

Received July 1990; revised September 1991.

[1]Supported by an SERC grant.

AMS 1991 *subject classifications*. Primary 03H05, 28E05, 60J65; secondary 28A35, 28C20, 51M05, 51N05, 60H05.

Key words and phrases. Wiener measure, Loeb measure, infinite-dimensional sphere.

Half a century later, interest in the sphere was revived when new work was done by Hida and Nomoto [5] and Hasegawa [4]. Many of the insights and results from this period were assembled by McKean in [8], which is still the most valuable source of information on this subject. The papers [5] and [4] gave rigorous treatments using the projective limit of $S^n(\sqrt{n})$. Their main interest was in applications to functional analysis. More recently, Morrow and Silverstein [9] utilized the sphere to construct the Ornstein–Uhlenbeck process by means of weak convergence arguments. (This was also discussed in Williams [12] concerning the Brownian sheet and Malliavin calculus.) However, these approaches are conceptually unsatisfactory. Indeed, they are clumsy to use and obscure the geometric content of $S^\infty(\sqrt{\infty})$.

Robinson's nonstandard analysis enables us to consider, for infinite $N \in {}^*\mathbb{N}$, a genuine sphere $S^{N-1}(\sqrt{N})$. Furthermore, the Loeb measure theory construction gives a genuine (standard) uniform probability on it. In this paper, for purely notational reasons, we find it more convenient to work with the sphere $S^{N-1}(1)$ of unit radius, since the coordinates of an element will then represent the increments of a Brownian path. We call $S^{N-1}(1)$ the "Wiener sphere." The main result is the construction of Wiener measure from the sphere. There are at least two other basic methods of constructing Wiener measure using nonstandard analysis. The first is Anderson's "discrete" construction from a hyperfinite random walk [2]. The second is that in [3], obtained from ${}^*\mathbb{R}^N$ equipped with hyperfinitely many copies of Gaussian measure—this is a "*continuous" construction. Our spherical construction is also *continuous. Viewed internally, Anderson's paths correspond to a hyperfinite number of lattice points on the sphere; on the other hand, most of the paths in [3] correspond to points that lie inside a thin shell surrounding the sphere. Thus the spherical model is somewhere in between these two earlier constructions. Perhaps the most significant difference is that the nonstandard increments on the sphere fail to be independent, and the surprising thing is that nevertheless the corresponding standard increments are independent.

Prerequisites on nonstandard analysis for this paper are kept to a minimum. Only a basic understanding of transfer (mostly overspill) and the saturation principle is needed. But we do assume a good knowledge of the Loeb measure construction. The reader may wish to consult [1] and [7]. (They provide enough probability background for this paper as well.)

The first section contains preliminary material. In Section 2 we verify the spherical construction of Wiener measure using results from [3]. Then we prove a weak convergence theorem for measures on finite-dimensional (standard) spheres.

Two applications of the Wiener sphere are obtained.

In Section 3 we consider the shift transformation of the sphere that corresponds to an external translation of Wiener space. We derive the Cameron–Martin formula from the Jacobian of this transformation. This geometric approach makes rigorous the intuition explained in McKean [8].

As a second application, we prove in Section 4 a Kakutani-style dichotomy theorem for the Loeb measure given by a shifted uniform measure. In particular, we obtain an infinite family of mutually orthogonal Loeb measures on the sphere.

1. Preliminaries and notation. We fix an infinite natural number N throughout this paper and write $\Delta t = N^{-1}$ and let $T = \{\Delta t, \dots, 1 = N\Delta t\}$ denote the hyperfinite time line. For $Y \in {}^*\mathbb{R}^T$ we adopt the convention that $Y_0 = 0$.

Given a measure λ, we write E_λ for its expectation (with the subscript dropped when there is no confusion). If λ is internal, we write λ_L for its Loeb extension.

We will work with the *Euclidean space ${}^*\mathbb{R}^N$ whose elements have the form $x = (x_1, \dots, x_N)$, with inner product $x \cdot y = \sum_{i=1}^N x_i y_i$ and norm $\|x\| = (x \cdot x)^{1/2}$. We identify the set $\{1, 2, \dots, N\}$ with T (via $k \leftrightarrow k\,\Delta t$) and hence ${}^*\mathbb{R}^N = {}^*\mathbb{R}^T$.

The Gaussian distribution with mean 0 and variance $u > 0$ is denoted by $\mathscr{N}(0, u)$. It has density function $\Phi(x; u) = (2\pi u)^{-1/2}\exp(-x^2/2u)$. The measure γ on ${}^*\mathbb{R}^N$ is defined as the N-fold product of $\mathscr{N}(0, \Delta t)$, that is,

$$\gamma(A) = (2\pi\,\Delta t)^{-N/2}\int_A \exp\left[-\frac{1}{2}N\sum_{i=1}^N x_i^2\right]dx_1 \cdots dx_N. \tag{1.1}$$

Note that $E[\|x\|^2] = N\Delta t = 1$; for $i \neq j$, $E[x_i^2 x_j^2] = E[x_i^2]E[x_j^2] = \Delta t^2$; and $E[x_i^4] = 3\,\Delta t^2$.

