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WALD'S EQUATION FOR A CLASS OF DENORMALIZED U-STATISTICS

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Under suitable conditions on a stopping time T and zero mean i.i.d. random variables $\{X_n, n \geq 1\}$, a Wald-type equation $ES_{k,T} = 0$ is obtained where $S_{k,n}$ is the sum of products of k of the X's with indices from 1 to n. This, in turn, is utilized to obtain information about the moments of $T_k = \inf\{n \geq k: S_{k,n} \geq 0\}$ and $W_c = \inf\{n \geq 2: S_{1,n}^2 > c\sum_{j=1}^n X_j^2\}$, $c > 0$.

1. Introduction. For any sequence $\{X, X_n, n \geq 1\}$ of i.i.d. random variables and integers $n \geq k \geq 1$, define

$$(1) \qquad S_{k,n} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} X_{i_1} \cdots X_{i_k}, \qquad S_{k,n}^* = \max_{k \leq j \leq n} |S_{k,j}|$$

and set $S_{1,n} = S_n$, $S_{0,n} = 1$, $n \geq 1$. Then, if $EX = 0$, for each $k > 1$, $U_{k,n} = \binom{n}{k}^{-1} S_{k,n}$ is a so-called degenerate U-statistic since the kernel $h(x_1, \dots, x_k) = \prod_{j=1}^k x_j$ is such that $E\{h(X_1, \dots, X_k) | X_1, \dots, X_j\} =_{\text{a.c.}} 0$ for $j = 1$ (also for $2 \leq j < k$).

For $k = 1$, Wald's equation (as generalized by Blackwell) asserts that $ES_T = 0$ whenever $EX = 0$ and T is a stopping time of $\{X_n\}$ with $ET < \infty$. Numerous extensions in a variety of directions have appeared over the years involving alternative moment conditions, higher moment analogues, martingales, Banach space random elements and so on. Closest in spirit to the current work are articles of Burkholder and Gundy [3] wherein it is shown that when $p = 2$, $EX = 0$, $E|X|^p < \infty$, $ET^{1/p} < \infty$ imply $ES_T = 0$ and Chow, Robbins and Siegmund [4] where this is extended to all p in $(1, 2]$.

Naturally, for $k > 1$, any Wald-type equation will involve the sums $S_{k,n}$ rather than the averages $U_{k,n}$. In particular, it will be shown for any p in $(1, 2]$ that $EX = 0$, $E|X|^p < \infty$ and $ET^{(k-1)/(p-1)} < \infty$ imply $ES_{k,T} = 0$ (Theorem 2) and this will be utilized to obtain information about the moments of $T_k = \inf\{n \geq k: S_{k,n} \geq 0\}$ (Theorem 3). The special case of T_2 has bearing on the behavior of $W_c = \inf\{n \geq 2: S_n^2 \geq c\sum_{j=1}^n X_j^2\}$, $c > 0$ (see Corollary 1). Mean convergence is discussed briefly in Theorem 4. Finally, some partial results involving a second moment analogue of Theorem 2 are given (Theorem 5).

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2. Mainstream. The analysis is facilitated by the simple but pivotal recursion relation

$$S_{k,n} = \sum_{j=k}^{n} X_j S_{k-1,j-1}, \qquad n \ge k \ge 1. \tag{2}$$

THEOREM 1. *Let $\{S_{k,n},\ n \ge k \ge 2\}$ be as in* (1) *where $\{X, X_n,\ n \ge 1\}$ are i.i.d. random variables with $EX = 0$, $E|X|^p < \infty$ and let T be a stopping time of $\{X_n;\ n \ge 1\}$ with $ET^q < \infty$, where $p \le 2$, $q \ge 1$ and $q(p-1) \ge r$ for some nonnegative integer r. If $\alpha = \alpha_r = (pq)/(q+r)$, then $1 \le \alpha \le p$ with $1 \le \alpha < p$ when $r > 0$ and moreover,*

$$E(S^*_{r+1,T\wedge n})^{\alpha_r} = O(1) \quad \textit{as } n \to \infty. \tag{3}$$

Note that $\alpha_r = p$ implies $r = 0$. The special case $\alpha_r = 1$ in conjunction with dominated convergence yields:

THEOREM 2. *If $\{X, X_n,\ n \ge 1\}$ are i.i.d. with $EX = 0$, $E|X|^p < \infty$, $1 < p \le 2$, then for any positive integer r and stopping time T,*

$$ES_{r+1,T} = 0 \quad \textit{if } ET^{r/(p-1)} < \infty. \tag{4}$$

THEOREM 3. *Let $\{S_{k,n},\ n \ge k \ge 2\}$ be defined by* (1), *where $\{X, X_n,\ n \ge 1\}$ are i.i.d. random variables with $EX = 0 < E|X|^p < \infty$, $1 < p \le 2$. Then if $T_k = \inf\{n \ge k\colon S_{k,n} \ge 0\}$, $ET_k^{(k-1)/(p-1)} = \infty$, $k \ge 2$.*

PROOF. Suppose $ET_k^{(k-1)/(p-1)} < \infty$ for some integer $k \ge 2$. Then via Theorem 2,

$$0 \le E(X_1 \cdots X_k)^+ \le ES_{k,T_k} = 0,$$

implying

$$(E|X|)^k = E|X_1 \cdots X_k| = 2E(X_1 \cdots X_k)^+ - EX_1 \cdots X_k = 0,$$

contradicting the hypothesis of nondegeneracy. □

COROLLARY 1. *Let $W_c = \inf\{n \ge 2\colon S_n^2 \ge c\Sigma_{j=1}^n X_j^2\}$, $c > 0$, where $\{X, X_n,\ n \ge 1\}$ and i.i.d. random variables with $EX = 0$, $EX^2 = 1$. Then $EW_c < \infty$ or $= \infty$ according as $0 < c < 1$ or $c \ge 1$.*

PROOF. For $c \ge 1$, the conclusion follows from Theorem 3 via $W_c \ge W_1 = T_2$. Let $0 < c < 1$ and suppose $EW_c = \infty$. Now if $V = W_c \wedge n$, then $EV < \infty$ and $EX_V^2 = o(EV)$ as $n \to \infty$ ([10] or Lemma 5.4.2 of [7]). Thus, by the second moment analogue of Wald's equation [5],

$$EV = ES_V^2 = E\left(S_{V-1}^2 + 2X_V S_{V-1} + X_V^2\right)$$
$$< cE\sum_{j=1}^{V} X_j^2 + 2\left[cEX_V^2 E\sum_{j=1}^{V} X_j^2\right]^{1/2} + EX_V^2 = cEV + o(EV),$$

yielding a contradiction as $n \to \infty$. □

REMARK. Clearly, $EW_c^{1/(p-1)} = \infty$, $c \ge 1$, if $X \in \mathscr{L}_p$ for some p in $(1,2)$ rather than $p = 2$.

PROOF OF THEOREM 1. For $n \ge k \ge 1$ and $n > m \ge 0$, define

$$_mS_{k,n} = \sum_{m<i_1<\cdots<i_k\le n} X_{i_1}\cdots X_{i_k}, \qquad {}_oS_{k,n} = S_{k,n},\ {}_mS_{0,n} \equiv 1,\ S_n = S_{1,n}$$

Then for $r \ge 2$ and $j > 2^i$,

$$S_{r,j-1} = S_{r,2^i} + \sum_{h_1=2^i+1}^{j-1} X_{h_1} S_{r-1,h_1-1}$$
$$= S_{r,2^i} + S_{r-1,2^i}\left({}_{2^i}S_{1,j-1}\right) + \sum_{h_1>2^i}^{j-1}\sum_{h_2>2^i}^{h_1-1} X_{h_1}X_{h_2}S_{r-2,h_2-1}$$
$$= \cdots = \sum_{h=0}^{r} S_{h,2^i}\left({}_{2^i}S_{r-h,j-1}\right),$$

and this also holds for $r = 1$. Hence for any stopping time T relative to $\mathscr{F}_n = \sigma(X_1,\ldots,X_n)$,

$$\sum_{j=r+1}^{2^{n+1}} X_j^2 S_{r,j-1}^2 I_{[T\ge j]} \le O(1)\sum_{i=\log_2(r+1)}^{n} I_{[T\ge 2^i]}\sum_{j=2^i+1}^{2^{i+1}} X_j^2 \sum_{h=0}^{r} S_{h,2^i}^2\left({}_{2^i}S_{r-h,j-1}^2\right)$$
$$= O(1)\sum_i I_{[T\ge 2^i]}\sum_{h=0}^{r} S_{h,2^i}^2 \sum_{j=2^i+1}^{2^{i+1}} X_j^2\left({}_{2^i}S_{r-h,j-1}^2\right),$$

implying for $1 \le \alpha \le 2$ that

$$J_{r,\alpha} =_{\text{def}} E\left(\sum_{j=r+1}^{2^{n+1}} X_j^2 S_{r,j-1}^2 I_{[T\ge j]}\right)^{\alpha/2}$$