We note that almost all points in ${}^*\mathbb{R}^N$ are near the unit sphere:

Proposition 1.1. *$\|x\| \approx 1$, a.s. γ_L.*

Proof.

$$\begin{aligned} E\left[\left(\|x\|^2 - 1\right)^2\right] &= E\left[\|x\|^4\right] - 2E\left[\|x\|^2\right] + 1 = E\left[\|x\|^4\right] - 1 \\ &= E\left[\sum_{i=1}^N x_i^4 + 2\sum_{i<j} x_i^2 x_j^2\right] - 1 \\ &= N3\,\Delta t^2 + (N-1)N\Delta t^2 - 1 = 2N^{-1} \approx 0. \end{aligned}$$ □

Remarks. 1. Let $w\colon [0,1] \to \mathbb{R}$ be differentiable with differential $\dot{w} \in L^1$. Let $x(w) \in {}^*\mathbb{R}^T$ so that $\sum_{s \le t} x(w)_s = {}^*w_t$. Then

$$\|x(w)\|^2 = \sum_{s \le 1} x(w)_s^2 \le \max_{s \le 1}|x(w)_s| \sum_{s \le 1}|x(w)_s| \approx 0.$$

So we can consider $\|x(w)\|^2$ as a nondifferentiability coefficient of w. (See [11], Section 4.)

2. It is possible with a little care to estimate the thickness of the shell on which γ_L is concentrated. We can show that

$$\lambda\left(\left\{x\colon 1 - (\log N)^{1/2}N^{-1/2} \le \|x\| \le 1 + MN^{-1/2}\right\}\right) \approx 1,$$

where M is any positive infinite number.

A bijection $\Delta: {}^*\mathbb{R}^T \to {}^*\mathbb{R}^T$ is defined by $\Delta Y_t = Y_t - Y_{t-\Delta t}$.

The measure Γ given in [3], Theorem 2.2, is related to γ by $\gamma \circ \Delta = \Gamma$, that is, Γ as a measure on paths Y on ${}^*\mathbb{R}^T$ is given by the measure γ on the increments (ΔY_t) of the paths. This is the connection that enables us to quote results from [3].

Write $\mathscr{C} = \{w \in C[0,1]: w_0 = 0\}$, that is, continuous paths starting from 0. As usual, $\mathscr{C}$ is equipped with the supremum norm. Recall that for S-continuous $Y \in {}^*\mathbb{R}^T$, $st(Y) = {}^\circ Y$ is the element $z \in \mathscr{C}$ so that $z_{{}^\circ t} \approx Y_t$ for all $t \in T$.

The inverse of Δ is given by the function $\Sigma: {}^*\mathbb{R}^T \to {}^*\mathbb{R}^T$, where $(\Sigma x)_t = \Sigma_{s \le t} x_s$. We will be thinking of points in ${}^*\mathbb{R}^T = {}^*\mathbb{R}^N$ as vectors of increments, so the following function from ${}^*\mathbb{R}^T$ to $\mathscr{C}$ is important. We define $\pi = st \circ \Sigma$; this has domain $\operatorname{dom}(\pi) = \Delta(st^{-1}(\mathscr{C})) = \{x \in {}^*\mathbb{R}^T: \Sigma x \text{ is } S\text{-continuous}\}$. More explicitly, we note that $\pi(x)_0 = 0$ and $\pi(x)_t = {}^\circ\Sigma_{s \le t} x_s = {}^\circ\Sigma_{i=1}^M x_i$ when $t = M/N \in T$.

Now let $\pi_\Omega = \pi \restriction \Omega$, the restriction of π to Ω, the unit sphere in ${}^*\mathbb{R}^N$. We see below [Proposition 1.2(i)] that π_Ω is measurable, hence we can define a Borel measure W on $\mathscr{C}$ by $W = \mu_L \circ \pi_\Omega^{-1}$, where μ is the uniform probability on Ω. We will see in Section 2 that this is the Wiener measure, and so $b = {}^\circ B$ is a Brownian motion, where B is the process $B: \Omega \times T \to {}^*\mathbb{R}$ given by $B(x,t) = (\Sigma x)_t$.

The proofs of the following basic properties of π are easy and thus omitted.

PROPOSITION 1.2. (i) *π is measurable w.r.t. the σ-algebra generated by the *Borel subsets of ${}^*\mathbb{R}^T$.*

(ii) *Given finite $\alpha \in {}^*\mathbb{R}$ and $x, y \in \operatorname{dom}(\pi)$, both $x + y$, $\alpha x \in \operatorname{dom}(\pi)$; moreover, $\pi(x+y) = \pi(x) + \pi(y)$ and $\pi(\alpha x) = {}^\circ\alpha\pi(x)$.*

(iii) *Let $z \in \mathscr{C}$ and $c = \Delta^* z$; that is, $c_t = {}^*z_t - {}^*z_{t-\Delta t}$ for $t \in T$. Then $\pi(c) = z$. In particular, π is onto.*

The following will be useful when we deal with unit spheres.

PROPOSITION 1.3. *If $\|x\| \approx 1$, then $x \in \operatorname{dom}(\pi)$ iff $x/\|x\| \in \operatorname{dom}(\pi)$; in either case, $\pi(x) = \pi(x/\|x\|)$.*

PROOF. Write $p(x) = x/\|x\|$. First note that $\Sigma x = \|x\| \Sigma p(x)$. So for $\|x\| \approx 1$, if one of Σx and $\Sigma p(x)$ is S-continuous, then $(\Sigma x)_t \approx (\Sigma p(x))_t$ for all $t \in T$; hence $\pi(x) = \pi(p(x))$. □

The unit sphere in $\mathbb{R}^m$ is written as $S^{m-1}(1) = \{x \in \mathbb{R}^m: x_1^2 + \cdots + x_m^2 = 1\}$. For infinite M, $S^{M-1}(1)$ denotes the corresponding sphere in ${}^*\mathbb{R}^M$.

So $S^{N-1}(1) = \{x \in {}^*\mathbb{R}^N: \|x\| = 1\}$. From [6], we have

$$(1.2)\quad \sigma_m = \operatorname{area}\left(S^{m-1}(1)\right) = \begin{cases} \pi^{m/2} m/\left(\frac{1}{2}m\right)!, & \text{if } m \text{ is even}, \\ 2^{(m+1)/2}\pi^{(m-1)/2}/(1 \cdot 3 \ldots (m-2)), & \text{if } m \text{ is odd and } > 1. \end{cases}$$

(For consistency of our notation, we could define $\sigma_1 = 2$.)

Throughout, we write $\Omega = S^{N-1}(1)$ as our main sample space. We define μ on Ω to be the unique internal uniform (i.e., invariant under rotations) probability measure on Ω, which is the same as the normalized *Lebesgue measure on $S^{N-1}(1)$. (The existence and uniqueness of these entities follow from the transfer of appropriate results on S^n for finite n.) The internal algebra $\mathscr{F}$ consists of *Lebesgue measurable subsets of $S^{N-1}(1)$.

$(\Omega, \mathscr{F}, \mu)$ is called the *Wiener sphere*.

PROPOSITION 1.4. *Let $A = \{x \in \Omega: (x_1 \cdots x_m) \in A'\}$ for some *Lebesgue measurable $A' \subseteq \{x \in {}^*\mathbb{R}^m: x_1^2 + \cdots + x_m^2 \le 1\}$. Then*

$$\operatorname{area}(A) = \sigma_{N-m} \int_{A'} (1 - x_1^2 - \cdots - x_m^2)^{(N-m-2)/2}\, dx_1 \cdots dx_m.$$

PROOF. Let $r = r(x_1, \ldots, x_m) = (1 - x_1^2 - \cdots - x_m^2)^{1/2}$ and let $dS = r^{-1}\, dx_1 \cdots dx_m$. Then by transfer of standard theory of surface integration,

$$\operatorname{area}(A) = \int_{A'} \sigma_{N-m} r^{N-m-1}\, dS = \sigma_{N-m} \int_{A'} r^{N-m-2}\, dx_1 \cdots dx_m. \qquad \square$$

PROPOSITION 1.5. (i) $E_\mu[x_i^2] = N^{-1}$.

(ii) *Suppose* $0 < H \le N^{1/2}$, *H is infinite and* $\|u\| = 1$. *Then* $\mu(\{x: |x \cdot u| < HN^{-1/2}\}) \approx 1$.

PROOF. By symmetry, $NE_\mu[x_i^2] = E_\mu[\|x\|^2] = 1$, so (i) holds. For (ii), using the rotational invariance of μ, we can assume that $u_1 = 1$ and $u_i = 0$ for $i > 1$. Then $x \cdot u = x_1$ and note that by Chebyshev's inequality,

$$\mu(\{x_1 \ge HN^{-1/2}\}) \le H^{-2} NE_\mu[x_1^2] = H^{-2} \approx 0. \qquad \square$$

2. Wiener sphere and weak convergence. In this section, we prove that the measure $W = \mu_L \circ \pi_\Omega^{-1}$ on the Wiener sphere defined in Section 1 is the Wiener measure. The proof given here is short, but relies on results from [3]. As a corollary, we also obtain a representation for Wiener integrals. A further corollary is a quick proof of a weak convergence result for measures induced by uniform probabilities on finite-dimensional unit spheres; this result is noted in [9].

THEOREM 2.1. $W = \mu_L \circ \pi_\Omega^{-1}$ *is the Wiener measure on* $\mathscr{C}$; *in particular,* Σx *is S-continuous a.s.* μ_L.

PROOF. The measure γ on ${}^*\mathbb{R}^N$ defined in Section 1 is clearly invariant under rotations about the origin, and so is the measure $\gamma \circ p^{-1}$ on Ω induced by the projection $p(x) = x/\|x\|$. Therefore, by transfer of standard results on finite-dimensional spheres, we have $\mu = \gamma \circ p^{-1}$, the unique *$\sigma$-additive probability on Ω invariant under rotations. Thus $W = \gamma_L \circ p^{-1} \circ \pi_\Omega^{-1}$.