$$\le O(1)E\sum_{i=\log_2(r+1)}^{n} I_{[T\ge 2^i]}\sum_{h=0}^{r}|S_{h,2^i}|^\alpha\left[\sum_{j>2^i}^{2^{i+1}} X_j^2\left({}_{2^i}S_{r-h,j-1}^2\right)\right]^{\alpha/2} \tag{5}$$
$$= O(1)\sum_{i=\log_2(r+1)}^{n} EI_{[T\ge 2^i]}\sum_{h=0}^{r}|S_{h,2^i}|^\alpha E\left(\sum_{j=r-h+1}^{2^i} X_j^2 S_{r-h,j-1}^2\right)^{\alpha/2},$$

whence for $r \geq 1$ and $1 \leq \alpha \leq 2$, recalling (2) and a martingale inequality of Burkholder [2] and Davis [8],

$$E|S^*_{r,T\wedge n}|^\alpha \leq O(1)E\left(\sum_{j=r}^{n} X_j^2 S_{r-1,j-1}^2 I_{[T\geq j]}\right)^{\alpha/2} \leq O(1)J_{r-1,\alpha}. \tag{6}$$

Let $X \in \mathscr{L}_p$. Since the special case $r = 0$, $\alpha = p = 1$, is included in (11′), suppose that $1 \leq \alpha < p \leq 2$, whence $r \geq 1$. By the martingale extension of the Marcinkiewicz–Zygmund inequality [2],

$$\begin{aligned} E|S_{r,n}|^p &\leq O(1)E\left(\sum_{j=r}^{n} X_j^2 S_{r-1,j-1}^2\right)^{p/2} \\ &\leq O(1)E\sum_{j=r}^{n}|X_j|^p|S_{r-1,j-1}|^p = O(1)\sum_{j=r}^{n} E|S_{r-1,j-1}|^p, \end{aligned}$$

implying $E|S_{r,n}|^p = O(n^r)$, $r \geq 1$, whence [2]

$$\begin{aligned} E\left(\sum_{j=r}^{n} X_j^2 S_{r-1,j-1}^2\right)^{\alpha/2} &\leq E^{\alpha/p}\left(\sum_{j=r}^{n} X_j^2 S_{r-1,j-1}^2\right)^{p/2} \\ &\leq O(1)E^{\alpha/p}|S_{r,n}|^p = O(n^{r\alpha/p}). \end{aligned}$$

Moreover, if $J_{s-1,\alpha} = O(1)$ as $n \to \infty$ for some positive integer s, then (6) ensures $E|S_{s,T}|^\alpha = O(1)$ whence (Corollary 7.4.6 of [7]) with probability 1,

$$E\{|S_{s,T}|^\alpha|\mathscr{F}_m\} \geq |S_{s,m}|^\alpha I_{[T\geq m]}, \qquad m \geq 1.$$

Thus, for $r \geq 1$ and $1 \leq \alpha < p \leq 2$, recalling (5),

$$\begin{aligned} J_{r,\alpha} &\leq O(1)\sum_{h=0}^{r}\sum_{i=1}^{\infty} EI_{[T\geq 2^i]}|S_{h,2^i}|^\alpha 2^{i(r-h+1)\alpha/p} \\ &\leq O(1)\sum_{h=0}^{r}\sum_{i=1}^{\infty} 2^{i(r-h+1)\alpha/p} EI_{[T\geq 2^i]}|S_{h,T}|^\alpha \\ &= O(1)\sum_{h=0}^{r} E|S_{h,T}|^\alpha T^{(r-h+1)\alpha/p}. \end{aligned} \tag{7}$$

Consequently, combining (6) and (7), the desired conclusion (3) is implied by

$$\sum_{h=0}^{r} E|S_{h,T}|^\alpha T^{(r-h+1)\alpha/p} < \infty,$$

or equivalently by

$$I_\alpha(h) =_{\text{def}} E|S_{h,T}|^\alpha T^{(r-h+1)\alpha/p} < \infty, \qquad h = 0, 1, \dots, r. \tag{8}$$

Since $S_{0,T} \equiv 1$ and $(r+1)\alpha/p = (q(r+1))/(q+r) \le q$, (8) holds for $h = 0$. Thus, it suffices to verify (8) for $h = 1, 2, \dots, r$.