Now by Propositions 1.1 and 1.3, $\pi(x) = \pi(p(x))$, a.s. γ_L, so $W = \gamma_L \circ \pi^{-1}$, hence $W = \Gamma_L \circ \Delta^{-1} \circ \pi^{-1} = \Gamma_L \circ st^{-1}$, by the discussion in Section 1. But $\Gamma_L \circ st^{-1}$ is the Wiener measure ([3], Theorem 2.2). □

REMARK. A direct verification of Theorem 2.1 is harder than the corresponding result for the Gaussian nonstandard model in [3], but will nevertheless follow from elementary calculations similar to those given in McKean [8].

COROLLARY 2.2. *b is a Brownian motion* (*where $b = {}^\circ B$, as defined in Section* 1).

We now have the following representation of Wiener integrals.

COROLLARY 2.3. *Let $F\colon T \to {}^*\mathbb{R}$ be an SL^2-lifting of some $f \in L^2[0,1]$. Then for $t \in T$,*

$$\sum_{s \le t} F_s x_s \approx \int_0^{{}^\circ t} f_u \, db_u, \quad a.s.\ \mu_L.$$

PROOF. The above proof shows $\mu = \gamma \circ p^{-1}$. By Proposition 1.1, $\|x\| \approx 1$, a.s. γ_L, therefore the result follows from the representation given by [3], Theorem 3.1. □

Another application of Theorem 2.1 shows that uniform probabilities on finite-dimensional unit spheres induce measures on $\mathscr{C}$ which converge weakly to Wiener measure.

Let μ_n be the uniform probability measure on $S^{n-1}(1)$. Define B_n: $S^{n-1}(1) \times [0,1] \to \mathbb{R}$ by $B_n(x,0) = 0$, $B_n(x, k/n) = \sum_{i=1}^{k} x_i$ and for $(k-1)/n < t \le k/n$, we interpolate $B_n(x,t)$ linearly. The Borel measures W_n on $\mathscr{C}$ are given by the following formula:

$$W_n(A) = \mu_n(\{x\colon B_n(x,\cdot) \in A\}). \tag{2.1}$$

Notice that $W_n(\mathscr{C}) = 1$. Let E_n denote E_{W_n}. The following result is similar to that of [2], Theorem 29.

THEOREM 2.4 (Weak convergence). *$W_n \to_w W$ as $n \to \infty$.*

PROOF. Let $f\colon \mathscr{C} \to \mathbb{R}$ be a bounded continuous function. We want to show that

$$\lim_{n\to\infty} E_n[f] = E_W[f]. \tag{2.2}$$

By construction, $E_n[f] = \int_{S^{n-1}(1)} f(B_n(x,\cdot))\, d\mu_n(x)$ for each n. For our fixed infinite N, we have $\Omega = S^{N-1}(1)$, $\mu_N = \mu$ and $E_N = E_\mu$. Therefore

$$E_N[{}^*f] = \int_\Omega {}^*f(B_n(x,\cdot))\, d\mu(x) \approx \int_\Omega f(\pi(x))\, d\mu_L(x) = E_W[f],$$

where $\approx$ follows from Loeb measure theory, continuity of f and the fact that $B_N(x,\cdot) \approx \pi(x)$ for $x \in \mathrm{dom}(\pi)$. The last equality follows from Theorem 2.1.

Since N is an arbitrary infinite natural number, $E_M[{}^*f] \approx E_W[f]$ for any $M \in {}^*\mathbb{N} \setminus \mathbb{N}$, that is, (2.2) holds. □

We remark that Corollary 2.2 gives a standard Brownian motion $b(x) = {}^\circ B(x,\cdot)$ with time set $[0,1]$. If the time set $[0,\infty)$ is desired, we could use $\{\Delta t, \ldots, N^2\Delta t\}$ and $S^{N^2-1}(\sqrt{N})$ in place of T and $S^{N-1}(1)$ respectively. The definition of B remains unchanged, and ${}^\circ B(x,\cdot)$ will still be a Brownian motion.

3. Shift transformations and the Cameron–Martin formula. It is well known that given a "nice" translation of the Wiener space, the resulting new measure is equivalent to the Wiener measure, with a density given by the Cameron–Martin formula. Using an appropriate shift transformation of the Wiener sphere, we derive this formula from an *internal Jacobian* (Theorem 3.5). Moreover, this transformation produces a measure which is internally equivalent to the uniform probability.

We will work with the following type of shift transformation of the Wiener sphere.

Let $c \in {}^*\mathbb{R}^N$ and $\|c\| < 1$. Define

$$\theta: \Omega \to \Omega$$

by $\theta(x) = (x-c)/\|x-c\| = p(x-c)$ (the projection). θ is well defined since $\|c\| < 1$.

PROPOSITION 3.1. (i) *$\theta: \Omega \to \Omega$ is an internal homeomorphism.*
(ii) $(\mu \circ \theta)_L = \mu_L \circ \theta$.

PROOF. (i) follows from $\|c\| < 1$.
θ induces an internal automorphism on the algebra of μ-measurable sets, hence (ii) follows from the construction of the Loeb measure. □

DEFINITION 3.2. We let $\mu^c = \mu \circ \theta$ be the measure on Ω carried by the transformation. (Then the above shows $\mu^c_L = \mu_L \circ \theta$.)

Let the translation of the Wiener measure by $z \in \mathscr{C}$ be defined as

$$W^z(A) = W(A-z).$$

Notice that we can always find $c \in {}^*\mathbb{R}^N$ with $\pi(c) = z$ and $\|c\| \approx 0$.

LEMMA 3.3. *Suppose $c \in {}^*\mathbb{R}^N$, $\|c\| \approx 0$ and $\pi(c) \doteq z \in \mathscr{C}$.*

(i) *Let $x \in \Omega$. Then $x \in \mathrm{dom}(\pi)$ iff $\theta(x) \in \mathrm{dom}(\pi)$ and $\pi(\theta(x)) = \pi(x) - z$.*
(ii) *For any $A \subseteq \mathscr{C}$ and $x \in \mathrm{dom}(\pi)$, $\pi(x) \in A$ iff $\pi(\theta(x)) \in A - z$.*

PROOF. Let $x \in \Omega$, since $\|c\| \approx 0$, $\|x-c\| \approx 1$ and hence (i) follows from Proposition 1.2(ii) and Proposition 1.3. (ii) is a consequence of (i). □

The relation between W^z and μ_L^c is given by the following result, which can be viewed as an extension of Theorem 2.1.

COROLLARY 3.4. $W^z = \mu_L^c \circ \pi_\Omega^{-1}$.

PROOF.

$$\begin{aligned} W^z(A) &= W(A - z) \\ &= \mu_L(\{y \in \Omega \colon \pi(y) \in A - z\}) \\ &= \mu_L(\{\theta(x) \colon \pi(\theta(x)) \in A - z\}) \\ &= \mu_L(\{\theta(x) \colon \pi(x) \in A\}) \quad \text{[by Lemma 3.3(ii)]} \\ &= \mu_L^c(\pi_\Omega^{-1}(A)). \end{aligned}$$

□

The density of μ^c w.r.t. μ is given by the following theorem. [We are *not* now assuming that $c \in \mathrm{dom}(\pi)$.

THEOREM 3.5. *Let $c \in {}^*\mathbb{R}^N$ and $\|c\| < 1$. Then we have the internal density*

$$\frac{d\mu^c}{d\mu}(x) = \frac{1 - x \cdot c}{\|x - c\|^N}.$$

PROOF. Let θ be the shift transformation associated with c as before. By rotating the sphere, we can assume the following on the coordinates of c:

$$c_1 = \|c\| \quad \text{and} \quad c_i = 0 \quad \text{for } i > 1. \tag{3.1}$$

(In particular, $x \cdot c = x_1 x_1$.) Now let $y = \theta(x)$. Then

$$y = (x - c)/\|x - c\|. \tag{3.2}$$

Let

$$J(x) = \left| \frac{\partial(y_1 \cdots y_{N-1})}{\partial(x_1 \cdots x_{N-1})} \right|$$

be the Jacobian of θ^{-1} (regarded as a transformation of the first $N - 1$ coordinates of points on the sphere). Let $A \subseteq \Omega$ be *Lebesgue measurable and without loss of generality, assume that $x_N \geq 0$ for each $x \in A$. As a consequence, we also have $(\theta(x))_N \geq 0$ for each $x \in A$. Moreover, the projection onto the first $N - 1$ coordinates is one to one on both A and $\theta(A)$. We denote the images by A' and $\theta(A)'$ respectively. Now by Proposition 1.4 (with $m = N - 1$), and (3.1) and (3.2) above,

$$\begin{aligned} \mu^c(A) = \mu(\theta(A)) = \frac{\mathrm{area}(\theta(A))}{\mathrm{area}(\Omega)} &= \frac{1}{2}\sigma_1 \int_{\theta(A)'} (\sigma_N |y_N|)^{-1} \, dy_1 \cdots dy_{N-1} \\ &= \frac{1}{2}\sigma_1 \int_{A'} \|x - c\| J(x) (\sigma_N |x_N|)^{-1} \, dx_1 \cdots dx_{N-1}, \end{aligned}$$

hence

$$\frac{d\mu^c}{d\mu}(x) = \|x - c\| J(x). \tag{3.3}$$

Since $\|x\| = 1$, for $x \in \Omega$, we have

$$\|x - c\| = \left(1 - 2x \cdot c + \|c\|^2\right)^{1/2}. \tag{3.4}$$

From this we obtain

$$\frac{\partial}{\partial x_i}\|x - c\| = \frac{-c_i}{\|x - c\|}.$$

Thus (3.1) and (3.2) give

$$\frac{\partial y_1}{\partial x_1} = \frac{1 - x_1 c_1}{\|x - c\|^3}, \qquad \frac{\partial y_i}{\partial x_i} = \frac{1}{\|x - c\|}$$

for $i > 1$ and $\partial y_i / \partial x_j = 0$ when $i \neq j > 1$. In particular, the Jacobian matrix is lower triangular, hence

$$J(x) = \left[\frac{1 - x_1 c_1}{\|x - c\|^3}\right]\left[\frac{1}{\|x - c\|}\right]^{N-2} = \frac{1 - x \cdot c}{\|x - c\|^{N+1}}.$$