To this end, define

$$\theta_h = \theta_h(r) = \frac{pq}{pq - \alpha(r - h + 1)} \tag{9}$$

and note that since $\theta_h = (q + r)/(q + h - 1)$, necessarily $\theta_h > 1$, $1 \le h \le r$. Thus, by Hölder's inequality,

$$I_\alpha(h) \le E^{1/\theta_h}|S_{h,T}|^{\alpha\theta_h} \cdot E^{1-1/\theta_h}T^q$$

and so (8) is implied by

$$E(S^*_{h,T\wedge n})^{\alpha_r\theta_h} = O(1) \quad \text{as } n \to \ , h = 1, \dots, r. \tag{10}$$

However, $\alpha_r\theta_h = qp/(q + h - 1) = \alpha_{h-1}$, whence

$$E(S^*_{h,T\wedge n})^{\alpha_{h-1}} = O(1) \quad \text{as } n \to \infty, h = 1, \dots, r, \tag{11}$$

ensures (10). Proceeding inductively, (3) or (11) is implied by

$$E(S^*_{T\wedge n})^{\alpha_0} = O(1) \quad \text{as } n \to \infty.$$

Now $\alpha_0 = p$, $S_{1,n} = S_n$ and since $1 \le p \le 2$,

$$\begin{aligned} E(S^*_{T\wedge n})^p &\le O(1)E\left(\sum_{j=1}^n X_j^2 I_{[T\ge j]}\right)^{p/2} \\ &\le O(1)E\left(\sum_{j=1}^n |X_j|^p I_{[T\ge j]}\right) \le O(1)E|X|^p ET < \infty, \end{aligned} \tag{11'}$$

completing the proof. □

A perusal of Theorem 1 reveals that when $r \ge 1$, necessarily $\alpha_r < 2$, so that it does not encompass the case of a moment of order 2.

In [13], a Marcinkiewicz–Zygmund type strong law is obtained for $S_{k,n}$. Specifically,

$$S_{k,n}/n^{k/p} \to_{\text{a.c.}} 0$$

if $E|X|^p < \infty$ for some p in $(0, 2)$ provided $EX = 0$ whenever $1 \le p < 2$. Under the same hypothesis, convergence in mean of order p obtains.

THEOREM 4. *Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $E|X|^p < \infty$ for some p in $(0,2)$ with $EX = 0$ whenever $1 \leq p < 2$. Then for $k = 1, 2 \cdots$,*

$$\lim_{n \to \infty} E\left|\frac{S_{k,n}}{n^{k/p}}\right|^p = 0. \tag{12}$$

PROOF. For $k = 1$ this was proved by Pyke and Root [12]. When $k > 1$ and $1 \leq p < 2$,

$$E\frac{|S_{k,n}|^p}{n^k} \leq \frac{O(1)}{n^k} E\left(\sum_{j=k}^{n} X_j^2 S_{k-1,j-1}^2\right)^{p/2} \leq \frac{O(1)}{n} \sum_{j=k}^{n} \frac{E|S_{k-1,j-1}|^p}{j^{k-1}} = o(1)$$

as $n \to \infty$ and (12) follows inductively. For p in $(0,1)$ the argument is similar via $|S_{k,n}|^p \leq \sum_{j=r}^{n} |X_j|^p |S_{k-1,j-1}|^p$. □

THEOREM 5. *Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX = 0$, $EX^2 = 1$ and T a stopping time relative to $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$. Then for $k = 2$,*

$$ES_{k,T}^2 = E\sum_{j=k}^{T} S_{k-1,j-1}^2 < \infty \tag{13}$$

provided $T \in \mathscr{L}_\rho$, $X \in \mathscr{L}_{2\rho/(\rho-1)}$ for some $\rho \geq 2$. Moreover, (13) holds for $k = 3$ if $T \in \mathscr{L}_\rho$, $X \in \mathscr{L}_{2\rho\beta/(\rho-1)}$ for some $\rho \geq 3$, $\beta > 2$ where

$$\rho = \frac{(2\beta - 1)\left[2\beta - 1 + (4\beta^2 - 8\beta + 1)^{1/2}\right] - 2\beta}{(\beta - 1)\left[2\beta - 1 + (4\beta^2 - 8\beta - 1)^{1/2}\right] - 2\beta}. \tag{14}$$

REMARK. When $k = 2$, the most natural parameter choices are $\rho = 2(T \in \mathscr{L}_2, X \in \mathscr{L}_4)$ or $\rho = 3(T \in \mathscr{L}_3, X \in \mathscr{L}_3)$, whereas for $k = 3$, one might select $\rho = 3 + 7^{1/2}$, $\beta = 1 + 7^{1/2}/2$ (T and X are both elements of $\mathscr{L}_{3+7^{1/2}}$) or $\rho = 3$, $\beta = 10/3$ ($T \in \mathscr{L}_3$, $X \in \mathscr{L}_{10}$) or $\rho = 4$, $\beta = 21/8(T \in \mathscr{L}_4, X \in \mathscr{L}_7)$.