Now the theorem follows from (3.3). □

We remark that there are transformations other than θ for which Corollary 3.4 still holds. For instance, one may consider $\phi\colon \Omega \to \Omega$, where ϕ^{-1} is given by

$$\phi^{-1}(y) = (y + c)/\|y + c\|.$$

Then it can be shown that the counterpart to Theorem 3.5 is the density

$$\frac{\left(c \cdot x + \left(1 + (c \cdot x)^2 - \|c\|^2\right)^{1/2}\right)^{N-1}}{\left(1 + (c \cdot x)^2 - \|c\|^2\right)^{1/2}},$$

and clearly θ is preferable.

The next lemma is the key to showing that μ^c and μ^d are equivalent if $c - d$ is sufficiently small.

Lemma 3.6. *Let $c, d \in {}^*\mathbb{R}^N$ such that ${}^\circ\|c\|, {}^\circ\|d\| < 1$. Suppose*

$$N\|c - d\|^2 < \infty.$$

Then $\delta = d\mu^c / d\mu^d$ is S-integrable w.r.t. μ^d.

Proof. Our aim is to show $E_{\mu^d}[\delta^2] < \infty$, then the result follows from Lindstrøm's lemma ([3], Lemma 1.5).

First recall the parallelogram law for Euclidean spaces:

$$\|a + b\|\,\|a - b\| \leq \|a\|^2 + \|b\|^2. \tag{3.5}$$

Then it follows that for $x \in \Omega$,

$$\|x - d\|\,\|x - 2c + d\| \leq \|x - c\|^2 + \|c - d\|^2. \tag{3.6}$$

Notice also if we pick a standard h with ${}^\circ\|c\|, {}^\circ\|d\| < h < 1$, then

$$\|x - c\| \geq 1 - \|c\| \geq 1 - h.$$

Hence if we let $\alpha = (1 - h)^{-2} < \infty$, then $\|x - c\|^{-2} \leq \alpha$. Now by (3.6),

$$\frac{\|x - d\|\,\|x - (2c - d)\|}{\|x - c\|^2} \leq 1 + \frac{\|c - d\|^2}{\|x - c\|^2} \leq 1 + \alpha\|c - d\|^2.$$

Therefore

$$\frac{\|x - d\|^2}{\|x - c\|^2} \leq \left(1 + \alpha\|c - d\|^2\right)\frac{\|x - d\|}{\|x - (2c - d)\|}. \tag{3.7}$$

On the other hand, Theorem 3.5 implies

$$\delta = \frac{1 - x \cdot c}{1 - x \cdot d} \cdot \frac{\|x - d\|^N}{\|x - c\|^N}. \tag{3.8}$$

Using (3.7), we obtain then

$$\delta^2 \leq \left[\frac{1 - x \cdot c}{1 - x \cdot d}\right]^2 \left(1 + \alpha\|c - d\|^2\right)^N \frac{\|x - d\|^N}{\|x - (2c - d)\|^N}. \tag{3.9}$$

Notice that ${}^\circ\|c\| < 1$, $\|c - d\| \approx 0$ and $\|2c - d\| \leq \|c\| + \|c - d\|$, so we have

$${}^\circ\|2c - d\| < 1. \tag{3.10}$$

Applying Theorem 3.5 to $2c - d$ and c, (3.9) gives

$$\delta^2 \leq \frac{(1 - x \cdot c)^2}{1 - x \cdot d}\,\frac{\left(1 + \alpha\|c - d\|^2\right)^N}{1 - x \cdot (2c - d)}\,\frac{d\mu^{2c-d}}{d\mu^d}. \tag{3.11}$$

Now

$$\frac{(1 - x \cdot c)^2}{(1 - x \cdot d)(1 - x \cdot (2c - d))} < \infty$$

and by $(1 + a/b)^b \leq e^a$ for all $a \geq 0$ and $b > 0$, we have

$$\left(1 + \alpha\|c - d\|^2\right)^N = \left(1 + \alpha N\|c - d\|^2/N\right)^N \leq \exp\left(\alpha N\|c - d\|^2\right) < \infty.$$

So for some $\beta < \infty$,

$$\delta^2 \leq \beta\frac{d\mu^{2c-d}}{d\mu^d}. \tag{3.12}$$

This shows that $E_{\mu^d}[\delta^2] \leq \beta < \infty$. □

COROLLARY 3.7. *Under the assumptions in the above lemma*, $\mu_L^c \equiv \mu_L^d$.

PROOF. $\mu^c(B) = \int_B \delta \, d\mu^d$, and since δ is S-integrable, $\mu^d(B) \approx 0 \Rightarrow \mu^c(B) \approx 0$, that is, $\mu_L^c \ll \mu^d{}_L$. By interchanging the role of c and d, we get $\mu_L^d \ll \mu_L^c$. □

We can obtain the Cameron–Martin formula for a translation of Wiener measure by showing that its density is the standard part of the internal density $d\mu^c/d\mu$.

LEMMA 3.8. *Suppose* $z \in \mathscr{C}$ *and* $\dot{z} = dz/dt \in L^2[0,1]$. *Let* $c \in {}^*\mathbb{R}^N$ *be such that* $c_i = {}^*z_{i/N} - {}^*z_{(i-1)/N}$. *Then* $d\mu^c/d\mu$ *is S-integrable w.r.t.* μ, *and for* μ_L*-a.a.* x,

$$\frac{d\mu^c}{d\mu}(x) \approx \exp\left(\int_0^1 \dot{z}_t \, db_t - \frac{1}{2}\int_0^1 \dot{z}_t^2 \, dt\right) = \rho(b), \quad say,$$

where $b = \pi(x)$ *is the Brownian motion given by* μ.

PROOF. By Proposition 1.2(iii), $\pi(c) = z$. We first establish the following interpretations of some geometric quantities:

$$N\|c\|^2 \approx \int_0^1 \dot{z}_t^2 \, dt, \tag{3.13}$$

$$Nx \cdot c \approx \int_0^1 \dot{z}_t \, db_t, \quad \text{a.s. } \mu_L, \tag{3.14}$$

$$\|x - c\|^{-N} \approx \rho(b), \quad \text{a.s. } \mu_L. \tag{3.15}$$

In order to prove (3.13)–(3.15), we let $a \in {}^*\mathbb{R}^T$ so that $a_{i/N} = c_i(\Delta t)^{-1}$. (Recall our convention that $a_0 = 0$.) Since $c_i = \int_{(i-1)/N}^{i/N} {}^*\dot{z}_t \, dt$, it follows from [3], Lemma 1.7, that a is an SL^2-lifting of $\dot{z}$. So $\int_0^1 \dot{z}_t^2 \, dt \approx \sum_{i=0}^N a_{i/N}^2 \, \Delta t$, which is just $N\|c\|^2$, and (3.13) follows. Similarly, we have $Nx \cdot c = \sum_{i=0}^N a_{i/N} x_i$, so (3.14) follows from Corollary 2.3. Notice that $Nx \cdot c$ is μ_L-a.e. finite. From (4) we obtain

$$\begin{aligned} \|x - c\|^{-N} &= \left(1 - (2Nx \cdot c - N\|c\|^2)/N\right)^{-N/2} \\ &\approx \exp\left(Nx \cdot c - \tfrac{1}{2}N\|c\|^2\right), \quad \text{a.s. } \mu_L \\ &\approx \rho(\pi(x)), \quad \text{a.s. } \mu_L, \text{ by (3.13) and (3.14)}, \end{aligned}$$

so (3.15) holds.

Note that (3.13) and the assumption on z imply that $\|c\| \approx 0$. Consequently, Theorem 3.5 applies. Since $|x \cdot c| \le \|c\| \approx 0$, from (3.15), we see that $d\mu^c/d\mu(x) \approx \rho(\pi(x))$, a.s. μ_L, that is, $d\mu^c/d\mu$ lifts ρ, as required. Lemma 3.6 (with $d = 0$) shows that $d\mu^c/d\mu$ is S-integrable w.r.t. μ. □

From Corollary 3.4 and Lemma 3.8 we have the following corollary.

COROLLARY 3.9 (Cameron–Martin formula). *Suppose* $x \in \mathscr{C}$ *and* $\dot{z} \in L^2[0,1]$. *Then* $W^z \ll W$ *and* $dW^z/dW = \rho$.

4. The dichotomy of shifted measures. We will show that the Loeb extension of a shifted uniform probability as in the last section is either equivalent or orthogonal to the Loeb extension of the uniform probability. In fact, the dichotomy theorem (Theorem 4.2) can be used to produce an infinite family of mutually orthogonal measures each arising from an internal measure equivalent to the uniform probability. (The maximal cardinality of such a family depends on how much saturation the nonstandard universe has.)

We first see that when the transformation moves too far, the measures become orthogonal.

LEMMA 4.1. *Let* $c, d \in {}^*\mathbb{R}^N$ *such that* $\|c\| < 1$, ${}^\circ\|d\| < 1$ *and* $N\|c-d\|^2$ *is infinite. Then* $\mu_L^c \perp \mu_L^d$.

PROOF. Let $\delta = d\mu^c/d\mu^d$. Then by Theorem 3.5,

$$\delta(x) = \frac{1 - x\cdot c}{1 - x\cdot d}\cdot\frac{\|x-d\|^N}{\|x-c\|^N}. \tag{4.1}$$

We define $A = \{x \in \Omega\colon (x-d)\cdot(c-d) \le \|c-d\|^2/3\}$ and will show

$$\mu_L^d(A) = 1, \tag{4.2}$$

$$\delta(x) \approx 0 \quad \text{whenever } x \in A. \tag{4.3}$$

Since (4.3) implies $\mu_L^c(A) = 0$, the lemma will follow.