PROOF. Since the proof of (6) and (7) in Theorem 1 carry over to the case $1 < \alpha = p < 2$,

$$ES_{r,T\wedge n}^{*2} \leq O(1)J_{r-1,2} \leq O(1)\left[ET^r + ET^{r-1}S_T^2 + \cdots + ETS_{r-1,T}^2\right] \tag{15}$$

and so if the right side of (15) is finite, dominated and monotone convergence applied to (13) with T replaced by $T \wedge n$ yield the desired conclusion.

Now for $h < \rho$,

$$ET^h S_T^2 \leq E^{h/\rho} T^\rho \cdot E^{(\rho-h)/\rho} |S_T|^{2\rho/(p-h)} \tag{16}$$

and so when $\rho = 2$, setting $h = 1$, (13) follows.

When $\rho = 3$, choosing $h = 2$ in (16), $ET^2S_T^2 < \infty$ since $2\rho/(\rho - 2) \le 2\rho\beta/(\rho - 1)$, or equivalently, $\rho \ge (2\beta - 1)/(\beta - 1)$ in view of (14). Moreover,

$$ETS_{2,T}^2 \le E^{1/\rho}T^\rho \cdot E^{(\rho-1)/\rho}|S_{2,T}|^{2\rho/(\rho-1)}$$

and for $\beta > \rho(\rho - 1)$,

$$\begin{aligned}
&E|S_{2,T\wedge n}|^{2,\rho/(\rho-1)} \\
&\quad \le O(1)E\left(\sum_{j=2}^{T\wedge n} X_j^2 S_{j-1}^2\right)^{\rho/(\rho-1)} \\
&\quad \le O(1) \cdot E\left(\sum_2^T |X_j|^{2\beta/(\beta-1)}\right)^{\rho(\beta-1)/\beta(\rho-1)} \left(\sum_2^T |S_{j-1}|^{2\beta}\right)^{\rho/(\beta(\rho-1))} \\
&\quad \le 0(1) \cdot E^{(\beta\rho-\rho-\beta)/(\beta(\rho-1))}\left(\sum_2^T |X_j|^{2\beta/(\beta-1)}\right)^{\rho(\beta-1)/(\beta\rho-\rho-\beta)} \\
&\qquad \times E^{\rho/(\beta(\rho-1))}\left(\sum_2^T |S_{j-1}|^{2\beta}\right),
\end{aligned}$$

where (see, e.g., the proof of Lemma 9 of [5])

$$E\sum_2^T |S_{j-1}|^{2\beta} \le ET|S_T|^{2\beta} \le E^{1/\rho}T^\rho \cdot E^{(\rho-1)/\rho}|S_T|^{2\beta\rho/(\rho-1)} < \infty$$

and for $\gamma > (\rho(\beta - 1))/(\beta\rho - \rho - \beta)$,

$$\begin{aligned}
&E\left(\sum_1^T |X_j|^{2\beta/(\beta-1)}\right)^{(\rho(\beta-1))/(\beta\rho-\rho-\beta)} \\
&\quad \le E\left(\sum_1^T |X_j|^{2\beta\gamma/(\beta-1)}\right)^{(\rho(\beta-1))/(\gamma(\beta\rho-\rho-\beta))} T^{(\rho(\beta-1))/(\beta\rho-\rho-\beta)(\gamma-1)/\gamma} \qquad (17)\\
&\quad \le E^{(\rho(\beta-1))/(\gamma(\beta\rho-\rho-\beta))}\sum_1^T |X_j|^{2\beta\gamma/(\beta-1)} \\
&\qquad \times E^{(\gamma(\beta\rho-\rho-\beta)-\rho(\beta-1))/(\gamma(\beta\rho-\rho-\beta))}T^{(\rho(\gamma-1)(\beta-1))/(\gamma(\beta\rho-\beta-\rho)-\rho(\beta-1))}.
\end{aligned}$$

The last expectation on the right side will be finite provided $(\gamma - 1)(\beta - 1) = \gamma(\beta\rho - \rho - \beta) - \rho(\beta - 1)$, that is, if

$$\rho = 1 + \frac{\beta\gamma}{(\beta - 1)(\gamma - 1)}. \qquad (18)$$

Furthermore, via the hypothesis and Wald's equation, the first term on the right of (17) is finite when $2\beta\gamma/(\beta - 1) = 2\beta\rho/(\rho - 1)$, which, recalling (18), is tantamount to

$$\gamma = \frac{\beta - 1}{2\rho}\left[2\beta - 1 + \left(4\beta^2 - 8\beta + 1\right)^{1/2}\right],$$

and the latter, in conjunction with (18), implies (14). □

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