For (4.2), let $\alpha = (3(1 + \|d\|))^{-1}$. Define

$$A' = \left\{y \in \Omega\colon y\cdot(c-d) \le \alpha\|c-d\|^2\right\}.$$

If $y \in A'$, then

$$y\cdot\frac{(c-d)}{\|c-d\|} \le \alpha\|c-d\| = (\alpha N^{1/2}\|c-d\|)N^{-1/2}.$$

Since ${}^\circ\alpha > 0$, $N^{1/2}\|c-d\|$ is infinite, so by Proposition 1.5(ii), $\mu(A') \approx 1$. Now let $p(u) = u/\|u\|$ be the projection and we show that $A' \subseteq p(A-d)$, from which it follows that $\mu^d(A) = \mu(p(A-d)) \approx 1$ and thus (4.2) holds.

Take $y \in A'$, let $x \in \Omega$ so that $y = p(x-d)$. Then

$$\frac{(x-d)}{\|x-d\|}\cdot(c-d) = y\cdot(c-d) \le \frac{\|c-d\|^2}{3(1+\|d\|)},$$

hence

$$(x-d)\cdot(c-d) \le \|c-d\|^2/3,$$

that is,

$$A' \subseteq p(A-d).$$

For (4.3), note that

$$\|x-c\|^2 = \|(x-d)-(c-d)\|^2 = \|x-d\|^2 + \|c-d\|^2 - 2(x-d)\cdot(c-d).$$

So for $x \in A$,

$$\|x - c\|^2 \geq \|x - d\|^2 + \|c - d\|^2 - 2\|c - d\|^2/3 = \|x - d\|^2 + \|c - d\|^2/3,$$

hence

$$\frac{\|x - c\|^N}{\|x - d\|^N} \geq \left[1 + \frac{\|c - d\|^2}{3\|x - d\|^2}\right]^{N/2} \geq \left(1 + 3\alpha^2\|c - d\|^2\right)^{N/2}, \quad \text{infinite}.$$

Notice that $|1 - x \cdot d| \leq 1 - \|d\|$, is noninfinitesimal, and using (4.1) we see that for $x \in A$, $\delta^{-1}(x)$ is infinite, that is, $\delta(x) \approx 0$, hence (4.3) holds. □

A little calculation actually shows that in the above lemma we can replace the condition ${}^\circ\|d\| < 1$ with the weaker one $\log(1 - \|d\|)/N\|c - d\|^2 \approx 0$ (still requiring that $N\|c - d\|^2$ is infinite). Combining Corollary 3.7 and Lemma 4.1, we obtain the following Kakutani-style classification for Loeb measures μ_L^c.

THEOREM 4.2 (Dichotomy). *Suppose* $c, d \in {}^*\mathbb{R}^N$ *and* ${}^\circ\|c\|, {}^\circ\|d\| < 1$. *If* $N\|c - d\|^2 < \infty$, *then* $\mu_L^c \equiv \mu_L^d$; *otherwise* $\mu_L^c \perp \mu_L^d$.

However tempting it appears, this theorem does not readily give the corresponding dichotomy theorem for W^z. Consider, for example, that if $c_i = (-1)^i N^{-1/2}$ for $i \leq N^{1/2}$ and $c_i = 0$ otherwise, then $N\|c\|^2$ is infinite, so $\mu_L^c \perp \mu_L$. But since $\pi(c) = 0$, $\mu_L^c \circ \pi_\Omega^{-1} = W = \mu_L \circ \pi_\Omega^{-1}$.

REFERENCES

[1] ALBEVERIO, S., FENSTAD, J., HØEGH-KROHN, R. and LINDSTRØM, T. (1986). *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*. Academic, New York.
[2] ANDERSON, R. M. (1976). A nonstandard representation for Brownian motion and Itô integration. *Israel J. Math.* **25** 15–46.
[3] CUTLAND, N. (1987). Infinitesimals in action. *J. London Math. Soc.* (2) **35** 202–216.
[4] HASEGAWA, Y. (1980). Lévy's functional analysis in terms of an infinite dimensional Brownian motion. I. *Proc. Japan Acad. Ser. A Math. Sci.* **56** 109–113.
[5] HIDA, T. and NOMOTO, H. (1964). Gaussian measure on the projective limit space of spheres. *Proc. Japan Acad. Ser. A Math. Sci.* **40** 301–304.
[6] LÉVY, P. (1951). *Problémes Concréts d'Analyse Fonctionelle*. Gauthier-Villars, Paris.
[7] LINDSTRØM, T. (1988). An invitation to nonstandard analysis. In *Nonstandard Analysis and Its Applications* (N. Cutland ed.). Cambridge Univ. Press.
[8] MCKEAN, H. P. (1973). Geometry of differential space. *Ann. Probab.* **1** 197–206.
[9] MORROW, G. J. and SILVERSTEIN, M. L. (1986). Two parameter extension of an observation of Poincaré. *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204** 396–418. Springer, New York.
[10] POINCARÉ, H. (1912). *Calcul des Probabilités*. Gauthier-Villars, Paris.
[11] WIENER, N. (1923). Differential space. *J. Math. Phys.* **2** 132–174.
[12] WILLIAMS, D. (1981). To begin at the beginning: ... *Stochastic Integrals. Lecture Notes in Math.* **851** 1–55. Springer, New York.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF HULL
HULL, HU6 7RX
UNITED KINGDOM